TORSION BOUNDS OF LOOP SPACES AND SUSPENSIONS

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Due to a result of Arkowitz-Curjel [1] we know that a connected finite type H-space X admits a map $g : X \to K_X$ with the following properties. The space K_X is a product of Eilenberg Mac Lane spaces with homotopy groups $\pi_n(K_X) = \pi_n(X)$ for $n \geq 1$ and

$$g_* : \pi_n(X) \to \pi_n(K_X) = \pi_n(X)$$

satisfies $g_*(a) = N_n \cdot a$ for $a \in \pi_n(X)$ where N_n is a natural number ≥ 1. The sequence N_* of numbers N_n is termed a π-torsion bound for X. We are interested in the following questions:

(A) *Does any connected H-space admit a π-torsion bound N_* ?*

(B) *Is there a sequence N_* which is a π-torsion bound for all connected finite type H-spaces?*

In fact, if (A) is true then also (B) is true.

Dually to the result of Arkowitz-Curjel a simply connected finite type co-H-space Y admits a map $f : M_Y \to Y$ with the following properties. The space M_Y is a one point union of Moore spaces with homology groups $H_n(Y) = H_n(M_Y)$ for $n \geq 1$ and

$$f_* : H_n(Y) = H_n(M_Y) \to H_n(Y)$$

satisfies $f_*(a) = M_n \cdot a$ for $a \in H_n(Y)$ where M_n is a natural number ≥ 1. The sequence M_* of numbers M_n is termed an H-torsion bound for Y. Similarly as in the problems (A) and (B) above we obtain the following questions.

(C) *Does any simply connected co-H-space admit an H-torsion bound M_*?*

(D) *Is there a sequence M_* of numbers which is an H-torsion bound for all simply connected finite type co-H-spaces?*

Again if (C) holds then also (D) holds.

All problems (A) ... (D) are unsolved. It was, however, observed by Arlettaz-Pointet [AP], [P2], [P1] that there exists a sequence N_* which is a π-torsion bound for a large class of loop spaces. In this paper we recall and sketch proofs of such results and we show that dual results hold. We prove that there exists a sequence M_* which is an H-torsion bound for a large class of suspensions.

These bounds have implications concerning the exponents of the kernel, resp. cokernel of the Hurewicz homomorphism; compare also [A5], [A3]. We are not interested in writing down formulas for such bounds since these depend on the method of proof and tend to be very large; compare [P2], [A4]. The interested reader, however, will be able to deduce from the proofs explicit formulas.

The results in this paper are nice examples of a duality in homotopy theory which is called $H\pi$-duality in [B3], [B2] and which is part of Eckmann-Hilton duality. In particular, the boundary invariants introduced in [B2] which are dual to Postnikov invariants play an essential role in the proofs. In fact, the application of boundary invariants was the motivation to consider the problems in this paper. For the convenience of the reader and in order to make the duality explicit we recall from the literature some results on Postnikov-invariants and π-torsion bounds which essentially are, with the exception of (3.14), due to Arlettaz and Pointet. The main new results in this paper are dealing with boundary invariants and H-torsion bounds of co-H-spaces and suspensions.

§ 1 Bounded classes of spaces

Let A be an abelian group. An Eilenberg-Mac Lane space $X = K(A,n)$ is a connected space with homotopy groups $\pi_n(X) = A$ and $\pi_i X = 0$ for $i \neq n$. A Moore space $Y = M(A,n)$ is a 1-connected space with reduced homology groups $H_n(Y) = A$ and $H_i Y = 0$ for $i \neq n$. We only consider spaces with base point of the homotopy type of a CW-complex and base point preserving maps. Let $[X,Y]$ be the set of homotopy classes of such maps $X \to Y$. Then

$$(1.1) \qquad \begin{cases} \pi_n(A,X) = [M(A,n),X] \\ H^n(X,A) = [X,K(A,n)] \end{cases}$$

are the homotopy groups, resp. cohomology groups of X with coefficients in A. These are groups since $\Omega K(A,n+1) = K(A,n)$ and $\Sigma M(A,n) = M(A,n+1)$ where Ω and Σ denote the loop space and the suspension respectively.

In the literature homotopy groups with coefficients are often denoted by $\pi_n(X,A)$. The comparison with the Hom-functor, however, shows that the notation $\pi_n(A,X)$ as above is more natural.Let

$$(1) \qquad H^n(X,A)_\pi \subset H^n(X,A) = [X,K(A,n)]$$

be the subgroup of all $f \in [X,K(A,n)]$ which induce the trivial map $0 = \pi_n f : \pi_n X \to A$. Similarly let

$$(2) \qquad \pi_n(A,X)_H \subset \pi_n(A,X) = [M(A,n),X]$$

be the subgroup of all $f \in [M(A,n),X]$ which induce the trivial map $0 = H_n f : A \to H_n X$.

Recall that a space X is <u>simple</u> if X is connected and if $\pi_1 X$ acts trivially on $\pi_n X$ for $n \geq 1$. This implies $\pi_1 X$ is abelian. Clearly simply connected spaces are simple. A simple space X has a <u>Postnikov decomposition</u> given by maps $(n \geq 2)$

$$X \xrightarrow{q} X_{(n)} \xrightarrow{p_n} X_{(n-1)}$$

where q induces isomorphisms $\pi_i(q)$ for $i \leq n$ and where $\pi_i X_{(n)} = 0$ for $i > n$. The map p_n is the fiber of a <u>k-invariant</u>

$$(1.2) \qquad k_n(X) \in H^{n+1}(X_{(n-1)}, \pi_n(X))_\pi$$

If all $k_n(X)$ are trivial then X is a product of Eilenberg-Mac Lane spaces. The Postnikov tower shows that all spaces X are 'twisted' products of Eilenberg-Mac Lane spaces. In the following definition we consider spaces for which all k-invariants are torsion elements. Such spaces are 'close' to products of Eilenberg-Mac Lane spaces.

For an abelian group A and a natural number $N \in \mathbb{N} = \{1, 2, \dots\}$ let $N : A \to A$ be the homomorphism, which carries $a \in A$ to $N \cdot a$.

(1.3) Definition. A class $\mathcal{X}$ of simple spaces is <u>π-bounded</u> if there exists a sequence $N_1, N_2, \dots$ of natural numbers such that for all $n \geq 1$ and $X \in \mathcal{X}$ one of the following equivalent conditions (1), (2), (3) is satisfied.

(1) The Hurewicz homomorphism h is embedded in a commutative diagram of homomorphisms between abelian groups

$$\begin{tikzcd} \pi_n X \arrow[rr, "N_n"] \arrow[dr, "h"'] & & \pi_n X \\ & H_n X \arrow[ur, "r"'] & \end{tikzcd}$$

(2) There exists a map

$$g_n : X \to K(\pi_n X, n)$$

inducing $(g_n)_* = N_n : \pi_n X \to \pi_n X$.

(3) the $n - th$ k-invariant $k_n(X)$ of X is an N_n -torsion element, that is $N_n \cdot k_n(X) = 0$.

