Infinitesimal Torelli theorem for surfaces of general type with certain invariants

Masaaki Murakami

Abstract

We prove the infinitesimal Torelli theorem for general minimal complex surfaces X's with the first Chern number 3, geometric genus 1, and irregularity 0 which have non-trivial 3-torsion divisors. We also show that the coarse moduli space for surfaces with the invariants as above is a 14-dimensional unirational variety.

1 Introduction

In the present paper, we will prove the infinitesimal Torelli theorem for general minimal complex surfaces X's with $c_1^2 = 3$, $\chi(\mathcal{O}) = 2$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$, where c_1, $\chi(\mathcal{O})$, and $\mathrm{Tors}(X)$ are the first Chern class, the Euler characteristic of the structure sheaf, and the torsion part of the Picard group of X, respectively. We will also show that all surfaces with the invariants as above are deformation equivalent to each other, and that their coarse moduli space $\mathcal{M}$ is a 14-dimensional unirational variety. Note, here, that the condition $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$ is a topological one; minimal surfaces with $c_1^2 = 3$ and $\chi(\mathcal{O}) = 2$ have the geometric genus $p_g = 1$ and the irregularity $q = 0$, hence the torsion group $\mathrm{Tors}(X)$ isomorphic to the first homology group $H_1(X, \mathbb{Z})$.

As is well known today, Torelli type theorems do not necessarily hold for surfaces. One of the most famous counter examples is surfaces of general type with $p_g = q = 0$. Although Torelli type theorems have been proved for many classes of surfaces, finding what conditions we should impose still remains as a problem. So it makes sense to study period maps for concrete classes of surfaces.

Let us recall some results on period maps for surfaces of general type with $p_g = 1$ and $q = 0$. In [1], Catanese proved the infinitesimal Torelli

---

Masaaki Murakami: Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan. E-mail: murakami@kusm.kyoto-u.ac.jp

2000 Mathematics Subject Classification: 14J29, 14C34.

theorem for general minimal surfaces with $c_1^2 = 1$, $p_g = 1$, and $q = 0$, while in [2] that the global period mapping has degree at least 2. He first showed that any such surface is essentially a weighted complete intersection of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,2,2,3,3)$, and used this complete description to study the period map for these surfaces. Meanwhile for the case $c_1^2 = 2$, $p_g = 1$, and $q = 0$, the torsion group is either 0 or $\mathbb{Z}/2$. Using a complete description for the case of $\mathbb{Z}/2$ by Catanese and Debarre [3], Oliverio studied in [8] the infinitesimal period maps for the case of non-trivial 2-torsion divisors by the same method as in [1].

Consider the case $c_1^2 = 3$. In this case, the order $\sharp\mathrm{Tors}(X)$ is at most 3 by a result in [6]. Moreover, in [7], the author showed that any surface X of this class with $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$ is essentially a quotient of a $(3,3)$-complete intersection in $\mathbb{P}^4$ by a certain free action by $\mathbb{Z}/3$. Using this complete description, we will show in the present paper the infinitesimal Torelli theorem for general X's by an argument similar to those in [1] and [8]. Here, general X means any surface corresponding to a point in a certain Zariski open subset of the coarse moduli space $\mathcal{M}$.

In Section 2, we state our main theorems of the present paper and recall our previous results given in [7]. In Section 3, we show the unirationality of the coarse moduli space $\mathcal{M}$. Finally in Section 4, we prove the infinitesimal Torelli theorem for our surfaces X's. Throughout this paper, we work over the complex number field $\mathbb{C}$.

Notation

Let S be a compact complex manifold of dimension 2. We denote by $p_g(S)$, $q(S)$, and K_S, the geometric genus, the irregularity and a canonical divisor of S, respectively. We denote by Tors(S) the torsion part of the Picard group, and call it the torsion group of S. For a coherent sheaf $\mathcal{F}$ on S, we denote by $h^i(\mathcal{F})$ the dimension of the i-th cohomology group $H^i(S,\mathcal{F})$. The sheaf $\mathcal{O}_S$, Ω_S^p, and Θ_S are the structure sheaf, the sheaf of germs of holomorphic p-forms, and that of germs of holomorphic vector fields on S, respectively. As usual, $\mathbb{P}^n$ is the projective space of dimension n. We denote by $\varepsilon = \exp(2\pi\sqrt{-1}/3)$ a third root of unity.

2 Statement of main results

In a previous paper [7], the author gave a complete description for minimal algebraic surfaces X's with $c_1^2 = 3$, $\chi(\mathcal{O}) = 2$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$, where c_1, $\chi(\mathcal{O})$, and $\mathrm{Tors}(X)$ are the first Chern class, the Euler characteristic of the structure sheaf, and the torsion part of the Picard group of X, respectively.

