REMARKS ON TRANSFER
IN
WHITEHEAD THEORY

by

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INTRODUCTION: Suppose X is a Poincaré complex and $\tilde{X}$ is a regular covering space of X with $\pi_1(\tilde{X}) = \pi$ and a finite group of deck transformations G, and such that $\tilde{X}$ is homotopy equivalent to a compact manifold W. In [AV] it is shown that the obstruction to find a finite complex K homotopy equivalent to X and such that the induced covering $\tilde{K}$ be π-simple homotpy equivalent to W lies in an abelian group $Wh_1^T(\pi_1(\tilde{X}) \xrightarrow{\subset} \pi_1(X))$. The construction of $Wh_1^T(\pi_1(\tilde{X}) \xrightarrow{\subset} \pi_1(X))$ is given via a Gothendieck-type construction on certain categories of projective modules, which shows that Wh_1^T is a functor from pairs of groups to abelian groups. Moreover, Wh_1^T is closely related to the functors Wh_1 and $\tilde{K}_0$ via a five-term exact sequence involving transfer homomorphisms.

In the following we give an alternative construction and a generalization of Wh_1^T via the fibre of a (geometric) transfer in the context of higher simple homotopy theory of Hatcher [H]. This expands the domain of definition of Wh_1^T where one naturally expects topological applications in a wider variety of problems.

SECTION 1 THE ALGEBRAIC WHITEHEAD TRANSFER

Let $(E):1 \longrightarrow \pi \longrightarrow \Gamma \underset{u}{\overset{\longrightarrow}{\leftarrow}} G \longrightarrow 1$ be an extension of groups and let u be a section (not a homomorphism necessarily). Here G is a finite group of order q and π and Γ are such that $\mathbb{Z}\pi$ and $\mathbb{Z}\Gamma$ satisfy the usual property that any two bases for a free finitely generated module (over $\mathbb{Z}\pi$ or $\mathbb{Z}\Gamma$) have the same cardinality.

Let P_0 be the category whose objects consist of pairs $(M,\mathcal{B})$, where M is a finitely generated Γ-projective module which is free over π and $\mathcal{B}$ is a π-basis for M. Let $(M_1,\mathcal{B}_1) \sim (M_2,\mathcal{B}_2)$ if there exists a Γ-isomorphism $f:M_1 \longrightarrow M_2$ such that f is π-simple with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$. The set of equivalence classes $P = P_0/\sim$ has a monoid structure under direct sum of modules and disjoint union of bases, and $(0,\emptyset)$ is the neutral element. Let T be the submonoid generated by $(\mathbb{Z}\Gamma,u(G))$.

Then $Wh_1^T(\pi \longrightarrow \Gamma)$ is defined to be the quotient monoid P/T.

<u>1.1 PROPOSITION.</u> $Wh_1^T(\pi \longrightarrow \Gamma)$ is an abelian group.

Cf. [AV] Proposition 1.1.

The forgetful functor $(M,\mathcal{B}) \longmapsto M$ induces a homomorphism $\beta:Wh_1^T(\pi \longrightarrow \Gamma) \longrightarrow \tilde{K}_0(\mathbb{Z}\Gamma)$. On the other hand, given $\tau \in Wh_1(\pi)$, we define $\alpha(\tau)$ to be the equivalence class of $(M,\mathcal{B})$ where $M \cong (\mathbb{Z}\Gamma)^k$ and $\mathcal{B}$ is obtained from twisting the standard basis $u(G)^k$ by τ, i.e., $id:(M,u(G)^k) \longrightarrow (M,\mathcal{B})$ has π-torsion τ. It follows that the sequence $Wh_1(\pi) \xrightarrow{\alpha} Wh_1^T(\pi \longrightarrow \Gamma) \longrightarrow \widetilde{K}_0(\mathbb{Z}\Gamma)$ is exact. In [AV], this sequence is extended to a five term exact sequence involving the transfers in Wh_1 and $\widetilde{K}_0$ where $\Gamma = \pi \times G$.

<u>1.2 PROPOSITION.</u> The following sequence is exact.

$$Wh_1(\Gamma) \xrightarrow{Tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \to \Gamma) \xrightarrow{\alpha} \widetilde{K}_0(\mathbb{Z}\Gamma) \xrightarrow{tr} \widetilde{K}_0(\mathbb{Z}\pi).$$

(Cf. [AV] Proposition 1.2).

SECTION 2. HOMOTOPIFICATION OF A FUNCTOR.

