Boundaries of π_1-Injective Surfaces

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ABSTRACT. According to a result of A. Hatcher, just finitely many boundary slopes (isotopy classes of simple closed curves) can be realized as boundaries of incompressible, ∂-incompressible surfaces in a closed, compact, orientable, irreducible 3-manifold with boundary a single torus. We consider, in this paper, proper maps of surfaces $(S, \partial S)$ into a 3-manifold $(M, \partial M)$ which are injective on π_1 and on relative π_1, and which are embeddings on ∂S. We show that there exists a 3-manifold M, with boundary a single torus, in which every boundary slope is realized by the boundary of such a map. We prove a result interpreting the significance of boundary slopes of such surfaces for Dehn filling. More generally, we consider maps of surfaces S which are injective on π_1 and on relative π_1 as before, and which embed each component of ∂S, but do not necessarily embed all of ∂S. We show that there exists a 3-manifold with boundary a single torus admitting such a map of a connected surface simultaneously realizing an arbitrary finite set of boundary slopes. We also give examples generalizing the preceding constructions to the case where ∂M is a surface of higher genus.

§1. Statement of Results.

We begin with definitions needed to state the results. We always assume M is an orientable, compact 3-manifold. If S is a compact surface, possible with boundary, we will say that a map $f: (S, \partial S) \to (M, \partial M)$ is π_1-*injective* if the induced map $\pi_1(S) \to \pi_1(M)$ is injective and if the induced map on $\pi_1(S, \partial S, p) \to \pi_1(M, \partial M, f(p))$ is injective for every choice of base point p in ∂S. Sometimes we shall refer to the injectivity of the relative fundamental group separately, for emphasis. We will say the map $f: (S, \partial S) \to (M, \partial M)$ is ∂-π_1-*injective* if the induced map on $\pi_1(S, \partial S, p) \to \pi_1(M, \partial M, f(p))$ is injective for every choice of base point p in ∂S. The maps of surfaces occurring in this paper will often be immersions; thus we will often refer to π_1-injective immersions.

Our first result is a strengthening of a result, proved independently and earlier by Mark Baker, showing that there is an example of a 3-manifold such that slopes p/q, $p \geq 1, q > p$, are realized as boundaries of π_1-injective immersed surfaces, see

[Ba]. This answers a question which has its origins in the result of Allen Hatcher, [H], which says that if M is an orientable, irreducible 3-manifold with boundary a single torus, then just finitely many isotopy classes of closed curves are realized as boundaries of (embedded) incompressible surfaces. Isotopy classes of closed curves in a torus are classified by "slopes" $r \in \mathbf{Q} \cup \{\infty\}$. Hatcher's result depended on the analysis in [FO] of incompressible surfaces, using branched surfaces.

Proposition 1. *There is a 3-manifold M, with boundary a single torus, such that for each $r \in \mathbf{Q} \cup \{\infty\}$ there exists a π_1-injective immersion $f_r: S_r \to M$ embedding every curve of ∂S_r in ∂M as a closed curve of slope r.*

Proposition 2. *Suppose V is an arbitrary finite subset of $\mathbf{Q} \cup \{\infty\}$. Then there is a 3-manifold M, with ∂M a single torus, and there exists an immersion $f: S \to M$ of a connected surface S, whose restriction to every boundary component is an embedding, such that for each $r \in V$, and only for $r \in V$, r is the slope of the embedding by f of some component of ∂S.*

Suppose now that M is a 3-manifold with boundary a connected surface of genus $g \geq 2$. We let $\mathcal{PL}(\partial M)$ denote the projective lamination space of ∂M. Points in $\mathcal{PL}(\partial M)$ represented by curve systems are called *rational points*. A variation of Hatcher's argument due to Floyd, [F], shows that points of projective lamination space $\mathcal{PL}(\partial M)$ realized as boundaries of embedded incompressible surfaces are restricted to a subcomplex of $\mathcal{PL}(\partial M)$, of dimension at most half of the dimension of the space. The following shows that there is no similar result for π_1-injective maps of surfaces.

Proposition 3. *For every $g \geq 2$, there is a 3-manifold M, whose boundary is a connected surface of genus g, having the property that for every rational point in $\mathcal{PL}(\partial M)$ there is a π_1-injective immersion $f: S \to M$ which is an embedding on ∂S with $f(\partial S)$ representing the point in $\mathcal{PL}(\partial M)$.*

Proposition 4. *For every $g \geq 2$ there is a 3-manifold M, with ∂M a connected surface of genus g, which allows π_1-injective immersions of surfaces as follows. Given any finite set of points $V = \{v_1, \ldots, v_n\}$ in $\mathcal{PL}(M)$, there is a connected surface S and a π_1-injective immersion $f: S \to M$, with the components of ∂S partitioned into n sets $C_1, \ldots, C_n$, such that f embeds each C_i in ∂M and $[f(C_i)] = v_i$ as a point in $\mathcal{PL}(\partial M)$.*

For completeness we shall explain the implications of the examples in this paper for Dehn surgery. For embedded incompressible surfaces, the relationship between incompressible, ∂-incompressible surfaces in the exterior of a knot K to the closed incompressible surfaces in a 3-manifold $M(r)$ or M_r obtained by doing Dehn surgery on a curve of slope r is reasonably well understood. Assuming all manifolds are irreducible, a closed incompressible surface in M_r yields either a closed incompressible surface in $M - \overset{\circ}{N}(K)$, or it yields an incompressible, ∂-incompressible surface in $M - \overset{\circ}{N}(K)$. The surface in $M - \overset{\circ}{N}(K)$ is obtained from an incompressible surface S in M_r by minimizing the number of meridian disc intersections of S with the the Dehn filling solid torus, then removing the solid torus. Whether an incompressible (∂-incompressible) surface in $M - \overset{\circ}{N}(K)$ yields an incompressible surface in M_r is a more delicate question, but there are many results addressing the question.

In attempting to understand the effect of Dehn filling on π_1-injective surfaces, one encounters a difficulty related to the simple loop conjecture for maps from surfaces to 3-manifolds. The conjecture says that given a map $f: S \to M$, where S is a closed orientable surface and M is a 3-manifold, and f does not induce an injection on π_1, then there exists a simple closed curve in S which is mapped to a homotopically trivial curve in M. Since the conjecture remains unproven, we must work our way around it. Given an orientable surface S, we shall say that a map $f: S \to M$ is *incompressible* if no essential simple loop in S is mapped to a homotopically trivial curve in M. If the simple loop conjecture is false, then this is a weaker notion than π_1-injectivity: A π_1-injective surface is incompressible, but an incompressible surface is not necessarily π_1-injective. If M has boundary, we say that the map $f: S \to M$ is *∂-incompressible* if no essential simple arc in S is mapped to an arc in M which is homotopic in M to an arc in ∂M.

Our definition of an "incompressible" map of a surface may not be generally accepted. Some authors apply the term "incompressible" loosely to π_1-injective maps of surfaces.