Clearly a space X is π -bounded if the class $\{X\}$ consisting only of the space X is π -bounded.

There is an obvious way to dualize the definition of a π -bounded class as follows. For this one has to consider the dual of the k-invariant $k_n(X)$ which is the <u>boundary invariant</u>

$$(1.4) \qquad \beta_n(X) \in \Gamma_{n-1}(H_n X, X)/\Delta\,\mathrm{im}\,(b_{n+1}X)_*$$

of a simply connected space X defined in 2.6.4 [B2]. The new concept of boundary invariant in [B2] is not yet common knowledge as is the concept of Postnikov invariant appearing in each reasonable text book of homotopy theory. There are various different ways to define the boundary invariant $\beta_n X$, see [B2], one of which is recalled below for the convenience of the reader. The striking duality between k-invariants and boundary invariants discussed in [B2], [B3] and the results of this paper should convince the reader that, in fact, boundary invariants are as useful as the classical Postnikov invariants. Boundary invariants are related to homology groups similarly as Postnikov invariants are related to homotopy groups. If all boundary invariants $\beta_n(X)$ are trivial then X is a one point union of Moore spaces.

The homology decomposition of a simply connected space X is given by maps $(n \geq 2)$

$$X_{n-1} \xrightarrow{i_n} X_n \xrightarrow{j} X \tag{1}$$

where j induce a homology isomorphism $H_r(j)$ for $r \leq n$. Moreover i_n is the cofiber of a map

$$k'_n \in \pi_{n-1}(H_n X, X_{n-1})_H \tag{2}$$

Here k'_n, however, is not an invariant of the homotopy type of X. The "invariant part" of k'_n is described by the boundary invariant $\beta_n(X)$ in (1.4). In fact, there is a canonical homomorphism

$$\lambda_A : \pi_{n-1}(A, X_{n-1})_H \to \Gamma_{n-1}(A, X)/\Delta\, \mathrm{im}(b_{n+1}X)_* \tag{3}$$

and $\beta_n(X)$ is the image of k'_n under this homomorphism, i.e. $\beta_n(X) = \lambda_A(k'_n)$ for $A = H_n X$. See section 2.7 [B2] and (1.9) (c) below.

Let $SP_\infty X$ be the infinite symmetric product of X and let ΓX be the homotopy fiber of $X \to SP_\infty X$. Then we define the group in (1.4) (3) by

$$\Gamma_{n-1}(A, X) = \pi_{n-1}(A, \Gamma X)$$

and $\Delta : \mathrm{Ext}(A, \pi_n \Gamma X) \to \pi_{n-1}(A, \Gamma X)$ is defined by the universal coefficient sequence. Moreover

$$(b_{n+1}X)_* : \mathrm{Ext}(A, \pi_{n+1} SP_\infty X) \to \mathrm{Ext}(A, \pi_n \Gamma X)$$

is obtained by the boundary homomorphism $b_{n+1}X : \pi_{n+1} SP_\infty X \to \pi_n \Gamma X$ associated to the fibration $\Gamma X \to X \to SP_\infty X$. Of course $b_{n+1}X$ coincides with the secondary boundary homomorphism $b_{n+1} : H_{n+1}X \to \Gamma_n X$ of J.H.C. Whitehead [WH].

There is the following alternative way to define $\Gamma_{n-1}(A, X)$ by use of the skeleta of X; see 2.2.3 [B2] and 7.5 [B3]. It is well known that

$$\pi_n \Gamma X = \Gamma_n X = \mathrm{image}\,\{\pi_n X^{n-1} \to \pi_n X^n\} \tag{4}$$

is the Γ-group of J.H.C. Whitehead [WH]. Now it is proved in [B2] that $\Gamma_n(A, X)$ can be obtained from $\pi_n(A; X^n)$ by the following diagram consisting of a pull back and a push out of abelian groups. Here $i : \Gamma_n X \to \pi_n X^n$ is the inclusion and $p : \pi_{n+1} X^n \to \Gamma_{n+1} X$ is the projection given by (4).

$$
\begin{array}{lccccc}
 & \mathrm{Ext}(A,\Gamma_{n+1}X) & \xleftarrow{p_*} & \mathrm{Ext}(A,\pi_{n+1}X^n) & \xrightarrow{1} & \mathrm{Ext}(A,\pi_{n+1}X^n) \\
 & \Delta\big\downarrow & \text{push} & \big\downarrow & & \big\downarrow\Delta \\
(5) & \Gamma_n(A,X) & \longleftarrow & P & \longrightarrow & \pi_n(A,X^n) \\
 & & & \big\downarrow & \text{pull} & \big\downarrow\mu \\
 & & & \mathrm{Hom}(A,\Gamma_n X^n) & \xrightarrow{i_*} & \mathrm{Hom}(A,\pi_n X^n)
\end{array}
$$

The column at the right hand side is the short exact sequence given by the universal coefficient formula.

The advantage of the boundary invariant $\beta_n X$ compared to the element k_n' in (2) is the naturality formula: Let $f : X \to Y$ be a map between simply connected spaces and let $\overline{H_n f} : M(H_n X, n-1) \to M(H_n Y, n-1)$ be any map between Moore spaces representing the homomorphism $H_n f : H_n X \to H_n Y$ induced by f in homology. Then the formula

$$
(\overline{H_n f})^* \beta_n(Y) = f_* \beta_n(X) \tag{6}
$$

holds. The elements k_n' in (2) do not satisfy such a naturality formula.

The homology decomposition shows that all simply connected spaces X are ‘twisted’ one point unions of Moore spaces. In the following definition we consider spaces for which all boundary invariants are torsion elements. Such spaces are ‘close’ to one point unions of Moore spaces.

(1.5) Definition. A class $\mathcal{X}$ of simply connected spaces is <u>H-bounded</u> if there exists a sequence $N_1, N_2, \dots$ of natural numbers such that for all $n \geq 1$ and $X \in \mathcal{X}$ one of the following equivalent conditions (1), (2), (3) is satisfied.

(1) The Hurewicz homomorphism h is embedded in a commutative diagram of homomorphisms between abelian groups

$$
\begin{array}{ccc}
 & \pi_n X & \\
s \nearrow & & \searrow h \\
H_n X & \xrightarrow{N_n} & H_n X
\end{array}
$$

(2) There exists a map

$$
g_n : M(H_n X, n) \to X
$$

inducing $(g_n)_* = N_n : H_n X \to H_n X$

(3) The $n-th$ boundary invariant $\beta_n(X)$ is an N_n -torsion element, that is $N_n \cdot \beta_n(X) = 0$

Using the Hurewicz homomorphism for spectra we can replace in (1.5) the class $\mathcal{X}$ of spaces by a class $\mathcal{X}$ of connected spectra and we define this way an H-bounded class of spectra.