In the present paper, we will prove the following two theorems:

Theorem 1. *All minimal algebraic surfaces X's with $c_1^2 = 3$, $\chi(\mathcal{O}) = 2$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$ are deformation equivalent to each other. Their coarse moduli space $\mathcal{M}$ is a 14-dimensional unirational variety.*

Theorem 2. *Let X be any general surface as in Theorem 1. Then the infinitesimal period map $\mu : H^1(\Theta_X) \to \mathrm{Hom}(H^0(\Omega_X^2), H^1(\Omega_X^1))$ is injective.*

Remark 1. The surfaces X's as in Theorem 1 have the geometric genus $p_g = 1$ and the irregularity $q = 0$. We refer the readers to [4] for the existence of the coarse moduli space $\mathcal{M}$. See also [5] for the infinitesimal period map.

In order to give proofs for the theorems above, let us first recall the main results given in [7]. See [7] for proofs of the following two theorems:

Theorem 3 ([7]). *Let X be a minimal algebraic surface with $c_1^2 = 3$, $\chi(\mathcal{O}) = 2$, and $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. Let $\pi : Y \to X$ be the unramified Galois triple cover corresponding to a non-trivial 3-torsion divisor. Then both the fundamental group $\pi_1(X)$ and the torsion group $\mathrm{Tors}(X)$ are isomorphic to the cyclic group $\mathbb{Z}/3$. Further, the canonical model Z of Y is a complete intersection in the 4-dimensional projective space $\mathbb{P}^4$ defined by two homogeneous polynomials $\tilde{F}_1$ and $\tilde{F}_2$ of degree 3 satisfying*

$$\tilde{F}_i(W_0, \varepsilon X_1, \varepsilon X_2, \varepsilon^{-1} Y_3, \varepsilon^{-1} Y_4) = \tilde{F}_i(W_0, X_1, X_2, Y_3, Y_4) \quad (i = 1, 2).$$

Here, $(W_0, X_1, \cdots, Y_4)$ is a homogeneous coordinate of $\mathbb{P}^4$, and the constant $\varepsilon = \exp(2\pi\sqrt{-1}/3)$ is a third root of unity.

Theorem 4 ([7]). *Let X be a surface as in Theorem 3. If X has an ample canonical divisor K_X, then $h^1(\Theta_X) = 14$ and $h^2(\Theta_X) = 0$, hence the Kuranishi space of X is smooth and of dimension 14.*

Remark 2. Explicit forms of the two polynomials in Theorem 3 are given by

$$\tilde{F}_i = a_0^{(i)} W_0^3 + W_0 \tilde{\gamma}_i(X_1, X_2, Y_3, Y_4) + \tilde{\alpha}_i(X_1, X_2) + \tilde{\beta}_i(Y_3, Y_4) \qquad (1)$$

for $i = 1, 2$, where

$$\begin{aligned}
\tilde{\gamma}_i &= a_1^{(i)} X_1 Y_3 + a_2^{(i)} X_1 Y_4 + a_3^{(i)} X_2 Y_3 + a_4^{(i)} X_2 Y_4, \\
\tilde{\alpha}_i &= a_5^{(i)} X_1^3 + a_6^{(i)} X_1^2 X_2 + a_7^{(i)} X_1 X_2^2 + a_8^{(i)} X_2^3, \\
\tilde{\beta}_i &= a_9^{(i)} Y_3^3 + a_{10}^{(i)} Y_3^2 Y_4 + a_{11}^{(i)} Y_3 Y_4^2 + a_{12}^{(i)} Y_4^3,
\end{aligned}$$

are homogeneous polynomials of $X_1, \cdots, Y_4$ with coefficients $a_j^{(i)} \in \mathbb{C}$.

Remark 3. The complete intersection Z is the image of the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^4$. We have a natural action on Z by the Gaolis group $G = \mathrm{Gal}(Y/X) \simeq \mathbb{Z}/3$ of Y over X. This action is given by

$$\tau_0 : (W_0 : X_1 : X_2 : Y_3 : Y_4) \mapsto (W_0 : \varepsilon X_1 : \varepsilon X_2 : \varepsilon^{-1} Y_3 : \varepsilon^{-1} Y_4), \qquad (2)$$

where τ_0 is a generator of the group G. Since this action on Z has no fixed points, the coefficients $a_j^{(i)}$'s satisfy the following three conditions:

i) at least one out of $a_0^{(1)}$ and $a_0^{(2)}$ are not equal to zero,

ii) two polynomials $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ have no common zeroes on $\mathbb{P}^1 = \{(X_1 : X_2)\}$,

iii) two polynomials $\tilde{\beta_1}$ and $\tilde{\beta_2}$ have no common zeroes on $\mathbb{P}^1 = \{(Y_3 : Y_4)\}$.

