KOBAYASHI GEODESICS IN $\mathcal{A}_g$

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ABSTRACT. We consider Kobayashi geodesics in the moduli space of abelian varieties A_g that is, algebraic curves that are totally geodesic submanifolds for the Kobayashi metric. We show that Kobayashi geodesics can be characterized as those curves whose logarithmic tangent bundle splits as a subbundle of the logarithmic tangent bundle of A_g.

Both Shimura curves and Teichmüller curves are examples of Kobayashi geodesics, but there are other examples. We show moreover that non-compact Kobayashi geodesics always map to the locus of real multiplication and that the $\mathbb{Q}$-irreducibility of the induced variation of Hodge structures implies that they are defined over a number field.

INTRODUCTION

Let Y be a non-singular projective curve defined over $\mathbb{C}$, let $Y_0 \subset Y$ be an open dense subscheme and let $\mathbb{W}_{\mathbb{Z}}$ be a polarized $\mathbb{Z}$-variation of Hodge structures on Y_0 of weight one. So $\mathbb{W}_{\mathbb{Z}}$ gives rise to a morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$, where $g = \frac{\mathrm{rk}(\mathbb{W})}{2}$.

It is our aim to characterize those φ that are totally geodesic submanifolds for the Kobayashi metric, in the sequel referred to as Kobayashi geodesics. Roughly speaking, the characterization will be in terms of the irreducible direct factors of $\mathbb{W} = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ or in terms of the splitting of the natural map

$$\varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \longrightarrow \Omega^1_Y(\log(Y \setminus Y_0)).$$

We now explain this characterization and the above notation in more detail and then why the main theorem is a unified treatment of the characterization of Shimura curves and Teichmüller curves - and beyond.

All the conditions we are interested in are preserved by étale coverings. Hence we will allow to replace Y_0 by an étale covering and $\mathbb{W}_{\mathbb{Z}}$ by its pullback. By abuse of notations $\mathcal{A}_g$ will denote a fine moduli scheme of polarized abelian varieties with a suitable level structure. $\overline{\mathcal{A}}_g$ will always denote a *Mumford compactification*, i.e. is one of the toroidal compactifications, constructed by Ash, Mumford, Rapoport and Tai and studied by Mumford in [Mu77]. We write $S_{\overline{\mathcal{A}}_g} = \overline{\mathcal{A}}_g \setminus \mathcal{A}_g$ for the boundary and $\varphi : Y \to \overline{\mathcal{A}}_g$ for the extension of $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ to Y.

One of the characterizations we are heading for uses the logarithmic Higgs bundle of a polarized complex variation of Hodge structures ([Si90], recalled in Section 1). Writing $(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta)$ for the logarithmic Higgs bundle of an irreducible direct factor $\mathbb{V}$ of $\mathbb{W}$, and we will say that (E, θ) (or $\mathbb{V}$) is *maximal Higgs* if $\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)$ is an isomorphism.

Then Theorem 1.3 states that $\varphi_0 : Y_0 \to \mathcal{A}_g$ is a Kobayashi geodesic if and only if *at least one* of the direct factors $\mathbb{V}$ of $\mathbb{W}$ is maximal Higgs.

The decomposition of $\mathbb{W}$ in irreducible direct factors is in fact defined over $\bar{\mathbb{Q}}$, hence induced by a decomposition of $\mathbb{W}_{\bar{\mathbb{Q}}} = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$. If one assumes in addition that the general fibre $f_0 : X_0 \to Y_0$ is simple, then for suitable elements

$$\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_\ell$$

of the Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ one can write

$$\mathbb{W} = \mathbb{V} \oplus \mathbb{V}^{\sigma_2} \oplus \cdots \oplus \mathbb{V}^{\sigma_\ell},$$

where $\mathbb{V}^{\sigma_i}$ denotes the conjugate of $\mathbb{V}$ under σ_i.

The condition "maximal Higgs" is not compatible with Galois conjugation. In particular the Higgs field of $\mathbb{V}^{\sigma_i}$ might be zero or equivalently $\mathbb{V}^{\sigma_i}$ might be unitary. By [VZ04] $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ is a Shimura curve, hence a totally geodesic submanifold for the Bergman-Siegel metric, if and only if *all* direct factors of $\mathbb{W}$ are either maximal Higgs or unitary.

Geodesics for the Kobayashi metric have been considered by the first named author in [Mö06] under the additional assumption, that $f_0 : X_0 \to Y_0$ is a family of Jacobians of a smooth family of curves. In this case $\varphi_0(Y_0)$ is a geodesic for the Kobayashi metric, if and only if the image of Y_0 in the moduli scheme M_g of curves of genus g with the right level structure is a geodesic for the Teichmüller metric, hence if and only if Y_0 is a Teichmüller curve. In particular Y_0 will be affine.

By [Mö06] Y_0 being a Teichmüller curve is equivalent to the existence of one direct factor $\mathbb{V}$ of $\mathbb{W}$ which is maximal Higgs. This implies that it is of rank two, and that its conjugates are neither unitary nor maximal Higgs. So $\mathbb{V}$ is the variation $\mathbb{L}$ of Hodge structures, defined in Section 1 starting from logarithmic theta characteristic. The variation $\mathbb{L}$ is defined over a totally real number field, and it looks like the uniformizing variation of Hodge structures on a modular curve, except that it has no $\mathbb{Q}$ structure. Adding the condition that the general fibre of f_0 is simple, the Teichmüller curve is determined by the Weil restriction of $\mathbb{L}$. In a quite sloppy way one could say that the variation of Hodge structures on a Teichmüller curve is like the one on "a modular curve without a $\mathbb{Q}$-structure".

The curves $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ satisfying the equivalent conditions of Theorem 1.3 include Teichmüller curves as well as Shimura curves, and again they could be seen as a generalization of Shimura curves, obtained by dropping the condition that the variations of Hodge structures are defined over $\mathbb{Q}$.

Examples of geodesics for the Kobayashi metric are given by curves Y_0 on Hilbert modular varieties U. By definition, the universal covering $\widetilde{U}$ is a product of disks Δ, and the property 'geodesic' just means that one of the projections $\widetilde{U} \to \Delta$ induces an isometry between the universal covering of Y_0 and Δ. In Section 5 we will show, that all affine geodesics for the Kobayashi metric are obtained in this way, again a result which is well known for Teichmüller curves.

In Section 6 we consider families of curves $\mathcal{Y}_0 \to U$ in $\mathcal{A}_g$, whose general fibre is a Kobayashi geodesic. Assuming that the corresponding variation of Hodge structures is irreducible over $\mathbb{Q}$, or equivalently that the general fibre of the induced family of abelian varieties is simple, we show that such families have to be locally analytically products $Y_0 \times U$, hence just the map to $\mathcal{A}_g$ might vary. In the affine case, the latter can not happen, which implies that affine Kobayashi geodesics are defined over number fields. In [Mö06] a similar result was stated for Teichmüller curves, but its proof, as pointed out by C. McMullen, was incomplete.

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1. A CHARACTERIZATION OF KOBAYASHI GEODESICS

As in the introduction $\mathcal{A}_g$ denotes a fine moduli scheme of polarized abelian varieties with a suitable level structures, $f_0 : X_0 \to Y_0$ is a family induced by a morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ and $\mathbb{W}_{\mathbb{Z}}$ is the induced polarized variation of Hodge structures. We denote the complement $Y \setminus Y_0$ by S.

If K is a subfield of $\mathbb{C}$ we write $\mathbb{W}_K = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ and usually $\mathbb{W} = \mathbb{W}_{\mathbb{C}}$. Replacing Y_0 by an étale covering, we may assume that the local monodromy operators around $s \in S$ are unipotent and in addition that $\deg(\Omega^1_Y(\log S))$ is even, hence that there exists a *logarithmic theta-characteristic*, i.e. an invertible sheaf $\mathcal{L}$ on Y with $\mathcal{L}^2 \cong \Omega^1_Y(\log S)$.