A simple space X is of <u>finite type</u> if all homotopy groups $\pi_i X$ are finitely generated abelian groups or equivalently if all homotopy groups $H_i X$ are finitely generated, $i \geq 1$. A <u>rational sphere</u> $S^n_{\mathbb{Q}}$ is the same as the Moore space $M(\mathbb{Q}, n)$ and $K(\mathbb{Q}, n)$ is termed a rational Eilenberg-Mac Lane space. We have $K(\mathbb{Q}, n) = S^n_{\mathbb{Q}}$ if n is odd and $K(\mathbb{Q}, n) = \Omega S^{n+1}_{\mathbb{Q}}$ if n is even. Let $X_{\mathbb{Q}}$ be the rationalization of the simple space X.

(1.6) Proposition. *Let X be a simple space*

(A) *If X is π -bounded then $X_{\mathbb{Q}} \simeq K_{\mathbb{Q}}$ where $K_{\mathbb{Q}}$ is a product of rational Eilenberg-Mac Lane spaces. Conversely,*

(B) *if X is of finite type and $X_{\mathbb{Q}} \simeq K_{\mathbb{Q}}$ then X is π -bounded.*

Proof. (A) is clear since $g : X \to K_X$ is obviously a rational equivalence. Moreover the assumption $X_{\mathbb{Q}} \simeq K_{\mathbb{Q}}$ in (B) shows that $k_n(X) \otimes \mathbb{Q}$ is trivial. Since X is of finite type we obtain therefore a natural number N_n with $N_n k_n(X) = 0$. q.e.d.

Hence a finite type simple space X is π -bounded if and only if $X_{\mathbb{Q}} \simeq K_{\mathbb{Q}}$. This, however, does not hold for spaces which are not of finite type as the following example of Arlettaz [AP] shows.

Example. Let n be even. Then $K = K(\mathbb{Q}, n)$ satisfies $H_{2n+1}K = 0$ and $H_{2n}K = \mathbb{Q}$ so that by the universal coefficient theorem

$$(*) \qquad H^{2n+1}(K; \mathbb{Z}) = \mathrm{Ext}\,(H_{2n}K, \mathbb{Z}) = \mathrm{Ext}\,(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$$

Let X be the space with exactly two non trivial homotopy groups $\pi_n X = \mathbb{Q}, \pi_{2n} X = \mathbb{Z}$ and k-invariant $k_{2n}X = \alpha$ a non trivial element in (*). Then α is not a torsion element but one readily checks $X_{\mathbb{Q}} \simeq K(\mathbb{Q}, n) \times K(\mathbb{Q}, 2n)$ since $\mathrm{Ext}(\mathbb{Q}, \mathbb{Q}) = 0$.

We now consider the $H\pi$ -dual of (1.6).

(1.7) Proposition. *Let X be a simply connected space.*

(A) *If X is H-bounded then $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}$ where $S_{\mathbb{Q}}$ is a one point union of rational spheres. Conversely,*

(B) *if X is of finite type and $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}$ then X is H-bounded.*

Proof. The proof is dual to the proof of (1.6). Again (A) is clear since $g : M_X \to X$ is obviously a rational equivalence and clearly $(M_X)_{\mathbb{Q}} = S_{\mathbb{Q}}$. Moreover the assumption $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}$ implies that $\beta_n(X) \otimes \mathbb{Q}$ is trivial. Since X is of finite type we obtain therefore a natural number M_n with $M_n \beta_n(X) = 0$. q.e.d.

Hence a finite type simply connected space X is H-bounded if and only if $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}$. Again this does not hold for spaces which are not of finite type as the following example shows.

Example. Let X be the simply connected space with exactly two non trivial homology groups $H_2 X = \mathbb{Z}$ and $H_3 X = \mathbb{Q}$ and with boundary invariant $\beta_3 X = \alpha$ where α is a non trivial element in

$$\begin{aligned} \Gamma_2(\mathbb{Q}, X)/\Delta \operatorname{im}(b_{n+1}X)_* &= \Gamma_2(\mathbb{Q}; X) \\ &= \mathrm{Ext}\,(\mathbb{Q}, \Gamma_3 X) = \mathrm{Ext}\,(\mathbb{Q}, \Gamma \pi_2 X) \\ &= \mathrm{Ext}\,(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}. \end{aligned}$$

Compare (2.2.3) [HH]. Then X has a homology decomposition given by

$$k'_3 = \alpha \in \pi_2(\mathbb{Q}, S^2) = \mathrm{Ext}\,(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$$

Hence $X_{\mathbb{Q}} \simeq S^2_{\mathbb{Q}} \vee S^3_{\mathbb{Q}}$ since $\mathrm{Ext}\,(\mathbb{Q}, \mathbb{Q}) = 0$. But the boundary invariant α of X is of infinite order.

The proposition above shows that for finite type spaces X the condition of X being π -bounded or H-bounded is equivalent to a condition on the rationalization $X_{\mathbb{Q}}$ of X. This clearly is not at all true for classes of spaces $\mathcal{X}$.

(1.8) Lemma. *See [AP]. The conditions (1), (2) and (3) defining a π -bounded class in (1.3) are equivalent.*

Proof. One obtains (2) $\Rightarrow$ (1) by the naturality of h applied to the map g_n and (1) $\Rightarrow$ (2) follows from the surjectivity of

$$H^n(X, \pi_n X) \to \mathrm{Hom}(H_n X, \pi_n X)$$

Moreover (2) $\Rightarrow$ (3) follows from the naturality of the k-invariant applied to g_n. Finally (3) $\Rightarrow$ (2) is a consequence of the fact that $X_{(n)}$ is the fiber of $k_n(X)$. q.e.d.

(1.9) Lemma. *The conditions (1), (2) and (3) defining an H-bounded class in (1.5) are equivalent.*

Proof. We again obtain (2) $\Rightarrow$ (1) by the naturality of the Hurewicz map h applied to g_n. Next we prove (1) $\Rightarrow$ (2) as follows. Let $0 \to B_n \to Z_n \to H_n X \to 0$ be a short free resolution of $H_n X$. Then $d : B_n \to Z_n$ defines a map $M(B_n, n) \to M(Z_n, n)$ with cofiber $M(H_n X, n)$. Now s in (1) yields $\alpha : Z_n \to H_n X \to \pi_n X$ with $\alpha d = 0$. Hence the map $\bar{\alpha} : M(Z_n, n) \to X$ given by α has an extension g_n over $M(H_n X, n)$.

Next we see that (2) $\Rightarrow$ (3) follows from the naturality of $\beta_n(X)$ applied to g_n; compare (2.6.9) (a) in [B2]. The proof (3) $\Rightarrow$ (2) is more complicated. Let

$$\beta \in \Gamma_{n-1}(H_n X, X) \tag{a}$$

be an element representing $\beta_n X$ in (1.4). Then we have $N \cdot \beta_n(X) = 0$ for some $N \geq 1$ if and only if there exists

$$\begin{cases} \alpha \in \mathrm{Ext}\,(H_n X, H_{n+1} X) \quad \text{with} \\ N\beta = \Delta(b_{n+1})_* \alpha \end{cases} \tag{b}$$

Here $b_{n+1} : H_{n+1} X \to \Gamma_n X$ is part of the exact sequence of J.H.C. Whitehead. We can choose a homomorphism α' such that the composite

$$B_n \xrightarrow{\alpha'} Z_{n+1} \twoheadrightarrow H_{n+1} X$$

represents α. By (2.7.5) [B2] we can choose a map β' associated to k'_n for which the following diagram commutes

(c)
$$
\begin{tikzcd}
 & X^n \\
 & X_{n-1} \arrow[u, phantom, "\cup"] \\
M(H_n, n-1) \arrow[ur, "k'_n"] \arrow[r, "\beta'"] & X^{n-1} \arrow[u, phantom, "\cup"] \\
M(Z_n, n-1) \arrow[u, phantom, "\cup"] \arrow[r] & X^{n-2} \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

Here k'_n is the map of the homology decomposition in (1.4) (2).

