THE ALGEBRAIC FUNDAMENTAL GROUP AND ABELIAN GALOIS COHOMOLOGY OF REDUCTIVE ALGEBRAIC GROUPS

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It is clear from the diagram that the composition

$$H^1(K,G) \longrightarrow \oplus H^1(K_v,G) \longrightarrow (M_\Gamma)_{tors}$$

is zero. Now let $\xi_A = \xi_\infty \times \xi_f \in \oplus H^1(K_v,G)$, where $\xi_\infty \in \prod_\infty H^1(K_v,G)$, $\xi_f \in \oplus_{\mathcal{V}_f} H^1(K_v,G)$. Suppose that $\mu(\xi_A) = 0$. Let h_A be the image of ξ_A in $\oplus H^1_{ab}(K_v,G)$. Then the image of h_A in $(M_\Gamma)_{tors}$ is zero, hence h_A is the image of some element $h \in H^1_{ab}(K,G)$. Consider the element $h \times \xi_\infty \in H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that $h \times \xi_\infty$ is contained in the fiber product over $\prod_\infty H^1_{ab}(K_v,G)$. By Theorem 5.12 $h \times \xi_\infty$ comes from $H^1(K,G)$. The theorem is proved.

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Introduction

Let G be a connected reductive group over a field K of characteristic 0. The aim of this paper is to "abelianize" the first Galois cohomology set $H^1(K,G)$.

Let G^{ss} denote the derived group of G. Let G^{sc} denote the universal covering of the semisimple group G^{ss}; the group G^{sc} is simply connected. Consider the canonical homomorphism

$$\rho : G^{sc} \longrightarrow G^{ss} \longrightarrow G .$$

Deligne ([De], 2.0.2) noticed that the quotient set $\rho(G^{sc}(K))\backslash G(K)$ has a natural structure of an abelian group. We regard this abelian group as *the abelianized 0-dimensional Galois cohomology* $H^0(K,G)^{abld}$ of G.

Inspired by the results of Kottwitz [Ko2], [Ko3], we try to abelianize the 1–dimensional Galois cohomology. Consider the abelianized cohomology set

$$H^1(K,G)^{abld} := \rho_* H^1(K,G^{sc}) \backslash H^1(K,G) .$$

This expression makes sense: we use twisting to define a certain equivalence relation on $H^1(K,G)$. We will show that $H^1(K,G)^{abld}$ can be canonically embedded into some abelian group $H^1_{ab}(K,G)$, *the first abelian Galois cohomology group.* Moreover, if K is a local field or a number field, then this embedding turns out to be a bijection; thus the set $H^1(K,G)^{abld}$ has in this case a natural structure of an abelian group. Following Kottwitz [Ko2], [Ko3], we compute this abelian group in the local case. We use these results to investigate and in a sense compute $H^1(K,G)$ when K is a number field.

Let $\overline{K}$ be an algebraic closure of K. We write $\overline{G}$ for $G_{\overline{K}}$. In Section 1 we define *the algebraic fundamental group* $\pi_1(\overline{G})$ as follows. Let $T \subset G$ be a maximal torus defined over K. We write $T^{(sc)}$ for $\rho^{-1}(T)$ and set

$$\pi_1(\overline{G}) = X_*(\overline{T})/\rho_* X_*(\overline{T}^{(sc)})$$

where X_* denotes the cocharacter group. The group $\pi_1(\overline{G})$ is a finitely generated abelian group endowed with a $\mathrm{Gal}(\overline{K}/K)$–action. If $K = \mathbb{C}$ then $\pi_1(\overline{G})$ is just the usual topological fundamental group $\pi_1^{top}(G(\mathbb{C}))$. For any K our algebraic fundamental group is connected with the invariant $Z(\hat{G})$ of Kottwitz [Ko2], where $\hat{G}$ is a connected dual Langlands group for G and $Z(\hat{G})$ is its center. Namely, $\pi_1(G)$ is the character group of the $\mathbb{C}$–group $Z(\hat{G})$.

In Section 2 we define the *abelian Galois cohomology groups*

$$H^i_{ab}(K,G) := \mathbb{H}^i(K,T^{(sc)} \longrightarrow T) \qquad (i \geq -1)\,.$$

Here $\mathbb{H}^i$ denotes the Galois hypercohomology of the complex

$$0 \longrightarrow \overset{-1}{T}{}^{(sc)} \longrightarrow \overset{0}{T} \longrightarrow 0$$

of tori, where -1 and 0 above the letters denote the degrees. We show that the abelian groups $H^i_{ab}(K,G)$ depend only on $\pi_1(\overline{G})$. A short exact sequence

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

of (connected) reductive K–groups gives rise to the short exact sequence

$$0 \longrightarrow \pi_1(\overline{G}_1) \longrightarrow \pi_1(\overline{G}_2) \longrightarrow \pi_1(\overline{G}_3) \longrightarrow 0$$

and the long cohomology exact sequence

$$0 \longrightarrow H^{-1}_{ab}(K,G_1) \longrightarrow H^{-1}_{ab}(K,G_2) \longrightarrow H^{-1}_{ab}(K,G_3) \longrightarrow H^0_{ab}(K,G_1) \longrightarrow \ldots$$

Thus π_1 is in a sense an exact functor and $(H^i_{ab})_{i\geq -1}$ is in a sense a cohomological functor.

In the third section we construct *the abelianization map*

$$ab^1 = ab^1_G : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$$

with kernel $\rho_* H^1(K,G^{sc})$. This map defines an embedding of the abelianized Galois cohomology $H^1(K,G)^{abld}$ into $H^1_{ab}(K,G)$. Observe that in the case of a semisimple group G we have

$$G = G^{sc}/\ker \rho, \ \ H^1_{ab}(K,G) = H^2(K,\ker \rho)$$

(where $\ker \rho$ is a finite abelian group), and ab^1 is in this case the connecting homomorphism $H^1(K,G) \longrightarrow H^2(K,\ker \rho)$. We generalize the construction of Kottwitz [Ko3], who constructs ab^1 in the case of a local field K. We also construct a homomorphism

$$ab^0 : G(K) \longrightarrow H^0_{ab}(K,G)$$

with kernel $\rho(G^{sc}(K))$; in the case of a local field K this map was constructed by Langlands [La1] (see also [Bo], 10.2).

In Section 4 we compute explicitely the groups $H^1_{ab}(K,G)$ for a local field K in terms of $\pi_1(\overline{G})$. We write Γ for $\mathrm{Gal}(\overline{K}/K)$ and M for $\pi_1(\overline{G})$. Then

$$H^1_{ab}(K,G)= \begin{cases} H^{-1}(\Gamma,M) & \text{if} \quad K = \mathbb{R} \\ \\ (M_\Gamma)_{tors} & \text{if} \quad K \text{ is non-archimedian,} \end{cases}$$

where $(M_\Gamma)_{tors}$ denotes the torsion subgroup of the group of coinvariants M_Γ. We then write an exact sequence connecting the groups $H^i_{ab}(K,G)$ $(i \geq 1)$ for a number field K and for its completions K_v. In particular, we compute $H^i_{ab}(K,G)$ for $i \geq 3$ and compute it in a sense for i = 2. For i = 1 we compute the group

$$Ш^1_{ab}(K,G) := \ker[H^1_{ab}(K,G) \to \prod_v H^1_{ab}(K_v,G)]$$

in terms of $\pi_1(\overline{G})$. All these results are of an abelian nature and generalize the Tate–Nakayama duality theory for tori. The results concerning the case i = 1 are essentially due to Kottwitz.

In Section 5 we prove that if K is a local or a number field, then the abelianization map ab^1 is surjective. For local fields this is very close to a result of Kottwitz [Ko3]. This surjectivity means, in particular, that for a local non–archimedean field K

$$H^1(K,G) \simeq (M_\Gamma)_{tors}$$

([Ko2], 6.4.1). In this case ab^1_G is not only surjective but also injective.

We use the surjectivity of ab^1 over local and number fields to investigate the usual, non–abelian Galois cohomology $H^1(K,G)$, where K is a number field.

Theorem 5.11. For any finite subset $\Xi \subset M^1(K,G)$ there exists a K–torus $j : T \hookrightarrow G$ such that $\Xi \subset j_* H^1(K,T)$.

In other words, for a number field K all the $H^1(K,G)$ comes from tori.

Further, we compute $H^1(K,G)$ in terms $H^1_{ab}(K,G)$ and the real cohomology:

Theorem 5.12. $H^1(K,G)$ is the fiber product of $H^1_{ab}(K,G)$ and $\prod_{\infty} H^1(K_v,G)$ over $\prod_{\infty} H^1_{ab}(K_v,G)$, where ∞ denotes the set of infinite places of K.

This result generalizes a theorem of the beautiful paper [Sa] of Sansuc (and is inspired by Sansuc's result).

From Theorem 5.12 we obtain

Theorem 5.13. The restriction of ab^1 to the Shafarevich–Tate kernel defines a bijection $Ш^1(K,G) \longrightarrow Ш^1_{ab}(K,G)$.

Thus we see again after Voskresenskii [Vo], Sansuc [Sa] and Kottwitz [Ko2], that $Ш(G)$ has a natural structure of an abelian group. Combining this bijection with the results of Section 4 we can compute $Ш(G)$ in terms of $\pi_1(\overline{G})$. The obtained formula is equivalent to a formula of Kottwitz [Ko2].

Remark 0.1. The results of this paper can be easily adapted to the case of any, not necessarily reductive, connected K–group. Let G^u denote the unipotent radical of G. We set $G^{red} = G/G^u$; this is a reductive group. We set

$$\pi_1(\overline{G}) = \pi_1(\overline{G}^{red}), \quad H^1_{ab}(K,G) = H^1_{ab}(K,G^{red})$$

and so on. With this notation almost all the results of the paper remain true for all connected K–groups.

Remark 0.2. In the case of a semisimple group G all the results of this paper were already known (cf. [Sa]). On the other hand for local fields our results are just a more functorial reformulation of results of Kottwitz [Ko2], [Ko3]. The contribution of the present paper is that we construct the abelian Galois cohomology and the abelianisation map for *any* reductive group over an *aribitrary* field of characteristic 0. This enables us to obtain new results concerning usual, non–abelian Galois cohomology of reductive groups over number fields.

Remark 0.3. Most of the results of this paper are relative, they describe the Galois cohomology of G modulo the Galois cohomology of G^{sc}. Thus our computations in Section 5 of Galois cohomology of reductive groups over number fields are based on the fundamental results on Galois cohomology of semisimple groups due to Kneser [Kn1], [Kn2] and Harder [Ha1], [Ha2].

Remark 0.4. Our algebraic fundamental group $\pi_1(\overline{G})$, abelianization map ab^1_G and so on, are functorial with respect to *any* homomorphism $\varphi : G \longrightarrow G'$ of reductive K–groups. Kottwitz [Ko2], [Ko3] computes everything in terms of the center $Z(\hat{G})$ of a connected Langlands group $\hat{G}$. The group $\hat{G}$ is functorial only with respect to *normal* homomorphisms $\varphi : G \longrightarrow G'$, i.e. such that $\varphi(G)$ is normal in G. Therefore the corresponding groups and maps of the papers [Ko2] and [Ko3] are functorial only with respect to normal homomorphisms; so his results look less functorial than ours. It should however be mentioned that the *methods* and *constructions* of [Ko2] and [Ko3] are completely functorial. It suffices just to substitute $Hom(\pi(\overline{G}),\mathbb{C}^*)$ for $Z(\hat{G})$ to make all the statements and proofs of the corresponding results of Kottwitz completely functorial with respect to all homomorphisms $G \longrightarrow G'$.

Acknowledgements.

It is clear from the introduction, that the present paper is inspired by the papers [Ko2] and [Ko3] of Kottwitz. I must add that in June of 1989 Robert Kottwitz explained me that my abelian Galois cohomology group $H^1_{ab}(K,G)$ (which had been previously defined in a rather aukward and non–functorial way) is in fact the Galois hypercohomology group of a complex of tori. This remark greatly influenced the exposition in Sections 2–4. For this I am extremely grateful to him.

This paper was written during my stay at the Max–Planck–Institut in Bonn. I use this opportunity to express my deep gratitude to the Institute for its generous hospitality, excellent working conditions and the creative atmosphere.

I thank M. Kapranov, H.–J. Baues and W. Soergel, who answered my numerous questions on homological algebra. I thank I. André and T. Zink for very helpful discussions. I am grateful to the participants of the Arbeitsgemeinschaft Geyer–Harder "Tamagawazahlen und die Selbergsche Spurformel" (Oberwolfach, October 8–14, 1989), and the participants of the Harder seminar on automorphic forms at Bonn University for their interest and valuable remarks. Last but not least, I would like to thank Miss Grau for the careful typing of the manuscript.

Notation

K is a field of characteristic 0, $\bar{K}$ is an algebraic closure of K. We write Γ for $Gal(\bar{K}/K)$. For an algebraic variety X over K we write $\bar{X}$ for $X_{\bar{K}}$.

When K is a number field, let $\mathcal{V} = \mathcal{V}(K)$, $\mathcal{V}_\infty$ and $\mathcal{V}_f$ denote the set of all places, the set of infinite (archimedian) places and the set of finite (non–archimedian) places of K, respectively. We often write just ∞ for $\mathcal{V}_\infty$. If $v \in \mathcal{V}$, we let K_v denote the completion of K at v.

We denote by μ_n the group or roots of unity of order dividing n, and set $\hat{\mathbb{Z}}(1) = \varprojlim \mu_n$.

G is a reductive K–group. By a reductive K–group we always mean a *connected* reductive K–group. Let G^{ss} denote the derived group of G. We set $G^{tor} = G/G^{ss}$. We denote by $Z(G)$ the center of G and set $G^{ad} = G/Z(G)$. Let G^{sc} denote the universal covering of the semisimple group G^{ss}. We have the canonical homomorphism

$$\rho : G^{sc} \longrightarrow G^{ss} \longrightarrow G \ .$$

Let S be a K–group of multiplicative type, e.g. a torus. We let $X^*(S)$ denote the character group $Hom(S,\mathbb{G}_m)$ and let $X_*(S)$ denote the cocharacter group $Hom(\mathbb{G}_m,S)$, where $\mathbb{G}_m$ is the multiplicative group. We usually consider $X^*(\bar{S})$ and $X_*(\bar{S})$.

For a reductive K–group G and a split maximal K–torus T we let $R(G,T)$ denote the root system of G with respect to T. We denote by $R^\vee(G,T)$ the system of coroots. By definition $R(G,T) \subset X^*(T)$ and $R^\vee(G,T) \subset X_*(T)$.

Let L be a torsion free abelian group. We write $L^\vee$ for $Hom(L,\mathbb{Z})$.

Let M be an abelian group. We let M_{tors} denote the torsion subgroup of M. We set $M_{tf} = M/M_{tors}$; this is the maximal torsion free quotient of M.

Let Δ be a group and M a Δ–module. We say that M is a finitely generated (resp. torsion free) Δ–module if M is finitely generated (resp. torsion free) as an abelian group.

Let M be a finitely generated Δ–module. By a short torsion free resolution of M we mean an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow L^{0} \longrightarrow M \longrightarrow 0$$

of finitely generated Δ–modules such that L^{-1} and L^{0} are torsion free. We write $L^{\cdot}$ for the complex $0 \longrightarrow L^{-1} \longrightarrow L^{0} \longrightarrow 0$.

Let M be a Δ–module. We write M^{Δ} and M_{Δ} to denote the subgroup of invariants and the group of coinvariants of M , respectively. We often consider the functors $(M_{\Delta})_{tors}$ and $(M_{\Delta})_{tf}$.

Let G be an algebraic group. As usual, we write $H^{i}(K,G)$ to denote the Galois cohomology $H^{i}(\Gamma,G(\overline{K}))$ (where $\Gamma = \mathrm{Gal}(\overline{K}/K)$). We denote by $Z^{i}(K,G)$ the set of i–cocycles and by $B^{i}(K,G)$ the set of i–cobords.

For any Γ–module M we write $H^{i}(K,M)$ for $H^{i}(\Gamma,M)$. Similarly if F/K is a Galois extension with the Galois group Δ and if M is a Δ–module, we write $H^{i}(F/K,M)$ for $H^{i}(\Delta,M)$ and $\hat{H}^{i}(F/K,M)$ for $\hat{H}^{i}(\Delta,M)$, where $\hat{H}^{i}$ are the Tate cohomology groups.

If K is a number field, we use the the notation *loc* to denote the localization maps

$$\mathrm{loc}_{v} : H^{1}(K,G) \longrightarrow H^{1}(K_{v},G)$$
$$\mathrm{loc}_{\infty} : H^{1}(K,G) \longrightarrow \prod_{v \in \mathcal{V}_{\infty}} H^{1}(K_{v},G)$$

and so on.

1. The algebraic fundamental group of a reductive group

In this section we define the algebraic fundamental group $\pi_1(G_{\bar{K}})$ of a reductive group G defined over a field K of characteristic 0.

1.1. Let G be a (connected) reductive K–group. First suppose that G is split. Choose a maximal split torus $T \subset G$. Consider the canonical morphism $\rho : G^{sc} \longrightarrow G$. We write $T^{(sc)}$ for $\rho^{-1}(T) \subset G^{sc}$. Set

$$\pi_1(G,T) = X_*(T)/\rho_* X_*(T^{(sc)}) \ .$$

It is a finitely generated abelian group.