For each integer $n \geq 0$, we have a natural isomorphism

$$H^0(\mathcal{O}_Y(nK_Y)) \simeq \bigoplus_{m=0,1,-1} H^0(\mathcal{O}_X(nK_X - mT_0)) \qquad (3)$$

corresponding to the action by G, where T_0 is a generator of the torsion group $\mathrm{Tors}(X)$. Note that this is a decomposition into homogeneous eigen spaces, and that, in Theorem 3, the sets $\{W_0\}$, $\{X_1, X_2\}$, and $\{Y_3, Y_4\}$ correspond to a base of $H^0(\mathcal{O}_X(K_X))$, respectively of $H^0(\mathcal{O}_X(K_X - T_0))$, and respectively of $H^0(\mathcal{O}_X(K_X + T_0))$. The polynomials $\tilde{F_1}$ and $\tilde{F_2}$ generate the linear space consisting of all the elements in $H^0(\mathcal{O}_{\mathbb{P}^4}(3H))$ vanishing along Z, where H is a hyperplane in $\mathbb{P}^4$.

3 Unirationality of the moduli space

In this section, we will give a proof for Theorem 1. We denote by $W = \mathbb{P}^4$ and $(W_0 : X_1 : X_2 : Y_3 : Y_4)$, the 4-dimensional complex projective space and its homogeneous coordinate, respectively.

Let $\tilde{B}$ be the set of all $(a_j^{(i)})_{0\leq j\leq 12}^{1\leq i\leq 2} \in \mathbb{C}^{26}$ satisfying the conditions i), ii) and iii) in Remark 3 such that two polynomials $\tilde{F_1}$ and $\tilde{F_2}$ given by (1) define in $W = \mathbb{P}^4$ a complete intersection with at most rational double points as its singularities. We denote by $\tilde{B_0}$ the set of points in $\tilde{B}$ corresponding to non-singular complete intersections. Note by [7, Remark 1], we have $\tilde{B_0} \neq \emptyset$, hence the spaces $\tilde{B}$ and $\tilde{B_0}$ are dense Zariski open subsets of $\mathbb{C}^{26}$. We have a flat family $\tilde{\mathcal{Y}} \to \tilde{B}$ whose fiber on each $(a_j^{(i)}) \in \tilde{B}$ is a complete intersection defined by $\tilde{F_1}$ and $\tilde{F_2}$ with $a_j^{(i)}$'s as their coefficients. This $\tilde{\mathcal{Y}}$ is a subvariety of $\tilde{B} \times W$ stable under the action by $G \simeq \langle \mathrm{id}_{\tilde{B}} \times \tau_0 \rangle \simeq \mathbb{Z}/3$, where τ_0 is an automorphism of W given by (2). Taking the quotient of $\tilde{\mathcal{Y}}$ by this action, we obtain a family $\tilde{\mathcal{X}} \to \tilde{B}$ whose fibers are the canonical models of surfaces X's as in Theorem 3. Note that both restrictions $\tilde{\mathcal{Y}}|_{\tilde{B}_0} \to \tilde{B}_0$ and $\tilde{\mathcal{X}}|_{\tilde{B}_0} \to \tilde{B}_0$ are analytic families.

Lemma 3.1. *Let X be an algebraic surface as in Theorem 3. Then there exist bases of $H^0(\mathcal{O}_X(K_X))$, $H^0(\mathcal{O}_X(K_X - T_0))$, and $H^0(\mathcal{O}_X(K_X + T_0))$ such that the polynomials $\tilde{F}_1$ and $\tilde{F}_2$ satisfy $a_0^{(1)} = 1$, $a_5^{(1)} = a_9^{(1)} = 1$, $a_7^{(1)} = a_8^{(1)} = a_{12}^{(1)} = 0$, $a_8^{(2)} = a_{12}^{(2)} = 1$, and $a_5^{(2)} = a_6^{(2)} = a_9^{(2)} = 0$.*

Proof. Take those bases and $\tilde{F}_i$'s in Theorem 3 in such a way that each $\tilde{\alpha}_i$ for $i = 1, 2$ has a zero of order at least 2 at $(X_1 : X_2) = (i - 1 : 2 - i)$ and that each $\tilde{\beta}_i$ for $i = 1, 2$ has a zero at $(Y_3 : Y_4) = (i - 1 : 2 - i)$. This is possible, since $(X_1 : X_2) \mapsto (\tilde{\alpha}_1(X_1, X_2) : \tilde{\alpha}_2(X_1, X_2))$ is a morphism of degree 3. Then, by the conditions ii) and iii) in Remark 3, we have $a_5^{(1)} \neq 0$, $a_8^{(2)} \neq 0$, $a_9^{(1)} \neq 0$ and $a_{12}^{(2)} \neq 0$. Now, by replacing the elements in these bases by their multiples by non-zero constants, and changing indices if necessary, we easily obtain the assertion. □