For a $\mathbb{C}$-subvariation of Hodge structures $\mathbb{V} \subset \mathbb{W}$ let $\mathcal{V}$ denote the Deligne extension of $\mathbb{V} \otimes \mathcal{O}_{Y_0}$ to Y and let (E, θ) be the induced logarithmic Higgs bundle, i.e. the graded bundle $E = E^{1,0} \oplus E^{0,1}$ with respect to the $\mathcal{F}$-filtration, together with the Higgs field $\theta : E \to E \otimes \Omega^1_Y(\log S)$ induced by the Gauss-Manin connection. So $\theta|_{E^{0,1}} = 0$ and $\theta(E^{1,0}) \subset E^{0,1} \otimes \Omega^1_Y(\log S)$ and we may consider θ as a morphism

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S).$$

Writing for a coherent sheaf $\mathcal{E}$ on Y

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})},$$

the Arakelov inequality, due to Faltings and Deligne (see [Fa83] and [De87]), says that

$$\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y(\log S)) = \deg(\Omega^1_Y(\log S)).$$

Lemma 1.1 ([VZ04])**.** *The following conditions are equivalent for a non-unitary irreducible subvariation $\mathbb{V}$ of Hodge structures in $\mathbb{W}$.*

   a. *(E, θ) (or $\mathbb{V}$) satisfies the* Arakelov equality, *i.e. $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$.*
   b. *(E, θ) is maximal Higgs.*
   c. *$E^{1,0}$ and $E^{0,1}$ are both stable, and θ is a morphism between polystable bundles of the same slope.*
   d. *There exists a rank two, weight one variation of Hodge structures $\mathbb{L}$ on Y_0 and an irreducible unitary local system $\mathbb{U}$ on Y_0, regarded as a variation of Hodge structures of bidegree $(0, 0)$ such that:*
       i. *$\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$.*
       ii. *The Higgs bundle of $\mathbb{L}$ is of the form*
       $$\left(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau : \mathcal{L} \xrightarrow{\cong} \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S) \right),$$
       where $\mathcal{L}$ is a logarithmic theta characteristic and τ the induced isomorphism.

As in the introduction $\overline{\mathcal{A}}_g$ denotes a Mumford compactification of $\mathcal{A}_g$. In particular $\overline{\mathcal{A}}_g$ is non-singular and the boundary $S_{\overline{\mathcal{A}}_g}$ a normal crossing divisor. Moreover, as shown in [Mu77, §3], the sheaf $\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ is nef. We will exclude isotrivial families, hence assume that $\varphi : Y \to \overline{\mathcal{A}}_g$ is finite.

Definition 1.2. Let M be a complex domain and W be a subdomain. W is a *totally geodesic submanifold for the Kobayashi metric* if the restriction of the Kobayashi metric on M to W coincides with the Kobayashi metric on W. If $W = \Delta$ we call Δ simply *a (complex) Kobayashi geodesic.*

We call a map $\varphi_0 : Y_0 \to A_g$ a *Kobayashi geodesic*, if its universal covering map

$$\tilde{\varphi}_0 : \widetilde{Y_0} \cong \Delta \to \mathbb{H}_g$$

is a Kobayashi geodesic. In particular a Kobayashi geodesic will allways be one-dimensional.

Theorem 1.3. *Keeping the notations and assumptions introduced above the following conditions are equivalent:*

   a. $\varphi_0 : Y_0 \to \mathcal{A}_g$ *is Kobayashi geodesic.*
   b. *The natural map* $\varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \to \Omega^1_Y(\log S)$ *splits.*
   c. $\mathbb{W}$ *contains a non-unitary irreducible subvariation of Hodge structures* $\mathbb{V}$ *which satisfies the Arakelov equality.*

Remark that by definition $\varphi_0 : Y_0 \to A_g$ is a Kobayashi geodesic if and only if $Y_0 \to \varphi_0(Y_0)$ is étale and $\varphi_0(Y_0) \subset \mathcal{A}_g$ a Kobayashi geodesic. Since the conditions b) and c) in Theorem 1.3 also are invariant under étale coverings we may allways assume that φ_0 is an embedding.

The proof that c) implies a) is quite easy, and it will be given in Section 3. There, assuming that $Y_0 \to \mathcal{A}_g$ is Kobayashi geodesic, we will also construct a candidate for the splitting in b) over Y_0. In Section 4 we will show that the splitting extends to a splitting of the sheaves of log-differential forms, finishing the proof that a) implies b).

We start in Section 2 with the algebraic part of this note, i.e. by showing that b) implies c). In fact, since it hardly requires any additional work, we will show directly that b) and c) are equivalent.

2. Slopes of Higgs bundles

Instead of studying the log differentials on Y_0 we consider the dual sheaves $T_Y(-\log S) = (\Omega^1_Y(\log S))^\vee$ and $T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) = (\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}))^\vee$ in this section, and the dual Higgs field

$$\theta' : E^{1,0} \otimes T_Y(-\log S) \longrightarrow E^{0,1}.$$

Recall that, writing (E', θ') for the logarithmic Higgs bundle of $\mathbb{W} = \mathbb{W}_Z \otimes_{\mathbb{Z}} \mathbb{C}$, the pullback of the logarithmic tangent sheaf of $(\overline{\mathcal{A}}_g, S_{\overline{\mathcal{A}}_g})$ to Y is given by

$$S^2(E'^{0,1}) \subset \mathcal{H}om(E'^{1,0}, E'^{0,1}),$$

where one uses the polarization to identify $E'^{0,1}$ and the dual of $E'^{1,0}$. We write $\mathbb{W} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_\ell$, where $\mathbb{V}_0$ is a unitary subsheaf of $\mathbb{W}$ and where $\mathbb{V}_1, \ldots, \mathbb{V}_\ell$ are irreducible non-unitary direct factors. The logarithmic Higgs bundle of $\mathbb{V}_j$ will be denoted by $(E_j = E_j^{1,0} \oplus E_j^{0,1}, \theta_j)$. The dual Higgs field on $\overline{\mathcal{A}}_g$ induces an

identification

$$\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) = S^2(E_0^{0,1} \oplus \cdots \oplus E_\ell^{0,1}) \cong \Big(\bigoplus_j S^2(E_j^{0,1})\Big) \oplus \Big(\bigoplus_{i<j} E_i^{0,1} \otimes E_j^{0,1}\Big)$$

$$(2.1) \qquad \subset \mathcal{H}om(E_0^{1,0} \oplus \cdots \oplus E_\ell^{1,0}, E_0^{0,1} \oplus \cdots \oplus E_\ell^{0,1}) \cong \bigoplus_{i,j} E_i^{1,0\vee} \otimes E_j^{0,1}.$$

Since $(E_j = E_j^{1,0} \oplus E_j^{0,1}, \theta_j)$ is a Higgs subbundle the composition

$$(2.2) \qquad T_Y(-\log S) \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \longrightarrow E_i^{1,0\vee} \otimes E_j^{0,1}$$

is zero, except for $i = j \neq 0$. The dual $E_j^{1,0\vee}$ is the complex conjugate of $E_j^{1,0}$ hence isomorphic to $E_\iota^{0,1}$ if $\mathbb{V}_\iota = \overline{\mathbb{V}}_j$. In this case the right hand side of (2.2) is identified with $E_\iota^{0,1} \otimes E_j^{0,1}$. If this holds for $\iota = j$, hence if $\mathbb{V}_j$ is defined over $\mathbb{R}$, the map in (2.2) factors through $S^2(E_j^{0,1})$.

In order not to be forced to distinguish different cases, we define for each of the non-unitary irreducible $\mathbb{C}$-subvariation of Hodge structures $\mathbb{V}_j$ in $\mathbb{W} = R^1 f_* \mathbb{C}_{X_0}$, with Higgs bundle (E_j, θ_j),

$$(2.3) \qquad \mathcal{T}_j = \begin{cases} S^2(E_j^{0,1}) & \text{if } \mathbb{V}_j \text{ is defined over } \mathbb{R} \\ E_j^{1,0\vee} \otimes E_j^{0,1} & \text{otherwise.} \end{cases}$$

The sheaf $\mathcal{T}_j$ is a direct factor of $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$, uniquely determined by $\mathbb{V}_j$. As we have seen above, the image of $T_Y(-\log S)$ lies in $\mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell$.

As in Theorem 1.3 c) we start with a non-unitary irreducible direct factors of $\mathbb{W}$, say $\mathbb{V}_1$. For simplicity we will drop the lower index $_1$ in the sequel. The Higgs bundle (E, θ) gives rise to a morphism

$$\eta = \eta_{(E,\theta)} : T_Y(-\log S) \longrightarrow \mathcal{T}$$

and since $\mathbb{V}$ is non-unitary, hence θ non-trivial, η is injective.

Lemma 2.1. *Let $\mathbb{V}$ be non-unitary and satisfying the Arakelov equality. Then for the direct factor $\mathcal{T}$ of $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$, induced by $\mathbb{V}$, the natural injection $T_Y(-\log S) \to \mathcal{T}$ splits.*

In particular the existence of such a direct factor $\mathbb{V}$ of $\mathbb{W}$ implies that

$$T_Y(-\log S) \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\mathcal{A}_g})$$

splits.

Proof. Using the notation from Lemma 1.1, d) one finds that $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$, with $\mathbb{U}$ unitary. So the logarithmic Higgs bundle of $\mathbb{U}$ is of the form $(\mathcal{U}, 0)$. The sheaf $T_Y(-\log S) \cong \mathcal{L}^{-2}$ is the subsheaf of

$$E^{1,0\vee} \otimes E^{0,1} \cong \mathcal{L}^{-1} \otimes \mathcal{U}^\vee \otimes \mathcal{L}^{-1} \otimes \mathcal{U},$$

induced by the homotheties in $\mathcal{H}om(\mathcal{U}, \mathcal{U})$. In particular one has a splitting of $\eta(T_Y(-\log S))$ as subsheaf of $E^{1,0\vee} \otimes E^{0,1}$, hence of $\mathcal{T}$. $\qquad\square$

As a next step we will show the converse of the first part of Lemma 2.1.