Lemma 1.2. For two split maximal tori $T,T' \subset G$, the groups $\pi_1(G,T)$ and $\pi_2(G,T')$ are canonically isomorphic.

Proof. Choose an element $g \in G(K)$ such that $T' = gTg^{-1}$. The isomorphism $\mathrm{int}(g) : T \longrightarrow T'$ induces an isomorphism $g_* : \pi_1(G,T) \longrightarrow \pi_1(G,T')$. We will show that g_* does not depend on the choice of g.

Let N denote the normalizer of T in G. It suffices to show that if $g \in N(K)$ then the automorphism g_* of $\pi_1(G,T)$ is trivial. The group $N(K)$ acts on T and on $\pi_1(G,T)$ through its quotient group $W := N(K)/T(K)$. One knows that the Weyl group W is generated by the reflections r_α corresponding to the roots $\alpha \in R(G,T)$. It remains to show that for $\alpha \in R(G,T)$ the reflection r_α acts on $\pi_1(G,T)$ trivially.

We have

$$r_\alpha(X) = X - \langle \alpha, X \rangle \alpha^\vee$$

for $X \in X_*(T)$, where $\alpha^\vee$ is the corresponding coroot. Since all the coroots come from $X_*(T^{(sc)})$, we see that

$$r_\alpha(X) \equiv X \mod \rho_* X_*(T^{sc}) \;,$$

thus r_α acts on $X_*(T)/\rho_* X_*(T^{(sc)})$ trivially. The lemma is proved.

Definition 1.3. Let G be a split reductive K–group. Let $T \subset G$ be a split maximal K–torus. We set $\pi_1(G) = \pi_1(G,T)$ and call this abelian group the algebraic fundamental group of G .

By Lemma 1.2 this definition is correct.

1.4. Now let G be any (not necessarily split) reductive K–group. By the algebraic fundamental group of G we mean $\pi_1(\overline{G})$ (recall that $\overline{G} = G_{\overline{K}}$).

The Galois group $\Gamma = \mathrm{Gal}(\overline{K}/K)$ acts on G and thus on $\pi_1(\overline{G})$. This action can be described as follows.

Choose a maximal torus $T' \subset \overline{G}$. For $\sigma \in \Gamma$ choose an element $g_\sigma \in G(\overline{K})$ such that $g_\sigma \cdot \sigma T' \cdot g_\sigma^{-1} = T'$. Then σ acts on $\pi_1(\overline{G},T')$ as the composition

$$\pi_1(\overline{G},T') \xrightarrow{\sigma_*} \pi_1(\overline{G},\sigma T') \xrightarrow{(g_\sigma)_*} \pi_1(\overline{G},T')$$

In particular, if $T \subset G$ is a maximal torus defined over K , then the action of Γ on $\pi_1(\overline{G})$ is the action on $\pi_1(\overline{T})/\rho_* X_*(\overline{T}^{(sc)})$ induced from $X_*(\overline{T})$.

Our algebraic fundamental group is a functor from the category of reductive K–groups and K–homomorphisms to the category of finitely generated Γ–modules. The following lemma shows that this functor is in a sense exact.

Lemma 1.5. Let $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ be an exact sequence of connected reductive K–groups. Then the sequence

$$0 \longrightarrow \pi_1(\overline{G}_1) \longrightarrow \pi_1(\overline{G}_2) \longrightarrow \pi_1(\overline{G}_3) \longrightarrow 0$$

is exact.

Proof. Left to the reader as an easy exercise.

1.6. Examples. (1) For a K–torus T we have $\pi_1(\overline{T}) = X_*(\overline{T})$.

(2) Suppose G^{ss} to be simply connected. Then the canonical homomorphism $\pi_1(\overline{G}) \longrightarrow \pi_1(\overline{G}^{tor})$ is an isomorphism, thus $\pi_1(\overline{G}) = X_*(\overline{G}^{tor})$.

(3) Let G be a semisimple group. Then $G = G^{sc}/\ker \rho$, where $\ker \rho$ is a finite abelian K–group. Let $T \subset G$ be a maximal torus defined over K . Then $T = T^{(sc)}/\ker \rho$. One can easily show that $\pi_1(\overline{G}) = (\ker \rho)(-1) := \mathrm{Hom}(\hat{\mathbb{Z}}(1),\ker \rho)$. Note that $\pi_1(\overline{G})$ and $\ker \rho$ are isomorphic as abelian groups, but are in general non–isomorphic as Γ–modules. E.g. if $G = PGL_n$, then $\ker \rho = \mu_n$ but $\pi_1(\overline{G}) = \mathbb{Z}/n\mathbb{Z}$.

Corollary 1.7. For any reductive K–group G we have an exact sequence

$$0 \longrightarrow (\ker \rho)(-1) \longrightarrow \pi_1(\overline{G}) \longrightarrow X_*(G_{\overline{K}}^{tor}) \longrightarrow 0 \ .$$

Proof. We consider the canonical exact sequence $1 \longrightarrow G^{ss} \longrightarrow G \longrightarrow G^{tor} \longrightarrow 1$ and apply Lemma 1.5 and the statements 1.6 (1,3).

Now let $z \in Z^1(K,G^{ad})$ be a cocycle. Consider the twisted form ${}^zG$ of G. By definition $({}^zG)_{\overline{K}} = G_{\overline{K}}$, but $\sigma \in \mathrm{Gal}(\overline{K}/K)$ acts on $({}^zG)_{\overline{K}}$ by $g \longmapsto z_\sigma \cdot \sigma g \cdot z_\sigma^{-1}$, where $g \longmapsto \sigma g$ is the action of σ on $G_{\overline{K}}$.

Lemma 1.8. Let $z \in Z^1(K,G^{ad})$ be a cocycle. Then the map $\pi_1(G_{\overline{K}}) \longrightarrow \pi_1(({}^zG)_{\overline{K}})$, induced by the canonical isomorphism $G_{\overline{K}} \longrightarrow ({}^zG)_{\overline{K}}$, is an isomorphism of Galois modules.

Proof. The assertion follows from the description 1.4 of the Galois action on $\pi_1(\overline{G})$.

In the remaining part of this section we prove some comparison results, which will not be used later.

1.9. Consider the functor $Z(\hat{G})$ of Kottwitz. Here $\hat{G}$ is a connected Langlands dual group for G, and $Z(\hat{G})$ is the center of $\hat{G}$ (cf. [Ko2]). By definition $\hat{G}$ is a connected reductive $\mathbb{C}$–group endowed with an algebraic action of $\Gamma = \mathrm{Gal}(\overline{K}/K)$. The group $Z(\hat{G})$ is an algebraic $\mathbb{C}$–group of multiplicative type; Γ acts on $Z(\hat{G})$ algebraically. The character group $X^*(Z(G))$ is a finitely generated Γ–module.

Proposition 1.10. The Γ–modules $\pi_1(\overline{G})$ and $X^*(Z(\hat{G}))$ are canonically isomorphic.

Proof. By definition (cf. [Ko2]) there is a maximal torus $\hat{T} \subset \hat{G}$ such that $X^*(\hat{T}) = X_*(T_{\overline{K}})$, where T is a maximal torus of G defined over K. Moreover $R(\hat{G},\hat{T}) = R^\vee(G_{\overline{K}},T_{\overline{K}})$, where R and $R^\vee$ denote the system of roots and the system of coroots, respectively. We have $Z(\hat{G}) = \cap \ker[\alpha^\vee : \hat{T} \longrightarrow G_{m\mathbb{C}}]$ where $\alpha^\vee$ runs through $R(\hat{G},\hat{T}) = R^\vee(G_{\overline{K}},T_{\overline{K}})$. Hence

$$X^*(Z(\hat{G})) = X^*(\hat{T})/\langle R(\hat{G},\hat{T})\rangle = X_*(T_{\overline{K}})/\langle R^\vee\rangle$$

where we write $R^\vee$ for $R^\vee(\overline{G},\overline{T})$ and we use $\langle\ \rangle$ to denote the subgroup of $X_*(T_{\overline{K}})$ generated by the set in brackets.

All the coroots $\alpha^\vee \in R^\vee \subset X_*(\overline{T})$ come from $X_*(\overline{T}^{(sc)})$; moreover the set $R^\vee \subset \rho_* X_*(\overline{T}^{(sc)})$ generates $\rho_* X_*(\overline{T}^{(sc)})$ (cf. [St2], Lemma 25). Thus $X^*(Z(G)) = X_*(\overline{T})/\rho_* X_*(\overline{T}^{(sc)}) = \pi_1(\overline{G})$, which was to be proved.

Remark 1.9.1. Let $\varphi : G_1 \longrightarrow G_2$ be a homomorphism of reductive K–groups. First suppose that φ is normal, i.e. $\varphi(G_1)$ is normal in G_2. Then one can define a homomorphism $\varphi^* : \hat{G}_2 \longrightarrow \hat{G}_1$ (cf. [Bo], [Ko2]). But if φ is not normal, then we cannot define φ^*. In other words, $\hat{G}$ is functorial with respect to normal homomorphisms only. Proposition 1.9 shows, however, that the center $Z(\hat{G})$ of $\hat{G}$ is functorial with respect to all homomorphisms.

Remark 1.9.2. (of personal nature). For me the fact that $\pi_1(\overline{G})$ is the character group of $Z(\hat{G})$ is not at all surprising. When defining $\pi_1(\overline{G})$ I wanted to define more functorially the functor $Z(\hat{G})$ of Kottwitz. On the contrary, I was surprised by the following result:

Proposition 1.10. Let $\overline{K}$ be $\mathbb{C}$ and let K be either $\mathbb{R}$ or $\mathbb{C}$. For a connected reductive K–group G there is a canonical isomorphism

$$\pi_1(G) \xrightarrow{\sim} \mathrm{Hom}(\pi_1^{top}(\mathbb{G}_m(\mathbb{C})),\ \pi_1^{top}(G(\mathbb{C})))$$

where π_1^{top} is the usual topological fundamental group.

For brevity we write $\pi_1(G(\mathbb{C}))$ for $\pi_1^{top}(G(\mathbb{C}))$ and $\pi_1(G(\mathbb{C}))(-1)$ for

$\mathrm{Hom}(\pi_1^{top}(\mathbb{G}_m(\mathbb{C})),\ \pi_1^{top}(G(\mathbb{C})))$.

We recall that in the case $K = \mathbb{R}$ the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\pi_1(G(\mathbb{C}))$ and (non–trivially) on $\pi_1(\mathbb{G}_m(\mathbb{C}))$. Since $\pi_1(\mathbb{G}_m(\mathbb{C}))$ is isomorphic to $\mathbb{Z}$ as a group, but not as a Γ–module, we see that $\pi_1(G(\mathbb{C}))$ and $\pi_1(G(\mathbb{C}))(-1)$ are isomorphic as groups, but in general not as Γ–modules.

In the case $K = \mathbb{C}$ we have $\Gamma=1$, and $\pi_1(G(\mathbb{C}))(-1)$ is isomorphic to $\pi_1(G(\mathbb{C}))$. To fix this isomorphism it suffices to fix an isomorphism $\pi_1(\mathbb{C}^\times) \xrightarrow{\sim} \mathbb{Z}$ (or a square root of -1 in $\mathbb{C}$).

Proposition 1.10 justifies the term "algebraic fundamental group". The proposition means that $\pi_1(\overline{G})$ is "the topological fundamental group, defined algebraically".

Proof. First we consider the case of a torus. Let T, T' be two K–tori. There is a canonical map

$$\mathrm{Hom}(T'_{\mathbb{C}},T_{\mathbb{C}}) \longrightarrow \mathrm{Hom}(\pi_1(T'(\mathbb{C})),\ \pi_1 T(\mathbb{C}))$$

This map is Γ–equivariant, and one can easily see that it is an isomorphism of groups. Taking $\mathbb{G}_m$ for T' we obtain the required isomorphism

$$\pi_1(T) = X_*(T_{\mathbb{C}}) \longrightarrow \pi_1(G(\mathbb{C}))(-1) \ .$$

In the general case we define the map $\pi_1(G) \longrightarrow \pi_1(G(\mathbb{C}))(-1)$ as follows. Choose a maximal torus $T \subset G$ defined over K ; then $\pi_1(\overline{G}) = X_*(T)/\rho_* X_*(T^{sc})$. We consider the composition

$$\alpha_T : X_*(T) \longrightarrow \pi_1(T(\mathbb{C}))(-1) \longrightarrow \pi_1(G(\mathbb{C}))(-1) \ .$$

One can easily check that $\alpha_T(\rho_*(X_*(T^{sc}))) = 0$, hence α_T induces an homomorphism

$$(\alpha_T)_* : \pi_1(\overline{G}) \longrightarrow \pi_1(G(\mathbb{C}))(-1)$$

It is not hard to check that $(\alpha_T)_*$ does not depend on the choice of T .

Now we have the commutative diagram

$$\text{(1.10.3)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\overline{G}^{ss}) & \longrightarrow & \pi_1(\overline{G}) & \longrightarrow & \pi(\overline{G}^{tor}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(G^{ss}(\mathbb{C}))(-1) & \longrightarrow & \pi_1(G(\mathbb{C}))(-1) & \longrightarrow & \pi_1(G^{tor}(\mathbb{C}))(-1) & \longrightarrow & 0 \end{array}$$

The upper row is exact by Proposition 1.5. The lower row comes from the exact sequence of the fiber bundle $G(\mathbb{C})$ over $G^{tor}(\mathbb{C})$.

We have already shown that the right vertical row in (1.10.3) is an isomorphism. The proposition 1.10 is well known for semisimple groups (cf.e.g. [V–O]), hence the left vertical arrow is an isomorphism. We conclude that the middle vertical arrow is an isomorphism. q.e.d.

1.11. Our definition of $\pi_1(\overline{G})$ uses explicitely the group structure of G . We are now going to show how to define $\pi_1(\overline{G})$ in a more "algebraic–geometrical" way. We make no further use of this construction here.

Let again K be any field of characteristic 0. Consider the algebraic–geometrical fundamental group $\pi_1^{Gr}(\overline{G})$ defined by Grothendieck [Grl] (see also [Mil]) (we take $1 \in \mathbb{G}(\overline{K})$ as the base point). Set $\pi_1^{Gr}(\overline{G})(-1) = \mathrm{Hom}(\hat{\mathbb{Z}}(1), \pi_1^{Gr}(\overline{G}))$. Note that $\hat{\mathbb{Z}}(1) = \pi_1(\mathbb{G}_{m\overline{K}})$. To any regural map $m : \mathbb{G}_{m\overline{K}} \longrightarrow G_{\overline{K}}$ such that $m(1) = 1$

we associate its class $m_* = C\ell(m) \in \pi_1^{Gr}(\overline{G})(-1) = (\mathrm{Hom}\ \pi_1^{Gr}(G_{m\overline{K}}), \pi_1^{Gr}(G_{\overline{K}}))$. Let $\pi_1^{Gr}(\overline{G})(-1)_{alg}$ denote the subset of such algebraic classes in $\pi_1^{Gr}(\overline{G})(-1)$.

Proposition 1.12. (i) $\pi_1^{Gr}(\overline{G})(-1)alg$ is a subgroup of the abelian group $\pi_1^{Gr}(\overline{G})(-1)$.

(ii) The map $m \longmapsto C\ell(M)$ induces an isomorphism of Γ–modules $\pi_1(\overline{G}) \xrightarrow{\sim} \pi_1^{Gr}(\overline{G})(-1)alg$.

(iii) $\pi_1^{Gr}(\overline{G})(-1)$ is isomorphic (as a Γ–module) to the completion of $\pi_1(\overline{G})$ with respect to the topology defined by the subgroups of finite index.

We omit the proof.

Remark 1.13. Let H be a connected K–subgroup of G . Consider the homogeneous space $X = H\backslash G$. It has a canonical base point, namely the image of the neutral element of G . In this case one can similarly define the algebraic fundamental group $\pi_1(\overline{X})$ as the set of algebraic classes in

$$\pi_1^{Gr}(\overline{X})(-1) = \mathrm{Hom}(\pi_1^{Gr}(G_{m\overline{K}}), \pi_1^{Gr}(X_{\overline{K}})) .$$

One can show that $\pi_1^{Gr}(\overline{X})$ is an abelian group and that $\pi_1(\overline{X}) = \pi_1^{Gr}(\overline{X})(-1)alg$ is a subgroup. In the case $\overline{K} = \mathbb{C}$ we have $\pi_1(\overline{X}) \simeq \pi_1^{top}(X(\mathbb{C}))(-1)$.

2. Abelian Galois cohomology

2.1. Let K be a field of Characteristic 0. We write Γ for $\mathrm{Cal}(\overline{K}/K)$. Let G be a (connected) reductive K–group. Choose a maximal torus $T \subset G$ (defined over K). We consider the complex of tori

$$T^{\cdot} = (T^{(sc)} \xrightarrow{\rho} T)$$

where T is in degree 0 and $T^{(sc)}$ is in degree –1. We define *the abelian Galois cohomology* of G as follows:

Definition 2.2. $H^i_{ab}(K,G) = \mathbb{H}^i(K,T^{\cdot})$.