In fact, this diagram yields for $A = H_n$ the homomorphism λ_A in (1.4) (3) which carries k'_n to the class $\lambda_A(k'_n)$ represented by β'. That is, with the operators in diagram (1.4) (5) we have for a choice of β' as above that $\mu(\beta') \in \operatorname{image} i_*$ so that β' represents an element $\beta' \in P$ in the pull back of (1.4) (5). The projection map $q : P \to \Gamma_{n-1}(A, X)$ yields $\lambda_A(k'_n) = q(\beta')$ and this defines λ_A.

Let $C_{n+1} = H_{n+1}(X^{n+1}, X^n)$ and let $f : M(C_{n+1}, n) \to X^n$ be the attaching map of $(n+1)$ -cells in X. Then f induces the operator b_{n+1} in (b) so that the equation in (b) is equivalent to the condition that the following diagram homotopy commutes with $H_n = H_n(X)$.

(d)
$$
\begin{tikzcd}
M(H_n, n-1) \arrow[r] \arrow[d, "N"'] & M(B_n, n) \arrow[r, "\alpha'"] & M(Z_{n+1}, n) \arrow[r] & M(C_{n+1}, n) \arrow[r] \arrow[d, "f"] & * \arrow[d] \\
M(H_n, n-1) \arrow[rr, "k'_n"] \arrow[d] & & X_{n-1} \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\
* \arrow[rr] & & X_n \arrow[r] & X^{n+1} \arrow[r] & X
\end{tikzcd}
$$

Adding the homotopies in this diagram yields the required map $g_n : M(H_n, n) = \Sigma M(H_n, n-1) \to X$ with the property in (2). q.e.d.

§ 2 Bounded classes of H-spaces and co-H-spaces

An <u>H-space</u> is a space X together with a multiplication $\mu : X \times X \to X$ for which the composite

(2.1)
$$X \vee X \to X \times X \xrightarrow{\mu} X$$

is homotopic to the folding map $(1, 1)$. For example a loop space ΩX is an H-space. A <u>co-H-space</u> is a connected space together with a comultiplication $\mu : X \to X \vee X$ for which the composite

$$X \xrightarrow{\mu} X \vee X \to X \times X \tag{2.2}$$

is homotopic to the diagonal map $(1,1)$. For example a suspension ΣX is a co-H-space. The map μ induces in both cases an operation $+$ on the set of homotopy classes $[X,X]$, that is $\alpha+\beta=\mu_*(\alpha,\beta)$ and $\alpha+\beta=\mu^*(\alpha,\beta)$ respectively. For a natural number N let

$$N \cdot 1_X : X \to X \tag{2.3}$$

be a map given by the N-fold sum $N \cdot 1_X = (\dots((1_X+1_X)+1_X)+\dots+1_X)$ in $[X,X]$. The induced maps

$$(N \cdot 1_X)_* = N : \pi_n X \to \pi_n N \quad \text{for an } H\text{-space } X$$
$$(N \cdot 1_X)_* = N : H_n X \to H_n X \quad \text{for a co-}H\text{-space } X$$

are multiplication by N.

It is well known that the rationalization of a connected H-space is a product of rational Eilenberg-Mac Lane spaces. Therefore (1.6) implies the following result of Arkowitz-Curjel [AC].

(2.4) Proposition. *A finite type connected H-space is π -bounded.*

Also the $H\pi$ -dual of this result holds by (1.7) since the rationalization of a simply connected co-H-space is known to be a one point union of rational spheres.

(2.5) Proposition. *A finite type simply connected co-H-space is H-bounded.*

Is it true that (2.4) and (2.5) still hold if one omits the finite type condition? Here the examples in (1.6) and (1.7) cannot be used.

Let $\mathcal{X}$ be a class of spaces and let F be a functor which carries spaces to spaces then $F\mathcal{X}$ denotes the class of all spaces FX with $X \in \mathcal{X}$.

(2.6) Proposition. *Let $\mathcal{X}$ be a class of connected H-spaces and assume $\Omega\Sigma\mathcal{X}$ is π -bounded. Then $\mathcal{X}$ is π -bounded.*

Proof. For an H-space X the suspension map $i : X \to \Omega\Sigma X$ admits a homotopy retraction r. Compare Rutter [R2]. Now the result follows from the naturality of the k-invariants since

$$i_* k_n(X) = i^* k_n \Omega\Sigma X$$

where i_* is injective since i admits a retraction r. Compare also 4.9 and 4.10 in [P1] and 3.3 in [AP]. q.e.d.

(2.7) Proposition. *Let $\mathcal{X}$ be a class of simply connected co-H-spaces and assume $\Sigma\Omega\mathcal{X}$ is H-bounded. Then $\mathcal{X}$ is H-bounded.*

Proof. For a co-H-space the evaluation map $\varepsilon : \Sigma\Omega X \to X$ admits a homotopy section s. Compare Rutter [R1]. Now the naturality of the boundary invariants shows

$$\varepsilon_* \beta_n(\Sigma\Omega X) = \varepsilon^* \beta_n X$$

where ε^* is injective since ε admits the section s. q.e.d.

The propositions (2.6) and (2.7) show that it suffices to prove (2.4) and (2.5) for loop spaces and suspensions respectively.

(2.8) Proposition. *Let $\mathcal{X}$ be a class of connected H-spaces. Then $\mathcal{X}$ is π -bounded if and only if there exists a sequence $R_1, R_2, \dots$ in $\mathbb{N}$ such that for all $n \geq 1$ and $X \in \mathcal{X}$ there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\bar{R}_n"] \arrow[d] & X \arrow[d] \\ K_n \arrow[r] & X_{(n)} \end{tikzcd}$$

Here $\bar{R}_n$ is a map which induces multiplication by R_n on homotopy groups, that is $(\bar{R}_n)_* = R_n : \pi_* X \to \pi_* X$. Moreover K_n is a product of Eilenberg-Mac Lane spaces and $X \to X_{(n)}$ is the Postnikov section of X. The space K_n can be chosen to be the product of all $K(\pi_i, i)$ with $i \leq n$ and $\pi_i = \pi_i(X)$.