Proposition 5. *Suppose M is an orientable 3-manifold with k torus boundary components, and suppose $M(r_1, \ldots, r_k)$ is a the manifold obtained by performing Dehn filling of slope r_i on the i-th boundary torus. Suppose S is a closed orientable surface. Given a π_1-injective (or an incompressible) map $f: S \to M(r_1, \ldots, r_k)$, there is an incompressible, ∂-incompressible map $\hat{f}: (\hat{S}, \partial \hat{S}) \to (M, \partial M)$, where $\hat{S}$ is a surface obtained from S by removing some number of discs and where components of $\partial \hat{S}$ mapped to the i-th boundary component are curves of slope r_i.*

I thank Mark Baker, who first asked me the question which led to this paper, and I thank John Luecke, whose repetition of the question motivated me to find the answer. Finally, I thank Allen Hatcher, who pointed out some errors in an earlier draft and also suggested improvements. I am grateful for the support and hospitality of the Max Planck Institute during the writing of this paper.

§2. Proofs.

The method of construction of the examples in this paper depends on the use of branched surfaces. Branched manifolds were first defined by R. Williams, see [W], and were used to study embedded incompressible surfaces in [FO]. We begin by giving quick definitions of branched surfaces and related notions.

A closed *branched surface* B with generic branch locus is a space with smooth structure locally modelled on the space X shown in Figure 1a. (A neighborhood of each point of B is diffeomorphic to a neighborhood of an "interior" point in X.) In a branched surface with boundary, we also allow points modelled on neighborhoods of points on the "boundary" of the model X. The 1-dimensional "boundary" of the model X is a model, ignoring corners, for the 1-dimensional analogue of the branched surface, called a *train track*. The *branch locus K* of B is the union of points of B none of whose neighborhoods are manifolds or manifolds with boundary. The branch locus is a 1-complex. The *sectors Z_i* of B are the completions (in a path metric coming from a Riemannian metric on B) of the components of $B - K$. In order to deal with surfaces mapped into M, we shall use *generic immersions* $b: B \to M$ of branched surfaces into 3-manifolds. These immersions are locally
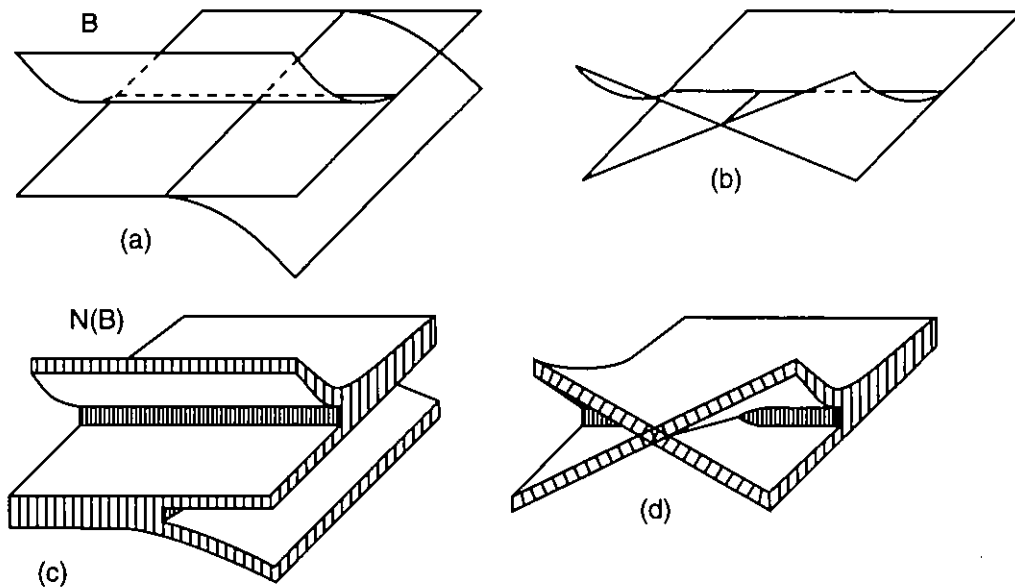
Figure 1

modelled on Figure 1b. In addition, although this behavior will not appear in this paper, a generic immersion should also be allowed to include triple points locally like triple points of immersions of surfaces, and also transverse intersections of a point in the interior of a sector with a 1-cell of the branch locus. For branched surfaces with boundary, generic immersions are immersions $b \colon (B, \partial B) \to (M, \partial M)$ locally modelled at ∂M on the obvious embedding of the model in Figure 1a in a cube. Transverse self-intersections of the boundary train track should also be allowed, as when the model of Figure 1b is embedded in a cube. We shall always assume in this paper that immersions are generic.

If a surface S and a branched surface B are given smooth structures, a map $g \colon S \to B$ is a *carrying map* if it is an immersion, and we say S is *carried* by B. If $b \colon B \to M$ is a generic immersion, we say the surface map $b \circ g \colon S \to M$ is *carried* by b. Given a carrying map of a surface, $g \colon S \to B$, there is an induced invariant weight vector for B. This is a vector of integer weights w_i, one for each sector Z_i, with w_i equal to the number of preimages in S under the map g of a point in $\overset{\circ}{Z}_i$. The surface map is *fully carried* if all the induced weights are positive. An induced invariant weight vector satisfies certain obvious *branch equations* of the form $w_1 + w_2 = w_3$, one equation for each 1-cell of the branch locus, and in fact an *invariant weight vector* is defined to be a vector of weights satisfying these equations.

As in the theory of embedded branched surfaces, it is convenient to use a neighborhood $N(b(B))$ to keep track of surfaces carried by b, see Figure 1c,d. The neighborhood is foliated by interval fibers, but fibers intersect where B has self-intersections, as shown in Figure 1d. Corresponding to the locus of self-intersection of B, we have a portion of $N(b(B))$ with two independent foliations by interval fibers. A surface S carried by B can be immersed in $N(b(B))$ transverse to fibers. Where there is more than one foliation by intervals, it is clear which fibers are meant. A triple point of the immersion of B corresponds to a solid box in $N(b(B))$ with 3 independent 1-foliations. A map $f \colon S \to N(b(B))$ transverse to fibers as

described above is also called a *carrying map*.

We denote by $\Sigma(b)$ the *singular locus* of b which is defined to be the union of the image of the branch locus of B and the locus of self-intersection of B in M when mapped by b. Completions of components of the complement in B of $\Sigma(b)$ are called *subsectors*. In this paper, $\overset{\circ}{N}(b(B))$ will always mean a relative version of the interior, i.e it is the interior of $N(b(B))$ union the interior of $N(b(B)) \cap \partial M$. The same applies to other regular neighborhoods. We use the notation $\partial_f N(b(B))$ to denote the frontier of the neighborhood, i.e. the closure of $\partial N(b(B)) - \partial M$.

It is fairly obvious that for a branched surface B embedded in a 3-manifold, invariant weight vectors on B with integer entries are in one-one correspondence with isotopy classes of surfaces embedded in M and carried by B, see [FO]. This is not the case for immersed branched surfaces carrying immersed surfaces. It is possible to construct examples of immersions of branched surfaces $b\colon B \to M$ with invariant weights not induced by a carrying map. Further, if an invariant weight vector on a branched surface B immersed as $b\colon B \to M$ is induced by a carrying map $g\colon S \to B$, then the homotopy class of the corresponding map $f = b \circ g$ does not depend uniquely on the invariant weight vector. This means that invariant weights on immersed branched surfaces are less useful for describing maps of surfaces into 3-manifolds than for describing embeddings. However, we shall see that they are nevertheless useful.

The maps of surfaces appearing in this paper are usually immersions. In fact, any π_1-injective map of a orientable surface into a 3-manifold can be replaced by an immersion whose induced map on π_1 is the same. This follows from the proof of Proposition 5.