Here $\mathbb{H}^i$ means that Galois hypercohomology of the complex $T^{(sc)}(\overline{K}) \longrightarrow T(\overline{K})$ of $\mathrm{Gal}(\overline{K}/K)$–modules. We may regard $H^{\cdot}_{ab}(K,G)$ as the hypercohomology of the double complex

(2.2.1)
$$\begin{array}{ccccc} 0 \to T(\overline{K}) & \longrightarrow & C^1(\Gamma,T(\overline{K})) & \longrightarrow & C^2(\Gamma,T(\overline{K})) \to \dots \\ \uparrow & & \uparrow & & \uparrow \\ 0 \to T^{(sc)}(\overline{K}) & \longrightarrow & C^1(\Gamma,T^{(sc)}(\overline{K})) & \longrightarrow & C^2(\Gamma,T^{(sc)}(\overline{K})) \to \dots \end{array}$$

where C^i are the usual groups of non–homogeneous continuous cochains. Note that the bidegree of $T^{sc}(\overline{K})$ is (–1,0).

We see that the groups $H^i_{ab}(K,G)$ do not depend of the choice of the algebraic closure $\overline{K}$ of K. We are going to show in this section that they neither depend on the choice of T. Moreover, they depend only on $\pi_1(\overline{G})$.

2.3. Short torsion free resolutions.

Let Δ be a finite group and M a finitely generated Δ–module.

Definition 2.3.1. A short torsion free resolution of M is an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow L^{0} \longrightarrow M \longrightarrow 0$$

of Δ modules such that L^{-1} and L^{0} are finitely generated and torsion free (over $\mathbb{Z}$).

We write $L^{\cdot}$ for the complex $(L^{-1} \longrightarrow L^{0})$. For brevity we shall speak of resolutions of M meaning short torsion free resolutions.

Let $L_1^{\cdot} \longrightarrow M$ and $L_2^{\cdot} \longrightarrow M$ be two resolutions. We say that the resolution $L_1^{\cdot}$ *dominates* $L_2^{\cdot}$ if there exists a surjective morphism $L_1^{\cdot} \longrightarrow L_2^{\cdot}$ of resolutions, i.e. a commutative diagram

$$\begin{array}{ccc} L_1^{\cdot} & \longrightarrow & M \\ \downarrow & & \| \\ L_2^{\cdot} & \longrightarrow & M \end{array}$$

such that the homomorphisms $L_1^{i} \longrightarrow L_2^{i}$ are surjective for $i = -1, \quad 0$.

Lemma 2.3.2. (i) For any finitely generated Δ–module M there exists a short torsion free resolution $L^{\cdot} \longrightarrow M$.

(ii) For any two resolutions $L_1^{\cdot} \longrightarrow M$ and $L_2^{\cdot} \longrightarrow M$ there exist a resolution $L_3^{\cdot} \longrightarrow M$

dominating both $L_1^{\cdot}$ and $L_2^{\cdot}$.

Proof (i). There exists an epimorphism $\mathbb{Z}[\Delta]^k \longrightarrow M$, where k is a natural number. We set $L^0 = \mathbb{Z}[\Delta]^k$, $L^{-1} = \ker [L^0 \longrightarrow M]$.
(ii) We take for $L^{\cdot}$ the fiber product of $L_1^{\cdot}$ and $L_2^{\cdot}$ over M. This means that $L^0 = L_1^0 \times_M L_2^0$, $L^{-1} = L_1^{-1} \oplus L_2^{-1}$.

Lemma 2.3.3. Let $\mu : M_1 \longrightarrow M_2$ be a morphism of Δ–modules.
(i) There exists a short torsion free reduction of μ, i.e. a commutative diagram

$$\begin{array}{ccc} L_1^{\cdot} & \longrightarrow & L_2^{\cdot} \\ \downarrow & & \| \\ M_1 & \xrightarrow{\mu} & M_2 \end{array}$$

where $L_1^{\cdot}$ and $L_2^{\cdot}$ are resolutions of M_1 and M_2 , respectively. Moreover, if μ is surjective, we can choose $L_1^{\cdot} \longrightarrow L_2^{\cdot}$ to be an epimorphism of complexes.
(ii) For any two resolutions of μ there exists a third one dominating both (in the above sense).

Proof. (i) Let $L_2^{\cdot} \longrightarrow M_2$ be a resolution of M_2 and let $L^{\cdot} \longrightarrow M_1$ be a resolution of M_1. We take for $L_1^{\cdot}$ the fiber product of $L^{\cdot}$ and $L_2^{\cdot}$ over M_2.
(ii) We construct the third resolution of μ as the fiber product over μ of the first and the second ones.

Lemma 2.3.4. Let

(M) $$0 \longrightarrow M_1 \xrightarrow{\lambda} M_2 \xrightarrow{\mu} M_3 \longrightarrow 0$$

be a short exact sequence of Δ–modules.

(i) There exist a short torsion free resolution of (M) , i.e. a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & L_1^{\bullet} & \longrightarrow & L_2^{\bullet} & \longrightarrow & L_3^{\bullet} & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with exact rows, where $L_i^{\bullet} \longrightarrow M_i$ is a resolution of M_i for $i = 1,2,3$.

(ii) For any two such resolutions of (M) there exists a third one that dominates both (in the obvious sense).

Proof. (i) By Lemma 2.3.3 there exists a resolution $(L_2^{\bullet} \longrightarrow L_3^{\bullet}) \longrightarrow (M_2 \longrightarrow M_3)$ of μ, such $L_2^{\bullet} \longrightarrow L_3^{\bullet}$ is an epimorphism of complexes. We set $L_1^{\bullet} = \ker[L_2^{\bullet} \longrightarrow L_3^{\bullet}]$.

(ii) We use the fiber product construction.

Now let D be any Δ–module. Choose a short torsion free resolution $L^{\bullet} \longrightarrow M$. We consider the complex

$$L^{\bullet} \underset{\mathbb{Z}}{\otimes} D = (L^{-1} \otimes D \longrightarrow L^0 \otimes D)$$

Definition 2.4. $\mathcal{H}^i(\Delta,M,D) = \mathbb{H}^i(\Delta,L^{\bullet} \otimes D)$.

To prove the correctness of Definition 2.4 we have to prove that $\mathbb{H}^i(\Delta, L^\cdot \otimes D)$ does not depend on the choice of the short torsion free resolution $L^\cdot$ of M.

First note that if a resolution $L_1^\cdot \longrightarrow M$ dominated a resolution $L_2^\cdot \longrightarrow M$, then the commutative diagram

$$\begin{array}{ccc} L_1^\cdot & \searrow & \\ \downarrow \alpha & & M \\ L_2^\cdot & \nearrow & \end{array}$$

defines a quasi–isomorphism $\alpha : L_1^\cdot \longrightarrow L_2^\cdot$ of complexes. Since torsion free $\mathbb{Z}$–modules are acyclic under the tensor product functor $\underset{\mathbb{Z}}{\otimes} D$, the morphism

$$\alpha \otimes D : L_1^\cdot \otimes D \longrightarrow L_2^\cdot \otimes D$$

is again a quasi–isomorphism. Any quasi–isomorphism $C_1^\cdot \longrightarrow C_2^\cdot$ of complexes of Δ–modules induces an isomorphism $\mathbb{H}^i(\Delta, C_1^\cdot) \xrightarrow{\sim} \mathbb{H}^i(\Delta, C_2^\cdot)$ on the hypercohomology. Thus in our case we have a canonical isomorphism

$$\alpha_* : \mathbb{H}^i(\Delta, L_1^\cdot \otimes D) \xrightarrow{\sim} \mathbb{H}^i(\Delta, L_2^\cdot \otimes D)$$

Now let $L_1^\cdot \longrightarrow M$ and $L_2^\cdot \longrightarrow M$ be two resolutions. Applying Lemma 2.3.2 (ii) we obtain that there is an isomorphism

$$\mathbb{H}^i(\Delta, L_1^\cdot \otimes D) \xrightarrow{\sim} \mathbb{H}^i(\Delta, L_2^\cdot \otimes D).$$

Applying Lemma 2.3.2 (ii) once more, we see that this isomorphism is canonical. Thus Definition 2.4 is correct.

2.5. Let $\mu : M_1 \longrightarrow M_2$ be a morphism of Δ–modules. Using Lemma 2.3.3 one can uniquely define the morphism

$$\mu_* : \mathscr{H}^i(\Delta,M_1,D) \longrightarrow \mathscr{H}^i(\Delta,M_2,D).$$

Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of Δ–modules. Using Lemma 2.3.4 one can uniquely define a family of connecting homomorphisms

$$\delta^i : \mathscr{H}^i(\Delta,M_3,D) \longrightarrow \mathscr{H}^{i+1}(\Delta,M_1,D)$$

such that the sequence

$$(2.5.1) \longrightarrow \mathscr{H}^i(\Delta,M_1,D) \longrightarrow \mathscr{H}^i(\Delta,M_2,D) \longrightarrow \mathscr{H}^i(\Delta,M_3,D) \xrightarrow{\delta} \mathscr{H}^{i+1}(\Delta,M_1,D) \longrightarrow \ldots$$

is exact.

We see that $\mathscr{H}^{\cdot}(\Delta,M,D)$ is a cohomological functor of M. Note that $\mathscr{H}^i(\Delta,M,D) = 0$ for $i \leq -2$.

Remark 2.5.2. In the language of derived categories we have just

$$\mathscr{H}^i(\Delta,M,D) = \mathbb{H}^i(\Delta, M \overset{L}{\underset{\mathbb{Z}}{\otimes}} D),$$

where $\overset{L}{\underset{\mathbb{Z}}{\otimes}}$ denotes the left derived functor of the tensor product.

Remark 2.5.3. We can also define the "Tate groups"

$$\hat{\mathscr{H}}^i(\Delta,M,D) := \hat{\mathbb{H}}^i(\Delta, L^\bullet \underset{\mathbb{Z}}{\otimes} M) \quad (i \in \mathbb{Z}) \ ,$$

where $L^\bullet \longrightarrow M$ is a short torsion free resolution. Here $\hat{\mathbb{H}}^\bullet$ denotes the hypercohomology of the double complex $\mathrm{Hom}(P^\bullet, L^\bullet)$, where $P^\bullet$ is a *complete resolution* for Δ (see e.g. [A–W]).

Proposition 2.6. Let $L^\bullet \longrightarrow M$ be a short torsion free resolution of M, and let D be a Δ–module. Then there is an exact sequence

(2.6.1) $$0 \longrightarrow \mathscr{H}^{-1}(\Delta,M,D) \longrightarrow H^0(\Delta, L^{-1} \otimes D) \longrightarrow H^0(\Delta, L^0 \otimes D) \longrightarrow \mathscr{H}^0(\Delta,M,D) \longrightarrow H^1(\Delta, L^{-1} \otimes D) \longrightarrow \dots$$

Proof. We consider the short exact sequence of complexes

$$0 \longrightarrow (0 \longrightarrow L^0 \otimes D) \longrightarrow L^\bullet \otimes D \longrightarrow (L^{-1} \otimes D \longrightarrow 0) \longrightarrow 0$$

and write down the corresponding long hypercohomology exact sequence

2.7. If Δ is a finite group and U is a normal subgroup of Δ, then we have inflation homomorphisms

$$\mathscr{H}^i(\Delta/U, M^U, D^U) \longrightarrow \mathscr{H}^i(\Delta, M, D)$$

Now let Γ be a pro–finite group and M a finitely generated (over $\mathbb{Z}$) discrete Γ–module. Let D be a discrete Γ–module. We set

$$\mathscr{H}^i(\Gamma, M, D) = \varinjlim_{U} \mathscr{H}^i(\Gamma/U, M^U, D^U) ,$$

where U runs over the open normal subgroup of Γ.

Let $L^{\cdot} \longrightarrow M$ be a *short torsion free resolution* of M, i.e. an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow L^0 \longrightarrow M \longrightarrow 0$$

of discrete Γ–modules, where L^{-1} and L^0 are finitely generated torsion free abelian groups. Let $\mathbb{H}^i(\Gamma, L^{\cdot}, D)$ denote the hypercohomology of the double complex

$$\begin{array}{ccccccc}
0 \longrightarrow & C^0(\Gamma, L^0 \otimes D) & \longrightarrow & C^1(\Gamma, L^0 \otimes D) & \longrightarrow & C^2(\Gamma, L^0 \otimes D) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & C^0(\Gamma, L^{-1} \otimes D) & \longrightarrow & C^1(\Gamma, L^{-1} \otimes D) & \longrightarrow & C^2(\Gamma, L^{-1} \otimes D) & \longrightarrow \cdots
\end{array}$$

where $C^i(\Gamma, \cdot)$ denotes the group of continuous non–homogeneous cochains. Since $M^U = M$ for sufficiently small U, we have

$$\mathscr{H}^i(\Gamma,M,D) = \mathbb{H}^i(\Gamma, L^\cdot \underset{\mathbb{Z}}{\otimes} D).$$

2.8. Let Γ again denote the Galois group $\mathrm{Gal}(\bar{K}/K)$. Let M be a discrete finitely generated Γ–module. We are interested in the groups $\mathscr{H}^i(\mathrm{Gal}(\bar{K}/K),M;\bar{K}^\times)$; for brevity we write just $\mathscr{H}^i(K,M,\bar{K}^\times)$.

Let $L^\cdot \longrightarrow M$ be a short torsion free resolution. Consider the complex $T^{-1} \longrightarrow T^0$ of K–tori such that $L^\cdot = (L^{-1} \longrightarrow L^0)$ is the complex $X_*(T^{-1}_{\bar{K}}) \longrightarrow X_*(T_{\bar{K}})$ of cocharacter groups of these tori. By definition

$$\mathscr{H}^i(K,M,\bar{K}^\times) = \mathbb{H}^i(K, L^{-1} \otimes \bar{K}^\times \longrightarrow L^0 \otimes \bar{K}^\times) = \mathbb{H}^i(K, T^{-1} \longrightarrow T^0)$$

Thus $\mathscr{H}^i(K,M,\bar{K}^\times)$ is the Galois hypercohomology of a complex of tori.

2.9 Examples. (1) If M is torsion free, then we set $L^{-1} = 0$, $L^0 = M$, $X_*(T^0) = M$. Thus $\mathscr{H}^1(K,M,\bar{K})^\times = \mathbb{H}^i(K,T)$.

(2) Suppose that M is finite. Choose a resolution $L^\cdot \longrightarrow M$ and define the complex $T^\cdot = T^{-1} \longrightarrow T^0$ as above. Then the homomorphism $T^{-1}(\bar{K}) \longrightarrow T^0(\bar{K})$ is surjective. Set $B = \ker[T^{-1} \longrightarrow T^0]$; it is a finite abelian K–group. Then the homomorphism

$$(B(\bar{K}) \longrightarrow 0) \longrightarrow (T^{-1}(\bar{K}) \longrightarrow T^0(\bar{K}))$$

of complexes is a quasi–isomorphism. Hence

$$\mathscr{H}^i(K,M,\overline{K}^\times) := \mathbb{H}^i(K,T^{-1}(\overline{K}) \longrightarrow T^0(\overline{K})) = \mathbb{H}^i(K,B(\overline{K}) \longrightarrow 0) = H^{i+1}(K,B).$$

Now let G be a connected reductive K–group.

Proposition 2.10. $H^i_{ab}(K,G) == \mathscr{H}^i(K,\pi_1(\overline{G}),\overline{K}^\times)$

Proof. Let $T \subset G$ be a maximal torus (defined over K). Set $L^0 = X_*(T_{\overline{K}})$, $L^{-1} = X_*(T^{(sc)})$. Then by definition of $\pi_1(\overline{G})$, $(L^{-1} \longrightarrow L^0) \longrightarrow \pi_1(\overline{G})$ is a resolution of $\pi_1(\overline{G})$. Hence, as it was shown in n°2.7, $\mathscr{H}^i(K,\pi_1(\overline{G}),\overline{K}^\times) = \mathbb{H}^i(K,T^{(sc)} \longrightarrow T)$. By definition $\mathbb{H}^i(K,T^{(sc)} \longrightarrow T) = H^i_{ab}(K,G)$. This proves the proposition.

We see from Proposition 2.10 that the groups $H^i_{ab}(K,G)$ depend only on the Galois module $\pi_1(\overline{G})$.

Corollary 2.11. Let $z \in H^1(K,G^{ad})$ be a cocycle. There are canonical isomorphisms $H^i_{ab}(K,{}^zG) \longrightarrow H^i_{ab}(K,G)$.

Proof. The assertion follows from Lemma 1.8 and Proposition 2.10.

Proposition 2.12. Let $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ be an exact sequence of connected reductive K–groups. Then there is a long abelian cohomology exact sequence

$$(2.12.1) \qquad 0 \longrightarrow H^{-1}_{ab}(K,G_1) \longrightarrow H^{-1}_{ab}(K,G_2) \longrightarrow H^{-1}_{ab}(K,G_3) \longrightarrow H^0(K,G_1) \longrightarrow \dots$$

Proof. The assertion follows from Lemma 1.5 and the results of n°2.5 (cf. (2.5.1)).