Proof. If the sequence $R_i, i \geq 1$, exists then the naturality of k-invariants shows that $\mathcal{X}$ is π -bounded. Now assume that $\mathcal{X}$ is π -bounded by $N_i, i \geq 1$. Then we construct R_i with the property in (2.8) as follows. Consider the homotopy commutative diagram with $N = N_n$.

$$\begin{tikzcd} X \arrow[r, "N\cdot 1"] \arrow[d] & X \arrow[d] \\ X_{(n)} \arrow[r, "N\cdot 1"] \arrow[d, "p_n"'] & X_{(n)} \arrow[d, "p_n"] \\ X_{(n-1)} \arrow[r, "N\cdot 1"] & X_{(n-1)} \end{tikzcd}$$

We use the fact that $X_{(n)}$ is an H-space and that the vertical arrows in the diagram are H-maps; see 9.4 [St]. Since p_n is a fibration the bottom square can be chosen to be commutative; see 9.4 [St]. Hence we obtain the factorization $N \cdot 1 : X_{(n)} \to (N \cdot 1)^* p_n \to X_{(n)}$. Here we have

$$(N \cdot 1)^* p_n \simeq X_{(n-1)} \times K(\pi_n, n)$$

since $N \cdot k_n(X) = 0$. Now we apply this inductively and we get for $R_n = N_1 \cdot N_2 \cdot \dots \cdot N_n$ the result. Compare also 1.2 in [AP]. q.e.d.

We do not know whether the dual of (2.8) holds for a class of simply connected co-H-spaces; but we are able to prove the following result.

(2.9) Proposition. *Let $\mathcal{X}$ be a class of suspensions $X = \Sigma Y$ of simply connected spaces Y. Then $\mathcal{X}$ is H-bounded if and only if there exist $R_1, R_2, \dots$ in $\mathbb{N}$ such that for all $n \geq 1$ and $X \in \mathcal{X}$ there is a homotopy commutative diagram*

$$\begin{tikzcd} X & X \arrow[l, "R_n \cdot 1_X"'] \\ M_n \arrow[u] & X_n \arrow[l] \arrow[u] \end{tikzcd}$$

where M_n is a one point union of Moore spaces and $X_n \to X$ is given by a homology decomposition of Y; that is $X_n = \Sigma Y_{n-1} \to \Sigma Y = X$. The space M_n can be chosen to be the one point union of all Moore spaces $M(H_i, i)$ with $i \le n$ and $H_i = H_i X$.

Proof. If the sequence $R_i, i \ge 1$, exists the naturality of boundary invariants shows that $\mathcal{X}$ is H-bounded. Now assume that $\mathcal{X}$ is H-bounded by $N_i, i \ge 1$. Then we have to construct R_i and diagrams as in (2.8). Since $X = \Sigma Y \in \mathcal{X}$ is a suspension we have

$$X^n = \Sigma Y^{n-1} \quad \text{and} \quad X_n = \Sigma Y_{n-1}$$

and all inclusion maps $X^n \subset X_n \subset X$ are suspensions. For $X \in \mathcal{X}$ we know $N_n \cdot \beta_n(X) = 0$. This implies that also the homomorphism

$$b_n(X) : H_n X \to \Gamma_{n-1} X \tag{1}$$

satisfies $N_n b_n(X) = 0$. The homomorphism is represented by the map b_n in the commutative diagram

$$\begin{array}{ccc} M(C_n, n-1) & \xrightarrow{f_n} & X^{n-1} \subset X^n = C(f_n) \\ \cup & & \cup \\ M(Z_n, n-1) & \xrightarrow{b_n} & X^{n-2} \end{array} \tag{2}$$

Here $C_n = H_n(X^n, X^{n-1})$ and f_n is the attaching map of n-cells. Moreover $B_n \subset Z_n \subset C_n$ denote the subgroups of boundaries and cycles in C_n. We have $C_n = Z_n \oplus B_{n-1}$ and the homology decomposition X_n of X is obtained by the cofiber sequence

$$M(B_{n-1}, n-1) \xrightarrow{\bar{f}_n} X^{n-1} \to X_{n-1} = C(\bar{f}_n) \tag{3}$$

where $\bar{f}_n$ is the restriction of f_n so that $X_{n-1} \subset X^n$. We obtain the diagrams in (2.9) inductively as follows. Let

$$M_n = \bigvee_{i=1}^{n} M(H_i X, i).$$

The inductive assumption is the existence of a homotopy commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{R_{n-1}} & X \\ \uparrow & & \uparrow \\ M_{n-1} & \xleftarrow{S_{n-1}} & X_{n-1} \end{array} \tag{4}$$

with the property that the composites

$$(5)\qquad \begin{cases} M(Z_{n+1},n) \xrightarrow{b_{n+1}} X^{n-1} \xrightarrow{I^{n-1}} X_{n-1} \xrightarrow{S_{n-1}} M_{n-1} \\ M(Z_n,n-1) \xrightarrow{b_n} X^{n-2} \xrightarrow{I^{n-1}i^{n-2}} X_{n-1} \xrightarrow{S_{n-1}} M_{n-1} \end{cases}$$

are null homotopic. This holds for $n=3$ since $M_2 = X_2$ is a point. Clearly X is 2-connected since Y is simply connected. We now consider the following diagram

$$(6)\qquad \begin{tikzcd}
X & X \arrow[l, "R_{n-1}"'] & X \arrow[l, "N_n"'] \\
 & X^n \arrow[u] & \\
M_{n-1} \arrow[uu] \arrow[ur, phantom, "\circledast"] & X_{n-1} \arrow[l, "f_{n-1}"'] \arrow[u, "i"'] & X_{n-1} \arrow[l, "N_n"'] \arrow[uu] \\
 & M(Z_{n+1},n) \arrow[u, "b_{n+1}"'] & \\
 & M(B_n,n) \arrow[u, "\alpha'"'] & M(H_n,n-1) \arrow[l, "q"'] \arrow[uu, "k'_n"']
\end{tikzcd}$$

Here we know by (1) and diagram (1.9) (d) that

$$\alpha'' = N_n k'_n - b_{n+1}\alpha' q$$

satisfies $i_*\alpha'' = 0$. Since

$$M(Z_n,n-1) \xrightarrow{b_n} X_{n-1} \xrightarrow{i} X^n$$

is a cofiber sequence this implies that

$$\alpha'' = b_n\alpha''' \quad \text{for some} \quad \alpha''' : M(H_n,n-1) \to M(Z_n,n-1)$$

Here we use the fact that X is 2-connected. Hence we see by (5) that

$$(7)\qquad f_{n+1}N_nk'_n \simeq 0$$

This shows that we have the following commutative diagram where P denotes a push out

$$(8)\qquad \begin{array}{ccccc} X & \xleftarrow{R_{n-1}N_n} & & & X \\ \uparrow & & & & \uparrow \\ P & \xleftarrow{\bar{S}_n} & & & X_n \\ \uparrow{\scriptstyle \bar{i}} & & & & \uparrow{\scriptstyle i_{n-1}} \\ M_{n-1} & \xleftarrow{S_{n-1}} & X_{n-1} & \xleftarrow{N_n} & X_{n-1} \end{array}$$