Example 1. Let us begin with the problem of constructing an infinite family of maps of surfaces into a solid torus M, each being an embedding on the boundary, and each realizing a different boundary slope. The technical problem of making surfaces π_1-injective will be ignored until later. The immersed branched surface B in Figure 2 is a natural candidate for carrying surfaces with many different boundary slopes; the branched surface is constructed by attaching to an annulus a rectangle with a half twist. This is an immersed branched surface, so strictly we should work in terms of an immersion $b\colon B \to M$, but we shall occasionally abuse notation by confusing B and its image $b(B)$. Assigning non-negative weights p, v, and $q = p + v$ to the rectangular sectors P, V, Q respectively, the weights on the embedded boundary train track determine embedded curves of slope $p/2q$, where $q = p + v$. This gives all rational slopes in the interval $[0, 1/2]$.

Since, as we mentioned before, our assignment of weights is certainly not sufficient to determine immersed surfaces carried by B we must describe a carrying map explicitly, or we must describe the map of a surface to the 3-manifold. We imagine the image of the surface in $N(B)$ transverse to fibers, immersed, and in general position. Notice that in our example, along the arc ξ of self-intersection of B, a surface S fully carried by B *must* intersect itself, and the intersections must extend to "tangential" intersections in the sector Q. We shall describe the carrying map for S by describing self-intersections of the image of the surface in the manifold. We require maps of surfaces which embed the boundary of the surface, so we cannot extend the arcs of self intersection to ∂M. One might be tempted to extend the

arcs of self-intersection along the center of the sector Q of B, but one easily checks that this is only possible if $p = v$, or trivially when $v = 0$ or $p = 0$.
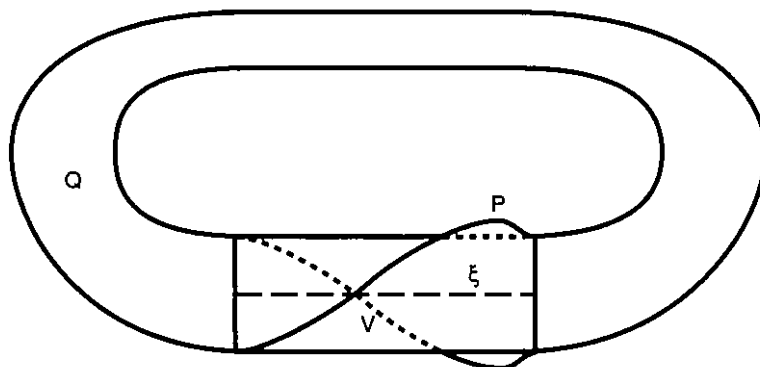

Figure 2

Thus the example must be further modified. We observe that if we remove a neighborhood in B of the arc ξ, then the weights on the remainder of the branched surface determine an embedded surface. The immersed surface we construct will coincide with this embedded surface outside the neighborhood of ξ. We shall attach a tube T to the given branched surface B, to obtain the branched surface in Figure 3, so that we can divert the curves of self intersection over the tube. Notice that the tube can be embedded in the solid torus without adding new self-intersections of B.
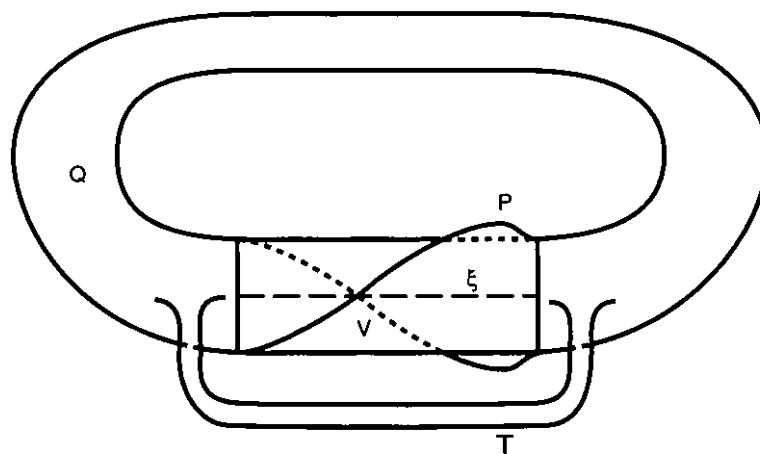

Figure 3

Now we can describe suitable surfaces carried by the branched surface B of Figure 3. Begin with the branched surface B of Figure 2 and punch two holes near the ends of ξ. We lead all arcs of intersection forced by the self-intersection of B at ξ to the punched holes. It is quite clear that the weights p, q, v on the branched surface of Figure 2 with two punched holes are indeed induced by a surface S carried by the punctured branched surface. We see on the annuli in $\partial N(b(B))$ corresponding to the two punched holes, a pattern of curves as shown in Figure 4, where $p = 3, v = 2$. These curves are images of boundary curves of the surface carried. We would like

to attach the tube T to the branched surface at the two punched holes, and extend the immersed surface so it is carried by the new branched surface. If the boundary patterns (shown in Figure 4) in the two boundary annuli of $N(B)$, corresponding to boundaries of punched holes, were mirror images of each other, one could simply glue the two annuli, and the boundary curves of the surface S carried would match. Although the patterns do not match, we can extend the surface S through a product $A^2 \times I$, where A^2 denotes an annulus, because the two mapped-in curve systems at the ends of the product are homotopic via a homotopy through curves transverse to fibers. This is because there is a correspondence between simple closed curves of the curve systems in the two annuli such that corresponding curves wrap around the annuli the same number of times. The homotopy gives a one-parameter family of curve systems C_t which define S in the product, such that $S \cap (A^2 \times t) = C_t$ is always transverse to fibers of the neighborhood. Thus we now have a surface S carried by the branched surface in Figure 3. It induces the weights $p, v, q = p + v$ on the sectors P, V, Q respectively, where now the sector Q includes the tube T. Thus the branched surface carries immersed surfaces realizing all boundary slopes $p/2q$ satisfying $q \geq p, p \geq 0$, i.e. all slopes in $[0, 1/2]$.
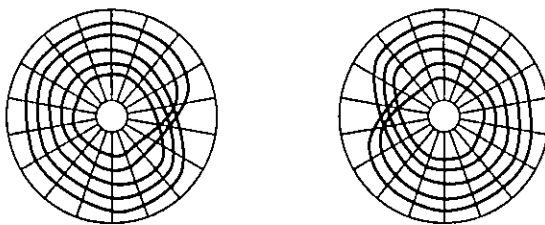

Figure 4

The surfaces S carried by B are far from π_1-injective. Our next task is to modify the manifold M, currently a solid torus, to guarantee that all surfaces carried by B are π_1-injective. The modification can be effected by replacing each component of the complement of an open regular neighborhood $\overset{\circ}{N}[(b(B)) \cup \partial M]$ by another manifold with incompressible boundary homeomorphic to the boundary of the replaced manifold. For technical reasons, before modifying the complementary manifolds, we shall arrange that all subsectors have non-trivial topology by locally adding a small compressible surface handle to each disc subsector so that the sector is replaced by a punctured torus. In fact, one can add as many handles as one wishes to any sector of the branched surface, before replacing complementary pieces. We apply the following proposition to prove that all surfaces carried by B are π_1-injective in the modified M.   ◇

Proposition 7, below, guarantees π_1-injectivity of surfaces carried by branched surfaces similar to the one constructed in Example 1. We remark that the proposition is certainly inadequate for dealing with arbitrary π_1-injective surfaces. A complete theory of π_1-injective maps of surfaces using branched surfaces would need to be much more subtle. Before proving the proposition, we state an elementary fact about train tracks.