Proposition 2.2. *Assume that $\eta(T_Y(-\log S))$ is non-zero and a direct factor of $\mathcal{T}$. Then $\mathbb{V}$ satisfies the Arakelov equality.*

Proof. The sheaf $\mathcal{T}$ is a direct factor of $E^{1,0\vee} \otimes E^{0,1}$, hence by assumption the invertible sheaf $T_Y(-\log S) \cong \eta(T_Y(-\log S))$ as well.

By Lemma 1.1 the conclusion of Proposition 2.2 would imply the stability of the sheaves $E^{1,0}$ and $E^{0,1}$. The strategy of the proof of the proposition will be to verify first the semistability of both sheaves. This will give the semistability of $E^{1,0\vee} \otimes E^{0,1}$, which implies in turn that the slope of the direct factor $T_Y(-\log S)$ has to be $\mu(E^{0,1}) - \mu(E^{1,0})$.

Let $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\alpha$ be the Harder-Narasimhan (HN) filtration of $E^{1,0\vee} \otimes E^{0,1}$.

Claim 2.3. For some $\ell > 0$ the sheaf $T_Y(-\log S)$ maps to a direct factor of $\mathcal{G}_\ell/\mathcal{G}_{\ell-1}$.

Proof. Since $\mathcal{T}$ is a direct factor of $E^{1,0\vee} \otimes E^{0,1}$ we can write

$$E^{1,0\vee} \otimes E^{0,1} = \mathcal{G}' \oplus T_Y(-\log S)$$

for some $\mathcal{G}'$. Let $0 = \mathcal{G}'_0 \subset \mathcal{G}'_1 \subset \cdots \subset \mathcal{G}'_{\alpha'} = \mathcal{G}'$ be the HN-filtration of $\mathcal{G}'$.

If $\mu(T_Y(-\log S)) < \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$ define $\mathcal{G}''_\nu = \mathcal{G}'_\nu$.

If $\mu(T_Y(-\log S)) = \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$ choose $\mathcal{G}''_\beta = \mathcal{G}'_\beta \oplus T_Y(-\log S)$ for all $\beta \geq \nu$.

If $\mu(\mathcal{G}'_{\nu-1}/\mathcal{G}'_{\nu-2}) > \mu(T_Y(-\log S)) > \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$ choose $\mathcal{G}''_\beta = \mathcal{G}'_{\beta-1} \oplus T_Y(-\log S)$ for all $\beta \geq \nu$.

Obviously $\mathcal{G}''_\bullet$ is a second HN-filtration of $E^{1,0\vee} \otimes E^{0,1}$, and the uniqueness of the HN-filtration implies the claim. $\square$

Let $\mathcal{F}^{1,0}_\bullet$ and $\mathcal{F}^{0,1}_\bullet$ be the HN-filtrations of $E^{1,0}$ and $E^{0,1}$, and write

$$\mathfrak{gr}_{\mathcal{F}^{1,0}_\bullet} = \bigoplus H^{1,0}_\eta, \quad \mathfrak{gr}_{\mathcal{F}^{0,1}_\bullet} = \bigoplus H^{0,1}_\eta.$$

So $\mathcal{G}_\nu/\mathcal{G}_{\nu-1}$ is the direct sum of copies of $H^{1,0\vee}_{\eta_1} \otimes H^{0,1}_{\eta_2}$, all of the same slope. Since those sheaves are semi-stable, we find some pairs (η_1, η_2) such that $T_Y(-\log S)$ is a direct factor of $H^{1,0\vee}_{\eta_1} \otimes H^{0,1}_{\eta_2}$. In particular

(2.4) $$\mu(H^{0,1}_{\eta_2}) = \mu(T_Y(-\log S)) + \mu(H^{1,0}_{\eta_1}).$$

If one replaces $\mathbb{V}$ by its complex conjugate, the role of $H^{1,0\vee}_{\eta_1}$ and $H^{0,1}_{\eta_2}$ is interchanged. So we may assume that

(2.5) $$\mu(H^{0,1}_{\eta_2}) \geq \mu(H^{1,0\vee}_{\eta_1}) = -\mu(H^{1,0}_{\eta_1}),$$

for at least one pair (η_1, η_2) for which (2.4) holds.

Consider next the restriction of the Higgs field to $\mathcal{F}^{1,0}_{\eta_1}$ with image

$$\mathcal{B} \otimes \Omega^1_Y(\log S) \subset E^{0,1} \otimes \Omega^1_Y(\log S).$$

Writing $\mathcal{K}$ for the kernel of $\mathcal{F}^{1,0}_{\eta_1} \to \mathcal{B} \otimes \Omega^1_Y(\log S)$, one gets

$$\deg(\mathcal{B}) - \operatorname{rk}(\mathcal{B}) \cdot \mu(T_Y(-\log S)) = \deg(\mathcal{B} \otimes \Omega^1_Y(\log S)) = \deg(\mathcal{F}^{1,0}_{\eta_1}) - \deg(\mathcal{K})$$

Both, $(\mathcal{K}, 0)$ and $(\mathcal{F}^{1,0}_{\eta_1} \oplus \mathcal{B}, \theta|_{\mathcal{F}^{1,0}_{\eta_1}})$ are Higgs subbundles of (E, θ). So Simpson's correspondence [Si90] implies that $0 \geq \deg(\mathcal{K})$, or equivalently

(2.6) $$\deg(\mathcal{B}) + \deg(\mathcal{F}^{1,0}_{\eta_1}) \geq 2 \cdot \deg(\mathcal{F}^{1,0}_{\eta_1}) + \operatorname{rk}(\mathcal{B}) \cdot \mu(T_Y(-\log S)),$$

and that $0 \geq \deg(\mathcal{B}) + \deg(\mathcal{F}_{\eta_1}^{1,0})$. Since obviously $\mathrm{rk}(\mathcal{B}) \leq \mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0})$ and since $\mu(T_Y(-\log S)) < 0$, this together with the inequality (2.6) implies that

$$(2.7) \qquad 0 \geq \deg(\mathcal{B}) + \deg(\mathcal{F}_{\eta_1}^{1,0}) \geq \mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0}) \cdot (2\mu(\mathcal{F}_{\eta_1}^{1,0}) + \mu(T_Y(-\log S))).$$

By definition of the HN-filtration one has

$$(2.8) \qquad \deg(\mathcal{F}_{\eta_1}^{1,0}) \geq \mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0}) \cdot \mu(H_{\eta_1}^{1,0}) \quad \text{or} \quad \mu(\mathcal{F}_{\eta_1}^{1,0}) \geq \mu(H_{\eta_1}^{1,0}),$$

and since we assumed (2.5) one finds that $2\mu(\mathcal{F}_{\eta_1}^{1,0}) \geq \mu(H_{\eta_1}^{1,0}) - \mu(H_{\eta_2}^{0,1})$ or

$$(2.9) \quad \mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0}) \cdot (2\mu(\mathcal{F}_{\eta_1}^{1,0}) + \mu(T_Y(-\log S))) \geq$$
$$\mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0}) \cdot (\mu(H_{\eta_1}^{1,0}) - \mu(H_{\eta_2}^{0,1}) + \mu(T_Y(-\log S))).$$

By (2.4) the left hand side in (2.9) is zero, hence (2.7) and (2.9) must both be equalities. The first one implies that $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{F}_{\eta_1}^{1,0})$ and that $0 = \deg(\mathcal{B}) + \deg(\mathcal{F}_{\eta_1}^{1,0})$. So $\mathcal{K} = 0$ and since we assumed $\mathbb{V}$ to be irreducible, $\mathcal{F}_{\eta_1}^{1,0} = E^{1,0}$ and $\mathcal{B} = E^{0,1}$. In addition (2.8) has to be an equality, so $E^{0,1}$ and $E^{0,1} \cong E^{1,0} \otimes T_Y(-\log S)$ are both semistable.

As said already at the beginning of the proof, the semistability implies the Arakelov equality. In fact we could also refer to the equality in (2.4) since we know that

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(H_{\eta_1}^{1,0}) - \mu(H_{\eta_2}^{0,1}).$$

$\square$

Proof of "b) $\iff$ c)" in Theorem 1.3. By Lemma 2.1 it only remains to show that b) implies c). Recall that one has a factorization

$$T_Y(-\log S) \longrightarrow \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$$

and that the composition with the projection $\mathrm{pr} : \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \to E_j^{1,0^\vee} \otimes E_j^{0,1}$ is induced by the Higgs field of $\mathbb{V}_j$.