This diagram is obtained by diagram $\circledast$ in (6) which we may choose to be commutative since $X_{n-1} \to X$ is a cofibration. By (7) we know that there is a homotopy equivalence

$$(9)\qquad \mu : P \simeq M_{n-1} \vee M(H_n, n) = M_n$$

Now we have to obtain condition (5) for n replaced by $(n+1)$. For this we consider the following composite with $N = N_{n+1} \cdot N_{n+2}$.

$$M(Z_{n+2}, n+1) \xrightarrow{b_{n+2}} X^n \xrightarrow{N} X^n \xrightarrow{i^n} X^{n+1}$$

which is null homotopic since $Nb_{n+2}(X) = 0$. Hence the cofiber sequence

$$M(C_{n+1}, n) \xrightarrow{f_n} X^n \xrightarrow{i^n} X^{n+1}$$

shows that there exists $\lambda : M(Z_{n+2}, n+1) \to M(C_{n+1}, n)$ such that

$$(10)\qquad f_n \lambda = N b_{n+2}$$

Moreover the diagram

$$(11)\qquad \begin{array}{ccc} M(C_{n+1}, n) & & = M(B_n, n) \vee M(Z_{n+1}, n) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle (0, i_{n-1} I^{n-1} b_{n+1})} \\ X^n & \xrightarrow{I^n} & X_n \end{array}$$

homotopy commutes with $i_{n-1} I^{n-1} : X^{n-1} \subset X_{n-1} \subset X_n$. Now let

$$(12)\qquad R_n = R_{n-1} \cdot N_n \cdot N_{n+1} \cdot N_{n+2}$$

Then (8), (9) show that we have a homotopy commutative diagram

$$(13)\qquad \begin{array}{ccc} X & \xleftarrow{R_n} & X \\ \uparrow & & \uparrow \\ M_n & \xleftarrow{S_n} & X_n \end{array}$$

with $S_n = (\mu\bar{S}_n)N$. Moreover we get the following equations.

$$\begin{aligned}
S_n I^n b_{n+2} &= (\mu\bar{S}_n)NI^n b_{n+2}, && (13)\\
&= \mu\bar{S}_n I^n N b_{n+2} \\
&= \mu\bar{S}_n I^n f_n \lambda, && (10)\\
&= \mu\bar{S}_n (0, i_{n-1}I^{n-1}b_{n+1})\lambda, && (11)\\
\mu\bar{S}_n i_{n-1}I^{n-1}b_{n+1} &= \mu\bar{i}S_{n-1}N_n I^{n-1}b_{n+1}, && (8)\\
&= \mu\bar{i}S_{n-1}I^{n-1}b_{n+1}N_n, \\
&= 0, \quad \text{see (5)}
\end{aligned}$$

Here we use the fact that b_{n+1} is a suspension since f_{n+1} is a suspension. This completes the proof that $S_n I^n b_{n+2} = 0$. Moreover we get

$$\begin{aligned}
S_n I^n i^{n-1} b_{n+1} &= \mu\bar{S}_n N I^n i^{n-1} b_{n+1}, && (13)\\
&= \mu\bar{S}_n I^n N i^{n-1} b_{n+1} \\
&= 0
\end{aligned}$$

since $N_{n+1}i^{n-1}b_{n+1} = 0$ by $N_{n+1}b_{n+1}(X) = 0$. This completes the proof of the inductive step; see (5). q.e.d.

§ 3 Examples of bounded classes of loop spaces and suspensions

We shall prove in this section the following two results (3.1) and (3.2) which are $H\pi$ -dual and we dscribe various further classes of loop spaces and suspensions which are π -bounded and H-bounded respectively.

(3.1) Theorem. *The class of all connected infinite loop spaces is π -bounded.*

(3.2) Theorem. *The class of all connected spectra is H-bounded.*

For the proof of (3.1) and (3.2) compare (3.13) below. See also [A1], [A2], [A4], [AP], [A7]. On the other hand we show:

(3.3) Theorem. *See [P1].*

(A) *The class of all connected finite type loop spaces ΩX with at most two non trivial homotopy groups is π -bounded.*

(B) *The class of all connected double loop spaces $\Omega\Omega X$ with at most two non trivial homotopy groups is π bounded.*

This result is due to Pointet-Tischler [P1] where part (A) is also stated for non-finite type X. The proof of this non-finite type case of (A), however, has a gap in 3.26 [P1]. Also the following $H\pi$ -dual of (3.3) holds.

(3.4) Theorem.

(A) *The class of all suspensions ΣX of simply connected finite type spaces X with at most two non trivial homology groups is H-bounded.*

(B) *The class of all double suspensions $\Sigma\Sigma X$ of simply connected spaces X with at most two non trivial homology groups is H-bounded.*

In (3.3) (B) and (3.4) (B) we do not need the assumption that spaces are of finite type.

(3.5) Definition. Let $\mathbf{Ab}_*$ be the category of graded abelian groups $\{A_n; n \geq 1\}$ and homomorphisms $h_* = \{h_n : A_n \to B_n; n \geq 1\}$. We say that a class $\mathcal{H}$ of morphisms in $\mathbf{Ab}_*$ is <u>bounded torsion</u> if there exists a sequence $N_1, N_2, \dots$ in $\mathbb{N}$ such that for all $h_* \in \mathcal{H}$ the image of $h_n : A_n \to B_n$ is N_n -torsion (that is $N_n h_n = 0$).

The results above rely on the following two basic facts (3.6) and (3.7) where we use the evaluation map $\Sigma\Omega K \to K$ and the suspension map $M \to \Omega\Sigma M$. For any space X and $n \geq 1$ let $\pi'_n(X)$, resp. $H'_n(X)$, be the kernel, resp. cokernel of the Hurewicz map $h : \pi_n(X) \to H_n(X)$. Hence we have the exact sequence

$$0 \to \pi'_n(X) \to \pi_n(X) \to H_n(X) \to H'_n(X) \to 0$$

(3.6) Lemma. *See [P1].*

(A) *Consider the class of graded homomorphisms*

$$\Omega_* = \{H_n(\Sigma\Omega K) \xrightarrow{\Omega_n} H_n(K) \to H'_n(K)\}_{n\geq 1}$$

induced by the evaluation map in homology where K is a product of Eilenberg-Mac Lane spaces. This class is bounded torsion.

(B) *Consider the class of graded homomorphisms*

$$\Omega^A_* = \{\Omega^A_n : H^n(K, A)_\pi \to H^n(\Sigma\Omega K, A)_\pi\}_{n\geq 1}$$

induced by the evaluation map in cohomology where K is a finite type product of Eilenberg-Mac Lane spaces and $A \in \mathbf{Ab}$. This class is bounded torsion.

This crucial lemma is due to Pointet-Tischler in [P1]; compare also the proof below where we consider also the following lemma which is dual to (3.6).

(3.7) Lemma.

(A) *Consider the class of graded homomorphisms*

$$\Sigma_* = \{\pi'_n(M) \to \pi_n(M) \xrightarrow{\Sigma_n} \pi_n(\Omega\Sigma M)\}_{n\geq 1}$$

induced by the suspension map on homotopy groups where M is a one point union of Moore spaces. This class is bounded torsion.