Lemma 6. *Suppose τ is a train track embedded in a closed surface F of genus $g \geq 1$ without complementary 0-gons or monogons. Then any embedded curve system carried by τ is essential, i.e., no simple closed curve of the system bounds a disc in F.*

Proposition 7. *Suppose $b : (B, \partial B) \to (M, \partial M)$ is an immersion of a branched surface modelled on Figure 3, but also possibly with triple points. Suppose no subsector is a disc. Suppose ∂B is a train track embedded in ∂M without complementary monogons or 0-gons. Finally suppose that the frontier surface of a regular neighborhood $N(b(B) \cup \partial M)$ is incompressible in $M - \overset{\circ}{N}(b(B) \cup \partial M)$. Then any carrying map $f : S \to N(b(B))$ is π_1-injective.*

Proof. We shall use N to denote a regular neighborhood $N(b(B) \cup \partial M)$. The closed neighborhood N is the union of C and of $N(b(B))$, where C is a collar of ∂M, $C = F \times I$, say, and where $I = [0, 1]$ and $F \times 1 = \partial M$. We use $\partial_f N$ to denote the frontier of N, i.e. $\partial_f N = \partial N - \partial M$.

We will construct a kind of handle-decomposition of N corresponding to the branched surface B. To begin with, $\Sigma(b)$ is already a 1-complex, after adding 0-cells to circle components. Its regular neighborhood can be given a handle structure using a 0-handle for each 0-cell and a 1-handle for each 1-cell. In addition, we will treat a collar neighborhood C of ∂M as a kind of 0-handle, corresponding to all 0-handles of $\Sigma(b) \cap \partial M$. Thus some 1-handle attachments are made to this collar. Finally, for every component of $B - \Sigma(b)$ we add a product $Z \times I$ which will be treated like a 2-handle. Handle attachements are almost of the same type as though all handles were genuine. For example, 1-handles are attached to 0-handles on round discs which we will call spots, 2-handles touch 1-handles and 0-handles on rectangles which we call strips. Exceptionally, a 2-handle may intersect the 0-handle C in an annulus which we call an annular strip, if ∂B has a component which is a circle. On the boundary of each 0-handle, 1-handle, or C, the complementary components of spots and strips have irregular shapes, and we refer to them as fields. For our unconventional handles, these fields may not be discs. Some fields are actually rectangular in shape. For example, all fields in the boundary of a 1-handle are rectangular, but they are still called fields, not strips. In what follows, the term "handle" will include all unconventional handles.

It will be convenient to divide N into two pieces. First, we have the union of the 2-handles, which we call Q. The complement is a product with handles, which we call P. These are separated by a collection of annuli whose union we denote by A. The 2-handles are attached to P along these annuli. We begin by proving an elementary claim about the topology of (P, A).

Claim 1: a) A is incompressible in P, and b) $\partial P - \overset{\circ}{A}$ is incompressible and ∂-incompressible in (P, A).

This depends on the fact that at least two 2-handles are attached to each 1-handle. It also depends on the fact that the boundary train track has no complementary monogons or 0-gons. To use the latter fact, notice that $\text{cl}(N - C)$ intersects C in a neighborhood of the train track, projected from $F \times 1 = \partial M$ to $F \times 0$, which is a union of spots, strips and annuli where handles are attached. We use the following properties of the train track $\partial B \subset \partial M = F = F \times 1$: If E

is a disc embedded in F with $\overset{\circ}{E} \cap \partial B \neq \emptyset$, then ∂E intersects ∂B at least twice. If this is not true, then ∂B must have a complementary 0-gon or monogon. All the proofs of statements in Claim 1 are done by considering a compressing disc or boundary compressing disc in P and simplifying intersections with frontier discs between handles. These are the spots mentioned above; we denote the union by R.

Suppose D is a compressing disc for A in P. Isotope D so it is transverse to R. Consider an innermost closed curve of intersection bounding a disc H in D. If H is contained in a 1-handle, we can easily eliminate the curve by isotopy. If H is in the collar C, we can isotope H out of C, since the product is irreducible. Now consider an innermost arc of intersection in D, cutting off a half-disc H. If H is in a 0-handle or 1-handle, ∂H is contained in the union of a spot and an adjacent strip, so H can be pushed out of the handle by isotopy. If H is in C, a priori it is possible that ∂H runs over a strip longitudinally, describing a non-trivial path in the pattern of spots and strips in $F \times 0$. However, this would yield a closed curve bounding a disc and intersecting the boundary train track just once, since the projection of H provides a null-homotopy for ∂H in $F \times 0$. After finitely many moves, we have D in a single handle or in C, with its boundary in a strip, or possibly in an annular strip. If its boundary is in a strip, we have proved incompressibility. If its boundary is essential in an annular strip in $F \times 0$, then projecting H to $F \times 0$ shows that ∂H bounds a disc in $F \times 0$, and we again obtain a disc in ∂M whose boundary does not intersect the train track but whose interior intersects the train track, a contradiction. This completes the proof of the incompressibility of A in P.

Now suppose D is a compressing disc for $\partial P - \overset{\circ}{A}$ in P, transverse to R. We remove innermost closed curves of intersection with R exactly as above. If an innermost arc cuts off a half-disc H, H lies in a 0- handle, 1-handle, or in C, with its boundary in the union of fields and spots. If it lies in a genuine handle, it can easily be isotoped out of the handle. If it lies in C we again must use the properties of the train track; ∂H could run non-trivially across a field and return to the same spot. But ∂H would then yield a curve intersecting the train track once and bounding a disc, which contradicts the properties of our train track. Eventually, we have D either in a handle or in C. In the first case, we have proved incompressibility, in the second case, ∂D would lie in a field in $F \times 0$, and by the properties of the train track, it must bound a disc in the field. This completes the proof that $\partial P - \overset{\circ}{A}$ is incompressible in P.

Finally, suppose D is a ∂-compressing disc in (P, A) for $\partial P - \overset{\circ}{A}$. Thus $\partial D = \alpha \cup \beta$ where $\alpha \subset A$ and $\beta \subset \partial P - A$. We can assume α is disjoint from R; whether it is essential in A or not, it can be homotoped to be contained in a strip on the boundary of a 1-handle, or into an annular strip in C. We deal with closed curves of intersection with R as before. Suppose H is an innermost half-disc cut from D and not containing α. The situation is the same as above, where we were dealing with a compressing disc, so we can eliminate the arc. Eventually we have D in a 1-handle or in C. If D is contained in a 1-handle, it is not a ∂-compression, since the boundary of each 1-handle contains at least two strips. If $D \subset C$, and is a ∂-compressing disc, then ∂D intersects the train track just once, violating the properties of the train track.

This completes the proof of Claim 1.

Notice that Q has properties similar to those of P: A is incompressible in Q and $\partial Q - \overset{\circ}{A}$ is incompressible in (Q, A). We conclude that A is incompressible in N.