The splitting $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \to T_Y(-\log S)$ of $T_Y(-\log S)$ defines a splitting $\hat{\eta} : \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \to T_Y(-\log S)$, and the the composition

$$T_Y(-\log S) \xrightarrow{\subset} \mathcal{T}_j \xrightarrow{\subset} \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \xrightarrow{\hat{\eta}} T_Y(-\log S)$$

is either zero, or an isomorphism. So one finds some $j > 0$ for which $T_Y(-\log S)$ is a direct factor of $\mathcal{T}_j$ and, by Proposition 2.2, $\mathbb{V}_j$ satisfies the Arakelov equality. $\square$

3. Geodesics and multidiscs

We recall some facts needed for a characterization of Kobayashi geodesics in $\mathcal{A}_g$. The universal covering of $\mathcal{A}_g$ is isomorphic to the Siegel half space. Via the Cayley transformation it also has a realization as a bounded symmetric domain

$$\mathcal{D}_g = \{Z \in M^{g \times g}(\mathbb{C}) \mid I_g - ZZ^* > 0\} \subset \Delta^{g^2} \subset \mathbb{C}^{g^2}$$

that will be more convenient to work with in the next section when we discuss boundary components. The intersection of $\mathcal{D}_g$ with the diagonal in $\mathbb{C}^{g^2}$ is isomorphic to Δ^g, a totally geodesic submanifold for the Bergman metric on $\mathcal{D}_g$. Submanifolds of $\mathcal{D}_g$ that are isomorphic to Δ^r for some r and totally geodesic for the Bergman metric are called *multidiscs*. It is well known that $r \leq g$, and the multidiscs of maximal dimension, hence isomorphic to Δ^g, will be called *polydiscs*.

Note that the Cayley transformation maps diagonal matrices to diagonal matrices and that hence polydiscs are given by diagonal matrices in $\mathbb{H}_g$ as well.

The maximal compact subgroup K in the isometry group of $\mathcal{D}_g$ acts on the set of polydiscs and
$$\mathcal{D}_g = \bigcup_{k \in K} k(\Delta^g).$$
In particular for any $v \in T_{0,\mathcal{D}_g}$ there is a unique polydisc in $\mathcal{D}_g$ tangent to v.

The *Carathéodory metric* c_M on M is defined by
$$c_M(x, y) = \sup_{p:M \to \Delta} \rho(p(x), p(y)),$$
where ρ is the Poincaré metric on Δ. One shows using the Arzela-Ascoli theorem that this supremum is attained for some map p.

Lemma 3.1. *If $\tilde{\varphi}_0 : \Delta \to \mathcal{D}_g$ is a Kobayashi geodesic, then there is a holomorphic map $p : \mathcal{D}_g \to \Delta$ such that $p \circ \tilde{\varphi}_0 = \mathrm{id}$. Moreover, there is polydisc $\Delta^g \subset \mathcal{D}_g$ such that the image $\tilde{\varphi}_0(\Delta)$ lies in Δ^g. Furthermore, we may choose the polydisc, such that $p|_{\Delta^g} = \mathrm{pr}_1$.*

Conversely, if there is a map $p : \mathcal{D}_g \to \Delta$ such that $p \circ \tilde{\varphi}_0$ is an isometry for the Kobayashi metric on Δ, then $\tilde{\varphi}_0$ is a Kobayashi geodesic.

Proof. The converse implication is immediate from the distance-decreasing property of the Kobayashi metric.

We choose a base point $\mathcal{D}_g \ni 0 = \tilde{\varphi}_0(0)$ and we write $T_{0,\tilde{\varphi}_0(\Delta)} = \langle v \rangle$. Choose

(3.1) $\Delta^g \subset \mathcal{D}_g \xrightarrow{\ i\ } \Delta^{g^2}$

as above such that the polydisc Δ^g is tangent to v.

We claim that $\tilde{\varphi}_0(\Delta) \subset \Delta^g$. One easily shows from the definitions that for products of discs we have

(3.2) $c_{\Delta^g}(x, y) = d_{\Delta^g}(x, y) = \max_{i=1,\ldots,g} \rho(x_i, y_i).$

Together with the distance-decreasing property of the Kobayashi metric this implies that both inclusions in (3.1) are totally geodesic for the Kobayashi metric and that Kobayashi and Caratheodory metric coincide on these domains. Moreover, in a strictly convex bounded complex domain there is a unique complex Kobayashi geodesic with prescribed tangent vector at a given point (see e.g. [Le81] Theorem 2). Consequently, there is a Kobayashi geodesic in $\psi : \Delta \to \Delta^g$ with tangent vector v at zero, and the Kobayashi geodesic $\tilde{\varphi}_0$ in $\mathcal{D}_g$ we started with, with tangent vector v at zero. Since $\Delta^g \subset \mathcal{D}_g$ is a totally geodesic submanifold, the uniqueness implies that $\psi(\Delta) = \tilde{\varphi}_0(\Delta) \subset \mathcal{D}_g$. This proves the claim.

For a given point $x \in \Delta$ write $x_{ij} = (\mathrm{pr}_{ij} \circ i \circ \tilde{\varphi}_0)(x)$. Chose $0 \neq x \in \Delta$. By the claim and equation 3.2
$$\rho(0, x) = d_{\mathcal{D}_g}(\tilde{\varphi}_0(0), \tilde{\varphi}_0(x)) = d_{\Delta^g}(\tilde{\varphi}_0(0), \tilde{\varphi}_0(x)) = \rho(0, x_{ii})$$
for some i. Renumbering the factors to have $i = 1$, we conclude that the map $p = \mathrm{pr}_{11}|_{\mathbb{H}}$ realizes the supremum in the definition of the Caratheodory metric at the pair of points $(\tilde{\varphi}_0(0), \tilde{\varphi}_0(x))$. By [Ve81] Proposition 3.3 this map realizes the supremum at every pair of points. Consequently, $p \circ \tilde{\varphi}_0$ is an isometry for the Poincaré metric. Postcomposing by a Möbius transformation we may suppose that it is the identity.   □

Let us return to the situation considered in Section 1. So the map $\varphi_0 : Y_0 \to \mathcal{A}_g$ is induced by the variation of Hodge structures $\mathbb{W}_{\mathbb{Z}}$, and the image $\tilde{\varphi}_0(\widetilde{Y_0})$ of the universal covering map is isomorphic to a disc.

Proof of "c) $\Longrightarrow$ a)" in Theorem 1.3.
By Lemma 1.1 d.) we know that $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$. At some point $y \in Y_0$ fix a basis $\{a, b\}$ of the fiber $\mathbb{L}_y$, a basis $\{e_1, \ldots, e_s\}$ of the fiber $\mathbb{U}$. Write $a_i = a \otimes e_i$, $b_i = b \otimes e_i$, and choose a symplectic basis $\{a_{s+1}, b_{s+1}, \ldots, a_g, b_g\}$ of the orthogonal complement $\mathbb{V}^{\perp}$ in $\mathbb{W}$. We identify the universal cover of $\mathcal{A}_g$ in this proof with $\mathbb{H}_g$. There, the basis extends uniquely to a basis of sections of $R^1 \tilde{f}_* \mathbb{Q}_A$, where $\tilde{f} : A \to \mathbb{H}_g$ is the universal family of abelian varieties over $\mathbb{H}_g$.

As usual in the construction of a period matrix for an abelian variety, there is a unique basis $\{s_1, \ldots, s_g\}$ of sections of $\tilde{f}_* \Omega^1_{A/\mathbb{H}}$ such that

$$\int_{b_i} s_j(\tau) = \delta_{ij} \quad \text{for} \quad \tau \in \mathbb{H}_g, \quad i, j = 1, \ldots, g.$$

Then the matrix Z with entries

$$Z_{ij} = \int_{a_i} s_j(\tau) \quad \text{for} \quad \tau \in \mathbb{H}_g$$

is the period matrix for the fibre A_τ. The point is that $\mathrm{pr}_{11} : Z \mapsto z_{11}$ is the period mapping for $\mathbb{L}$. Consequently, $\mathrm{pr}_{11} \circ \tilde{\varphi}_0$ is an isometry and by Lemma 3.1 the map φ_0 is a Kobayashi geodesic. $\qquad\square$

Next we will use that Y_0 is a Kobayashi geodesic to construct a splitting of the map $\varphi_0^* \Omega^1_{\mathcal{A}_g} \to \Omega^1_{Y_0}$. Equivalently, writing $\pi : \widetilde{Y_0} \to Y_0$ for the universal covering and identifying $\widetilde{Y_0}$ with its image in $\mathcal{D}_g$ we will construct a $\pi_1(Y_0, *)$-invariant invertible subsheaf of $\Omega_{\mathcal{D}_g}|_{\widetilde{Y_0}} = \pi^* \Omega^1_{\mathcal{A}_g}$ which splits the restriction map $\Omega_{\mathcal{D}_g}|_{\widetilde{Y_0}} \to \Omega^1_{\widetilde{Y_0}}$.