(B) *Consider the class of graded homomorphisms*

$$\Sigma^A_* = \{\Sigma^A_n : \pi_n(A, M)_H \to \pi_n(A, \Omega\Sigma M)_H\}_{n\geq 1}$$

induced by the suspension map for homotopy groups with coefficients where M is a finite type one point union of Moore spaces and $A \in \mathbf{Ab}$. This class is bounded torsion.

Proof of (3.6) and (3.7). Since the functors Σ, Ω, π_n and H_n are compatible with filtered direct limits it suffices to prove (3.6) (A) and (3.7) (A) for spaces K and M respectively which are of finite type. Moreover since the maps Ω_n and Σ_n carry cross effects to zero it in addition suffices to prove (3.6) (A) and (3.7) (A) for $K = K(C,M)$ and $M = M(C,m)$ respectively where $C = \mathbb{Z}$ or $C = \mathbb{Z}/p^r$ with p a prime and $r \geq 1$. In this case (3.6) (A) is a consequence of the work of Cartan [C]; compare 3.34 Pointet-Tischler [P1]. Moreover for $M = M(\mathbb{Z}, m)$ lemma (3.7) (A) is a well known property of homotopy groups of spheres. For $M = M(\mathbb{Z}/p^r, m)$ lemma (3.7) (A) is a consequence of the fact that the suspension map

$$(*) \qquad M(\mathbb{Z}/p^r, m) \to \Omega\Sigma M(\mathbb{Z}/p^r, m)$$

factors through the space $S^{m+1}\{p^r\}$ which is the fiber of the map $p^r \cdot 1_{S^{m+1}} : S^{m+1} \to S^{m+1}$. This is a result of Cohen-Moore-Neisendorfer in 11.6 [CMN].

For a proof of (3.6) (B) we refer to the proof of Pointet-Tischler where, however, we restrict to the finite type case so that we do not need 3.26 [P1]. For the proof of (3.7) (B) we again use (*) above and the fact that Σ_n^A carries cross effects to zero. Hence again it suffices to prove (3.7) (B) for $M = M(C,n)$ and this is a consequence of (*). q.e.d.

Proof of (3.3) and (3.4). One obtains (3.3) (A) as a consequence of (3.6) (B) and one obtains (3.4) (A) as a consequence of (3.7) (B). Next we prove (3.3) (B) and (3.4) (B). For this we only use the universal coefficient formulas and (3.6) (A) and (3.7) (A) respectively. In fact let $\pi, G \in Ab$ and consider the following commutative diagram where Ω is the loop map and $K = K(\pi, m), K' = \Omega K, K'' = \Omega\Omega K$.

$$\begin{tikzcd}
 & Ext(H_{n-2}K', G) \arrow[r, "(\sigma\Omega_{n-2})^*"] \arrow[d] & Ext(H_{n-3}K'', G) \arrow[d] \\
H^n(K,G)_\pi \arrow[r, "\Omega"] \arrow[d] & H^{n-1}(K',G)_\pi \arrow[r, "\Omega"] \arrow[d] & H^{n-2}(K'',G)_\pi \\
Hom(H_nK, G) \arrow[r, "(\sigma\Omega_n)^*"'] & Hom(H_{n-1}K', G) &
\end{tikzcd}$$

Here σ denotes the suspension isomorphism $\sigma : H_{n-1}X \cong H_n\Sigma X$ and Ω_n, Ω_{n-2} are given by (3.6) (A). The column in the middle of the diagram is short exact. If $N\Omega_n = 0$ and $M\Omega_{n-2} = 0$ then clearly the diagram shows $N \cdot M(\Omega\Omega) = 0$. Moreover since $\Omega k_n(X) = k_{n-1}(\Omega X)$ we see that (3.6) (A) implies (3.3) (B). The proof of (3.4) (B) is strictly dual. For this we consider the suspension map Σ in the following commutative diagram where $M = M(\pi, m), M' = \Sigma M(\pi, m)$ and $M'' = \Sigma\Sigma M(\pi, m)$.

$$\begin{array}{lccccc}
 & & Ext(G,\pi_{n+2}M') & \xrightarrow{(\omega\Sigma_{n+2})^*} & Ext(G,\pi_{n+3}M'') \\
 & & \downarrow & & \downarrow \\
(2) & \pi_n(G,M)_H & \xrightarrow{\Sigma} & \pi_{n+1}(G,M')_H & \xrightarrow{\Sigma} & \pi_{n+2}(G,M'')_H \\
 & \downarrow & & \downarrow & & \\
 & Hom(G,\pi_n M) & \xrightarrow{(\omega\Sigma_n)^*} & Hom(G,\pi_{n+1}M') & &
\end{array}$$

Here ω denotes the canonical isomorphism $\omega : \pi_n\Omega X = \pi_{n-1}X$ and Σ_n and Σ_{n+2} are given by (3.7) (A). The column in the middle is short exact. Hence $N\Sigma_n = 0$ and $M\Sigma_{n+2} = 0$ implies $N \cdot M(\Sigma\Sigma) = 0$ and this yields (3.4) (B) since we can use the homology decomposition of X. q.e.d.

(3.9) Theorem. *Let $\mathcal{X}$ be a π -bounded class of connected H-spaces. Then (A) and (B) hold.*

(A) *The class of all double loop maps*

$$\Omega^2_* = \{\Omega^2 : H^n(X,A)_\pi \to H^{n-2}(\Omega^2 X, A)_\pi\}_{n\geq 1}$$

*with $X \in \mathcal{X}$ and $A \in$ **Ab** is bounded torsion.*

(B) *The class of all loop maps*

$$\Omega_* = \{\Omega : H^n(X,A)_\pi \to H^{n-1}(\Omega X, A)_\pi\}_{n\geq 1}$$

*with $X \in \mathcal{X}$ and X of finite type and $A \in$ **Ab** is bounded torsion.*

Proof. This follows from (2.8), (3.8) (1) and (3.6) (B). q.e.d.

(3.10) Theorem. *Let $\mathcal{X}$ be an H-bounded class of suspensions $X = \Sigma Y$ of simply connected spaces Y. Then (A) and (B) hold.*

(A) *The class of all double suspension maps*

$$\Sigma^2_* = \{\Sigma^2 : \pi_n(A,X)_H \to \pi_{n+1}(A,\Sigma^2 X)_H\}_{n\geq 1}$$

*with $X \in \mathcal{X}$ and $A \in$ **Ab** is bounded torsion.*

(B) *The class of all suspension maps*

$$\Sigma_* = \{\Sigma : \pi_n(A,X)_H \to \pi_{n+1}(A,\Sigma X)_H\}_{n\geq 1}$$

*with $X \in \mathcal{X}$ and X of finite type and $A \in$ **Ab** is bounded torsion.*

Proof. This follows from (2.9), (3.8) (2) and (3.7) (B). q.e.d.