Suppose that $f: S \to M$ is a map transverse to fibers of $N(b(B))$. Suppose that $d: D \to M$ is a null-homotopy for a map of a curve γ to S, so ∂d factors as $\partial d = f \circ \gamma$, where γ is a curve $\gamma: \partial D \to S$. Suppose that d is transverse to $\partial_f N$. Consider an innermost curve of $d^{-1}(\partial_f N)$ bounding a disc E in D. There are two cases. If E is mapped to $M - \overset{\circ}{N}$, then the incompressibility of $\partial_f N$ in $M - \overset{\circ}{N}$ implies that the map restricted to E can be replaced by a map to N.

If $E \subset N$, we shall show that ∂E is mapped to a curve trivial in $\partial_f N$ using a sequence of homotopies of $e : (E, \partial E) \to (N, \partial_f N)$, where e is the restriction of d to E. We consider intersections with A, supposing e is transverse to A. We know that A is incompressible in N, so we can remove innermost close curves of intersection. From Claim 1, we also know that $\partial P - \overset{\circ}{A}$ (or $\partial Q - \overset{\circ}{A}$) is ∂-incompressible in (P, A) (or (Q, A)), hence we can remove innermost arcs of intersection. Then E is mapped to P or to a 2-handle. By the claim, if E is in P, ∂e is null-homotopic in $\partial P - A$ and we are done. Otherwise, E is mapped to a 2-handle, with ∂E mapped to the complement of A in the boundary of the 2-handle, so we are also done, since the handle is a product.

Thus we can reverse all of the homotopies done on e, and we know that ∂E is mapped by d to a trivial curve in $\partial_f N$. Once again, we can remove the curve of intersection ∂E by replacing the map on E by a map taking E to $\partial_f N$, and then pushing a little further into $M - \overset{\circ}{N}$. Repeating the above homotopies, we can remove all curves of $d^{-1}(\partial_f N)$, so that $d(D) \subset N$.

We now continue to use the "handle" structure to simplify the map d. We shall use homotopies of $(d, \partial d)$ with ∂d always factoring through the map $f: S \to M$. In order to perform these homotopies, we need to use the following claim. In proving the claim, we used the "handle structure" induced on S by the map f. Thus S is divided into 0-handles, 1-handles and 2-handles, where the 0-handles include collars of ∂S. The collars are mapped to vertical annuli in C, and i-handles of S are mapped to i-handles in M. In fact, the restriction of f to each of the induced handles is an embedding and the images in genuine 0-handles and 1-handles are normal discs, see [FO]. This means that the boundary of the image of a 0-handle in a 0-handle of M intersects each strip on the boundary of the 0-handle at most once. Likewise the image of a 1-handle of S has boundary intersecting each strip or spot in the boundary of a 1-handle of M at most once. These properties are a direct consequence of the fact that f is a carrying map for B.

Claim 2: The restriction of f to $f^{-1}(P)$ is π_1-injective and ∂-π_1-injective as a map of pairs to (P, A).

Let $\hat{S}$ denote $f^{-1}(P)$ and let $\hat{f}$ denote the restriction of f to this surface. First suppose $e: E \to P$ is a map from a disc E to P with ∂e factoring as $\hat{f} \circ \epsilon$ where $\epsilon: S^1 \to \hat{S}$ is a closed curve in $\hat{S}$. Make e transverse to R and suppose an innermost curve of $e^{-1}(R)$ bounds a disc H in E. As in the proof of Claim 1, we can eliminate the curve ∂H by homotopy. Now suppose H is cut from E by an innermost arc of intersection with R. The boundary of H is mapped as two paths, one to the image of one of the handles of $\hat{S}$, the other to a component of R. If H is in a 0-handle

or 1-handle, then clearly e can be homotoped out of the handle. Otherwise, H is mapped to C, with one arc mapped to the image of a collar of ∂S in C and the other arc to R. If an arc of ∂H wrapped essentially around the collar, then a component of ∂S would be null-homotopic in ∂M, which contradicts Lemma 6. Thus the arc of intersection can be removed in this case as well. Eventually $e^{-1}(R)$ is empty and the image of E is contained in a handle or in C again. In both cases, we see ϵ is null-homotopic in $\hat{S}$. When E is mapped to C we use Lemma 6 again to show that ϵ is null-homotopic.

Now suppose E is a half-disc, with $e: E \to M$ potentially a null-homotopy for an element of $\pi_1(\hat{S}, \partial\hat{S})$. The boundary, ∂E, consists of complementary arcs α and β, with $e: (E, \alpha) \to (P, A)$ a map to M with $e|_\beta$ factoring as $f \circ \epsilon$ where $\epsilon: \beta \to \hat{S}$ is an arc in $\hat{S}$. We may suppose that α is mapped to C or to a 1-handle, since $\hat{f}$ maps components of $\partial\hat{S}$ to cores of components of A. Make e transverse to R and eliminate innermost closed curves as above. Eliminate innermost arcs as above as well. Then E is mapped to a handle, or to C. If E is mapped to a genuine handle, ϵ is homotopic in $\hat{S}$ to an arc in $\partial\hat{S}$, otherwise we contradict the normality of the image discs in the handle; if E is mapped to C, ϵ is homotopic in $\hat{S}$ to an arc in $\partial\hat{S}$ using the fact that f maps S vertically into C.

This completes the proof of Claim 2.

We return to our null-homotopy $(d, \partial d): (D, \partial D) \to M$ for a map of a curve γ to S, so ∂d factors as $\partial d = f \circ \gamma$, where γ is a curve $\gamma: \partial D \to S$. Recall that we have already homotoped d such that $d(D) \subset N$. Homotope d to make it transverse to A. Suppose the disc $H \subset D$ is bounded by an innermost curve of $d^{-1}(A)$. The map d restricted to H has image in P or in Q. By the incompressibility of A in N, we can eliminate such an innermost curve of intersection. Suppose the disc $H \subset D$ is cut by an innermost arc of $d^{-1}(A)$ from D. Using Claim 2, we can eliminate the arc if H is mapped to P. If H is mapped to a 2-handle, the ∂-π_1-injectivity relative to A of f restricted to $f^{-1}(Q)$ is obvious, so again we can eliminate the arc of intersection. Ultimately, our map has image in P or in a 2-handle. By Claim 2, γ is null-homotopic in $\hat{S}$ if the image of D is in P, otherwise γ is null-homotopic in $f^{-1}(Q)$ because each component is mapped to a core of a 2-handle, a product $Z \times I$, and the null-homotopy d can be projected to this component of $f^{-1}(Q)$.

Now we need an argument to deal with the ∂-π_1-injectivity of f. We let D be a disc with $\partial D = \alpha \cup \beta$, where α and β are complementary arcs in ∂D. We consider a map $(d, \delta): (D, \alpha) \to (M, \partial M)$ with the property that ∂d restricted to β factors as $\partial d|_\beta = f \circ \gamma$, where γ is a curve $\gamma: \beta \to S$. One simplifies intersections with A as above using Claim 2, always using innermost arcs in D disjoint from α, and finally one obtains d mapping D to P. Next, proceeding as in the proof of Claim 2, one homotopes d to eliminate intersections with R. Ultimately, then, one obtains d mapping D to C. The fact that $f^{-1}(C)$ consists of collars of ∂S mapped vertically to C ensures that γ is boundary parallel in S.   ◇

In order to obtain all boundary slopes, we need another example.