Consider the case of X for which $\tilde{F}_i$'s as in Lemma 3.1 satisfy $a_0^{(2)} \neq 0$. In this case, we replace X_2 and Y_4 by their multiples by a non-zero constant such that the equality $a_0^{(2)} = 1$, as much as the equalities in the lemma above, holds. Then the defining polynomials $F_i = \tilde{F}_i$'s of Z in W are given by

$$F_i = W_0^3 + W_0 \gamma_i(X_1, X_2, Y_3, Y_4) + \alpha_i(X_1, X_2) + \beta_i(Y_3, Y_4), \tag{4}$$

for $i = 1, 2$, where

$$\begin{aligned}
\gamma_1 &= a^{(1)} X_1 Y_3 + b^{(1)} X_1 Y_4 + c^{(1)} X_2 Y_3 + d^{(1)} X_2 Y_4, \\
\gamma_2 &= a^{(2)} X_1 Y_3 + b^{(2)} X_1 Y_4 + c^{(2)} X_2 Y_3 + d^{(2)} X_2 Y_4, \\
\alpha_1 &= X_1^3 + e^{(1)} X_1^2 X_2, \\
\alpha_2 &= g^{(2)} X_1 X_2^2 + X_2^3, \\
\beta_1 &= Y_3^3 + h^{(1)} Y_3^2 Y_4 + l^{(1)} Y_3 Y_4^2, \\
\beta_2 &= h^{(2)} Y_3^2 Y_4 + l^{(2)} Y_3 Y_4^2 + Y_4^3
\end{aligned}$$

are homogeneous polynomials with coefficients in $\mathbb{C}$. We have a natural inclusion $\mathbb{C}^{14} = \{(a^{(1)}, b^{(1)}, \cdots, l^{(2)})\} \hookrightarrow \mathbb{C}^{26} = \{(a_j^{(i)})\}$, since the F_i's above are special cases of $\tilde{F}_i$'s. We put $B_0 = \mathbb{C}^{14} \cap \tilde{B}_0$, and denote by $\psi : \mathcal{Y}_0 \to B_0$ and $\varphi : \mathcal{X}_0 \to B_0$, the pull-back of $\tilde{\mathcal{Y}} \to \tilde{B}$ and that of $\tilde{\mathcal{X}} \to \tilde{B}$, respectively. Now we are ready to prove Theorem 1. We use the same method as in [3, Theorem 2. 11] and [2, Theorem 2.3].

Proof of Theorem 1. Let $\mathcal{M}$ be the coarse moduli space for surfaces X's as in Theorem 3. By Theorem 3, any surface X as in Theorem 3 corresponds to a fiber of the family $\tilde{X} \to \tilde{B}$, where $\tilde{B}$ is a non-empty Zariski open subset in $\mathbb{C}^{26}$. Thus by Tjurina's results on resolution of singularities ([9]), all surfaces X's as in Theorem 1 are deformation equivalent, and their moduli space $\mathcal{M}$ is irreducible. Meanwhile by the universality of the coarse moduli space, we have a natural morphism $B_0 \to \mathcal{M}$ corresponding to the family $\varphi : \mathcal{X}_0 \to B_0$. This morphism is dominant by Lemma 3.1 and its succeeding argument. By this together with Theorem 4, we obtain the unirationality of $\mathcal{M}$, since B_0 is a Zariski open subset in $\mathbb{C}^{14}$. $\square$

4 The infinitesimal period map

In this section, we will give a proof for Theorem 2. Let $\psi : \mathcal{Y}_0 \to B_0$ and $\varphi : \mathcal{X}_0 \to B_0$ be the two analytic families given in Section 3. For each $t = (a^{(1)}, b^{(1)}, \cdots, l^{(2)}) \in B_0$, the fibers $X = \varphi^{-1}(t)$ and $Y = \psi^{-1}(t)$ are a surface with invariants as in Theorem 1 and its universal cover, respectively. Note that Y is a complete intersection in $W = \mathbb{P}^4$ defined by F_1 and F_2 as in (4). We denote by $\pi : Y \to X$ and $\iota : Y \to W$ the natural projection and the natural inclusion, respectively.

Let $T_{B_0,t}$ be the holomorphic tangent space at $t \in B_0$. We denote by $\rho : T_{B_0,t} \to H^1(\Theta_X)$ and $\rho' : T_{B_0,t} \to H^1(\Theta_Y)$ the Kodaira-Spencer map of φ and that of ψ, respectively. In order to prove Theorem 2, it only suffices, by Theorem 4 and the equality $\dim T_{B_0,t} = 14$, to show the injectivity of $\mu \circ \rho$ for general $t \in B_0$, where μ is the morphism given in Theorem 2. Note that the composite $\mu \circ \rho$ corresponds to the infinitesimal period map of φ. Let $\omega \in H^0(\Omega_X^2)$ be a non-zero holomorphic 2-form on X such that $\pi^*\omega$ corresponds to the section $W_0 \in H^0(\Omega_Y^2)$ in Remark 3. Since $p_g(X) = 1$, the kernel of $\mu \circ \rho$ is equal to that of the morphism $T_{B_0,t} \ni \xi \mapsto ((\mu \circ \rho)(\xi))(\omega) \in H^1(\Omega_X^1)$. Meanwhile we have the following commutative diagram:

$$\begin{array}{ccccc}
T_{B_0,t} & \xrightarrow{\rho} & H^1(\Theta_X) & \xrightarrow{\omega\times} & H^1(\Omega_X^2 \otimes \Theta_X) \simeq H^1(\Omega_X^1) \\
\downarrow \| \mathrm{id} & & \downarrow & & \downarrow \\
T_{B_0,t} & \xrightarrow{\rho'} & H^1(\Theta_Y) & \xrightarrow{W_0\times} & H^1(\Omega_Y^2 \otimes \Theta_Y) \simeq H^1(\Omega_Y^1),
\end{array}$$

where the vertical morphisms $H^1(\Theta_X) \to H^1(\Theta_Y)$ and $H^1(\Omega_X^2 \otimes \Theta_X) \to H^1(\Omega_Y^2 \otimes \Theta_Y)$ are natural inclusions induced by the decomposition of $\pi_*\Theta_Y$ and $\pi_*(\Omega_Y^2 \otimes \Theta_Y) \simeq \pi_*\Omega_Y^1$ associated with the action of the Galois group $G = \mathrm{Gal}(Y/X)$. Note that $((\mu \circ \rho)(\xi))(\omega) = ((\omega\times) \circ \rho)(\xi)$ for any $\xi \in T_{B_0,t}$. Thus in order to prove the injectivity of $\mu \circ \rho$, we only need to show that of the morphism $(W_0\times) \circ \rho' : T_{B_0,t} \to H^1(\Omega_Y^2 \otimes \Theta_Y)$.

Let us prove the injectivity of $(W_0\times)\circ\rho'$ for general $t\in B_0$. We denote by $R\simeq\oplus_{n=0}^{\infty}R_n$ and R_n the graded ring $\mathbb{C}[W_0,X_1,X_2,Y_3,Y_4]/\langle F_1,F_2\rangle$ and its homogeneous part of degree n, respectively. This graded ring R is naturally isomorphic to the canonical ring of Y. For each $m=0,1,-1$, we denote by $R_n^{(m)}$ the set of all $F\in R_n$ satisfying

$$F(W_0,\varepsilon X_1,\varepsilon X_2,\varepsilon^{-1}Y_3,\varepsilon^{-1}Y_4)=\varepsilon^m F(W_0,X_1,X_2,Y_3,Y_4).$$

This space $R_n^{(m)}$ corresponds to the eigenspace $H^0(\mathcal{O}_X(nK_X-mT_0))$ via the isomorphism (3).

We have a natural exact sequence $0\to\Theta_Y\to\iota^*\Theta_W\to\mathcal{O}_Y(3)^{\oplus 2}\to 0$ of $\mathcal{O}_Y$-modules. By the similar argument as in Catanese [1] and Oliverio [8], we obtain, from this short exact sequence, the following commutative diagram:

$$\begin{array}{ccccccccc}
R_1^{\oplus 5} & \longrightarrow & R_3^{\oplus 2} & \longrightarrow & H^1(\Theta_Y) & \longrightarrow & 0 & & \\
\Big\downarrow{\scriptstyle W_0\times} & & \Big\downarrow{\scriptstyle W_0\times} & & \Big\downarrow{\scriptstyle W_0\times} & & \Big\downarrow & & \\
R_2^{\oplus 5} & \xrightarrow{\delta} & R_4^{\oplus 2} & \longrightarrow & H^1(\Omega_Y^2\otimes\Theta_Y) & \longrightarrow & \mathbb{C} & \longrightarrow & 0,
\end{array}\tag{5}$$

where both of the horizontal sequences are exact, and the morphisms $R_1^{\oplus 5}\to R_3^{\oplus 2}$ and $\delta: R_2^{\oplus 5}\to R_4^{\oplus 2}$ are given by the matrix

$$\begin{pmatrix}\frac{\partial F_1}{\partial W_0} & \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial Y_3} & \frac{\partial F_1}{\partial Y_4}\\ \frac{\partial F_2}{\partial W_0} & \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \frac{\partial F_2}{\partial Y_3} & \frac{\partial F_2}{\partial Y_4}\end{pmatrix}.$$

Let $A': T_{B_0,t}\to R_3^{\oplus 2}$ be the morphism given by $\frac{\partial}{\partial t}\mapsto(\frac{\partial}{\partial t}F_1,\frac{\partial}{\partial t}F_2)$, that is, the morphism giving the infinitesimal displacement of the deformation $\psi:\mathcal{Y}_0\to B_0$ of the submanifold $Y\subset W$. Since the composite $(W_0\times)\circ A'$ maps $T_{B_0,t}$ into the subspace $R_4^{(0)\oplus 2}\subset R_4^{\oplus 2}$, we obtain a restriction $A: T_{B_0,t}\to R_4^{(0)\oplus 2}$ of $(W_0\times)\circ A'$. We put $V=R_2^{(0)}\oplus R_2^{(1)}\oplus R_2^{(1)}\oplus R_2^{(-1)}\oplus R_2^{(-1)}\subset R_2^{\oplus 5}$, and denote by $C: V\to R_4^{(0)\oplus 2}$ the restriction of $\delta: R_2^{\oplus 5}\to R_4^{\oplus 2}$ to this subspace. Then from the commutative diagram (5), we infer the equality $\ker((W_0\times)\circ\rho')=A^{-1}(C(V))$, where $C(V)$ is the image of the morphism C.