Let $\mathcal{X}$ be a class of spaces. Then $\mathcal{X}_F$ denotes the class of fibers $F(f)$ of maps $f : X \to Y$ with the following properties:

- $X, Y \in \mathcal{X}$ and X is an H-space,
- $\pi_i X \neq 0$ implies $\pi_{i+1} Y = 0$,
- $f_* : \pi_* X \to \pi_* Y$ is trivial, $f_* = 0$.

Moreover $\mathcal{X}_F^{ft}$ is the subclass of $\mathcal{X}_F$ consisting of all $F(f)$ for which X is of finite type.

Dually let $\mathcal{X}_C$ be the class of cofibers $C(f)$ of maps $f : X \to Y$ with the following properties:

- $X, Y \in \mathcal{X}$ and Y is a suspension $Y = \Sigma Y'$ of a simply connected space Y',
- $H_i X \neq 0$ implies $H_{i+1} Y = 0$,
- $f_* : H_* X \to H_* Y$ is trivial, $f_* = 0$.

Here H_* denotes the reduced homology. Let $\mathcal{X}_C^{ft}$ be the subclass of $\mathcal{X}_C$ consisting of all $C(f)$ for which Y is of finite type.

Using this notation we get the following lemma which is needed for the induction in the proofs of theorem (3.12), (3.14), (3.15) below.

(3.11) Lemma.

(A) *If $\mathcal{X}$ is a π -bounded class of connected spaces then $\Omega \mathcal{X}_F^{ft}$ and $\Omega^2 \mathcal{X}_F$ are π -bounded classes.*

(B) *If $\mathcal{X}$ is an H-bounded class of simply connected spaces then $\Sigma \mathcal{X}_C^{ft}$ and $\Sigma^2 \mathcal{X}_C$ are H-bounded classes.*

Proof. The proof of (A) is strictly dual to the proof of (B). We therefore prove only (B) as follows. For $C(f) \in \mathcal{X}_C$ we have the short exact sequence

$$0 \to H_i Y \xrightarrow{i} H_i C(f) \xrightarrow{j} H_{i-1} X \to 0$$

where either i or j is an isomorphism; compare the definition of $\mathcal{X}_C$. If $H_i Y \neq 0$ we can use (1.5) (2) for Y and hence for $C(f)$. Now assume $H_i Y = 0$ and $H_{i-1} X \neq 0$ and let $M = M(H_{i-1} X, i-1)$. Then we can use (1.5) (2) for X and we get a map $M \to X$ inducing multiplication by $N \in \mathbb{N}$ on homology H_{i-1}. For the composite $\alpha : M \to X \to Y$ we can use (3.10) (A), resp. (3.10) (B). Hence we obtain $N' \in \mathbb{N}$ with $N' \Sigma^2 \alpha = 0$, resp. $N' \Sigma \alpha = 0$. Moreover we have the canonical map

$$C(N'\alpha) \to C(f)$$

which induces multiplication by $N' \cdot N$ on homology H_i. Hence we get also maps

$$\begin{aligned} &\Sigma^3 M \to \Sigma^2 C(N'\alpha) \to \Sigma^2 C(f), \quad \text{resp.} \\ &\Sigma^2 M \to \Sigma C(N'\alpha) \to \Sigma C(f) \end{aligned}$$

inducing multiplication by $N' \cdot N$ on homology H_{i+1}, resp. H_{i+1}. Using again (1.5) we obtain the proposition in (3.11) (B). q.e.d.

(3.12) Definition. We say that a space X is Ω -stable if X is an Eilenberg-Mac Lane space or if there exists $r \geq 2$ such that X is $(r-1)$ -connected and $\pi_i X = 0$ for $i \geq 2r-1$. Moreover X is Σ -stable if there exists $r \geq 2$ such that X is $(r-1)$ -connected and $\dim(X) \leq 2r-1$. In particular Moore spaces are Σ -stable.

(3.13) Theorem.

(A) *The class of products of Ω -stable spaces is π -bounded.*

(B) *The class of one point unions of Σ -stable spaces is H-bounded.*

Here a product of CW-complexes is defined by use of the CW-topology of the product set.

Proof of (3.13). Let X be a Ω -stable space. Then there exists for all $n \geq 1$ a space Y with $X = \Omega^n Y$. Hence we can use the Postnikov decomposition of Y and (3.11) (A) to see that (3.13) (A) holds. Now let X be a Σ -stable space. Then for $H_i X \neq 0$ the Hurewicz homomorphism $h : \pi_i X \to H_i X$ is isomorphic to $h : \pi_{i+n}\Sigma^n X \to H_{i+n}\Sigma^n X$ for all $n \geq 0$. Hence it suffices to show that the class of all $\Sigma^n X$ with n sufficiently large is H-bounded. But this follows from the homology decomposition of X and (3.11) (B) inductively. q.e.d.

(3.14) Theorem. *Let $n \geq 0$ and $r \geq 0$ and consider the following classes:*

(A) *the class of all n-connected $2r$-fold loop spaces $X = \Omega^{2r}Y$ with $\pi_i X = 0$ for $i > (r+2)(n+2r) - 2r$, and*

(B) *the class of all n-connected finite type r-fold loop spaces $X = \Omega^r Y$ with $\pi_i X = 0$ for $i > (r+2)(n+r) - r$.*

Then these classes (A) and (B) are π -bounded.

This is an improvement of the corresponding results in [P1]. We also give a more direct proof of the result relying on the "principal reduction theorem".

Proof. In case (B) we see that Y is $m = n + r$ connected. Hence by principal reduction the Postnikov tower $Y_{(i)}$ of Y admits fiber sequences

$$Y_{(jm)} \to Y_{(jm-m)} \to K_j$$

for $j \geq 2$ with $Y_{(2m)} = \Omega K_2$; compare 2.3.11 [B1]. All spaces $K_j, j \geq 2$, are Ω -stable. Hence using (3.13) (A) we can apply (3.11) (A) inductively to see that the class of all $\Omega^r Y_{((r+2)m)}$ is π -bounded if Y is of finite type. In case (A) have to use $\Omega^{2r} Y_{((r+2)m)}$ with $m = n + 2r$. q.e.d.

(3.15) Theorem. *Let $n \geq 2$ and $r \geq 0$ and consider the following classes:*

(A) *the class of all $2r$-fold suspensions $X = \Sigma^{2r} Y$ of $(n-1)$ -connected spaces Y with $\dim(Y) \leq (r+2)n - 1$, and*

(B) *the class of all r-fold suspensions $X = \Sigma^r Y$ of $(n-1)$ -connected finite type spaces Y with $\dim(Y) \leq (r+2)n - 1$.*

Then these classes (A) and (B) are H-bounded.

Proof. Since Y is $(n-1)$ -connected the skeleta Y^i satisfy principal reduction in the sense of 2.3.13 [B1]. Hence for $j \geq 2$ there exist cofibration sequences

$$K_j \to Y^{(j-1)n-1} \to Y^{jn-1}$$

where K_j is Σ -stable with $X^{2n-1} = \Sigma K_2$. Now we can apply (3.11) (B) inductively and we see that (A) and (B) hold. q.e.d.

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