Example 2: Once again, we begin with a solid torus M and build a branched surface in it. Figure 5a shows a portion of an immersed branched surface, including a disc which will be a meridian disc for the solid torus. (One could construct the

branched surface of Figure 5a as the union of the flat meridian disc and another disc in the form of a saddle, which intersects the flat disc in the horizontal dashed arc.) Once again, for surfaces carried by this branched surface, there are intersections forced along the arc of self intersection, and we must attach a tube in order to be able to extend the curves of intersection without hitting ∂M, see Figure 5b. We begin with a longitudinal annulus in the solid torus, as in the first example. Then we insert the branched surface unit of Figure 5b into the the annulus. In addition we insert the mirror image of the branched surface of Figure 5b as shown in Figure 6. Notice that the tube for each inserted unit must intersect the annulus. The first insertion would yield only positive boundary slopes, but with both insertions we obtain all boundary slopes. If we assign to the sectors P_1, P_2, and Q the weights p_1, p_2, and q respectively, the reader can easily check that the corresponding surface has boundary slope $(p_1 - p_2)/2q$.
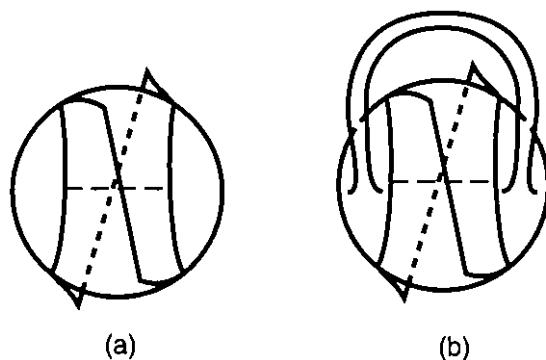

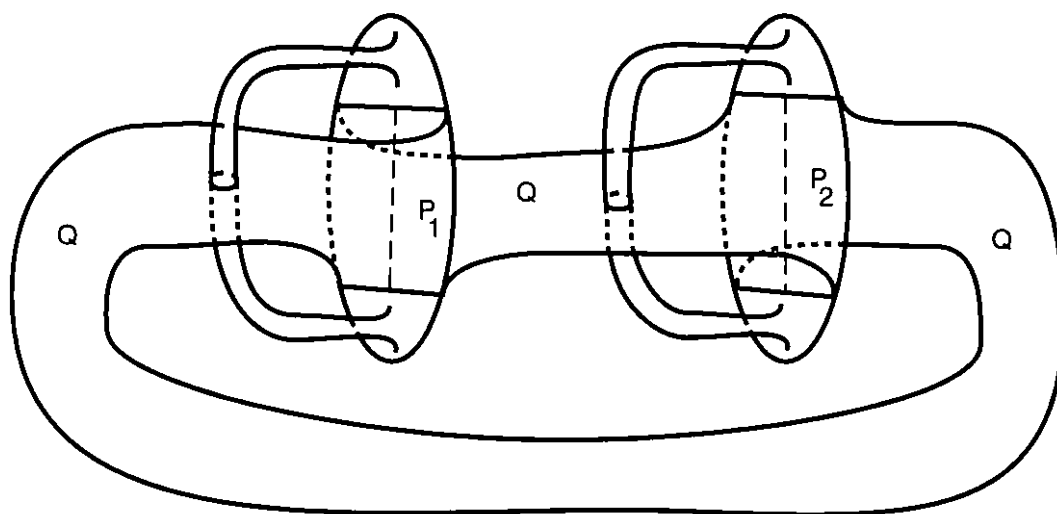(a)   (b)

Figure 5

Figure 6

As in the previous example, we modify subsectors which are discs by attaching handles, then we modify the complementary pieces of $b(B)$ to ensure that their

boundaries are incompressible. If we wish, we can modify subsectors by adding more handles to ensure that all subsectors contain essential non-separating curves. The latter property of the subsectors will be needed in the proof of Proposition 2.
◇

Proposition 7, applied to a branched surface constructed in Example 2, proves Proposition 1.

Proof of Proposition 2. Example 2 allows us to construct a connected surface whose boundary realizes an arbitrary finite subset of all slopes. We suppose that the branched surface has been modified such that every sector contains a non-separating curve. From the previous construction, for every slope r in the subset, we have a connected surface S_r realizing the slope as a boundary curve. The union of these surfaces is carried by B. It only remains to modify the union to obtain a connected surface. This is easily achieved by cut and paste on closed curves. More precisely, the modification is effected using a sequence of modifications of the following type. Choose an essential non-separating embedded closed curve γ in a subsector; then cut all surfaces carried on the pull-back curves of γ. Finally, reglue after suitably permuting the boundary curves of the cut surfaces in order to achieve connectedness. The new surface is still carried by the branched surface B satisfying the conditions of Proposition 7, hence it is π_1-injective.   ◇

The following lemma will be needed in the proof of Proposition 3. The original proof of Proposition 3 was less elegant; Allen Hatcher suggested this lemma as a way to simplify the proof.
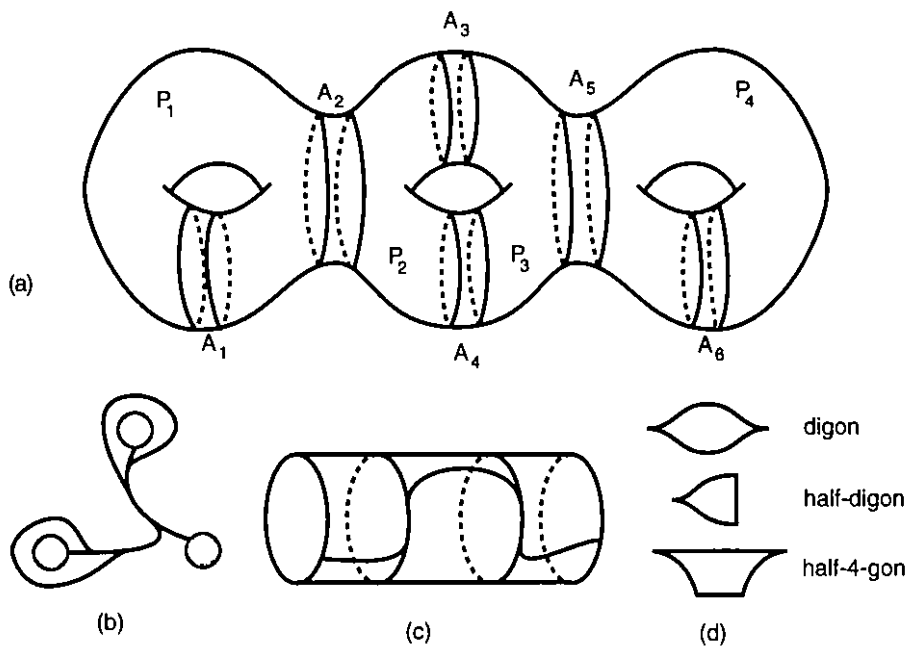

Figure 7

Lemma 8. *Suppose F is a closed surface of genus $g \geq 2$. Then there is a train track τ in F, without complementary 0-gons or monogons, which carries all embedded curve systems in F.*

Proof. Figure 7 shows how to construct the train track. First one decomposes the surface F into a union of pairs P_i of pants and annuli A_j as shown in Figure 7a.

In each pair of pants P_i one embeds the train track shown in Figure 7b. Notice that the train track carries all essential arc systems in the pair of pants. The complementary regions of the train track are: two half-4-gons and a half-digon as shown in Figure 7d.

In each annulus A_j one embeds the train track shown in Figure 7c. Notice that this train track carries both the essential closed curve in A_j and the only essential arc. In addition it carries all images of the essential arc under Dehn twists in the annulus. The complementary regions in A_j of the train track are: two half-digons and one digon as shown in Figure 7d.