The variation of Hodge structures $\mathbb{W}$ is given by a representation of $\pi(Y_0, *)$, hence by a homomorphism $\rho : \pi(Y_0, *) \to \mathrm{Sp}(2g, \mathbb{R})$. Its image lies in the subgroup

$$H = \{\sigma \in \mathrm{Sp}(2g, \mathbb{R}); \ \sigma(\widetilde{Y_0}) = \widetilde{Y_0}\}^0,$$

where ^{0} indicates that we are taking the connected component of the identity. We fix again a base point $\mathcal{D}_g \ni 0 = \tilde{\varphi}_0(0)$. If $\widetilde{Y_0}$ is Kobayashi geodesic, Lemma 3.1 provides us with a polydisc Δ^g containing $\widetilde{Y_0}$. We choose the parametrization such that $\widetilde{\varphi}_0(0)$ is the point $(0, \ldots, 0) \in \Delta^g$.

Lemma 3.2. *Let Δ^g be a polydisc containing $\widetilde{Y_0}$. Then there exists an H-invariant multidiscs $\Delta^r = \Delta^r \times \{0\}^{g-r} \subset \Delta^g$ with $\widetilde{Y_0} \subset \Delta^r$.*

Fixing an isometry $\Delta \cong \widetilde{Y_0}$ for some $1 \leq \ell \leq r$ the map $\Delta \cong \widetilde{Y_0} \to \Delta^r$ is given by the identity on the first ℓ factors, whereas the projection to the remaining $r - \ell$ factors is not an isometry. In particular the inclusion $\Delta^\ell \times \{0\}^{r-\ell} \to \Delta^r$ is invariant under H.

Proof. We start with the polydisc Δ^g containing $\widetilde{Y_0}$. Then

$$(0, \ldots, 0) \in \widetilde{Y_0} \subset h(\Delta^g)$$

for all $h \in H$. All the polydiscs $h(\Delta)$ for $h \in H$ are totally geodesic submanifolds for the Bergman metric. Consequently, the intersection $\bigcap_{h \in H} h(\Delta)$ is again totally

geodesic for the Bergman metric. Hence it is again a multidisc, of the form $\Delta^r \times \{0\}^{g-r}$ after permuting the coordinates suitably.

By Lemma 3.1, for at least one of the coordinate projections $\Delta^g \to \Delta$ the composition with $\widetilde{Y}_0$ is an isometry. Necessarily, at least of them is among the coordinate projections $\Delta^r \to \Delta$. There might be several projections with the isometry property, and we permute the coordinates in Δ^r such that the composition

$$\delta \cong \widetilde{Y}_0 \xrightarrow{\;\subset\;} \Delta^g \xrightarrow{\;\mathrm{pr}_i\;} \Delta$$

is an isometry for $i = 1, \cdots, \ell$ but not for $i > \ell$. As in Lemma 3.1 this composition is the restriction of a morphism $p_i : \mathcal{D}_g \to \Delta$. Again, postcomposing by a Möbius transformation we may suppose that p_i is the identity, for $i = 1, \ldots, \ell$, and hence

$$\widetilde{Y}_0 \longrightarrow \Delta^\ell \xrightarrow{\;\subset\;} \Delta^r \xrightarrow{\;\mathrm{pr}\;} \Delta^\ell$$

is the diagonal embedding of $\widetilde{Y}_0 \cong \Delta$. Since H acts on $\mathcal{D}_g$ by isometries, and since it leaves $\widetilde{Y}_0$ invariant, $\Delta^\ell \times \{0\}^{r-\ell}$ is again invariant under H. $\qquad\square$

Construction 3.3. We keep the notation from Lemma 3.2. Choose coordinates τ_{ij} on $\mathcal{D}_g$ and hence $\tau_{11}, \ldots, \tau_{\ell\ell}$ for the disc Δ^ℓ. Then $\Omega^1_{\Delta^\ell} = \langle d\tau_{11}, \ldots, dx_{\ell\ell} \rangle_{\mathcal{O}_{\Delta^\ell}}$ is a quotient sheaf of $\Omega^1_{\mathcal{D}_g}|_{\Delta^\ell}$. At the same time, the projections p_i allow to consider $\Omega^1_{\Delta^\ell}$ as a subsheaf of $\Omega^1_{\mathcal{D}_g}|_{\Delta^\ell}$.

On $\widetilde{Y}_0$ the diagonal subsheaf $\Omega' = \langle d\tau_{11} + \cdots + d\tau_{\ell\ell} \rangle_{\mathcal{O}_{\widetilde{Y}_0}}$ of $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0}$ satisfies:

1. Ω' is invariant under the action of H and hence $\pi(Y_0, *)$ on $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0}$.
2. The composition with the natural restriction map

$$\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0} \longrightarrow \Omega^1_{\widetilde{Y}_0}$$

is an isomorphism, compatible with the action of H and hence of $\pi(Y_0, *)$.

The first condition implies that Ω' descends to an invertible subsheaf Ω_0 of $\Omega_{\mathcal{A}_g}|_{Y_0}$. Together we obtain:

3. $\Omega' = \pi^* \Omega_0$ for an invertible subsheaf Ω_0 of $\varphi_0^* \Omega_{\mathcal{A}_g}$ and the composition

$$\Omega_0 \subset \varphi_0^* \Omega^1_{\mathcal{A}_g} \longrightarrow \Omega^1_{Y_0}$$

is an isomorphism.

This completes the proof of Theorem 1.3 in the case $Y = Y_0$.

4. Splitting of the log-differentials

We summarize what we need on toroidal compactifications of $\mathcal{A}_g$ as used in [Mu77], following the notation of loc. cit. whenever possible.

The degree g' of a point Z on the boundary of $\mathcal{D}_g$ is defined as the rank of $1 - ZZ^*$. Up to the action of $\mathrm{Sp}(2g, \mathbb{R})$ a point of degree g' is equivalent to a point in a standard boundary component

$$F_{g'} = \left\{ \begin{pmatrix} 1_{g-g'} & 0 \\ 0 & Z \end{pmatrix}, \quad Z \in \mathcal{D}_{g'} \right\} \subset \mathcal{D}_{g'}.$$

The sets $M \cdot F_{g'}$ for $0 \le g' < g$ and $M \in \mathrm{Sp}(2g, \mathbb{R})$ are called *boundary components*. A boundary component is called *rational* if $M \in \mathrm{Sp}(2g, \mathbb{Q})$, provided that the

identification of the universal covering $\widetilde{\mathcal{A}_g} \cong \mathcal{D}_g$ has been done using a rational basis of the (relative) homology. The stabilizer of $F_{g'}$ is the group
(4.1)

$$N(F_{g'}) = \{M \in \mathrm{Sp}(2g, \mathbb{R}), MF_{g'} = F_{g'}\} = \left\{ \begin{pmatrix} u & * & * & * \\ 0 & A' & * & B' \\ 0 & 0 & (u^t)^{-1} & 0 \\ 0 & C' & * & D' \end{pmatrix} \right\} \subset \mathrm{Sp}(2g, \mathbb{R})$$

We denote the unipotent radical of $N(F)$ by $W(F)$ and denote the center of $W(F)$ by $U(F)$. See [Na80] for explicit matrix realizations and examples.

A neighborhood of a boundary component F

$$D_F = \bigcup_{M \in U(F)_{\mathbb{C}}} M \cdot F$$

admits Siegel domain coordinates, given for a standard boundary component $F = F_{g'}$ by

$$D_{F_{g'}} \cong U(F_{g'})_{\mathbb{C}} \times (W(F_{g'})/U(F_{g'}))_{\mathbb{C}} \times F_{g'}$$

$$\begin{pmatrix} Z'' & Z''' \\ (Z''')^t & Z' \end{pmatrix} \mapsto (Z'', Z''', Z')$$

A toroidal compactification of $\mathcal{A}_g$ is constructed by gluing $\mathcal{A}_g$ with toroidal compactifications of the quotients of rational boundary components F by $U(F)$. We won't need the details.

Write x_{ij} $(i \leq j)$ for the coordinate functions of $U(F)_{\mathbb{C}} \cong M^{(g-g') \times (g-g')}(\mathbb{C})$, write y_{ij} for those of $(W(F)/U(F))_{\mathbb{C}} \cong M^{g' \times (g-g')}$ and write t_{ij} $(i \leq j)$ for the coordinate functions of $F_{g'} \subset M^{g' \times g'}(\mathbb{C})$. By [Mu77, Proposition 3.4] the pullback of sections of $\Omega^1_{\overline{\mathcal{A}_g}}(\log S_{\overline{\mathcal{A}_g}})$ to $\mathcal{D}_g$ in the neighborhood a boundary component of degree g' to are generated by $dx_{ij}, dy_{ij}, dt_{ij}$ as $\mathbb{C}[x_{ij}, y_{ij}, t_{ij}]$-module.

Proof of "a) $\implies$ b)" in Theorem 1.3.
It remains to show that the sheaf Ω_0 in the Construction 3.3 extends to a subsheaf $\Omega^1 \subset \varphi^*\Omega_{\overline{\mathcal{A}_g}}(\log S_{\overline{\mathcal{A}_g}})$, isomorphic to $\Omega^1_Y(\log S)$.