The exact sequence (2.12.1) can be defined more explicitly as follows. Let $T_2 \subset G_2$ be a maximal torus. Let T_3 be the image of T_2 in G_3, and let T_1 be the inverse image of T_2 in G_1. We have the short exact sequence

$$0 \longrightarrow (T_1^{(sc)} \longrightarrow T_1) \longrightarrow (T_2^{(sc)} \longrightarrow T_2) \longrightarrow (T_3^{(sc)} \longrightarrow T_3) \longrightarrow 0$$

of complexes of tori. Then (2.12.1) is the corresponding long hypercohomology exact sequence.

2.13 Examples. (1) G is a torus. Then $(T^{(sc)} \longrightarrow T) = (1 \longrightarrow G)$, and $H^i_{ab}(K,G) = H^i(K,G)$.

(2) Suppose that G^{ss} is simply connected. By 1.6(2) the homomorphism $\pi_1(\overline{G}) = \pi_1(\overline{G}^{tor})$ is an isomorphism, hence $H^i_{ab}(K,G) = H^i(K,G^{tor})$.

(3) Let G be a semisimple group, $G = G^{sc}/\ker \rho$. Then $\ker(T^{(sc)} \longrightarrow T) = \ker \rho$, and by 2.9 (2) $H^i_{ab}(K,G) = H^{i+1}(K,\ker \rho)$. Recall that $\ker \rho$ is a finite abelian K–group.

(4) For any G we have $H^{-1}_{ab}(K,G) = (\ker \rho)(K)$. This follows from the definition (the reader should look at the double complex (2.2.1)).

Proposition 2.14. Let G be a connected reductive K–group. Let $T \subset G$ be a maximal K–torus. Then there are exact sequences

(2.14.1) $\ldots \longrightarrow H^{i+1}(K,\ker \rho) \longrightarrow H^i_{ab}(K,G) \longrightarrow H^i(K,G^{tor}) \longrightarrow H^{i+2}(K,\ker \rho) \longrightarrow \ldots$

(2.14.2) $\ldots \longrightarrow H^i(K,T^{(sc)}) \longrightarrow H^i(K,T) \longrightarrow H^i_{ab}(K,G) \longrightarrow H^{i+1}(K,T^{(sc)} \longrightarrow \ldots$

Proof. Consider the short exact sequence

$$1 \longrightarrow G^{ss} \longrightarrow G \longrightarrow G^{tor} \longrightarrow 1$$

Applying Proposition 2.12 and calculations 2.13 (1,3), we obtain (2.14.1). We obtain (2.14.2) from Proposition 2.10 and Proposition 2.6.

3. The abelianization map

In this section we construct the abelianization maps

$$ab^0 : G(K) = H^0(K,G) \longrightarrow H^0_{ab}(K,G)$$
$$ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$$

for a reductive group G over a field K of characteristic 0. We follow closely the construction of Kottwitz [Ko3].

3.1. For any K–torus T we have canonical isomorphisms

$$H^i(K,T) \xrightarrow{\sim} H^i_{ab}(K,T) \quad . \tag{3.1.1}$$

These isomorphisms are isomorphisms of functors $T \longmapsto H^i(K,T)$ and $T \longmapsto H^i_{ab}(K,T)$ from the category $\mathcal{T}$ of K–tori to (the category of) abelian groups.

We consider the category $\mathcal{G}$ of connected reductive K–groups G and (all) their K–homomorphisms. Let $\mathcal{G}_0$ denote the full subcategory of $\mathcal{G}$ whose objects are reductive K–groups G such that G^{ss} is simply connected.

Theorem 3.2. The isomorphisms (3.1.1) for $i = 0,1$ can be uniquely prolonged to morphisms of functors

$$ab^0 : G(K) = H^0(K,G) \longrightarrow H^0_{ab}(K,G)$$

(from $\mathcal{G}$ to abelian groups) and

$$ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$$

(from $\mathcal{G}$ to sets).

We prove Theorem 3.2 in Subsections 3.3–3.5.

3.3. First we extend (3.1.1) (for $i = 0,1$) to $\mathcal{G}_0$. Let G be a K–group such that G^{ss} is simply connected. The diagram

$$\begin{array}{ccc} H^i(K,G) & \longrightarrow & H^i(K,G^{tor}) \\ \downarrow ab^i_G & & \wr\downarrow ab^i_{G^{tor}} \\ H^i_{ab}(K,G) & \xrightarrow{\sim} & H^i_{ab}(K,G)^{tor} \end{array} \qquad (i=0,1)$$

is commutative, and we are forced to define ab^i_G $(i=0,1)$ as the composition

$$H^i(K,G) \longrightarrow H^i(K,G^{tor}) = H^i_{ab}(K,G)$$

The map ab^0_G is a homomorphism of groups. Since $G \longmapsto G^{tor}$ is a functor (in our case from $\mathcal{G}_0$ to $\mathcal{T}$), we see that ab^0 and ab^1 are morphisms of functors.

To extend ab^0 and ab^1 to $\mathcal{G}$ we need *z–extensions*. The notion of a z–extension was introduced by Langlands [La1], [La2] and extensively used by Kottwitz. We collect in this section definitions and lemmas from a number of papers ([Ko1], [M–Sh], [Ko2], [Ko3]).

3.4. z–Extensions

Definition 3.4.1. Let G be a connected reductive K–group. A central extension

$$1 \longrightarrow Z \longrightarrow H \longrightarrow G \longrightarrow 1$$

of G is called a z–extension if H^{ss} is simply connected and Z is a product of tori of the form $R_{F/K}\mathbb{G}_m$ for finite extensions F/K.

Consider the canonical covering $\rho' = G^{sc} \times Z(G)^o \longrightarrow G$, where $Z(G)^o$ is the connected component of the center Z(G) of G, and the map ρ' is defined by $(g,z) \longmapsto \rho(g)\cdot z$ for $g \in G^{sc}$, $z \in Z(G)^o$. Set $A = \ker \rho'$; it is a finite abelian group.

Lemma 3.4.2. Let F/K be a finite Galois extension such that $\mathrm{Gal}(\overline{F}/F)$ acts on $X^*(A)$ trivially. Then there exists a z–extenson $H \longrightarrow G$ with kernel Z such that $Z \simeq (R_{F/K}\mathbb{G}_m)^n$ for some natural n.

Remark 3.4.2.1. This result was proved by Milne and Shih [M–Sh] with the additional hypothesis that F splits G.

Proof of Lemma 3.4.2. Set $\Delta = \mathrm{Gal}(F/K)$. There exists a surjective homomorphism $s : L \longrightarrow X^*(A)$ of Δ–modules, where L is a $\mathbb{Z}[\Delta]$–free module. Set Z be a K–torus such that $X^*(Z) = L$; it is a torus of the form $(R_{F/K}\mathbb{G}_m)^n$. Since s is surjective, the induced homomorphism $s^* : A \longrightarrow Z$ is injective. We set

$$H = (G^{sc} \times Z(G)^o \times Z)/A$$

and define $\alpha_H : H \longrightarrow G = (G^{sc} \times Z(G)^o)/A$ to be the epimorphism induced by the projection

$$G^{sc} \times Z(G)^o \times Z \longrightarrow G^{sc} \times Z(G)^o .$$

Then $\ker \alpha_H \simeq Z$ (because $A \hookrightarrow G^{sc} \times Z(G)^0$ is injective) and $H^{ss} \simeq G^{sc}$ (because $A \hookrightarrow Z$ is injective). The lemma is proved.

We need a special kind of z–extensions, namely, *ξ–lifting z–extensions* .

Definition 3.4.3. Let $\xi \in H^i(K,G)$ $(i = 0,1)$. A z–extension $\alpha : H \longrightarrow G$ is called a ξ–lifting z–extension if ξ comes from $H^i(K,H)$.

We observe that in the case $i = 0$ any z–extension is ξ–lifting for any $\xi \in H^0(K,G) = G(K)$. In the case $i = 1$ there is

Lemma 3.4.4 (Kottwitz [Ko3]). Let F/K be finite Galois extension such that $\mathrm{Res}_{K/F} : H^1(K,G) \longrightarrow H^1(F,G)$ takes ξ to 1. Let $Z \hookrightarrow H \longrightarrow G$ be a z–extension whose kernel Z is of the form $(R_{F/K}\mathbb{G}_m)^n$. Then $H \longrightarrow G$ is a ξ–lifting extension.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} H^1(K,H) & \longrightarrow & H^1(K,G) & \longrightarrow & H^2(K,Z) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F,H) & \longrightarrow & H^1(F,G) & \longrightarrow & H^2(F,Z) \end{array}$$

with exact rows. Since $Z \simeq (R_{F/K}\mathbb{G}_m)^n$, the restriction homomorphism $H^2(K,Z) \longrightarrow H^2(F,Z)$ is injective. We see from the diagram that the image of ξ in $H^2(K,Z)$ is trivial. Hence ξ comes from $H^1(K,Z)$, which was to be proved.

By definition any element $\xi \in H^1(K,G)$ comes from $H^1(F/K,G)$ for some finite Galois extension F/K . Then $\mathrm{Res}_{K/F}\xi = 1$. Thus we get

Corollary 3.4.5 (Kottwitz [Ko3]). For any $\xi \in H^1(K,G)$ there exists a ξ–lifting z–extension $H \longrightarrow G$.

The corollary follows from Lemma 3.4.2 and Lemma 3.4.4.

3.5. Now we can extend the maps ab^0 and ab^1 from $\mathcal{G}_0$ to $\mathcal{G}$.

3.5.1. Let $\xi \in H^i(K,G)$ $(i = 0,1)$. Choose a ξ–lifting z–extension $Z \hookrightarrow H \xrightarrow{\alpha} G$ and consider the commutative diagram

$$\begin{array}{ccc}
H^i(K,Z) & = & H^i(K,Z) \\
\downarrow & & \downarrow \\
H^i(K,H) & \xrightarrow{ab_H^i} & H^i_{ab}(K,H) \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_{ab}} \\
H^i(K,G) & \overset{ab_G^i}{\dashrightarrow} & H^i_{ab}(K,G)
\end{array}$$

The element $\xi \in H^1(K,G)$ is the image of some element $\eta \in H^1(K,H)$. We are forced to set $ab_G^i(\xi) = \alpha_{ab}(ab_H^i(\eta))$. Recall that the map ab_H^i has been defined before (because H^{ss} is simply connected).

In the case $i = 1$ we have $H^i(K,Z) = 0$, hence η is unique and $ab_G^i(\xi)$ is defined uniquely. In the case $i = 0$ the lifting η of ξ is not unique, but one can easily see from the diagram that $\alpha_{ab}(ab_H^0(\eta))$ does not depend on the choice of η. It is clear that ab_G^0 is a group homomorphism.

3.5.2. We have to prove that the above defined element $ab_G^i(\xi)$ does not depend on the choice of the z–extension $H \longrightarrow G$.

Let $\alpha_1 : H_1 \longrightarrow G$ and $\alpha_2 : H_2 \longrightarrow G$ be two z–extensions. We say that α_1

dominates α_2 if there exists a surjective morphism of z–extensions

$$\begin{array}{ccc} H_1 & \longrightarrow & H_2 \\ & {\scriptstyle \alpha_1} \searrow \quad \swarrow {\scriptstyle \alpha_2} & \\ & G & \end{array}$$

Lemma 3.5.3. Let $\xi \in H^i(K,G)$ $(i = 0;1)$. Let $\alpha_1 : H_1 \longrightarrow G_1$ and $\alpha_2 : H_2 \longrightarrow G$ be two ξ–lifting z–extensions. Then there exists a third one $\alpha : H \longrightarrow G$, dominating both.

Proof. We set $H = H_1 \times_G H_2$ (fiber product). Then $\alpha : H \longrightarrow G$ is surjective and $\ker \alpha = \ker \alpha_1 \times \ker \alpha_2$. We see that α is a z–extension. In the case $i = 1$ the set of cocycles $Z^1(K,H)$ is the fiber product of $Z^1(K,H_2)$ and $Z^1(K,H_2)$. Since α_1 and α_2 are ξ–lifting extensions, we conclude that α is also a ξ–lifting extension. In the case i=0 any z–extension is ξ–lifting. The lemma is proved.

3.5.4. We prove that the construction of $ab^i_G(\xi)$ does not depend on the choice of z–extension $H \longrightarrow G$. Let $\alpha_1 : H_1 \longrightarrow G$ and $\alpha_2 : H_2 \longrightarrow G$ be two ξ–lifting z–extensions. First suppose that α_1 dominates α_2 . Then we have commutative diagrams

$$\begin{array}{ccc} H_1 & \xrightarrow{\ \beta\ } & H_2 \\ & {\scriptstyle \alpha_1} \searrow \quad \swarrow {\scriptstyle \alpha_2} & \\ & G & \end{array}
\qquad\qquad
\begin{array}{ccc} H^i(K,H_1) & \xrightarrow{ab^i_1} & H^i_{ab}(K,H_1) \\ \downarrow {\scriptstyle \beta_*} & & \downarrow {\scriptstyle \beta_{ab}} \\ H^i(K,H_2) & \xrightarrow{ab^i_2} & H^i_{ab}(K,H_2) \\ \downarrow {\scriptstyle \alpha_{2*}} & & \downarrow {\scriptstyle \alpha_{2ab}} \\ H^i(K,G) & & H^i_{ab}(K,G) \end{array}$$

Let $\eta_1 \in H^i(K,H_1)$ be an element such that $\alpha_{1*}(\eta_1) = \xi$. Set $\eta_2 = \beta_*(\eta_1) \in H^i(K,H_2)$. Then $\alpha_{2*}(\eta_2) = \xi$. Since ab^i is a morphism of functors on $\mathcal{G}_0$, the rectangle in the diagram of cohomology above is commutative, and therefore $ab_2^i(\eta_2) = \beta_{ab}(ab_1^i(\eta_1))$. We conclude that $\alpha_{1ab}(ab_1^i(\eta_1)) = \alpha_{2ab}(ab_2^i(\eta_2))$. Thus in this case $\alpha_1 : H_1 \longrightarrow G$ and $\alpha_2 : H_2 \longrightarrow G$ gives us the same element $ab_G^i(\xi)$.

Now let α_1 and α_2 be any two ξ–lifting z–extensions. Using Lemma 3.5.1 we reduce the assertion to be proved to the already considered case when α_1 dominates α_2. Thus we have proved that the definition of $ab_G^i(\xi)$ does not depend on the choice of the z–extension $H \longrightarrow G$.

3.5.5. We have defined the map $ab_G^i : H^i(K,G) \longrightarrow H^i_{ab}(K,G)$. One can easily check that ab_G^0 is a homomorphism of groups. We must now prove that ab_G^i is a morphism of functors. To do it we need

Lemma 3.5.6 [Ko3]. Let $\beta: G_1 \longrightarrow G_2$ be a homomorphism of connected reductive K–groups. Let $\xi_1 \in H^1(K,H)$. Then there exists a ξ_1–lifting z–extension of β, i.e. a commutative diagram

(3.5.6.1)
$$\begin{array}{ccc} H_1 & \xrightarrow{\beta_H} & H_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ G_1 & \xrightarrow{\beta} & G_2 \end{array}$$

such that α_1 is a ξ_1–lifting z–extension and α_2 is a z–extension.

Proof. Set $\xi_2 = \beta_*\xi_1$. Let $\alpha : H \longrightarrow G_1$ be any ξ_1–lifting z–extension and let $\alpha_2 : H_2 \longrightarrow G_2$ be any ξ_2–lifting z–extension. Let H_1 be the fiber product of H and H_2 over G_2. We have canonical homomorphisms $\alpha_1 : H_1 \longrightarrow G_1$ and

$\beta_H : H_1 \longrightarrow H_2$. One can easily see that α_1 is a z–extension with kernel $\ker \alpha \times \ker \alpha_2$. Since $Z^1(K,H_1)$ is the fiber product of $Z^1(K,H)$ and $Z^1(K,H_2)$ over $Z^1(K,G_2)$, we see that α_1 is a ξ_1–lifting extension.

We will need later the following version of Lemma 3.5.6.

Lemma 3.5.7. Let β be a surjective homomorphism of connected reductive K–groups. Let $\xi_2 \in H^1(K,G_2)$. Then there exists a z–extension (3.5.6.1) of β such that β_H is surjective and α_2 is a ξ_2–lifting extension.

Proof. Let $\alpha_2 : H_2 \longrightarrow G_2$ be a ξ_2–lifting z–extension. Let $\alpha : H \longrightarrow G_1$ be any z–extension. We set $H_1 = H \times_{G_2} H_2$.

We prove that ab^i $(i = 0,1)$ is a morphism of functors. We consider the case $i = 1$; the case $i = 0$ can be proved similarily. Let $G_1, G_2, H_1, H_2, \xi_1, \xi_2$ be as in Lemma 3.5.6. We have the commutative diagram

$$\begin{array}{ccccccc}
H^i(K,G_1) & \longleftarrow & H^i(K,H_1) & \longrightarrow & H^i_{ab}(K,H_1) & \longrightarrow & H^i_{ab}(K,G_1) \\
\beta_* \downarrow & & \downarrow (\beta_H)_* & & \downarrow & & \downarrow \beta_{ab} \\
H^i(K,G_2) & \longleftarrow & H^i(K,H_2) & \longrightarrow & H^i_{ab}(K,H_2) & \longrightarrow & H^i_{ab}(K,G_2)
\end{array}$$

where the commutativity of the central rectangle follows from the already proved functoriality of ab^i_G on $\mathcal{G}_0$. Let $\eta_1 \in H^1(K,H_1)$ be a lifting of ξ_1 ; then $(\beta_H)_*(\eta) \in H^1(K,H_2)$ is a lifting of ξ_2 (because the left rectangle is commutative). Now from the commutativity of the other two rectangles we see that

$\beta_{ab}(ab^1_{G_1}(\xi_2)) = ab^1_{G_2}(\xi_2)$.

q.e.d.