Let $F: C_1 \longrightarrow C_2$ be a functor between two categories of topological spaces. "The homotopification of F" is the functor hF from the category of topological spaces to itself. It is constructed as follows (cf. [W] page 57). Let Δ_∞ be the standard (semisimplicial) category with morphisms consisting of boundaries and degeneracies ∂_i and s_i:

$$\Delta^0 \;\begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\end{array}\; \Delta^1 \;\begin{array}{c}\vdots\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \vdots\end{array}\; \Delta^2 \;\longrightarrow\; \cdots$$

Given a topological space X, we have the associated category $\Delta(X)$ where morphisms are $\partial_i \times id_X$, $s_i \times id_X$, etc.

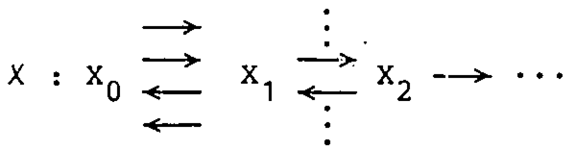
$$\Delta^0 \times X \;\begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\end{array}\; \Delta^1 \times X \;\begin{array}{c}\vdots\\ \longrightarrow\\ \longleftarrow\\ \vdots\end{array}\; \cdots$$

Applying the functor F we get the category $F\Delta(X)$:

$$F(\Delta^0 \times X) \;\begin{array}{c}\vdots\\ \longrightarrow\\ \longleftarrow\\ \vdots\end{array}\; F(\Delta^1 \times X) \;\begin{array}{c}\vdots\\ \longrightarrow\\ \longleftarrow\\ \vdots\end{array}\; \cdots$$

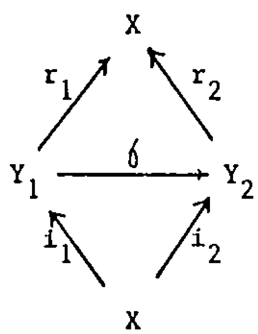
$hF(X)$ is defined to the classifying space of this category (Cf. [Q] section 1).

<u>2.1 REMARK.</u> In the semisimplicial set up



where X_i are topological spaces and the morphisms are homotopy equivalences, the classifying space of the category $\mathcal{X}$, $B\mathcal{X}$ is homotopy equivalent to each X_i.

For each topological space X, Hatcher has defined in [H] a category $Wh^{PL}(X)$ such that $\pi_0(BWh^{PL}(X)) \cong Wh_1\pi_1(X))$. We are interested in a modified version of Hatcher's construction which yields a functor $\mathcal{H}^{sp}$:top. spaces $\longrightarrow$ categories. We call $\mathcal{H}^{sp}(X)$ (respectively, $B\mathcal{H}^{sp}(X)$) the special Hatcher-Whitehead category (respectively, space) of X. The objects of $\mathcal{H}^{sp}(X)$ consist of $X \underset{i}{\overset{r}{\leftrightarrows}} Y$ where i is inclusion and r is a deformation retraction, (Y,X) is a relative finite CW complex, and all the maps are cell-like; that is, inverse images of points are contractible. A morphism f between $X \underset{i_1}{\overset{r_1}{\leftrightarrows}} Y_1$ and $X \underset{i_2}{\overset{r_2}{\leftrightarrows}} Y_2$ consists of a (strictly) commutative diagram (of cell-like maps)



In the combinatorial version of this category, we have simplicial complexes and simplicial maps such that inverse image of every simplex is contractible.

The classifying space of the special Hatcher-Whitehead category is not the Whitehead space of X and the functor $\Psi : B \longrightarrow B\mathcal{H}^{sp}(X)$ is not a homotopy functor. However, one has the following.

2.2 PORPOSITION. The homotopification $h\Psi$ of $\Psi : X \mapsto B\mathcal{H}^{sp}(X)$ is the functor $X \longmapsto Wh^{PL(X)}$,as defined by Hatcher, up to homotopy. (Cf. Hatcher [H] for the definition of $Wh^{PL}(X)$).

PROOF: Let $\mathcal{H}^{H}$ be the Hatcher's Whitehead category. Then the forgetful functor which forgets the retraction $r : Y \to X$ in $\mathcal{H}^{sp}(X)$, yields a functor $\Phi : \mathcal{H}^{sp}(X) \longrightarrow \mathcal{H}^{H}(X)$. This defines $\phi : B\mathcal{H}^{sp}(X) \longrightarrow Wh^{PL}(X)$ and one has the diagram

$$\begin{array}{ccc} B\mathcal{H}^{sp}(X) & \xrightarrow{\phi} & Wh^{PL}(X) \\ \downarrow & & \downarrow \\ hB\mathcal{H}^{sp}(X) & \xrightarrow{h\phi} & hWh^{PL}(X) \end{array}$$

where the bottom row is obtained by applying the homotopification functor to the upper row and the vertical lines are induced by "inclusions". The map α is a homtopy equivalence (see Remark 2.1), and $h\phi$ is a homotopy equivalence according to Waldhausen [W].

2.3 COROLLARY. ϕ factors through $hBH^{sp}(X)$.

SECTION 3 TRANSFER FOR COMPACT ANR FIBRATIONS.