Let M' be a 26-dimensional subspace of $R_4^{(0)\oplus 2}$ spanned by the following linearly independent elements:

$$\begin{aligned}
&(W_0^4,0),\quad (W_0X_1X_2^2,0),\quad (W_0Y_3^3,0),\quad (W_0Y_4^3,0),\quad (X_1^2Y_3^2,0),\\
&(X_1^2Y_3Y_4,0),\quad (X_1^2Y_4^2,0),\quad (X_1X_2Y_3^2,0),\quad (X_1X_2Y_3Y_4,0),\\
&(X_1X_2Y_4^2,0),\quad (X_2^2Y_3^2,0),\quad (X_2^2Y_3Y_4,0),\quad (X_2^2Y_4^2,0),\\
&(0,W_0^4),\quad (0,W_0X_1^2X_2),\quad (0,W_0Y_3^3),\quad (0,W_0Y_4^3),\quad (0,X_1^2Y_3^2),\\
&(0,X_1^2Y_3Y_4),\quad (0,X_1^2Y_4^2),\quad (0,X_1X_2Y_3^2),\quad (0,X_1X_2Y_3Y_4),\\
&(0,X_1X_2Y_4^2),\quad (0,X_2^2Y_3^2),\quad (0,X_2^2Y_3Y_4),\quad (0,X_2^2Y_4^2).
\end{aligned}\tag{6}$$

Then, denoting the image of $A : T_{B_0,t} \to R_4^{(0)\oplus 2}$ by M, we have $R_4^{(0)\oplus 2} = M \oplus M'$. Thus there exist two morphisms $D : V \to M$ and $D' : V \to M'$ such that $C = D + D'$. Note that $C(V) \cap M \simeq D(\ker D')$. By this together with the injectivity of A, we obtain

$$\ker((W_0 \times) \circ \rho') = A^{-1}(C(V)) \simeq D(\ker D').$$

Meanwhile we have $\dim V = 25$ and

$$(W_0^2, W_0X_1, W_0X_2, W_0Y_3, W_0Y_4) \in \ker C = \ker D \cap \ker D'.$$

Thus in order to prove the injectivity of $(W_0\times) \circ \rho' : T_{B_0,t} \to H^1(\Omega_Y^1 \otimes \Theta_Y)$, we only need to show the equality $\mathrm{rank} D' = 24$.

So, in what follows, we will show $\mathrm{rank} D' = 24$ for general $t \in B_0$. We employ the following base of V:

$$\begin{array}{lllll}
(W_0^2)_1, & (X_1Y_3)_1, & (X_1Y_4)_1, & (X_2Y_3)_1, & (X_2Y_4)_1, \\
(W_0X_1)_2, & (W_0X_2)_2, & (Y_3^2)_2, & (Y_3Y_4)_2, & (Y_4^2)_2, \\
(W_0X_1)_3, & (W_0X_2)_3, & (Y_3^2)_3, & (Y_3Y_4)_3, & (Y_4^2)_3, \\
(W_0Y_3)_4, & (W_0Y_4)_4, & (X_1^2)_4, & (X_1X_2)_4, & (X_2^2)_4, \\
(W_0Y_3)_5, & (W_0Y_4)_5, & (X_1^2)_5, & (X_1X_2)_5, & (X_2^2)_5,
\end{array} \tag{7}$$

where, for each $u \in R_2$ and $1 \le i \le 5$, we denote by $(u)_i$ the element $(v_1, v_2, v_3, v_4, v_5) \in R_2^{\oplus 5}$ given by $v_i = u$, $v_j = 0$ $(j \neq i)$. Let L_1 be the 26×25 matrix of D' corresponding to the bases (7) of V and (6) of M': i.e.

$$L_1 = \begin{pmatrix} L_{1,1}^{(1)} & L_{1,2}^{(1)} & L_{1,3}^{(1)} \\ L_{2,1}^{(1)} & L_{2,2}^{(1)} & L_{2,3}^{(1)} \end{pmatrix},$$

where 13×5 matrixes $L_{1,1}^{(1)}$, $L_{2,1}^{(1)}$, and 13×10 matrixes $L_{1,2}^{(1)}$, $L_{2,2}^{(1)}$, $L_{1,3}^{(1)}$, $L_{2,3}^{(1)}$ are given by