Theorem 3.2 is proved. In the remaining part of this section we prove three propositions that complete Theorem 3.2.

Proposition 3.6 [Ko3]. Ker $ab^i_G = \rho_* H^i(K,G^{sc})$ (i = 0,1) .

Proof. First suppose that G^{ss} is simply connected. Then ab^i_G is just the map $H^i(K,G) \longrightarrow H^i(K,G^{tor})$ induced by the canonical homomorphism $G \longrightarrow G^{tor}$. In this case the assertion follows from the exact cohomology sequence

$$\dots \longrightarrow H^i(K,G^{ss}) \xrightarrow{\rho_*} H^i(K,G) \longrightarrow H^i(K,G^{tor}) \longrightarrow \dots$$

In the general case we have the diagram

$$\begin{array}{ccc} H^i(K,G^{sc}) & \longrightarrow & H^i_{ab}(K,G^{sc}) = 0 \\ {\scriptstyle\rho_*}\downarrow & & \downarrow \\ H^i(K,G) & \longrightarrow & H^i_{ab}(K,G) \end{array}$$

which is commutative because ab^i is a morphism of functors. From this diagram it is clear that $\rho_* H^1(K,G^{sc}) \subset \ker ab^i_G$.

Now let $\xi \in \ker ab^i_G$. Choose a ξ–lifting z–extension $Z \hookrightarrow H \xrightarrow{\alpha} G$. We have the commutative diagram

$$\begin{array}{ccc} H^i(K,Z) & = & H^i_{ab}(K,Z) \\ \downarrow & & \downarrow \\ H^i(K,H) & \xrightarrow{ab^i_H} & H^i_{ab}(K,H) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow \\ H^i(K,G) & \xrightarrow{ab^i_G} & H^i_{ab}(K,G) \end{array}$$

with exact columns. Let $\eta \in H^i(K,H)$ be an element such that $\alpha_*(\eta) = \xi$. In the case $i = 1$ we have $H^1(K,Z) = 0$, hence η is unique and $ab^1_H(\eta) = 0$. In the case $i = 0$ we may choose η such that $ab^0_H(\eta) = 0$. In both cases $\eta \in \ker ab^i_H$, hence η comes from $H^i(K,H^{ss}) = H^i(K,H^{sc})$. Taking in account the commutative diagram

$$\begin{array}{ccc} H^{sc} & \longrightarrow & H \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \alpha} \\ G^{sc} & \xrightarrow{\rho} & G \end{array}$$

we conclude that ξ comes from $H^i(K,G^{sc})$, which was to be proved.

3.7. By Theorem 3.2 the map ab^0 is a group homomorphism. We want to show that the map ab^1 has also a certain multiplicativity property.

Let $z \in Z^1(K,G)$. We consider the twisted form ${}^zG$ of G. Let

$$t_z : H^1(K,{}^zG) \longrightarrow H^1(K,G)$$

denote the canonical map defined by $C\ell(z') \longmapsto C\ell(z'z)$ for $z' \in H^1(K,{}^zG)$, where $C\ell$ denotes the cohomological class. Note that if G is abelian, then ${}^zG$ can be identified with G and in this case t_z is $\xi \longrightarrow \xi + C\ell(z)$.

Proposition 3.8. Let $z \in H^1(K,G)$. Then the diagram

(3.8.1)
$$\begin{array}{ccc} H^1(K, {}^zG) & \xrightarrow{\ t_z\ } & H^1_{ab}(K, G) \\ \big\downarrow{\scriptstyle ab^1} & & \big\downarrow \\ H^1_{ab}(K, {}^zG) = H^1_{ab}(K,G) & \xrightarrow{x \mapsto x+\alpha(z)} & H^1_{ab}(K, G) \end{array}$$

commutes, where $\alpha(z) = ab^1_G(C\ell(z))$ and we identify the abelian groups $H^1_{ab}(K,{}^zG)$ and $H^1_{ab}(K,G)$ using Corollary 2.11.

Proof. Let $\beta : G \longrightarrow G'$ be a homomorphism of connected reductive K–groups. For $z \in Z^1(K,G)$ set $z' = \beta_* z \in Z^1(K,G')$. It is clear that the diagram

(3.8.2)
$$\begin{array}{ccc} H^1(K, {}^zG) & \xrightarrow{\ t_z\ } & H^1(K,G) \\ \big\downarrow & & \big\downarrow \\ H^1(K, {}^{z'}G') & \xrightarrow{\ t_{z'}\ } & H^1(K,G') \end{array}$$

commutes.

Now suppose that G^{ss} is simply connected, and take G^{tor} for G' . Then $H^1(K,G') = H^1(K,G^{tor}) = H^1_{ab}(K,G)$, $t_{z'} = (x \longmapsto x + C\ell(z'))$ and the diagram (3.8.2) becomes the diagram (3.8.1). This proves the proposition for $G \in \mathcal{G}_0$.

To treat the general case we need

Lemma 3.8.3. Let $z \in Z^1(K,G)$ and $\xi \in H^1(K,{}^zG)$. Then there exists a z–extension $\alpha : H \longrightarrow G$, a cocycle $w \in Z^1(K,H)$ such that $\alpha_* w = z$, and a cohomology class $\eta \in H^1(K,{}^wH)$ such that $({}^w\alpha)_*\eta = \xi$.

Proof. Choose a Galois extension F/K trivialising both $C\ell(z) \in H^1(K,G)$ and $\xi \in H^1(K,{}^zG)$. By Lemma 3.4.2 there exists a z–extension $\alpha : H \longrightarrow G$ whose kernel Z is isomorphic to $(R_{F/K}\mathbb{G}_m)^n$. By Lemma 3.4.4 α is a $C\ell(z)$–lifting extension. Moreover, since α is surjective, any cobord $b \in B^1(K,G)$ can be lifted to $B^1(K,H)$. Using twisting, we obtain that z is the image of some cocycle $w \in Z^1(K,H)$.

Consider the twisted homomorphism ${}^w\alpha : {}^wH \longrightarrow {}^zG$. It is clear that ${}^w\alpha$ is a z–extension with kernel $Z \simeq (R_{F/K}\mathbb{G}_m)^n$. By Lemma 3.4.4 ${}^w\alpha$ is a ξ–lifting extension. Thus Lemma 3.8.3 is proved.

We prove Proposition 3.8 in the general case. Let $\alpha : H \longrightarrow G$, w and z be as in Lemma 3.8.3. Since the diagram

$$\begin{array}{ccc} H^1(K,{}^wH) & \xrightarrow{t_w} & H^1(K,H) \\ \downarrow ({}^w\alpha)_* & & \downarrow \alpha_* \\ H^1(K,{}^zG) & \xrightarrow{t_z} & H^1(K,G) \end{array}$$

commutes and $\alpha_{ab}(ab^1_H(C\ell(w))) = ab^1_G(C\ell(z))$, the assertion to be proved is reduced to the already proved assertion concerning H, w and η . The proposition is proved.

Using Proposition 3.8 we can compute the fibers of the map ab^1_G .

Corollary 3.9. For $z \in Z^1(K,G)$ set $\xi = C\ell(z)$. Let ${}^z\rho : {}^zG^{sc} \longrightarrow {}^zG$ denote the twist of $\rho : G^{sc} \longrightarrow G$. Then

$$(ab^1)^{-1}(ab^1(\xi)) = t_z({}^z\rho_* H^1(K,{}^zG^{sc})) \ .$$

The corollary follows from Proposition 3.8 and Proposition 3.6.

Remark 3.9.1. We see that any fiber of ab^1_G is the image of the Galois cohomology of some twisted form of G^{sc}.

Remark 3.9.2. Corollary 3.9 shows that the map $ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$ induces the embedding

$$H^1(K,G)^{abld} = \rho_* H^1(K,G^{sc}) \backslash H^1(K,G) \hookrightarrow H^1_{ab}(K,G)$$

mentioned in the Introduction.

The following proposition shows that the maps ab^i define morphisms of cohomology exact sequences.

Proposition 3.10. (i) [Ko3]. Let

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be an exact sequence of connected reductive K–groups. Then the diagram

$$\begin{array}{ccccccccccccc} 0 \to & G_1(K) & \to & G_2(K) & \to & G_3(K) & \to & H^1(K,G_1) & \to & H^1(K,G_2) & \to & H^1(K,G_3) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^0_{ab}(K,G_1) & \to & H^0_{ab}(K,G_2) & \to & H^0_{ab}(K,G_3) & \to & H^1_{ab}(K,G_1) & \to & H^1_{ab}(K,G_2) & \to & H^1_{ab}(K,G_3) \end{array}$$

commutes.

(ii) If moreover G_1 is a torus, then the diagram

$$\begin{array}{ccccc} H^1(K,G_2) & \longrightarrow & H^1(K,G_3) & \longrightarrow & H^2(K,G_1) \\ \downarrow & & \downarrow & & \downarrow \\ H^1_{ab}(K,G_2) & \longrightarrow & H^1_{ab}(K,G_3) & \longrightarrow & H^2(K,G_1) \end{array}$$

commutes.

Proof. First suppose that $G_1, G_2, G_3 \in \mathcal{G}_0$. The morphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G_1^{tor} & \longrightarrow & G_2^{tor} & \longrightarrow & G_3^{tor} & \longrightarrow & 1 \end{array}$$

of short exact sequences defines a morphism of cohomology exact sequences, which proves the assertion in this case.

To treat the general case we need

Lemma 3.10.1. Let

$$\text{(G)} \qquad 1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be a short exact sequence, and let $\xi_3 \in H^1(K,G_3)$ be a cohomology class. Then there exists a ξ_3–lifting z–extension of (G), i.e. a morphism of short exact sequences

(3.10.1.1)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \end{array}$$

such that $H_\ell \longrightarrow G_\ell$ is a z–extension for $\ell = 1,2,3$, and moreover $H_3 \longrightarrow G_3$ is a ξ_3–lifting z–extension.

Proof. Choose a z–extension $H \longrightarrow G_2$ and a ξ_3–lifting z–extension $H_3 \longrightarrow G_3$. We set $H_2 = H \times_{G_3} H_3$. Let H_1 be the kernel of $H_2 \longrightarrow H_3$. We obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \end{array} .$$

Since $H_2 \longrightarrow G_2$ is surjective, the homomorphism $H_1 \longrightarrow G_1$ is also surjective. Since H_1 is a normal subgroup of H_2, H^{ss} is simply connected. Since $\ker[H_1 \longrightarrow G_1] = \ker[H \longrightarrow G_2]$, we conclude that $H_1 \longrightarrow G_1$ is a z–extension. The lemma is proved.

We prove Proposition 3.10 in the general case. To prove assertions (i) and (ii) it suffices to prove the commutativity of the diagrams

$$\begin{array}{ccc} H^0(K,G_3) & \xrightarrow{\delta} & H^1(K,G_1) \\ \downarrow & & \downarrow \\ H^0_{ab}(K,G_3) & \xrightarrow{\delta} & H^1_{ab}(K,G_1) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(K,G_3) & \xrightarrow{\delta} & H^2(K,G_1) \\ \downarrow & & \downarrow \\ H^1_{ab}(K,G_3) & \xrightarrow{\delta} & H^2_{ab}(K,G_1) \end{array}$$

respectively. Let $\xi_3 \in H^i(K,G_3)$, where $i = 0$ or 1. By Lemma 3.10.1 there exists a morphism (3.10.1.1) of exact sequences such that $H_\ell \longrightarrow G_\ell$ are z–extensions for $\ell = 1,2,3$ and that ξ can be lifted to an element $\eta \in H^i(K,H_3)$. Thus the assertion is reduced to the already considered case of a short exact sequence in $\mathcal{G}_0$. Proposition 3.10 is proved.

We observe that the maps ab_G^0 and ab_G^1 are isomorphisms for tori and are well known for groups G such that G^{ss} is simply connected. The following remark shows that these maps are also well known for semisimple groups.

Remark 3.11. Let G be a semisimple group, $G = G^{sc}/\ker \rho$. Then for $i = 0,1$ the diagram

(3.11.1)
$$\begin{array}{ccc} H^i(K,G) & \xrightarrow{\delta} & H^{i+1}(K,\ker \rho) \\ \| & & \downarrow \wr \\ H^i(K,G) & \xrightarrow{ab^i} & H^i_{ab}(K,G) \end{array}$$

commutes. Here δ is the connecting homomorphism and the right vertical arrow is the isomorphism of Example 2.13 (3). We omit the proof (cf. [Ko2] Remark 6.5, [Ko3] Lemma 1.8).

4. Computation of abelian Galois cohomology

In Section 3 we have defined the abelianisation map $ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$. By Proposition 2.10 $H^1_{ab}(K,G) = \mathcal{H}^1(K,M,\bar{K}^\times)$. In this section we try to calculate $\mathcal{H}^1(K,M,\bar{K}^\times)$ for $i \geq 1$. We compute $\mathcal{H}^1(K,M,\bar{K}^\times)$ for local fields. For a number field K we compute $\mathcal{H}^1(K,M,\bar{K}^\times)$ for $i \geq 2$. For $i = 1$ we compute the kernel and the cokernel of the localization map $\mathcal{H}^1(K,M,\bar{K}^\times) \longrightarrow \oplus\, \mathcal{H}^1(K_v,M,\bar{K}_v^\times)$.

All this stuff is a kind of Tate–Nakayama theory. The results in the most interesting case $i = 1$ are essentially due to Kottwitz.

4.1. In this section K is a local or global field of characteristic 0, $\Gamma = \mathrm{Gal}(\bar{K}/K)$, M is a finitely generated Γ–module .

Proposition 4.1. Let K be a non–archimedean local field. There are canonical isomorphisms:

(i) $\lambda_v : \mathcal{H}^1(K,M,\bar{K}^\times) \xrightarrow{\sim} (M_\Gamma)_{tors}$

(ii) $\mathcal{H}^2(K,M,\bar{K}^\times) \xrightarrow{\sim} (M_\Gamma)_{tf} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$

(iii) $\mathcal{H}^i(K,M,\bar{K}^\times) = 0$ for $i \geq 3$.

Recall that $(M_\Gamma)_{tf} = M_\Gamma/(M_\Gamma)_{tors}$.

4.1.1. We prove (iii). Let $L^\cdot \longrightarrow M$ be a short torsion free resolution, where $L^\cdot = (L^{-1} \longrightarrow L^0)$. In the exact sequence (2.6.1)

$$\ldots \longrightarrow H^i(K, L^0 \otimes \bar{K}^\times) \longrightarrow \mathscr{H}^i(K, M, \bar{K}^\times) \longrightarrow H^{i+1}(K, L^{-1} \otimes \bar{K}^\times) \longrightarrow \ldots$$

we have $H^i(K, L^0 \otimes \bar{K}^\times) = 0$, $H^{i+1}(K, L^{-1} \otimes \bar{K}^\times) = 0$ for $i \geq 3$ (cf. e.g. [Mi], 1.11). Hence $\mathscr{H}^1(K, M, \bar{K}^\times) = 0$, which proves (iii).

4.1.2. We begin proving (i) and (ii). Let $L^{\cdot} \longrightarrow M$ be short torsion free resolution. We consider the dual complex

$$L^{\cdot\vee} = \underline{\mathrm{Hom}}(L^{\cdot}, \mathbb{Z}) = (L^{0\vee} \longrightarrow L^{-1\vee})$$

(recall that $\vee$ denotes $\mathrm{Hom} \cdot \mathbb{Z}$). Here $L^{0\vee}$ is in degree 0 and $L^{-1\vee}$ is in degree +1.

We have by definition

$$\mathscr{H}^i(K, M, \bar{K}^\times) = \mathbb{H}^i(K, L^{\cdot} \otimes \bar{K}^\times).$$

The cup product pairing

$$\mathbb{H}^i(K, L^{\cdot} \otimes \bar{K}^\times) \otimes \mathbb{H}^{2-i}(K, L^{\cdot\vee}) \longrightarrow H^2(K, \bar{K}^\times) = \mathrm{Br}(K)$$

defines canonical homomorphisms

(4.1.2.1) $$\mathscr{H}^i(K, M, \bar{K}^\times) = \mathbb{H}^{2-i}(K, L^{\cdot\vee})^B,$$

where ^{B} denotes $\mathrm{Hom}(\cdot, \mathrm{Br}(K))$.

Lemma 4.1.3. Homomorphisms (4.1.2.1) are isomorphisms for $i \geq 1$.

Proof. If M is torsion free then this is the Tate–Nakayama duality theorem. In the general case we can write down the exact sequence (2.6.1) and the corresponding commutative diagram. Applying the five–lemma we obtain the desired result.