Fix a point $s \in S \subset Y$. The universal covering factors as

$$\widetilde{Y_0} \cong \Delta \to \Delta^* \to Y_0,$$

where the first map is $z \mapsto q = e^{2\pi i}$ and s corresponds to $q = 0 \in \Delta^*$. A local generator of $\Omega^1_Y(\log S)$ around s is thus given by dq/q or equivalently by dz near in a rational boundary point, say 1, of Δ.

We thus need to check that the local generator $d\tau_{11} + \ldots d\tau_{\ell\ell}$ of Ω' is in the pullback of $\Omega^1_{\overline{\mathcal{A}_g}}(\log S_{\overline{\mathcal{A}_g}})$ to $\mathcal{D}_g$ near $\varphi(s)$. This is a matter of understanding a base change.

Any two identifications of $\widetilde{\mathcal{A}_g} \cong \mathcal{D}_g$ differ by postcomposition with the action of some $M \in \mathrm{Sp}(2g, \mathbb{R})$. This applies in particular to the normalization of the multi-disc in Lemma 3.1 and the one where the $F_{g'}$ are rational boundary components.

Since $\tilde{\varphi}$ is the diagonal embedding onto the multidisc of dimension ℓ, the point $\varphi(s)$ lies on a boundary component of degree $g' \leq g - \ell$. More precisely, we may permute the remaining $g - \ell$ components of the polydisc and apply an element of

$\mathrm{Sl}(2,\mathbb{R})^{g-\ell-g'}$ to arrange the polydisc embedding $\widetilde{\varphi}$ on the middle $g-\ell-g'$ to have the convenient form

$$\widetilde{\varphi}(1) \;=\; \begin{pmatrix} 1_{g-g'} & 0 \\ 0 & Z(1) \end{pmatrix} \in \overline{D_g},$$

still with respect to the identification $\widetilde{\mathcal{A}}_g \cong \mathcal{D}_g$ of Lemma 3.1.

This implies that the base change $M \in N(F_{g'})$. Recall the action of a symplectic matrix on D_g is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = \frac{(A - IC)(Z+1) + (B - ID)I(Z-1)}{(A + IC)(Z+1) + (B + ID)I(Z-1)}$$

Using this, one quickly calculates, writing $M \in N(F_{g'})$ in blocks as in equation (4.1),

$$(dM^*) \begin{pmatrix} (d\tau_{ij}) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u \cdot (d\tau_{ij}) & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $(dM^*)(d\tau_{11} + \ldots d\tau_{\ell\ell})$ is a $\mathbb{C}$-linear combination of dx_{ij}, what we needed to show. $\qquad\square$

5. Affine geodesics for the kobayashi metric

In this section we will assume that Y_0 is affine, hence that $S \neq \emptyset$, and we will show that the image of Y_0 in $\mathcal{A}_g$ lies in a Hilbert modular surface U. If the general fibre of f_0 is simple, U will be indecomposable, in the sense that no finite étale cover can be a product.

Theorem 5.1. *Let Y_0 be an affine curve, and let $\varphi_0 : Y_0 \to \mathcal{A}_g$ be a geodesic for the Kobayashi metric, induced by a family $f_0 : X_0 \to Y_0$ of polarized abelian varieties. Assume that the induced variation of Hodge structures $\mathbb{W}_\mathbb{Q} = R^1 f_* \mathbb{Q}_{X_0}$ is irreducible, or equivalently that the general fibre of f_0 is simple.*

Then there exists a totally real number field $\sigma : K \subset \mathbb{C}$ of degree g over $\mathbb{Q}$ and a rank 2 variation of Hodge structures $\mathbb{L}_K$ of weight one, such that

(1) *$\mathbb{L} = \mathbb{L}_K \otimes_K \mathbb{C}$ is maximal Higgs. In particular the Higgs bundle of $\mathbb{L}$ is of the form*

$$\left(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau : \mathcal{L} \xrightarrow{\ \cong\ } \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S) \right),$$

where $\mathcal{L}$ is a logarithmic theta characteristic and τ the induced isomorphism.

(2) *If $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_g : K \to \mathbb{C}$ are the different embeddings of K in $\mathbb{C}$, and if $\mathbb{L}_i = \mathbb{L}_K^{\sigma_i}$ is the corresponding conjugate of $\mathbb{L}$, then*

$$\mathbb{W} = \mathbb{W}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g.$$

(3) *None of the $\mathbb{L}_i$ is unitary.*

Corollary 5.2. *In Theorem 5.1 there exists a g-dimensional Hilbert modular variety $Z_0 \subset \mathcal{A}_g$ such that $\varphi_0 : Y_0 \to \mathcal{A}_g$ factors through $Z_0 \subset \mathcal{A}_g$. The polydisk in Lemma 3.1 coincides with the minimal invariant multidisc in Lemma 3.2 and both are just the universal covering of Z.*

Proof of Theorem 5.1. Several of the arguments used in the proof are taken from [VZ04, Section 3 and 4].

By Theorem 1.3 and by Lemma 1.1 $\mathbb{W}$ contains a direct factor $\mathbb{V}$ which is of the form $\mathbb{L} \otimes \mathbb{U}$ with $\mathbb{U}$ unitary and with $\mathbb{L}$ maximal Higgs of rank two, hence induced by a logarithmic theta characteristic $\mathcal{L}$.

The local system $\mathbb{U}^{\otimes 2}$ is a subsystem of $\mathbb{W}^{\otimes 2}$. So on one hand, the local monodromy in $s \in S$ is unipotent, on the other hand it is unitary, of finite order. Then the local residues of $\mathbb{U}^{\otimes 2}$ and hence of $\mathbb{U}$ in $s \in S$ are trivial and $\mathbb{U}$ extends to a unitary local system on Y, again denoted by $\mathbb{U}$. Writing $\mathcal{U} = \mathbb{U} \otimes \mathcal{O}_Y$ the residues of $\mathbb{L} \otimes \mathbb{U}$ in $s \in S$ are given by

$$\mathcal{L} \otimes \mathcal{U} \xrightarrow{\ \tau\ } \mathcal{L}^{-1} \otimes \mathcal{U} \otimes \Omega_Y^1(\log S)) \xrightarrow{\ \text{res}\ } \mathcal{L}^{-1} \otimes \mathcal{U}|_s,$$

hence one finds:

Claim 5.3. The residues of $\mathbb{L} \otimes \mathbb{U}$ in $s \in S$ are isomorphisms $\mathcal{L} \otimes \mathcal{U}|_s \to \mathcal{L}^{-1} \otimes \mathcal{U}|_s$.

Since the rank two variations of Hodge structures, given by a theta characteristics, are unique up to the tensor product with local systems induced by two-division points, we can write

$$\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{T} \quad \text{with} \quad \mathbb{W}_1 = \mathbb{L} \otimes \mathbb{U} \oplus \mathbb{V}'$$

where $\mathbb{T}$ is the maximal unitary local subsystem, and where none of the direct factors of $\mathbb{V}'$ is maximal Higgs. By abuse of notations we write $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$, allowing $\mathbb{V}$ to be reducible.

Claim 5.4. The unitary part $\mathbb{T}$ is zero.

Proof. Assume the contrary. By [VZ04, Lemma 3.3] the first decomposition is defined over a number field. As above the residues of $\mathbb{T}$ in $s \in S$ are zero. This being invariant under Galois conjugation, the same holds true for all conjugates of $\mathbb{T}$ in $\mathbb{W}$. Since we assumed that $\mathbb{W}$ is irreducible over $\mathbb{Q}$, the same holds for $\mathbb{W}$, contradicting Claim 5.3. $\qquad\square$

Claim 5.5. The direct sum decomposition $\mathbb{W} = \mathbb{V} \oplus \mathbb{V}'$ is defined over a number field.

Proof. If not, as in the proof of [VZ04, Lemma 3.3], one obtains a non-trivial family $\mathbb{V}'_t$ of local systems over the disk Δ with $\mathbb{V}'_0 = \mathbb{V}'$. The non-triviality of the family forces the composition

$$\mathbb{V}'_t \to \mathbb{W} \longrightarrow \mathbb{L} \otimes \mathbb{U}$$

to be non-zero, for some t arbitrarily close to zero. The complete reducibility of local systems, coming from variations of Hodge structures, implies that some local system $\mathbb{V}''$ lies in both, $\mathbb{V}'_t$ and $\mathbb{L} \otimes \mathbb{U}$. Since for the second one the Higgs fields are isomorphism, the same holds true for $\mathbb{V}''$. On the other hand, for t sufficiently small, $\mathbb{V}'_t$ will not contain any local subsystem with isomorphisms as Higgs fields. $\qquad\square$

Since $\mathbb{V}$ is defined over a number field, we can apply [VZ04, Corollary 3.7] and we find that $\mathbb{L}$ and $\mathbb{U}$ can be defined over a number field, as well.