$$L_{1,1}^{(1)} = \begin{bmatrix}
3 & & & & \\
0 & & & & \\
0 & & & & \\
0 & & & & \\
 & a^{(1)} & & & \\
 & b^{(1)} & a^{(1)} & & \\
 & & b^{(1)} & & \\
 & c^{(1)} & & a^{(1)} & \\
 & d^{(1)} & c^{(1)} & b^{(1)} & a^{(1)} \\
 & & d^{(1)} & & b^{(1)} \\
 & & & c^{(1)} & \\
 & & & d^{(1)} & c^{(1)} \\
 & & & & d^{(1)}
\end{bmatrix}, \quad
L_{2,1}^{(1)} = \begin{bmatrix}
3 & & & & \\
0 & & & & \\
0 & & & & \\
0 & & & & \\
 & a^{(2)} & & & \\
 & b^{(2)} & a^{(2)} & & \\
 & & b^{(2)} & & \\
 & c^{(2)} & & a^{(2)} & \\
 & d^{(2)} & c^{(2)} & b^{(2)} & a^{(2)} \\
 & & d^{(2)} & & b^{(2)} \\
 & & & c^{(2)} & \\
 & & & d^{(2)} & c^{(2)} \\
 & & & & d^{(2)}
\end{bmatrix},$$

$$L_{1,2}^{(1)} = \begin{bmatrix}
-3 & & & & & -e^{(1)} & & & & \\
& 2e^{(1)} & & & & & & & & \\
-3 & & a^{(1)} & & & -e^{(1)} & 0 & c^{(1)} & & \\
& & & & b^{(1)} & & & & & d^{(1)} \\
& & 3 & & & & & e^{(1)} & & \\
& & & 3 & & & & & e^{(1)} & \\
& & & & 3 & & & & & e^{(1)} \\
& & 2e^{(1)} & & & & & & & \\
& & & 2e^{(1)} & & & & & & \\
& & & & 2e^{(1)} & & & & & \\
& & & & 0 & & & & & \\
& & & & 0 & & & & & \\
& & & & 0 & & & & &
\end{bmatrix},$$

$$L_{2,2}^{(1)} = \begin{bmatrix}
& -g^{(2)} & & & & & -3 & & & \\
& & & & & 2g^{(2)} & & & & \\
& & a^{(2)} & & & & & c^{(2)} & & \\
0 & -g^{(2)} & & & b^{(2)} & & -3 & & & d^{(2)} \\
& & & & & & & 0 & & \\
& & & & & & & 0 & & \\
& & & & & & & 0 & & \\
& & & & & & & 2g^{(2)} & & \\
& & & & & & & & 2g^{(2)} & \\
& & & & & & & & & 2g^{(2)} \\
& & g^{(2)} & & & & & 3 & & \\
& & & g^{(2)} & & & & & 3 & \\
& & & & g^{(2)} & & & & & 3
\end{bmatrix},$$

$$L_{1,3}^{(1)} = \begin{bmatrix}
& & -a^{(1)} & & -c^{(1)} & & & -b^{(1)} & & -d^{(1)} \\
& & & c^{(1)} & * & & & & d^{(1)} & * \\
3 & & -a^{(1)} & & & h^{(1)} & 0 & -b^{(1)} & & \\
& l^{(1)} & & & -c^{(1)} & & & & & -d^{(1)} \\
& & 3 & & & & & h^{(1)} & & \\
& & 2h^{(1)} & & & & & 2l^{(1)} & & \\
& & l^{(1)} & & & & & & & \\
& & & 3 & & & & & h^{(1)} & \\
& & & 2h^{(1)} & & & & & 2l^{(1)} & \\
& & & l^{(1)} & & & & & & \\
& & & & 3 & & & & & h^{(1)} \\
& & & & 2h^{(1)} & & & & & 2l^{(1)} \\
& & & & l^{(1)} & & & & &
\end{bmatrix},$$

$$
L_{2,3}^{(1)} = \begin{bmatrix}
 & & -a^{(2)} & & -c^{(2)} & & & -b^{(2)} & & -d^{(2)} \\
 & & * & a^{(2)} & & & & * & b^{(2)} & \\
 & & -a^{(2)} & & & h^{(2)} & & -b^{(2)} & & \\
0 & l^{(2)} & & & -c^{(2)} & & 3 & & & -d^{(2)} \\
 & & & & & & & h^{(2)} & & \\
 & & 2h^{(2)} & & & & & 2l^{(2)} & & \\
 & & l^{(2)} & & & & & 3 & & \\
 & & & & & & & & h^{(2)} & \\
 & & & 2h^{(2)} & & & & & 2l^{(2)} & \\
 & & & l^{(2)} & & & & & 3 & \\
 & & & & & & & & & h^{(2)} \\
 & & & & 2h^{(2)} & & & & & 2l^{(2)} \\
 & & & & l^{(2)} & & & & & 3
\end{bmatrix}.
$$

Here empty entries are zero. For general $t \in B_0$, we strike off the rows and the columns of L_1 meeting the following entries by doing the operations in this order: $(3,16)$,$(1,6)$,$(17,22)$,$(14,12)$, $(2,7)$,$(4,17)$,$(15,11)$,$(16,21)$. Then we see $\mathrm{rank} L_1 = \mathrm{rank} L_2 + 8$, where L_2 is the 18×16 matrix obtained by removing from L_1 the following: i) the rows and columns meeting the 8 entries given above, and ii) the first column. Thus we only need to show $\mathrm{rank} L_2 = 16$ for general $t \in B_0$.