4.1.4. We compute $\mathbb{H}^0(K,L^{\cdot\vee})^B$. By definition

$$\mathbb{H}^0(K,L^{\cdot\vee})^B = \ker[(L^{0\vee})^\Gamma \longrightarrow (L^{-1\vee})^\Gamma)^B = \operatorname{coker}[(L^{-1\vee})^{\Gamma B} \longrightarrow (L^{0\vee})^{\Gamma B}]$$

We have

$$(L^{0\vee})^\Gamma = \operatorname{Hom}_\Gamma(L^0,\mathbb{Z}) = \operatorname{Hom}(L^0_\Gamma,\mathbb{Z}) = \operatorname{Hom}((L^0_\Gamma)_{tf},\mathbb{Z}) = (L^0_\Gamma)^\vee_{tf}$$

Hence $(L^{0\vee})^{\Gamma B} = (L^0_\Gamma)_{tf} \underset{\mathbb{Z}}{\otimes} Br(K) = L^0_\Gamma \underset{\mathbb{Z}}{\otimes} Br(K)$.

Similarly

$$(L^{-1\vee})^{\Gamma B} = L^{-1}_\Gamma \underset{\mathbb{Z}}{\otimes} Br(K)$$

Further

$$\operatorname{coker}[(L^{-1\vee})^{\Gamma B} \longrightarrow (L^{0\vee})^{\Gamma B}] = \operatorname{coker}[L^{-1}_\Gamma \underset{\mathbb{Z}}{\otimes} Br(K) \longrightarrow L^0_\Gamma \otimes Br(K)]$$
$$= \operatorname{coker}[L^{-1}_\Gamma \longrightarrow L^0_\Gamma] \underset{\mathbb{Z}}{\otimes} Br(K) = M_\Gamma \underset{\mathbb{Z}}{\otimes} Br(K) = (M_\Gamma)_{tf} \underset{\mathbb{Z}}{\otimes} Br(K)$$

There is a canonical isomorphism $\mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Now 4.1 (ii) follows from Lemma 4.1.3.

4.1.5. We compute $\mathbb{H}^1(K,L^{\cdot\vee})^B$. Following an idea of Kottwitz [Ko2], we consider the short exact sequence

$$0 \longrightarrow L^{\cdot\vee} \longrightarrow L^{\cdot\vee} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow L^{\cdot\vee} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

which gives rise to the hypercohmology exact sequence

$$\mathbb{H}^0(K,L^{\cdot\vee} \otimes \mathbb{Q}) \longrightarrow \mathbb{H}^0(K,L^{\cdot\vee} \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathbb{H}^1(K,L^{\cdot\vee}) \longrightarrow 0$$

(because $L^{\cdot\vee} \otimes \mathbb{Q}$ is a complex of injective Γ–modules).

We observe that

$$L^{\cdot\vee} \otimes \mathbb{Q} = \mathrm{Hom}(L^{\cdot},\mathbb{Q}), \quad L^{\cdot\vee} \otimes (\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(L^{\cdot},\mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are $\mathbb{Z}$–injective, the sequences

$$0 \longrightarrow \mathrm{Hom}(M,\mathbb{Q}) \longrightarrow \mathrm{Hom}(L^0,\mathbb{Q}) \longrightarrow \mathrm{Hom}(L^{-1},\mathbb{Q}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(L^0,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(L^{-1},\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

are exact. Thus

$$\mathbb{H}^0(K, L^{\cdot\vee} \otimes \mathbb{Q}) = \mathbb{H}^0(K, \mathrm{Hom}(L^{\cdot}, \mathbb{Q})) = H^0(K, \mathrm{Hom}(M, \mathbb{Q})) = \mathrm{Hom}_\Gamma(M, \mathbb{Q}) =$$
$$\mathrm{Hom}(M_\Gamma, \mathbb{Q}) = \mathrm{Hom}((M_\Gamma)_{tf}, \mathbb{Q})$$

and similarly

$$\mathbb{H}^0(K, L^{\cdot\vee} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_\Gamma(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(M_\Gamma, \mathbb{Q}/\mathbb{Z})$$

We see that

$$\mathbb{H}^1(K, L^{\cdot\vee}) = \mathrm{coker}[\mathrm{Hom}(M_\Gamma)_{tf}, \mathbb{Q}) \longrightarrow \mathrm{Hom}(M_\Gamma, \mathbb{Q}/\mathbb{Z})] =$$
$$= \mathrm{coker}[\mathrm{Hom}((M_\Gamma)_{tf}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(M_\Gamma, \mathbb{Q}/\mathbb{Z})] =$$
$$\mathrm{Hom}(\ker[M_\Gamma \longrightarrow (M_\Gamma)_{tf}], \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}((M_\Gamma)_{tors}, \mathbb{Q}/\mathbb{Z})$$

Using the canonical isomorphism $\mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$, we conclude that

$$\mathbb{H}^1(K, L^{\cdot\vee})^B = \mathrm{Hom}((\mathrm{Hom}(M_\Gamma)_{tors}, \mathbb{Q}/\mathbb{Z}), \mathrm{Br}(K)) \simeq (M_\Gamma)_{tors},$$

Now 4.1 (i) follows from Lemma 4.1.3.

Proposition 4.1 is proved.

The exposition in the remaining part of the section is somewhat sketchy.

Proposition 4.2. For $K = \mathbb{R}$ there are canonical isomorphisms

$$\lambda_{\mathbb{R}}: \mathscr{H}^i(\mathbb{R},M,\mathbb{C}^\times) \xrightarrow{\sim} \hat{H}^{i-2}(\mathbb{R},M) \quad \text{for } i \geq 1.$$

In particular

$$\mathscr{H}^i(\mathbb{R},M,\mathbb{C}^\times) \simeq \begin{cases} H^1(\mathbb{R},M) & \text{if } i \text{ is odd} \\ \hat{H}^0(\mathbb{R},M) & \text{if } i \text{ is even } (i > 0). \end{cases}$$

Proof. Similar to that of Proposition 4.1.

4.3. Now let K be a number field. Set $\overline{A} = A \otimes_K \overline{K}$, where A is the adèle ring of K. We set $\overline{C} = \overline{A}^\times/\overline{K}^\times$.

Let M be a finitely generated Γ–module. Let $L^\cdot \longrightarrow M$ be a short torsion free resolution. We consider the short exact sequences

$$1 \longrightarrow \overline{K}^\times \longrightarrow \overline{A}^\times \longrightarrow \overline{C} \longrightarrow 1$$

$$0 \longrightarrow L^\cdot \otimes \overline{K}^\times \longrightarrow L^\cdot \otimes \overline{A}^\times \longrightarrow L^\cdot \otimes \overline{C} \longrightarrow 0$$

and the corresponding long exact sequence

(4.3.1) $$\ldots \longrightarrow \mathscr{H}^1(K,M,\overline{K}^\times) \longrightarrow \mathscr{H}^1(K,M,\overline{A}^\times) \longrightarrow \mathscr{H}^1(K,M,\overline{C}) \longrightarrow \ldots$$

We would like to compute this exact sequence.

Proposition 4.4. There are canonical isomorphisms

(i) $\lambda : \mathscr{H}^1(K,M,\overline{C}) \xrightarrow{\sim} (M_\Gamma)_{tors}$

(ii) $\mathscr{H}^2(K,M,\overline{C}) \xrightarrow{\sim} (M_\Gamma)_{tf} \otimes \mathbb{Q}/\mathbb{Z}$

(iii) $\mathscr{H}^i(\dot{K},M,\overline{C}) = 0$ for $i \geq 3$.

Proof. The same as that of Proposition 4.1.

Lemma 4.5. There is a canonical isomorphism

$$\mathrm{loc} : \mathscr{H}^i(K,M,\overline{A}^\times) \simeq \bigoplus_{v \in \mathscr{V}} \mathscr{H}^i(K_v,M,\overline{K}_v^\times) \quad \text{for } i \geq 1.$$

Proof. The embedding $\oplus(K_v \underset{K}{\otimes} \overline{K}) \hookrightarrow \overline{A}^\times$ induces the homomorphism

$$\oplus\, \mathscr{H}^i(K_v \otimes \overline{K})^\times) \longrightarrow \mathscr{H}^i(K,M,\overline{A}).$$

By Shapiro's lemma

$$\mathscr{H}^i(K,M,(K_v \otimes \overline{K})^\times) = \mathscr{H}^i(K_v,M,\overline{K}_v^\times).$$

Thus we obtain a homomorphism

$$\oplus\, \mathscr{H}^i(K,M,\overline{K}_v^\times) \longrightarrow \mathscr{H}^i(K,M,\overline{A}^\times).$$

We must prove that it is an isomorphism. Using the exact sequence (2.6.1) we reduce the assertion to the well known case of a torsion free module M. The lemma is proved.

Corollary 4.6. For any $h \in \mathscr{H}^i(K,M,\bar{K}^\times)$ $(i \geq 0)$ there exists a finite set $S \subset \mathscr{V}(K)$ such that $loc_v(h) \in \mathscr{H}^i(K_v,M,\bar{K}_v^\times)$ is zero for $v \notin S$.

Proof. It follows from the proof of Proposition 4.5 that for any $\xi \in \mathscr{H}^i(K,M,\bar{A}^\times)$ there exits a finite set $S \subset \mathscr{V}$ such that ξ comes from $\mathscr{H}^1(K, M, \underset{S}{\oplus}(K_v \underset{K}{\otimes} \bar{K})^\times)$. This implies the proposition.

Corollary 4.7. For $i \geq 3$ the localization map

$$(4.7.1) \qquad loc_\infty : \mathscr{H}^i(K,M,\bar{K}^\times) \longrightarrow \prod_{\infty} \mathscr{H}^i(K_v,M,\bar{K}_v^\times)$$

is an isomorphism (where we write ∞ for $\mathscr{V}_\infty(K)$).

Proof. This follows from the exact sequence (4.3.1) and Propositions 4.1(iii) and 4.4(iii).

Corollary 4.8. (Tate–Poitou). If $i = 2$ and M is finite then (4.7.1) is an isomorphism.

Proof. This follows from the exact sequence (4.3.1) and Propositions 4.1 (ii) and 4.4 (ii).

Proposition 4.9. The canonical isomorphisms

$tf_* : \mathcal{H}^2(K,M,\bar{K}^\times) \longrightarrow H^2(K,M_{tf} \otimes \bar{K}^\times)$

$loc_\infty : \mathcal{H}^2(K,M,\bar{K}^\times) \longrightarrow \prod_\infty \mathcal{H}^2(K_v,M,\bar{K}_v^\times)$

define an isomorphism of $\mathcal{H}^2(K,M,\bar{K}^\times)$ onto the fiber product of $H^2(K,M_{tf} \otimes \bar{K}^\times)$ and $\prod_\infty \mathcal{H}^2(K_v,M,\bar{K}_v^\times)$ over $\prod_\infty H^2(K_v,M_{tf} \otimes \bar{K}^\times)$.

Let T_M be the K–torus such that $X_*(T) = M_{tf}$. We have computed $\mathcal{H}^2(K,M,K^\times)$ in terms of the Galois cohomology $H^2(K,T_M)$ of this torus and of the real cohomology groups $\mathcal{H}^2(K,M,\bar{K}_v^\times) \simeq \hat{H}^0(K_v,M)$. Observe that the homomorphism

$$loc_\infty : H^2(K,M_{tf} \otimes \bar{K}^\times) \longrightarrow \prod_\infty H^2(K_v,T_M)$$

is surjective, but the homomorphism

$$tf_{*\infty} : \prod_\infty \mathcal{H}^2(K_v,M,\bar{K}^\times) \longrightarrow \prod_\infty H^2(K_v,T_M)$$

in general is not surjective.

Proof. Consider the canonical short exact sequence

$$0 \longrightarrow M_{tors} \xrightarrow{\ i\ } M \xrightarrow{\ tf\ } M_{tf} \longrightarrow 0$$

and the corresponding commutative diagram

$$\begin{array}{ccccccc}
H^1(K,T_M) & \xrightarrow{\delta} & H^3(K,M_{tors}(1)) & \xrightarrow{i_*} & \mathscr{H}^2(K,M,\bar{K}^\times) & \xrightarrow{tf_*} & H^2(K,T_M) \\
\downarrow{\scriptstyle loc_\infty} & & \downarrow{\scriptstyle loc_\infty} & & \downarrow{\scriptstyle loc_\infty} & & \downarrow{\scriptstyle loc_\infty} \\
\prod_\infty H^1(K_v,T_M) & \xrightarrow{\delta} & \prod_\infty H^3(K,M_{tors}(1)) & \xrightarrow{i_*} & \prod_\infty \mathscr{H}^2(K,M,\bar{K}^\times) & \xrightarrow{tf_*} & \prod_\infty H^2(K,T_M)
\end{array}$$

with exact rows. It is clear that

$$tf_* \times loc_\infty : \mathscr{H}^2(K,M,\bar{K}^\times) \longrightarrow H^2(K,T_M) \times \prod_\infty \mathscr{H}^2(K_v,M,\bar{K}_v^\times)$$

define a homomorphism j from $\mathscr{H}^2(K,M,\bar{K}^\times)$ into the fiber product over $\prod_\infty H^2(K_v,T_M)$.

We prove that j is injective. Suppose $\xi \in \ker j$. Then $\xi \in \ker tf_*$, hence $\xi = i_*(\eta)$ for some $\eta \in H^3(K,M(1))$. Now, since $\xi \in \ker loc_\infty$, $i_*(loc_\infty(\eta)) = 0$, hence $loc_\infty(\eta) = \delta(\zeta_\infty)$ for some $\zeta_\infty \in \prod_\infty H^1(K_v,M_T)$. Since the map

$$loc_\infty^1 : H^1(K,M_T) \longrightarrow \prod_\infty H^1(K_v,M_T)$$

is surjective ([Ha],II,A.1.2, see also [Sa], 1.8), there exists $\zeta \in H^1(K,T_M)$ such that $\zeta_\infty = loc_\infty(\zeta)$. We see that $loc_\infty(\delta(\zeta)) = loc_\infty(\eta)$. By Corollary 4.7 the map $loc_\infty^3 : H^3(K,M_{tors}(1)) \longrightarrow \prod_\infty H^3(K_v,M_{tors}(1))$ is bijective, hence $\delta(\zeta) = \eta$. By construction $\xi = i_*(\eta)$. We conclude that $\xi = 0$. This proves the injectivity of j.

The proof of the surjectivity of j is left to the reader.

We are going to consider $\mathcal{H}^1$ which is the most interesting case.

4.10. We write $H^{-1}(K,M)$ for $(M_\Gamma)_{tors}$ and, if $v \in \mathcal{V}_f$, write $H^{-1}(K_v,M)$ for $(M_{\Gamma_v})_{tors}$. For $v \in \mathcal{V}_f$ we have obvious corestriction homomorphisms

$$\mathrm{cor}_v : H^{-1}(K_v,M) = (M_{\Gamma_v})_{tors} \longrightarrow (M_\Gamma)_{tors} = H^{-1}(K,M)$$

For $v \in \mathcal{V}_\infty$ we also have corestriction homomorphisms

$$\mathrm{cor}_v : H^{-1}(\bar{K}_v/K,M) \hookrightarrow (M_{\Gamma_v})_{tors} \longrightarrow (M_\Gamma)_{tors} = H^{-1}(K,M).$$

Proposition 4.11. The following diagram commutes

$$\begin{array}{ccc} \mathcal{H}^1(K,M,\bar{A}^\times) & \longrightarrow & \mathcal{H}^1(K,M,\bar{C}) \\ {\scriptstyle \oplus\lambda_v}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \bigoplus_{v\in \mathcal{V}} H^{-1}(\Gamma_v,M) & \xrightarrow{\ \sum \mathrm{cor}_v\ } & H^{-1}(K,M) \end{array}$$

where the vertical arrows λ_v and λ are the isomorphisms of Propositions 4.1, 4.2 and 4.4.

Idea of proof. We reduce the assertion to the case of torsion free M. For such M the assertion is well known (as the compatibility the local and the global Tate–Nakayama dualities for tori).

Corollary 4.12. The localization map

$$\mathrm{loc}_\infty : \mathscr{H}^1(K,M,\overline{K}^\times) \longrightarrow \prod_\infty \mathscr{H}^1(K_v,M,\overline{K}_v^\times)$$

is surjective.

Idea of proof: We consider the exact sequence similar to the exact sequence 4.3.1, but for a sufficiently large *finite* Galois extension F/K. This exact sequence is partly computed, see Proposition 4.11. We obtain the desired assertion by applying Chebotarev's density theorem.

We can as well choose a short torsion free resolution $L^\bullet \longrightarrow M$ and reduce the assertion to the case of torsion free M .