It is easy to verify that the train track τ obtained by glueing the train tracks in annuli A_j and pairs of pants P_i does not have complementary 0-gons or monogons. This is done using the information we have about complementary regions in the A_j's and P_i's.

It is also easy to verify that the train track carries all essential curve systems. This follows from the fact that any curve system can be isotoped so that it intersects each P_i only in essential arcs.   ◇

Proof of Proposition 3. Given $g \geq 2$ we must construct a manifold M containing an immersed branched surface which carries surfaces realizing all curve systems in $\partial M = F$, up to equivalence as points in $\mathcal{PL}(\partial M)$. (In fact, we do not quite show that all curve systems can be realized.)

We work with the train track τ of the previous lemma. Corresponding to the train track we construct a branched surface B in the manifold M, which will be a handlebody initially. In fact, the branched surface will be constructed in a collar neighborhood of ∂M. We can decompose the train track into finitely many pieces, each of which is a train track with boundary, and each having just one switch and three segments, see Figure 8a. Corresponding to this portion of the train track, we construct a portion of the branched surface as shown in Figure 8ab. Initially we construct a branched surface with two boundary-parallel tunnels meeting tangentially on a rectangle W as shown in Figure 8a. This branched surface has an arc of self-intersection with ends at x and y as shown. At the point x we have a transverse self-intersection, which we change to a tangential intersection as shown in Figure 8b by introducing a half-disc of contact. To construct the branched surface B we then combine the branched surface units corresponding to each piece of τ. After the introduction of half-discs of contact, we have arcs of self-intersection of B similar to those in our previous examples; along each arc one sector does a half-twist relative to the other.

Clearly, given a curve system $\tau(w)$ carried by τ determined by a weight vector w, assigning the same weights to the corresponding sectors of B (excluding sectors which are half-discs of contact) determines a surface $S = B(w)$ carried by B with the property that ∂S represents $\tau(2w)$, i.e. a curve system isotopic to $\tau(w)$, except that each curve is repeated twice. Thus all rational points in $\mathcal{PL}(F)$ are realized as boundaries of surfaces carried by B.

We modify the branched surface B as before. First we punch holes at the ends of the arcs of self-intersection and join the holes with tubes. Each tube must intersect

the branched surface in at least one trivial closed curve. Next we modify B to ensure that no subsector is a disc. In the proof of Proposition 4, we shall use the same branched surface, and we shall require subsectors which have been modified such that they contain non-separating curves, but this is also easy to achieve. Finally, we modify the 3-manifold by replacing complementary regions of $N(\partial M \cup b(B))$ by manifolds with incompressible boundary.

Applying Proposition 7, we prove that all surfaces S carried by b are π_1-injective. We have already observed that all rational points of $\mathcal{PL}(F)$ were realized of boundaries of surfaces S carried by B before B was modified; this remains true after the modifications.   ◇
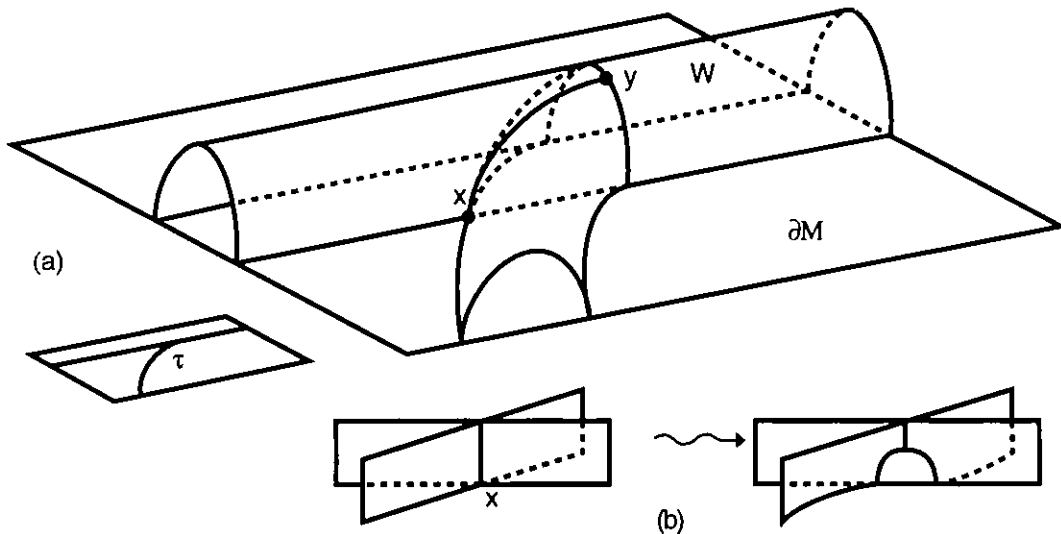

Figure 8

Proof of Proposition 4. A surface with the given curve systems as boundary is constructed in the same way as the surface realizing an arbitrary finite set of slopes. The branched surface and 3-manifold are the same as in the previous proof, but we must be sure that all sectors contain non-separating simple closed curves. For each rational point v_i in the finite set V, we include a surface S_i carried by B with $\partial S_i = C_i$, such that C_i represents the rational point $v_i \in \mathcal{PL}(\partial M)$. The union of the S_i's carried by B can be replaced by a connected surface carried by B using the usual cut-and-paste operations.   ◇

We note that there is an analogue of Proposition 4 in the case of a 3-manifold M with several torus boundary components. Hatcher showed that even for such a manifold, boundary curve systems of embedded incompressible and ∂-incompressible surfaces are constrained to lie in a lower-dimensional subspace of $\mathcal{PL}(\partial M)$. A slight modification of the proof of Proposition 2 shows that all curve systems in the boundary of such a manifold M can be realized as boundaries of π_1-injective maps of surfaces.

We now turn to the proof of the Proposition dealing with π_1-injective surfaces and Dehn filling.

Proof of Propostion 5. Assume $M(r_1, \ldots, r_k)$ is irreducible, and suppose the map $f\colon S \to M(r_1, \ldots, r_k)$ is incompressible. In particular, this would be the case if f were π_1-injective. Make f transverse to the cores of filling tori, R_i, and assume that the image of f interesects the R_i in meridian discs of the R_i. We minimize this number of intersections among maps $f\colon S \to M(r_1, \ldots, r_k)$ inducing the same map on π_1. The claim is that the restriction $\hat{f}$ to $\hat{S} = S - f^{-1}(\cup_{i=1}^k R_i)$ is incompressible and ∂-incompressible.

If there is an essential simple closed curve γ in $\hat{S}$ which is mapped to a curve null-homotopic in M, then by the incompressibility of f, γ bounds a disc D in S. The disc D must intersect the $\cup_i R_i$. Since $f|_\gamma$ is null-homotopic in M, the null-homotopy h can be regarded as a map on D which agrees with f on ∂D and whose image is disjoint from $\cup_i R_i$. We replace the map $f|_D$ by the map h to obtain a new map f which is still incompressible and which intersects $\cup_i R_i$ in fewer discs, contradicting our assumption. We have proved incompressibility, but it remains to prove ∂-incompressibility.