Consider next the sublocal system $\mathbb{V}^{\otimes 2}$ of $\mathbb{W}^{\otimes 2}$. Since $\mathbb{L}$ is self dual, one can identify $\mathbb{L}^{\otimes 2}$ with $\mathbb{End}(\mathbb{L})$, hence one obtains a trivial direct factor and its complement, denoted by $\mathbb{L}_0^{\otimes 2}$ in the sequel. So $\mathbb{V}^{\otimes 2}$ decomposes as a direct sum of $\mathbb{L}_0^{\otimes 2} \otimes \mathbb{U}^{\otimes 2}$ and $\mathbb{U}^{\otimes 2}$. For the first one the residues are again given by isomorphisms, whereas for the second one the residues are zero.

This property is preserved under Galois conjugation. Since for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ one has $\mathbb{V}^\sigma = \mathbb{L}^\sigma \otimes \mathbb{U}^\sigma$, and since the weight is zero, either $\mathbb{L}^\sigma$ or $\mathbb{U}^\sigma$ must be unitary. The first case can not occur, since the residues are isomorphisms. So the maximal unitary subbundle $\mathbb{T}'$ of $\mathbb{W}^{\otimes 2}$ is the Weil restriction of $\mathbb{U}^{\otimes 2}$. So it is defined over $\mathbb{Q}$ and inherits a $\mathbb{Z}$ structure from the one of $\mathbb{W}^{\otimes 2}$. This implies that $\mathbb{T}'$ trivializes after replacing Y_0 by an étale covering, and hence $\mathbb{U}^{\otimes 2}$ as well. In particular writing $\mathbb{U} = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_s$ for the decomposition of U in $\mathbb{C}$-irreducible direct factors, one finds

$$\mathbb{U}^{\otimes 2} = \bigoplus_{i,j} \mathbb{U}_i \otimes \mathbb{U}_j \cong \bigoplus \mathbb{C},$$

which is only possible if all $\mathbb{U}_j$ are isomorphic and of rank one, hence trivial.

Up to now we verified the existence of a direct sum decomposition of $\mathbb{W}$, satisfying the conditions (1), (2) and (3), over some number field L. For the next step, we will not need that Y_0 is affine.

Claim 5.6. Assume that over some curve Y_0 and for some number field L, there exists a direct sum decomposition, satisfying the conditions (1), (2) and (3) stated in Theorem 5.1, and assume that L is minimal with this property. Then $[L : \mathbb{Q}] = g$, and L is either totally real, or an imaginary quadratic extension of a totally real number field K.

Proof. Let L be the field of moduli of $\mathbb{L}_1$, i.e. L is the fixed field of all $\sigma \in \mathrm{Gal}_{\bar{\mathbb{Q}}/\mathbb{Q}}$ such that $\mathbb{L}_1^\sigma \cong \mathbb{L}_1$. Obviously one finds that $[L : \mathbb{Q}]$ is equal to the number of conjugates, hence equal to g.

Let K be the trace field of $\mathbb{L}_1$. Since $\mathbb{L}_1$ is isomorphic to a Fuchsian representation, K is real. It is well known (e.g. [Ta69]) that a rank two local system can be defined over an extension of degree at most two over its trace field. In particular, $[L : K] \leq 2$.

We claim that L injects into the endomorphism ring of the family f_0. In fact, let F be the Galois closure of $L/\mathbb{Q}$ and choose coset representatives τ_i of $\mathrm{Gal}(F/\mathbb{Q})/\mathrm{Gal}(F/L)$. Then for $x \in F$,

$$\sum_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \sigma(x) \cdot id_{\mathbb{L}^\sigma} = \sum_{\tau_i} \tau_i(\mathrm{tr}_L^F(x)) \cdot id_{\mathbb{L}^{\tau_i}}$$

is in $\mathrm{End}_{\mathbb{Q}}(\mathbb{W}_Q)$ of bidegree $(0,0)$, hence an endomorphism of f_0. Specializing to $x \in L$ proves the claim.

By Albert's classification ([BL04] §5.5) of endomorphism rings and since the general fibre of f_0 is simple, L is totally real or an imaginary quadratic extension of a totally real field K. $\qquad\square$

To finish the proof of Theorem 5.1 it remains to show that the minimal field L in Claim 5.6 has to be totally real, if Y_0 is affine. If $K = L$, there is nothing left to show.

We distinguish cases according to what the full endormorphism ring $W_{\mathbb{Q}}$ is, following notations in [BL04] §5.5. First, suppose the endromophism ring is of the second kind, i.e. the Rosati involution fixes L. Then since $[K : \mathbb{Q}] = \mathrm{rk}\mathbb{W}$ and since all the $\mathbb{L}_i$ have non-trivial Higgs field, the endomorphism ring is in fact an indefinite quaternion algebra B, containing L ([BL04] 9.10.Ex. (4), [Sh63] Proposition 18).

Second, suppose that the endomorphism ring is a totally definite quaternion algebra. Then (by [BL04] 9.10.Ex (1) or [Sh63] Proposition 15) the general fibre of f_0 is not simple.

It remains thus only to discuss the third case, a totally indefinite quaternion algebra B. Let Z be the ("PEL"-) Shimura variety of abelian varieties with $B \subset \text{End}(\mathbb{W}_{\mathbb{Q}})$. To construct Z, let G be the $\mathbb{Q}$-algebraic group with $G(\mathbb{Q}) = B^*$, fix a maximal compact subgroup K and form the quotient of G/K by an arithmetic lattice Γ. For the universal family A over $\Gamma \backslash G/K$ we have, by construction, a homomorphism $B \to \text{End}_Q(A)$, injective since B is simple. For dimension reasons we conclude that $Z = \Gamma \backslash G/K$ and that Z is compact by [Sh71] Proposition 9.3. Thus the map $Y_0 \to Z$ would extend to $Y \to Z$, contradicting the maximality of the Higgs field. $\qquad\square$

Remark 5.7. If in Claim 5.6 $L \neq K$, then Y_0 is projective. For $K = \mathbb{Q}$ examples are the moduli schemes of *false elliptic curves,* i.e. of polarized abelian surfaces Z with $\text{End}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ a totally indefinite quaternion algebra over $\mathbb{Q}$.

In a similar way there should exist compact *false Teichmüller curves* in $\mathcal{A}_g$. The corresponding variation of Hodge structures $\mathbb{W}_{\mathbb{Q}}$ should have a K irreducible direct factor $\mathbb{T}_K$ of rank 4, for some totally real number field K, and $\mathbb{T} = \mathbb{T}_K \otimes \mathbb{C}$ should be the direct sum of two complex variations of Hodge structures of rank 2.

6. Families of Kobayashi geodesics

Let k be an algebraically closed subfield of $\mathbb{C}$, let U be an irreducible non-singular variety, defined over k and let $h : \mathcal{Y} \to U$ be a smooth family of projective curves. We consider a relative normal crossing divisor $\mathcal{S} \in \mathcal{Y}$ and write $\mathcal{Y}_0 = \mathcal{Y} \setminus \mathcal{S}$.

We assume in the sequel that $\tilde{\varphi}_0 : \mathcal{Y}_0 \to \mathcal{A}_g$ is a generically finite morphism, defined over k, which extends to a morphism $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$. Here we consider $\overline{\mathcal{A}}_g$ as a k-variety. So $\tilde{\varphi}$ and h define a morphism

$$\varphi' : \mathcal{Y} \to \overline{\mathcal{A}}_{g\,U} := \overline{\mathcal{A}}_g \times U.$$

We want that all fibres of $h|_{\mathcal{Y}_0}$ are Kobayashi geodesics. However, since we do not want to study the behavior of this property in families, we use instead an extension of condition b) in Theorem 1.3 to the relative case, hence the natural morphism

$$(6.1) \qquad \Psi : \varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}),$$

where $S_{\overline{\mathcal{A}}_{g\,U}} = S_{\overline{\mathcal{A}}_g} \times U = \overline{\mathcal{A}}_{g\,U} \setminus \mathcal{A}_g \times U$.

Lemma 6.1. *We keep the notations introduced above and assume that the morphism Ψ in (6.1) splits, that $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$ is generically finite and that the general fibre of the pullback to $\mathcal{Y}$ of the universal family of abelian varieties is simple. Then*

    i. *The family $\mathcal{Y}_0 \to U$ is locally trivial, i.e. locally in the analytic topology a product.*

    ii. *If for $u \in U$ in general position the fibre $Y = h^{-1}(u)$ is affine, then U is a point.*

Proof. The splitting of the map Ψ in (6.1) implies that on all fibres Y of h one has a splitting of

$$(6.2) \qquad \varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \longrightarrow \Omega^1_Y(\log S).$$

So by Theorem 1.3 all fibres of h are Kobayashi geodesics. Let Ω denote a subsheaf of $\varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}}))$ for which

$$\Psi|_\Omega : \Omega \to \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S})$$

an isomorphism. Since

$$\varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) = \varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \oplus h^*\Omega^1_U$$

the subsheaf $\Omega \oplus \{0\}$ defines a splitting of the composition

$$\varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}).$$

This composition factors through

$$\varphi'^*\Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}}(\log S_{\mathcal{Y}}) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}).$$

Hence the exact sequence

(6.3) $$0 \to h^*\Omega^1_U \to \Omega^1_{\mathcal{Y}}(\log S_{\mathcal{Y}}) \to \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}) \to 0$$

splits as well. Recall that the Kodaira-Spencer map

$$\rho_u : T_{u,U} \longrightarrow H^1(Y, T_Y(-\log S)),$$

controlling the infinitesimal deformations of $Y \setminus S$ for $S = \mathcal{S}_u$ and $Y = h^{-1}(u)$, is given by edge morphism of the dual exact sequence of (6.3), restricted to $u \in U$ (see [Ka78], for example). So the splitting of (6.3) implies that ρ_u is zero, for all $u \in U$. By [Ka78, Corollary 4] this implies that the family $\mathcal{Y}_0 \to U$ is locally a product, as claimed in i).