4.13. Let F/K be a finite Galois extension such that Gal $\overline{K}(F)$ acts on M trivially. We set $\Delta = \mathrm{Gal}(F/K)$. Then M is a Δ–module. Consider the cokernel

$$c_1(F/K,M) = \mathrm{coker}[\bigoplus_v H_1(\Delta_v,M) \xrightarrow{\sum \mathrm{cor}_v} H_1(\Delta,M)]$$

where cor_v is the corestriction map, and Δ_v is a decomposition group of v in F. One can show that $c_1(F/K,M)$ does not depend on the choice of F. We write $c_1(K,M)$ for $c_1(F/K,M)$. We set

$$\text{Ш}^1_{\mathscr{H}}(K,M) = \ker[\mathrm{loc} : \mathscr{H}^1(K,M,\overline{K}^\times) \longrightarrow \bigoplus_v \mathscr{H}^1(K_v,M,\overline{K}_v^\times).$$

Proposition 4.14. There is a canonical isomorphism

$$c_1(K,M) \xrightarrow{\sim} Ш^1_{\mathscr{H}}(K,M)$$

Idea of proof. One can show that $Ш^1_{\mathscr{H}}(K,M)$ is canonically isomorphic to

$$Ш^1_{\mathscr{H}}(F/K,M) := \ker[\mathscr{H}^1(F/K,M,F^\times) \longrightarrow \mathscr{H}^1(F/K,M,(A \underset{K}{\otimes} F)^\times)] \ ,$$

where F/K is as in 4.13. We write Δ for $\mathrm{Gal}(F/K)$. This kernel is the cokernel of

$$\hat{\mathscr{H}}^0(\Delta,M,(A \underset{K}{\otimes} F)^\times) \longrightarrow \hat{\mathscr{H}}^0(\Delta,M,(A\otimes F)^\times/F^\times) .$$

(see Remark 2.5.3 for the definitions of the groups $\hat{\mathscr{H}}^i$). Then we compute these groups and the homomorphism by methods of the proof of Propositions 4.1, 4.4 and Lemma 4.5. We show that this homomorphism is

$$\oplus\, H_1(\Delta_v,M) \xrightarrow{\sum \mathrm{cor}_s} H^1(\Delta,M).$$

This proves the assertion.

5. Galois cohomology over local and number fields

In this section we apply the results of Sections 3 and 4 to the study of the usual (non–abelian) Galois cohomology of connected reductive groups over local and (especially) number fields.

5.0. We shall need the following fundamental results on Galois cohomology over local and global fields.

Theorem 5.0.1. ([Kn1], [Kn3]). Let G be a simply connected group over a non–archimedean local field K. Then $H^1(K,G) = 1$.

Another proof of this result appeared in [Br–T].

5.0.2. Let K be a number field. A K–group is said to satisfy the Hasse principle, if

$$Ш(G) := \ker[H^1(K,G) \longrightarrow \prod_{v \in \mathcal{V}} H^1(K_v,G)] = 0 \ .$$

Theorem 5.0.3 (Kneser–Harder–Chernousov). For any semisimple simply connected group G over a number field K, the map

$$H^1(K,G) \longrightarrow \prod_{\infty} H^1(K_v,G)$$

is bijective.

In particular, the Hasse principle is valid for such a group.

The classical groups were treated by Kneser (cf. [Kn2], [Kn3]), and the exceptional ones, excepting E_8, by Harder [Ha1]. The proof in the most difficult case E_8, initiated by Harder [Ha1], has recently been completed by Chernousov [Ch].

We begin with proving that the maps ab^0 and ab^1 are in some cases surjective.

Proposition 5.1. Let K be a non–archimedean local field. Then for any connected reductive group G the homomorphism $ab^0 : G(K) \longrightarrow H^0_{ab}(K,G)$ is surjective.

Proof. First suppose that G^{ss} is simply connected. Then in the exact cohomology sequence

$$G^{sc}(K) \longrightarrow G(K) \longrightarrow G^{tor}(K) \longrightarrow H^1(K,G^{sc})$$

we have $H^1(K,G^{sc}) = 0$ by Theorem 5.0.1. Thus in this case the map

$$ab^0 : G(K) \longrightarrow G^{tor}(K) = H^0_{ab}(K,G)$$

is surjective.

In the general case choose a z–extension $Z \hookrightarrow H \longrightarrow G$. We have the commutative diagram

$$\begin{array}{ccccc} H(K) & \longrightarrow & G(K) & \longrightarrow & H^1(K,Z) \\ \downarrow{\scriptstyle ab^0_H} & & \downarrow{\scriptstyle ab^0_G} & & \downarrow \\ H^0_{ab}(K,H) & \longrightarrow & H^0_{ab}(K,G) & \longrightarrow & H^1(K,Z) \end{array}$$

with exact rows. Since ab^0_H is surjective and $H^1(K,Z) = 0$, we conclude that ab^0_G is also surjective. q.e.d.

Remark 5.1.1. For $K = \mathbb{R}$ the homomorphism ab^0 is in general non–surjective. For example let $\mathfrak{A}$ denote the algebra of the Hamilton quaternions over $\mathbb{R}$. Set $G = \mathfrak{A}^\times$; then G^{ss} is simply connected and $G^{tor} = \mathbb{G}_m$. Hence

$$ab^0 : G(\mathbb{R}) \longrightarrow H^0_{ab}(\mathbb{R},G) = \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$$

is the reduced norm

$$Nm_{\mathfrak{A}/\mathbb{R}} : \mathfrak{A}^\times \longrightarrow \mathbb{R}^\times$$

We see that

$$\mathrm{im}\ ab^0_G = \mathbb{R}^\times_+ \neq \mathbb{R}^\times = H^0_{ab}(\mathbb{R},G) .$$

Corollary 5.2. If K is a non–archimedian local field, then $H^0_{ab}(K,G) = G(K)/\rho(G^{sc}(K))$.

To prove the surjectivity of ab^1 for local and global fields we need the notion of a fundamental torus.

5.3. Fundamental tori (a survey). Let K be a local field and let G be a connected reductive K–group.

Definition 5.3.1 [Ko3]. A *fundamental torus* $T \subset G$ is a maximal torus of minimal K–rank.

There is a one–to–one correspondence between the maximal K–tori of G and maximal K–tori of G^{sc} :

$$T \subset G \longmapsto T^{(sc)} \subset G^{sc}$$
$$T' \subset G^{sc} \longmapsto \rho(T') \cdot Z(G)^{\circ}$$

where $Z(G)^{\circ}$ is the connected component of the center of G . We see that a maximal torus $T \subset G$ is fundamental in G if and only if $T^{(sc)}$ is fundamenal in G^{sc} .

Proposition 5.3.2 ([Kn1] , II, p. 271). If $T \subset G$ is a fundamental torus of a semisimple group over a non–archimedian field, then T is anisotropic.

In other words, in this case G contains anisotropic maximal tori.

Lemma 5.3.3 [Ko3] . Let T be a fundamental torus of a simply connected semisimple group G over a local field K . Then $H^2(K,T) = 0$.

Proof. If K is non–archimedian, then T is anisotropic, and by Tate–Nakayama duality $H^2(K,T) = 0$. Now suppose $K = \mathbb{R}$. Then T is isomorphic to a product of a compact torus and a torus of the form $(R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)^n$ (cf. e.g. [Ko3] , Lemma 10.4), hence $H^2(\mathbb{R},T) = 0$.

Lemma 5.3.4 ([Ko3] , 10.1, see also [Brv1]). Let $T \subset G$ be a fundamental torus of a reductive $\mathbb{R}$–group. Then the map $H^1(\mathbb{R},T) \longrightarrow H^1(\mathbb{R},G)$ is surjective.

Theorem 5.4. If K is a local field, then the map $ab^1_G : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$ is surjective.

This result is essentially due to Kottwitz [Ko3] .

Proof. It suffices to find a maximal torus $T \subset G$ such that the map

$$H^1(K,T) \longrightarrow H^1_{ab}(K,G) = \mathbb{H}^1(K,T^{(sc)} \longrightarrow T)$$

is surjective. Let T be a fundamental torus of G ; then $T^{(sc)}$ is a fundamental torus of G^{sc} . From the exact sequence (2.14.2)

$$H^1(K,T) \longrightarrow H^1_{ab}(K,G) \longrightarrow H^2(K,T^{(sc)}) \ ,$$

where $H^2(K,T^{(sc)}) = 0$ by Lemma 5.3.3, we see that for such T the map $H^1(K,T) \longrightarrow H^1_{ab}(K,G)$ is surjective. The theorem is proved.

Corollary 5.4.1. If K is a non–archimedian local field, then the map ab^1_G of Theorem 5.4 is bijective.

Proof. By Corollary 3.9 any fiber of ab^1_G comes from $H^1(K,{}^zG^{sc})$ for some cocycle $z \in Z^1(K,G)$. Since ${}^zG^{sc}$ is simply connected, by Theorem 5.0.1 $H^1(K,{}^zG^{sc}) = 1$. Hence the map ab^1_G is injective. By Theorem 5.4 ab^1_G is surjective. Thus ab^1_G is bijective.

q.e.d.

Corollary 5.5 [Ko3]. Let G be a connected reductive group over a local field K . Set $M = \pi_1(\overline{G})$.

(i) If K is non–archimedian, then there is a canonical, functorial in G bijection $H^1(K,G) \longrightarrow (M_\Gamma)_{tors}$, where $\Gamma = Gal(\overline{K}/K)$.

(ii) If $K = \mathbb{R}$, then there is a canonical, functorial in G surjective map

$$H^1(\mathbb{R},G) \longrightarrow \hat{H}^{-1}(\mathbb{R},M) = H^1(\mathbb{R},M)$$

Proof. (i) By Corollary 5.4.1 the map ab_G^1 is bijective. By Proposition 4.1 (i) $H^1_{ab}(K,G) = (M_\Gamma)_{tors}$. The assertion (i) is proved.

(ii) By Theorem 5.4 ab_G^1 is surjective, and by Proposition 4.2 $H^1_{ab}(\mathbb{R},G) = \hat{H}^{-1}(\mathbb{R},M) = H^1(\mathbb{R},M)$, which proves the assertion (ii).

5.6. To investigate Galois cohomology over number fields we need some lemmas. Throughout this subsection K is a number field.

Lemma 5.6.1 (Kneser–Harder). Let G be a connected K–group. Then the map

$$\mathrm{loc}_\infty : H^1(K,G) \longrightarrow \prod_\infty H^1(K_v,G)$$

is surjective.

Proof. See [Ha1], II, 5.5.1. See also [Kn3].

Lemma 5.6.2 (Kneser–Harder). Let T be a K–torus. Suppose that there is a place v_0 of K such that T is anisotropic over K_{v_0}. Then

$$Ш^2(K,T) := \ker[H^2(K,T) \longrightarrow \prod_{v\in\mathscr{V}} H^2(K_v,T)] = 0 .$$

Proof. See [Ha1], II, p. 408, or [Kn3], 3.2, Thm. 7, p. 58, or [Sa], 1.9.3.

Lemma 5.6.3 (Harder). Let G be a K–group. Let $\Sigma \subset \mathscr{V}$ be a finite set of places of K. For any $v \in \Sigma$ let $T_v \subset G_{K_v}$ be a maximal torus. Then there exists a maximal torus $T \subset G$ such that T_{K_v} is conjugate to T_v under $G(K_v)$ for any $v \in \Sigma$.

Proof. See [Ha], II, Lemma 5.5.3.

Lemma 5.6.4. Let G be a semisimple simply connected K–group. Let $j : T \hookrightarrow G$ be a maximal torus of G such that for every $v \in \mathscr{V}_\infty$ the torus T_{K_v} is fundamental in G_{K_v}. Then the map

$$j_* : H^1(K,T) \longrightarrow H^1(K,G)$$

is surjective.

Proof. Let $\xi \in H^1(K,G)$. By Lemma 5.3.4 the map $j_* : H^1(K_v,T) \longrightarrow H^1(K_v,G)$ is surjective for $v \in \mathscr{V}_\infty$. Hence for any $v \in \mathscr{V}_\infty$ there exists an element $\eta_v \in H^1(K_v,T)$ such that $j_*(\eta_v) = \mathrm{loc}_v(\xi)$. By Lemma 5.6.1 the homomorphism $\mathrm{loc}_\infty : H^1(K,T) \longrightarrow \prod_\infty H^1(K_v,T)$ is surjective. Hence there is an element $\eta \in H^1(K,T)$ such that $\eta_v = \mathrm{loc}_v(\eta)$ for all $v \in \mathscr{V}_\infty$. We see that $\mathrm{loc}_\infty(j_*(\eta)) = \mathrm{loc}_\infty(\xi)$. By Theorem 5.0.3 it follows that $\xi = j_*(\eta)$. The lemma is proved.

Lemma 5.6.5. Let G be a semisimple simply connected K–group and let $\Sigma \subset \mathscr{V}(K)$ be a finite set of places of K. Then there exists a maximal K–torus $j : T \hookrightarrow G$ with the following properties:

(i) $H^2(K_v,T) = 0$ for $v \in \Sigma$;

(ii) $Ш^2(K,T) = 0$;

(iii) the map $j_* : H^1(K,T) \longrightarrow H^1(K,G)$ is surjective.

Proof. We may and will assume that $\Sigma \supset \mathscr{V}_\infty$ and that Σ contains at least one non–archimedian place v_0 of K. For every place $v \in \Sigma$ choose a fundamental torus $T_v \subset G_{K_v}$. By Lemma 5.6.3 there exists a K–torus $T \subset G$ such that T_{K_v} is conjugate to T_v for all $v \in \Sigma$. We see that T_{K_v} is fundamental for any $v \in \Sigma$. Hence by Lemma 5.3.3 $H^2(K_v,T) = 0$, which proves (i). The torus T is fundamental over K_{v_0}, where $v_0 \in \mathscr{V}_f(K)$, hence by Lemma 5.3.2 T is K_{v_0}–anisotropic. By Lemma 5.6.2 $Ш^2(K,T) = 0$, which proves (ii). Since $\Sigma \supset \mathscr{V}_\infty$, the assertion (iii) follows from Lemma 5.6.4. The lemma is proved.

Lemma 5.6.6 ([M–Sh], 3.1). Let H be a reductive K–group such that H^{ss} is simply connected. Then

$$\ker[H^1(K,H) \longrightarrow H^1(K,H^{tor}) \times \prod_\infty H^1(K_v,M)] = 1 .$$

Proof. Let η be an element of the kernel. Consider the cohomology exact sequence

$$H^{tor}(K) \xrightarrow{\delta} H^1(K,H^{ss}) \xrightarrow{i_*} H^1(K,H) \xrightarrow{j_*} H^1(K,H^{tor}) .$$

It is clear that η is the image of some element $\zeta \in H^1(K,H^{ss})$. Since $\mathrm{loc}_\infty(\eta) = 1$,

$$\mathrm{loc}_v(\zeta) \in \ker[i_* : H^1(K_v,H^{ss}) \longrightarrow H^1(K_v,H)]$$

for all $v \in \mathscr{V}_\infty$. Hence for any $v \in \mathscr{V}_\infty$ there is $t_v \in H^{tor}(K_v)$ such that $\mathrm{loc}_v(\zeta) = \delta(t_v)$. By the real approximation theorem (cf. e.g. [Sa], 3.5) the group $H^{tor}(K)$ is dense in $\prod_\infty H^{tor}(K_v)$, and therefore there exists $t \in H^{tor}(K)$ such that

$loc_v(\delta(t)) = \delta(t_v)$ for all $v \in \mathcal{V}_\infty$. Thus $loc_\infty(\delta(t)) = loc_\infty(\zeta)$. By Theorem 5.0.3 $\zeta = \delta(t)$. It follows that the image η of ζ in $H^1(K,H)$ is trivial.

q.e.d.

Now we can prove an analogue of theorem 5.4 for number fields.

Theorem 5.7. Let G be a connected reductive group over a number field K . Then the map $ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$ is surjective.

Proof. Let $h \in H^1_{ab}(K,G)$. It suffices to construct a torus $T \subset G$ such that the image of $H^1(K,T)$ in $H^1(K,T^{(sc)} \longrightarrow T) = H^1_{ab}(K,G)$ contains h .

By Corollary 4.6 there exists a finite set S of places of K such that $loc_v(h) = 0$ for $v \notin S$. Let $T' \subset G^{sc}$ be a maximal torus such as in Lemma 5.6.5. We set $T = \rho(T^{(sc)}) \cdot Z(G)^\circ$; then $T^{(sc)} = T'$. Consider the exact sequence (2.14.2)

$$\ldots \longrightarrow H^1(K,T) \longrightarrow H^1_{ab}(K,G) \xrightarrow{\ \delta\ } H^2(K,T^{(sc)}) \longrightarrow \ldots$$

Set $\eta = \delta(h)$; then $loc_v(\eta) = 0$ for $v \notin S$. Since $H^2(K_v,T^{(sc)}) = 0$ for $v \in S$ by 5.6.5 (i), we see that $loc_v(\eta) = 0$ for $v \in S$ as well. Thus $\eta \in Ш^2(K,T^{(sc)})$. By 5.6.5 (ii) $Ш^2(K,T^{(sc)}) = 0$. We conclude that $\eta = 0$. Hence h comes from $H^1(K,T)$. The theorem is proved.

Remark 5.7.1. Theorems 5.4 and 5.7 show if K is a local or a number field, then the canonical embedding

$$H^1(K,G)^{abld} = \rho_* H^1(K,G^{sc}) \backslash H^1(K,G) \hookrightarrow H^1_{ab}(K,G)$$

mentioned in Introduction (see also Remark 3.9.1) is a bijection.

We shall apply Theorems 5.4 and 5.7 to prolong non–abelian cohomology exact sequences.