If there is a homotopy in M of a simple essential arc $(\beta, \partial\beta) \subset (\hat{S}, \partial\hat{S})$ (rel endpoints) to an arc in ∂M, then there is a homotopy of pairs $(\beta, \partial\beta)$ in $(M, \partial\hat{S})$ to a simple arc in ∂M. There are two cases to consider. In the first case, see Figure 9a, the arc β joins two discs in S mapped to two meridian discs in some R_i with opposite orientations. In this case, f can be homotoped to reduce the number of intersections with $\cup_i R_i$, a contradiction. In the second case, the arc β joins two discs in S mapped to two meridian discs E_1 and E_2 in some R_i with the same orientations, see Figure 9b. In this case, the map f can be homotoped such that the image of a neighborhood of $E_1 \cup E_2 \cup \beta$ is as shown in Figure 9c. The homotopy can be done in such a way that $f(S)$ is otherwise unchanged in a neighborhood of $\cup_i R_i$. So we have replaced two intersection with $\cup_i R_i$ by one "cone singularity" which we label P.

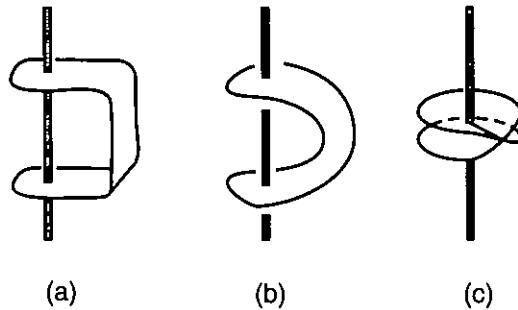(a)   (b)   (c)

Figure 9

In order to finish the proof, we must do some global modifications of the map f.

Applying the main technical result in [Wh], we can assume that the image of f is locally modelled as shown in Figure 10. At each point of the image, it is locally modelled as an embedding, as a transverse intersection with an arc of intersection, as a triple point, or as a "cone singularity" which is the cone on a figure eight. We may assume that the surface S is transversely oriented; in the figures we draw

intersections so that the angle from the "+-side" of one sheet of S to the +-side of the other sheet of S is obtuse.

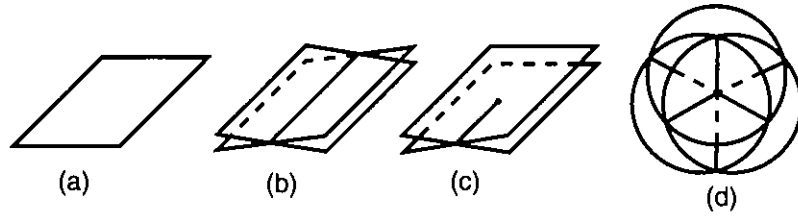(a)   (b)   (c)   (d)

Figure 10

The idea of the remainder of the proof is to eliminate cone singularities using homotopies starting at a cone singularity. Given an arc of self-intersection with at least one end a singularity, there is an obvious homotopy which moves the singularity along the arc of intersection while shortening the arc of intersection, see Figure 11. The effect is similar to the effect of a zipper, but four rather than two sheets of cloth come together along the zipper or arc of intersection. Also, our zipping operation has a different effect from the usual zipper. The usual zipper cuts a surface or rejoins it; our zipper cuts two surfaces and rejoins them locally in the opposite way. The singularity corresponds to the zipper head. This zipping operation is described in Bing's monograph, [Bi]. In the proof of the simple loop conjecture [G] and in [GK] this zipping operation was referred to as "the calculus of double curves."
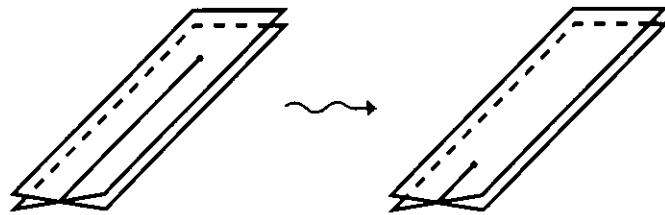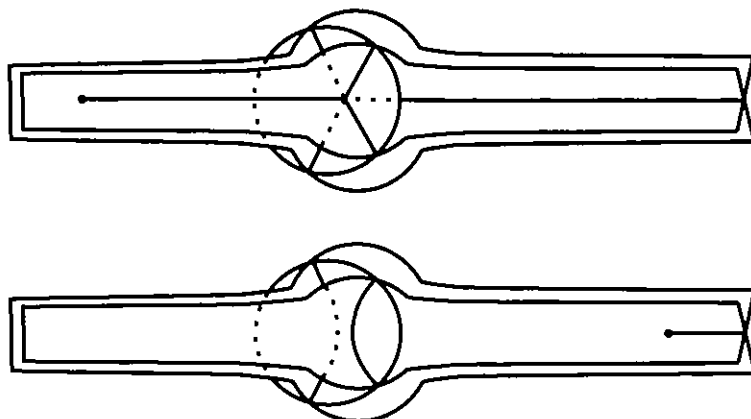

Figure 11

Figure 12

We begin by performing zipping operations with the goal of eliminating the cone singularity P. We choose any *other* cone singularity and begin zipping. If a triple point is encountered, we continue to zip straight through the triple point as shown in Figure 12. Clearly the curves of self-intersection of S at the triple point are altered when the singularity passes through the triple point; there is a surgery operation on the double curves. Notice that in our figures we use a standard model for a triple point, as shown in Figure 10d. This explains our way of drawing Figure 12. After passing through a triple point, the zipper may return to what was the triple point once more, or the zipping path could return to the same point a second time as well.

There is just one possible obstruction which makes it impossible to continue zipping: namely, the arc of intersection must end at another singularity as shown in Figure 13. We cut on the arc of intersection and paste to obtain a surface which is embedded in a neighborhood of the arc, see Figure 13. The final surgery respects orientation. In terms of the topology of the surface S, this final cancellation of cone singularities locally has the effect of replacing an annulus of the source surface by two discs; thus it performs a surgery. But in our situation, the core of the annulus is a closed curve embedded in S hence it must bound a disc in S by the π_1-injectivity of the map f. It follows that the final cancellation splits off a sphere, which we discard. If the zipping ends at P, then we have clearly removed the singularity, and we have reduced the number of intersections of the image of S under f with the filling tori by at least one, as compared with the number of intersections before the introduction of the singularity P. If the zipping ends at another singularity, then we simply begin zipping at another singularity, but not at P. Removing all singularities by zipping shows that there are an even number, so this is always possible. Eventually we must eliminate the singularity P, and reduce intersections of the map with the filling tori. ◇
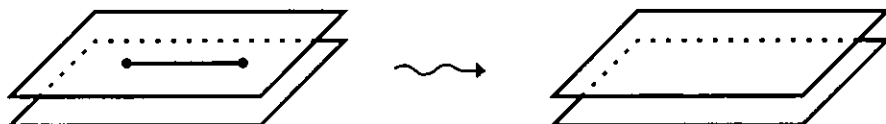

Figure 13

The following is a remark, due to Michel Boileau, concerning the phenomenon described in Proposition 1: only for finitely many values r can it be the case that S_r is orientable with both S_r and ∂S_r (with induced orientation) representing nontrivial homology classes in $H_2(M, \partial M)$ and $H_1(\partial M)$ respectively, for a surface representing homology in this way could be converted to an embedded incompressible, ∂-incompressible surface with non-empty boundary of slope r using cut-and-paste followed by surgery. Thus Hatcher's result shows that r can only take finitely many values for surfaces S_r of this type.

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