We define a pretransfer for the functor H^{sp} in the category of compact ANR fibrations.

3.1 DEFINITION. Let C be a subcategory of the category of topological spaces. $(\eta;E \xrightarrow{\pi} B)$ is called a compact ANR fibration in C if $\pi:E \longrightarrow B$ is a morphism in C with the following properties: (i) π is proper (ii) $E \xrightarrow{\pi} B$ is a Hurewicz fibration (iii) all the fibres are compact ANR. (In the combinatorial categories we assume that the fibres are finite simplicial complexes.

3.2 EXAMPLES

1. Let K be a finite simplicial complex. Then the trivial fibration $B \times K \longrightarrow B$ is a compact ANR fibration.

2. Any fibre bundle with compact fibres, in particular finite covering spaces are compact ANR fibrations.

Given $Y \underset{i}{\overset{r}{\rightleftarrows}} B$ in $H^{sp}(B)$ representing an object χ, we define Pretr (χ) by the pull back $(r^*\eta;E' \xrightarrow{\pi} Y)$. Then one has the natural inclusion $i_E:E \longrightarrow E'$ and retraction $r':E' \longrightarrow E$ and retraction $r':E' \longrightarrow E$ covering

$r: Y \longrightarrow B$. Moreover, r' is cell-like since r is cell like and F is a compact ANR. In addition, (E',E) is a relative finite CW complex (simplicial complex if F is a finite simplicial complex and C is the combinatorial category). Thus we have defined a functor $Petr: H^{sp}(B) \longrightarrow H^{sp}(E)$. Applying the homotopification procedure to this functor, we obtain "h Petr" which we call "the transfer" and denote it by Tr (or $Tr(\eta)$ if emphasis on η is needed). By Proposition 2.2 we have defined $Tr: Wh^{PL}(B) \longrightarrow Wh^{PL}(E)$. Delooping Tr, we get $Tr: \mathbf{Wh}(B) \longrightarrow \mathbf{Wh}(E)$ where $\mathbf{Wh}$ is the delooping of Wh^{PL}. Let $\mathbf{Tr}(\eta)$ be the fibre of this natural transformation (see [Q] section 1). Thus we have the fibration $\mathbf{Tr}(\eta) \longrightarrow \mathbf{Wh}(B) \longrightarrow \mathbf{Wh}(E)$.

<u>3.3 DEFINITION</u> $\pi_0(\mathbf{Tr}(\eta)) = Wh_0(\pi_1 E \longrightarrow \pi_1 B)$.

Let E be the category whose objects are extensions

$$(e): 1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$ where G is a finite group, and G be the category of compact ANR fibrations. Let R be the functor which sends the extension (e) to the compact ANR fibration

$$(\eta) \quad G \longrightarrow B\pi \longrightarrow B\Gamma$$

Thus one has two functors naturally defined on E ; namely, $(e) \longmapsto Wh_1^T(\pi \subseteq \Gamma)$ and $(e) \longmapsto \pi_0(\mathbf{Tr}(R(e))) = Wh_0(\pi \rightarrow \Gamma)$. It turns out that these functors are equivalent.

<u>3.4 THEOREM.</u> Suppose $(\eta; E \longrightarrow B)$ is a compact ANR fibration such that B and E are connected and $\pi_1 E \longrightarrow \pi_1 B$ is injective. Then

$$\pi_0(\mathbf{Tr}(\eta)) \cong Wh_1^T(\pi_1 E \longrightarrow \pi_1 B).$$

<u>3.5 COROLLARY.</u> There exists a long exact sequence of Whitehead groups extending the five-term sequence of Proposition 1.2.

In fact the long exact sequence of homotopy groups of the fibration $\mathbf{Tr}(\eta) \longrightarrow \mathbf{Wh}(B) \longrightarrow \mathbf{Wh}(E)$ for the fibration $\eta : G \longrightarrow B\pi \longrightarrow B\Gamma$ corresponding to the extension (E) of Section 1 ends with the five-term sequence of Proposition 1.2.

<u>3.6 AN EXAMPLE.</u> Let π be a geometric group, i.e. the fundamental group of an aspherical manifold, and let $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow \pi' \longrightarrow 1$ be an extension of groups. Then the fibration $B\pi \longrightarrow B\Gamma \longrightarrow B\pi'$ has compact fibres. In many cases the obstructions for converting this fibration to a compact ANR fibration vanish, for instance if $\Gamma = \pi \times \pi'$. Thus one can define the transfer and consequently $Wh_0(\pi \longrightarrow \Gamma)$ in this situation. Since π' can be an infinite group, the algebraic method of [AV] to define $Wh_1^T(\pi \longrightarrow \Gamma)$ does not apply to this case.

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