Let L_3 be the 18×16 matrix obtained by specializing L_2 by $e^{(1)} = g^{(2)} = 0$. We can strike off the rows and the columns of L_3 meeting the following entries: $(1,5)$,$(2,6)$,$(3,7)$,$(16,8)$,$(17,9)$,$(18,10)$. Thus we see $\mathrm{rank} L_2 \geq \mathrm{rank} L_3 = \mathrm{rank} L_4 + 6$ for general $t \in B_0$, where L_4 is the 12×10 matrix obtained by removing from L_3 the rows and columns meeting the 6 entries above. Hence we only need to show $\mathrm{rank} L_4 = 10$ for general $t \in B_0$.

It now suffices to show $\det L_5 \neq 0$ for general $t \in B_0$, where the 10×10 matrix

$$
L_5 = \begin{bmatrix}
c^{(1)} & & a^{(1)} & & & 3 & & & h^{(1)} & \\
d^{(1)} & c^{(1)} & b^{(1)} & a^{(1)} & & 2h^{(1)} & & & 2l^{(1)} & \\
 & d^{(1)} & & b^{(1)} & & l^{(1)} & & & & \\
 & & c^{(1)} & & & & 3 & & & h^{(1)} \\
 & & d^{(1)} & c^{(1)} & & & 2h^{(1)} & & & 2l^{(1)} \\
b^{(2)} & a^{(2)} & & & 2h^{(2)} & & & 2l^{(2)} & & \\
 & b^{(2)} & & & l^{(2)} & & & 3 & & \\
c^{(2)} & & a^{(2)} & & & & & & h^{(2)} & \\
d^{(2)} & c^{(2)} & b^{(2)} & a^{(2)} & & 2h^{(2)} & & & 2l^{(2)} & \\
 & d^{(2)} & & b^{(2)} & & l^{(2)} & & & 3 &
\end{bmatrix} \tag{8}
$$

is the one obtained by removing from L_4 its 6-th and 7-th rows. But, when we compute $\det L_5$ by the definition of the determinant, the monomial $(d^{(2)})^2(a^{(1)})^2(l^{(2)})^2h^{(2)}(h^{(1)})^2l^{(1)}$ appears only once, i.e., from the term passing the entries $(9,1)$, $(10,2)$, $(1,3)$, $(2,4)$, $(7,5)$, $(3,6)$, $(5,7)$, $(6,8)$, $(8,9)$, and $(4,10)$ of L_5. Thus, for general $t \in B_0$, we have $\det L_5 \neq 0$, and hence $\mathrm{rank} D' = 24$, which completes the proof of Theorem 2.

Acknowledgment

The author performed all the computations included in this article during his six-months stay at Universitá di Padova from October 2003, financially supported by INDAM-GNSAGA. He revised the old version of this manuscript during his six-months stay at Max-Planck Institut für Mathematik in Bonn from October 2004, financially supported by the institute. He expresses his deep gratitude to these institutions for the wonderful environment, hospitality, and financial supports, which he received during his stay.

References

[1] Catanese, F. Surfaces with $K^2 = p_g = 1$, and their period mapping, *in Algebraic Geometry, Lect. Notes in Math.*, **732** (1979), 1–29.

[2] Catanese, F. The moduli and the global period mapping of surfaces with $K^2 = p_g = 1$: A counter example to the global Torelli problem, *Compositio Math.*, **41**, (1980), no.3, 401–414.

[3] Catanese, F., Debarre, O., Surfaces with $K^2 = 2$, $p_g = 1$, $q = 0$, *Crelle's Journal reine angew. Math.*, **395**, (1989), 1–55.

[4] Gieseker, D., Global moduli for surfaces of general type, *Invent. Math.*, **43**, (1977), no. 3, 233–282.

[5] Griffiths, P., Periods of integrals on algebraic manifolds, I, II, *Amer. J. Math.*, **90**, (1968), 568–626, 805–865.

[6] Murakami, M., A bound for the orders of the torsion groups of surfaces with $c_1^2 = 2\chi - 1$, *preprint*, (2003),

[7] Murakami, M., Minimal algebraic surfaces of general type with $c_1^2 = 3$, $p_g = 1$ and $q = 0$, which have non-trivial 3-torsion divisors, *J. Math. Kyoto Univ.*, **43**, (2003), no.1, 203–215.

[8] Oliverio, P., On the period map for surfaces with $K_S^2 = 2$, $p_g = 1$, $q = 0$ and torsion $\mathbb{Z}/2$, *Duke Math. J.*, **50**, (1983), no. 3, 561–571.

[9] Tjurina, G. N., Resolution of singularities of flat deformations of rational double points, *Funkcional. Anal. i Prilozen*, **4**, (1970), no.1, 77–83.