From now on we will use the analytic topology, hence assume by abuse of notations that $\mathcal{Y} = Y_0 \times U$. Part ii) of the Lemma will follow, if we show that the morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ is rigid. Using the desription of the infinitesimal deformations of subvarieties of $\mathcal{A}_g$ in [Fa83], one has to verify that $\mathrm{End}(R^1 f_* \mathbb{Q}_X) = \mathrm{End}(\mathbb{W}_{\mathbb{Q}})$ is concentrated in bidegree $(0,0)$.

Remark under the assumption in ii) Theorem 5.1 allows to write

$$\mathbb{W} = \mathbb{L}_1^{\oplus \nu_1} \oplus \cdots \oplus \mathbb{L}_\ell^{\oplus \nu_\ell},$$

where $\mathbb{L}_1 = \mathbb{L}$ satisfies the Arakelov equality, where $\mathbb{L}_1, \ldots, \mathbb{L}_\ell$ are pairwise non-isomorphic, and where all the $\mathbb{L}_j$ are irreducible of rank 2. Then there are no global homomorphisms $\mathbb{L}_i \to \mathbb{L}_j$, except for $i = j$, and $\mathrm{End}(\mathbb{L}_j)^{0,0} = \mathrm{End}(\mathbb{L}_j) \cong \mathbb{C}$. Hence

$$\mathrm{End}(\mathbb{W}) = \bigoplus_{j=i}^{\ell} M(\nu_j, \mathrm{End}(\mathbb{L}_j)) = M(\nu_j, \mathbb{C}).$$

is equal to $\mathrm{End}(\mathbb{W})^{0,0}$. $\qquad\square$

Corollary 6.2. *Let $\varphi_0 : Y_0 \to \mathcal{A}_g$ be an affine Kobayashi geodesic, such that the induced variation of Hodge structures $\mathbb{W}_{\mathbb{Q}}$ is $\mathbb{Q}$-irreducible. Then $\varphi_0 : Y_0 \to \mathcal{A}_g$ can be defined over a number field.*

Proof. Let $\overline{\mathcal{A}}$ be a Mumford compactification, defined over $\overline{\mathbb{Q}}$. Recall that we assumed $\mathcal{A}_g$ to be a fine moduli scheme, and that Y_0 is a Kobayashi geodesic, if its image in $\mathcal{A}_g$ has this property. So we may assume that $\varphi_0 : Y_0 \to \mathcal{A}_g$ is injective.

The $\mathbb{C}$-morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ extends to a morphism $\varphi : Y \to \overline{\mathcal{A}}_g$, and by Theorem 1.3 there exists a subsheaf $\Omega \subset \varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ splitting the natural projection

$$\varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \to \Omega^1_Y(\log S).$$

Choose a field $K \subset \mathbb{C}$, finitely generated over $\bar{\mathbb{Q}}$ such that Y, S, $\varphi : Y \to \overline{\mathcal{A}}_g$, and Ω are defined over K. By abuse of notations we will use the same letters for the objects, as schemes, morphisms or sheaves over K.

Let U denote a non-singular quasi-projective $\bar{\mathbb{Q}}$-variety with $K = \bar{\mathbb{Q}}(U)$. Choosing U small enough, we can assume (step by step):

- $Y \to \mathrm{Spec}(K)$ is the general fibre of a projective morphism $h : \mathcal{Y} \to U$.
- $\varphi : Y \to \overline{\mathcal{A}}_g$ extends to a morphism $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$.
- h is smooth and $\mathcal{S} := \tilde{\varphi}^{-1}(S_{\overline{\mathcal{A}}_g})_{\mathrm{red}}$ consists of disjoint sections of h.
- $\Omega \subset \varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ is induced by a subsheaf $\tilde{\Omega} \subset \tilde{\varphi}^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$.
- $\tilde{\Omega}$ defines a splitting of the natural map

$$\tilde{\varphi}^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) = (\tilde{\varphi} \times h)^* \Omega^1_{\overline{\mathcal{A}}_g \times U/U}(\log(S_{\overline{\mathcal{A}}_g} \times U)) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}).$$

Writing $\mathcal{Y}_0 = \mathcal{Y} \setminus \mathcal{S}$ we claim that $\dim(\tilde{\varphi}(\mathcal{Y}_0)) = 1$. Otherwise we may replace U by the intersection of $\dim(\tilde{\varphi}(\mathcal{Y}_0)) - 2$ general hyperplane section, hence assume that besides of the properties stated above U is a curve and the morphism $\tilde{\varphi}$ generically finite, contradicting the Lemma 6.1.

If $\dim(\tilde{\varphi}(\mathcal{Y}_0)) = 1$ choose any $\bar{\mathbb{Q}}$-valued point in U. Then

$$\tilde{\varphi}(h^{-1}(u) \cap \mathcal{Y}_0) = \tilde{\varphi}(\mathcal{Y}_0)$$

and Y_0 is obtained by base extension from $h^{-1}(u) \cap \mathcal{Y}_0$. $\square$

Remark 6.3. As stated in Lemma 6.1 the proof of Corollary 6.2 is based on two observations. Firstly the family $\mathcal{Y}_0 \to U$ constructed in the proof of the Corollary is locally a product, and secondly one has rigidity of the embedding of a fixed curve as a Kobayashi geodesics.

In [Mö06] Corollary 6.2 was formulated for Teichmüller curves, but the first part of the argument was omitted.

Remark 6.4. Without a bit of classification of the possibe variations of Hodge structures we are not able to give a criterion for rigidity of compact Kobayashi geodesics. As for Shimura curves the existence of a large unitary subbundle in $\mathbb{W}$ implies non-rigidity. However there might be other criteria.

7. EXAMPLES AND COMMENTS

Let $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$ be the trace field of the uniformizing group $Y_0 = \mathbb{H}/\Gamma$. Moreover let $d = [K : \mathbb{Q}]$ and $B = K \cdot \Gamma$ be the subalgebra of $\mathrm{GL}_2(\mathbb{C})$ generated by Γ.

Among the curves, satisfying the equivalent conditions in Theorem 1.3, one finds:

i) Shimura curves ([VZ04]). In this case the $\mathbb{C}$ local system $\mathbb{W}$ decomposes as a direct sum $\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_\ell$ and each $\mathbb{V}_i$ is either unitary or maximal Higgs. The algebra B is unramified at all but one of the infinite places of K. The curve Y_0 can be both compact and non-compact in this case.

ii) Teichmüller curves. By [VZ06] and [Mö06] these are the Kobayashi geodesic

curves $Y_0 \subset \mathcal{A}_g$ that lie in the image of M_g. In this case $\ell = 1$ by [Mö06] and if $d > 1$ then $\mathbb{V}^\perp$ is not unitary. B is ramified at all the infinite places of K. The curve Y_0 is not compact in this case.

iii) There examples of compact Kobayashi geodesics Y_0 that are not Shimura curves. In fact, by [DM86] (see also [CW90]) all triangle groups $\Gamma = \Delta(l, m, n)$ arise as uniformizing groups of Kobayashi geodesic curves $Y_0 \subset \mathcal{A}_g$. If $l, m, n < \infty$ are chosen not to be in Takeuchis's finite list of arithmetic triangle groups ([Ta75]), then Y_0 is neither a Shimura curve nor a Teichmüller curve.

iv) Not all Kobayashi geodesics in $\mathcal{A}_g$ with $d > 1$, with Y_0 not compact and B ramified at all the infinite places of K are Teichmüller curves in M_g. In fact, the Prym construction of [McM06] gives Teichmüller curves in M_4, whose family of Jacobian splits. The Prym-antiinvariant part together with $1/2$ the pullback polarization gives such a Kobayashi geodesic in $\mathcal{A}_2$.

Given this list of examples, the following questions remain open:

1) Are there compact Kobayashi geodesics in $\mathcal{A}_g$, that are neither Shimura curves nor uniformized by a group commensurable to a triangle group?

2) Can one classify all non-Shimura Kobayshi geodesics on the image of a Hilbert modular surface in A_2 for a given discriminant?

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