Proposition 5.8. Let

(5.8.1) $$1 \longrightarrow G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \longrightarrow 1$$

be an exact sequence of connected reductive K–groups. Suppose that the maps $ab^1_{G_2}$ and $ab^1_{G_3}$ are surjective. Then the sequence

(5.8.2) $$H^1(K,G_2) \xrightarrow{j_*} H^1(K,G_3) \xrightarrow{\Delta} H^2_{ab}(K,G_1) \longrightarrow H^2_{ab}(K,G_2)$$

is exact, where the connecting homomorphism Δ is the composition

$$H^1(K,G_3) \xrightarrow{ab^1} H^1_{ab}(K,G_3) \xrightarrow{\delta} H^2_{ab}(K,G_1) .$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
H^1(K,G_2) & \xrightarrow{j_*} & H^1(K,G_3) & & & & \\
\downarrow{\scriptstyle ab_2} & & \downarrow{\scriptstyle ab_3} & & & & \\
H^1_{ab}(K,G_2) & \xrightarrow{j_*} & H^1_{ab}(K,G_3) & \xrightarrow{\delta} & H^2_{ab}(K,G_1) & \xrightarrow{i_*} & H^2_{ab}(K,G_2)
\end{array}$$

with exact bottom row. Since ab_3 is surjective, the sequence (5.8.2) is exact in the term

$H^2_{ab}(K,G_1)$. It is clear from the diagram that the composition

$$H^1(K,G_2) \xrightarrow{j_*} H^1(K,G_3) \xrightarrow{\Delta} H^2_{ab}(K,G_1)$$

is trivial.

Now let $\xi_3 \in H^1(K,G_3)$ lie in the kernel of $\Delta : H^1(K,G_3) \longrightarrow H^2_{ab}(K,G_1)$. We want to prove that $\xi_3 \in \text{im } j_*$. Since ab_2 is surjective, there exists $\xi_2 \in H^1(K,G_2)$ such that $ab_3(j_*\xi_2) = ab_3(\xi_3)$. Let $z \in Z^1(K,G_2)$ be a cocycle representing ξ_2 . Twisting the short exact sequence (5.8.1) by z_2 and applying Proposition 3.8 and Corollary 3.9, we reduce the assertion to be proved to the case $\xi_2 = 0$. Then $ab_3(\xi_3) = 0$. By Proposition 3.6 there exists $\eta_3 \in H^1(K,G_3^{sc})$ such that $\xi_3 = \rho_*\eta_3$. Since the exact sequence of semisimple simply connected groups

$$1 \longrightarrow G_1^{sc} \longrightarrow G_2^{sc} \longrightarrow G_3^{sc} \longrightarrow 1$$

splits, the map $H^1(K,G_2^{sc}) \longrightarrow H^1(K,G_3^{sc})$ is surjective. Hence η_3 is the image of some cohomology class $\eta_2 \in H^1(K,G_2^{sc})$. Set $\xi_2 = \rho_*\eta_2 \in H^1(K,G_2)$; then $\xi_3 = j_*\xi_2$.

q.e.d.

Using Proposition 5.8 we can compute the fibers of the connecting map

$$\Delta : H^1(K,G_3) \longrightarrow H^2_{ab}(K,G_1) \ .$$

Corollary 5.9. With the assumptions and notation of Proposition 5.8, for any $w \in Z^1(K,G_3)$ we have

$$\Delta^{-1}(\Delta(C\ell(w)) = t_w(\mathrm{im}\,[{}^wj_* : H^1(K,{}^wG_2) \longrightarrow H^1(K,{}^wG_3)])$$

Proof. We apply twisting by z .

Applying Proposition 5.8 to the case of local and number fields, we obtain

Corollary 5.10. If K is a local or a number field then the sequence (5.8.2) of Proposition 5.8 is exact.

Proof. The assertion follows from Theorems 5.4 and 5.7.
Recall that if $K = \mathbb{R}$ then $H^2_{ab}(K,G) = \hat{H}^0(\mathbb{R},\pi_1(\overline{G}))$.

When proving Theorem 5.7 we have actually proved that any $h \in H^1_{ab}(K,G)$ comes from some torus $T \subset G$. We shall prove that a similar result holds for usual, non–abelian cohomology $H^1(K,G)$.

Theorem 5.11. Let G be a reductive group over a number field K . For any finite set $\Xi \subset H^1(K,G)$ there exists a torus $T \overset{j}{\hookrightarrow} G$ such that $\Xi \subset j_*H^1(K,T)$.

Remark 5.11.1. Steinberg ([St1]) proved for arbitrary field K that if G is quasi–split and $\xi \in H^1(K,G)$, then there is a torus $j : T \hookrightarrow G$ such that $\xi \in j_*H^1(K,G)$. Theorem 5.11 shows that for a *number* field a similar (and even more stronger) assertion holds for *any* group, not necessarily quasi–split. Of cause we use Steinberg's theorem when we use the Hasse principle for simply connected groups.

Proof of Theorem 5.11. Since Ξ is finite, there exists by Corollary 4.6 a finite set Σ of places of K such that $\mathrm{loc}_v(ab^1(\xi)) = 0$ for any $\xi \in \Xi$ and any $v \notin \Sigma$. We construct a maximal torus $T' \subset G^{sc}$ as in Lemma 5.6.5. We set $T = \rho(T') \cdot Z(G)^\circ$; then $T^{(sc)} = T'$. We denote by j the inclusion $T \hookrightarrow G$. We will prove that $j_*(H^1(K,T)) \supset \Xi$.

Let $\xi \in \Xi$. Set $h = ab^1(\xi) \in H^1_{ab}(G)$. When proving Theorem 5.7 we have proved that there exists $\eta \in H^1(K,T)$ such that h is the image of η , i.e. $ab^1(j_*(\eta)) = h = ab^1(\xi)$. Thus $j_*(\eta)$ and ξ lie in the same fiber of ab^1 .

Choose a cocycle $z \in Z^1(K,T)$ representing η . By Corollary 3.9. ξ "differs" from $j_*(\eta)$ by a certain cohomology class coming from $H^1(K,{}^zG^{sc})$. Since z comes from T , we have an embedding ${}^zj : T \hookrightarrow {}^zG$. For any $v \in \mathscr{V}_\infty$ the torus $T^{(sc)}_{K_v}$ is fundamental in $G^{sc}_{K_v}$ (by construction), and it is not hard to show that $T^{(sc)}_{K_v}$ is fundamental in ${}^zG^{sc}_{K_v}$ as well. By Lemma 5.6.4 the map $H^1(K,T^{(sc)}) \longrightarrow H^1(K,{}^zG^{sc})$ is surjective. Thus there exists an element $\zeta \in H^1(K,T^{(sc)})$ such that the image of the cohomology class $\eta + \rho_*(\zeta) \in H^1(K,T)$ in $H^1(K,G)$ is ξ . The theorem is proved.

Now using Theorem 5.7 we shall compute the first non–abelian Galois cohomology in terms of abelian cohomology and real cohomology.

Theorem 5.12. Let G be a reductive group over a number field K . Then

(i) the diagram

$$(5.12.1)\quad H^1(K,G) \xrightarrow{ab^1 \times loc_\infty} H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G) \rightrightarrows \prod_\infty H^1_{ab}(K_v,G)$$

is exact;

(ii) both the projections $loc_\infty : H^1_{ab}(K,G) \longrightarrow \prod_\infty H^1_{ab}(K_v,G)$ and $ab^1_\infty : \prod_\infty H^1(K_v,G) \longrightarrow \prod_\infty H^1_{ab}(K_v,G)$ are surjective.

Here the exactness of the diagram (5.12.1) means that $H^1(K,G)$ is the fiber product of $H^1_{ab}(K,G)$ and $\prod_\infty H^1(K_v,G)$ over $\prod_\infty H^1_{ab}(K_v,G)$.

Remark 5.12.2. For semisimple groups this assertion was proved by Sansuc [Sa].

Proof of Theorem 5.12. By Theorem 5.4 the map $ab^1_\infty : \prod_\infty H^1(K_v,G) \longrightarrow \prod_\infty H^1_{ab}(K_v,G)$ is surjective. By Corollary 4.12 the homomorphism $loc_\infty : H^1_{ab}(K,G) \longrightarrow \prod_\infty H^1_{ab}(K_v,G)$ is also surjective. Thus the assertion (ii) is proved.

We prove the injectivity of

$$H^1(K,G) \longrightarrow H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G) \quad . \tag{5.12.4}$$

Let ξ lie in the kernel. Choose a ξ–lifting z–extension $Z \hookrightarrow H \longrightarrow G$. Then ξ is the image of some element $\eta \in H^1(K,H)$. From the commutative diagram

$$\begin{array}{ccccc}
1 & & 1 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^1(K_v,H) & \xleftarrow{loc_v} & H^1(K,H) & \longrightarrow & H^1_{ab}(K,H) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(K_v,G) & \xleftarrow{loc_v} & H^1(K,G) & \longrightarrow & H^1_{ab}(K,G)
\end{array}$$

one sees that η lies in the kernel of

$$H^1(K,H) \longrightarrow H^1_{ab}(K,H) \times \prod_\infty H^1(K_v,H) \ .$$

Now by Lemma 5.6.6 $\eta = 0$. Hence $\xi = 0$. We have proved that the kernel of (5.12.4) is trivial. Using twisting (and applying Proposition 3.8 and Corollary 3.9) we obtain the injectivity of (5.12.4).

We prove the exactness at the term $H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that the image of (5.12.4) is contained in the kernel of the double arrow. Conversely, let

$$h \times \xi_\infty \in H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$$

be in the kernel of the double arrow, i.e. $loc_\infty(h) = ab^1(\xi_\infty)$. We want to show that $h \times \xi_\infty$ comes from $H^1(K,G)$.

By Theorem 5.7 $h = ab^1(\eta)$ for some $\eta \in H^1(K,G)$. Then $ab^1(loc_\infty(\eta) = ab^1(\xi_\infty)$. Let $z \in Z^1(K,G)$ be a cocycle representing η . By Corollary 3.9 $loc_\infty(\eta)$ and ξ_∞ "differ" by an element of the form ${}^z\rho_*(\zeta_\infty)$ where $\zeta_\infty \in \prod_\infty H^1(K_v,{}^zG^{sc})$. To be more precise, $\xi_\infty = t_z({}^z\rho_*(\zeta_\infty))$. By Lemma 5.6.1 there exists a cohomology class $\zeta \in H^1(K,{}^zG^{sc})$ such that $loc_\infty(\zeta) = \zeta_\infty$. We set $\xi = t_z({}^z\rho_*(\zeta))$. Then $ab^1(\xi) = ab^1(\eta) = h$ and $loc_\infty(\xi) = t_z({}^z\rho_*(\zeta_\infty)) = \xi_\infty$. The theorem is proved.

Theorem 5.13. Let G be a connected reductive K–group. The abelianiasation map $ab^1 : H^1(K,G) \longrightarrow H^1_{ab}(K,G)$ induces a canonical, functorial in G bijection of the Shafarevich–Tate kernel $Ш(G)$ onto the abelian group $Ш^1_{ab}(G)$.

Recall that by definition

$$Ш(G) = \ker[H^1(K,G) \longrightarrow \prod_{v \in \mathcal{V}} H^1(K_v,G)]$$

Proof. From the commutative diagram

$$(5.13.1) \qquad \begin{array}{ccc} H^1(K,G) & \xrightarrow{ab^1} & H^1_{ab}(K,G) \\ \downarrow{\scriptstyle loc_v} & & \downarrow{\scriptstyle loc_v} \\ H^1(K_v, G) & \xrightarrow{ab^1_v} & H^1_{ab}(K_v, G) \end{array}$$

it is clear that ab^1 takes $Ш(G)$ into $Ш^1_{ab}(G)$. Write temporarily $ab_Ш$ for the restriction of ab^1_G to $Ш(G)$.

We prove the injectivity of $ab_Ш$. By Theorem 5.12 the map

$$ab^1_G \times loc_\infty : H^1(K,G) \longrightarrow H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$$

is injective. Since $loc_\infty(Ш(G)) = 1$, we conclude that the restriction $ab_Ш$ of ab^1_G to $Ш(G)$ is injective.

We prove the surjectivity of $ab_Ш$. Let $h \in Ш^1_{ab}(G) \subset H^1_{ab}(K,G)$. Then $loc_\infty(h) = 1 \in \prod_\infty H^1_{ab}(K_v,G)$. Hence the element

$$h \times 1 \in H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$$

lies in the fiber product over $\prod_\infty H^1_{ab}(K_v,G)$. By Theorem 5.12 $h \times 1$ is the image of some element $\xi \in H^1(K,G)$. We will show that $\xi \in Ш(G)$.

We observe that $loc_\infty(\xi) = 1$. Now let $v \in \mathcal{V}_f$; consider the element

$loc_v(\xi) \in H^1(K_v,G)$. Since $\xi \in H^1(K,G)$, we see from the diagram (5.13.1) that $ab^1_v(loc_v(\xi)) = 0$. By Corollary 5.4.1 the map $ab^1_v : H^1(K_v,G) \longrightarrow H^1_{ab}(K_v,G)$ is bijective. Hence $loc_v(\xi) = 1$ for any $v \in \mathcal{V}_f$. We conclude that $\xi \in Ш(G)$. The theorem is proved.

Corollary 5.14 [Ko3]. With the notation of 4.13 we have a canonical, functorial in G bijection $Ш(G) \xrightarrow{\sim} c_1(K,\pi_1(\overline{G}))$.

Remark 5.14.1. Voskresenskii [Vo] was first to prove that $Ш(G)$ has a canonical structure of abelian group. Sansuc [Sa] showed that this abelian group structure is functorial in G . He computed $Ш(G)$ in terms of the arithmetic Brauer group $Br_a G$. Our formula is equivalent to the formula (4.2.2) of [Ko2]. Concerning the functoriality see Remark 0.4 in the Introduction.

5.15. Corollary 5.14 shows that the kernel of the localisation map

$$H^1(K,G) \longrightarrow \prod_{v \in \mathcal{V}} H^1(K_v,G) \qquad (5.15.1)$$

has a natural structure of an abelian group and can be computed in terms of $\pi_1(\overline{G})$. We show that a similar assertion holds for the cokernel of (5.15.1) as well.

Set $M = \pi_1(\overline{G})$. Set $\Gamma = Gal(\overline{K}/K)$, $\Gamma_v = Gal(\overline{K}_v/K_v)$, $H^{-1}(K,M) = (M_\Gamma)_{tors}$, $H^{-1}(K_v,M) = (M_{\Gamma_v})_{tors}$ for $v \in \mathcal{V}_f$. Consider the canonical corestriction homomorphisms $cor_v : H^{-1}(K_v,M) \longrightarrow H^{-1}(K,M)$. We define the compositions

$$\mu_v : H^1(K_v,G) \xrightarrow{ab^1} H^1_{ab}(K_v,G) = H^{-1}(K_v,M) \xrightarrow{cor_v} H^{-1}(K,M)$$

Let $\underset{v}{\oplus} H^1(K_v,G)$ denote the subset of the direct product consisting of the families $(\xi_v)_{v\in\mathcal{V}}$ such that $\xi_v = 1$ for v outside some finite set. We consider the map

$$\mu = \Sigma\mu_v : \underset{\mathcal{V}}{\oplus} H^1(K_v,G) \longrightarrow (M_\Gamma)_{tors}$$

The map μ is functorial in G.

Theorem 5.16 [Ko3]. The sequence

$$0 \longrightarrow Ш(G) \longrightarrow H^1(K,G) \longrightarrow \oplus H^1(K_v,G) \xrightarrow{\mu} (\pi_1(\overline{G})_\Gamma)_{tors}$$

is exact.

Proof. We have to prove only the exactness in the term $\oplus H^1(K_v,G)$. Consider the commutative diagram

(5.16.1)
$$\begin{array}{ccccc} H^1(K,G) & \longrightarrow & \oplus H^1(K_v,G) & & \\ \downarrow{\scriptstyle ab} & & \downarrow{\scriptstyle \oplus\, ab_v} & & \\ H^1_{ab}(K,G) & \longrightarrow & \oplus H^1_{ab}(K_v,G) & \longrightarrow & (\pi_1(\overline{G})_\Gamma)_{tors} \end{array}$$

Set $M = \pi_1(\overline{G})$; then using Proposition 4.11 we see that the lower row of the diagram is the exact sequence (4.3.1)

$$\mathcal{H}^1(K,M;\overline{K}^\times) \longrightarrow \mathcal{H}^1(K,M,\overline{\mathbb{A}}^\times) \longrightarrow \mathcal{H}^1(K,M,\overline{\mathbb{C}}) ,$$

hence the lower row of (5.16.1) is exact.

It is clear from the diagram that the composition

$$H^1(K,G) \longrightarrow \oplus H^1(K_v,G) \longrightarrow (M_\Gamma)_{tors}$$

is zero. Now let $\xi_A = \xi_\infty \times \xi_f \in \oplus H^1(K_v,G)$, where $\xi_\infty \in \prod_\infty H^1(K_v,G)$, $\xi_f \in \oplus_{\mathcal{V}_f} H^1(K_v,G)$. Suppose that $\mu(\xi_A) = 0$. Let h_A be the image of ξ_A in $\oplus H^1_{ab}(K_v,G)$. Then the image of h_A in $(M_\Gamma)_{tors}$ is zero, hence h_A is the image of some element $h \in H^1_{ab}(K,G)$. Consider the element $h \times \xi_\infty \in H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that $h \times \xi_\infty$ is contained in the fiber product over $\prod_\infty H^1_{ab}(K_v,G)$. By Theorem 5.12 $h \times \xi_\infty$ comes from $H^1(K,G)$. The theorem is proved.

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