TWISTED BURNSIDE THEOREM FOR TYPE II$_1$ GROUPS: AN EXAMPLE

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Abstract. The purpose of the present paper is to discuss the following conjecture, which is a generalization of the classical Burnside theorem.

Let G be a countable discrete group, ϕ its automorphism, $R(\phi)$ the number of ϕ-conjugacy classes, $S(\phi) = \#\operatorname{Fix}(\widehat{\phi})$ the number of ϕ-invariant equivalence classes of irreducible unitary representations. If one of $R(\phi)$ and $S(\phi)$ is finite, then it is equal to the other.

This conjecture plays a very important role in the theory of twisted conjugacy classes having a long history (see [7], [3]) and has very serious consequences in Dynamics, while its proof needs rather fine results from Functional and Non-commutative Harmonic Analysis. It was proved for finitely generated groups of type I in [5].

In the present paper this conjecture is disproved for non-type I groups. More precisely, an example of a group and its automorphism is constructed such that the number of fixed irreducible representations is greater than the Reidemeister number. But the number of fixed finite-dimensional representations (i.e. the number of invariant finite-dimensional characters) in this example coincides with the Reidemeister number.

The directions for search of an appropriate formulation are indicated (another definition of the dual object).

Let G be a semidirect product of $\mathbb{Z}^2$ and $\mathbb{Z}$ by Anosov automorphism α with the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It consists by the definition of triples $((m,k),n)$ of integers with the following multiplication low:

$$((m,k),n) * ((m',k'),n') = ((m,k) + \alpha^n(m',k'), n+n').$$

In particular,

$$((m,k),0) * ((0,0),n) = ((m,k),n).$$

The inverse of $((m,k),n)$ is $(-\alpha^{-n}(m,k),-n)$. Indeed,

$$((m,k),n) * (-\alpha^{-n}(m,k),-n) = ((m,k) - \alpha^n\alpha^{-n}(m,k), n-n) = ((0,0),0).$$

The group G is a solvable (hence, amenable) group which is not of type I. Its regular representation is factorial. The irreducible representations can be obtained from ergodic orbits of the action of α on the torus $\mathbb{T}^2$ which is dual for the normal subgroup $\mathbb{Z}^2$ using appropriate cocycles [2, Sec. II.4], [1, Ch. 17, § 1].

Let us define an automorphism $\phi : G \to G$ by

$$\phi((m,k),n) = ((k,-m),-n),$$

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i.e. the action on $\mathbb{Z}^2$ is defined by automorphism μ with the matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and on $\mathbb{Z}$ by $n \mapsto -n$. The map ϕ is clearly a bijection,

$$\begin{aligned}\phi(((m,k),n) * ((m',k'),n') &= \phi((m,k) + \alpha^n(m',k'), n+n')) \\ &= \phi((k,-m) + \mu\alpha^n(m',k'), -n-n'),\end{aligned}$$

$$\begin{aligned}\phi((m,k),n) * \phi(((m',k'),n') &= ((k,-m),-n) * ((k',-m'),-n') \\ &= ((k,-m) + \alpha^{-n}(k',-m'), -n-n').\end{aligned}$$

Hence, to prove that ϕ is an automorphism, we need $\mu\alpha^n = \alpha^{-n}\mu$. This follows from $\mu\alpha = \alpha^{-1}\mu$. The further results in this direction can be found in [6].

Let us find the Reidemeister classes of ϕ, i.e. the classes of the equivalence relation $h \sim gh\phi(g^{-1})$. For $h = ((m,k),n)$ and $g = ((x,y),z)$ the right hand side of the relation takes the following form:

$$((x,y) + \alpha^z(m,k), z+n) * (-\mu\alpha^{-z}(x,y), z) =$$

$$((x,y) + \alpha^z(m,k) - \alpha^{z+n}\mu\alpha^{-z}(x,y), 2z+n) =$$

$$= (\alpha^z\{(m,k) + (\mathrm{Id} - \alpha^n\mu)\alpha^{-z}(x,y)\}, 2z+n). \tag{1}$$

Let us call *level* n (of G) the coset L_n of $\mathbb{Z}^2 \subset G$ of all elements of the form $((m,k),n)$. Let us first take an element $((m,k),0)$ from the level 0 and describe elements from the same level, being equivalent to it. By (1) in this case $z = n = 0$ and they have the form

$$((m,k) + (\mathrm{Id} - \mu)(x,y), 0) = ((m + (x-y), k + (x+y)), 0),$$

where $\mathrm{Id} - \mu$ has the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Hence, the level 0 has intersections with 2 Reidemeister classes, say, B_1 and B_2. The first intersection $B_1 \cap L_0$ is formed by elements $((u,v),0)$ with odd $u+v$, and $B_2 \cap L_0$ — with even $u+v$. The elements from the other levels, which are equivalent to $((m,k),0)$, have the form

$$(\alpha^z\{(m,k) + (\mathrm{Id} - \mu)\alpha^{-z}(x,y)\}, 2z). \tag{2}$$

This means that B_1 and B_2 enter only even levels. Also, since α is an automorphism, we can rewrite (2) as

$$(\alpha^z\{(m,k) + (\mathrm{Id} - \mu)(u,v)\}, 2z). \tag{3}$$

with arbitrary integers u and v. This means, that the intersections $B_i \cap L_{2z}$ have the form $\alpha^z(B_i)$, $i = 1,2$. In particular, the other Reidemeister classes do not enter even levels.

In a similar way, the elements of L_1 equivalent to $((m,k),1)$ have the form

$$((m,k) + (\mathrm{Id} - \alpha\mu)(x,y), 1) = ((m + (2x-2y), k+x), 1).$$

This means, that L_1 enters 2 classes: B_3 is formed by elements with even first coordinate, and B_4 — with odd one. The elements from the other levels, which are equivalent to $((m,k),1)$, have the form

$$(\alpha^z\{(m,k) + (\mathrm{Id} - \alpha\mu)\alpha^{-z}(x,y)\}, 2z+1). \tag{4}$$

Since α is an automorphism, we can rewrite (4) as

$$(\alpha^z\{(m,k) + (\mathrm{Id} - \alpha\mu)(u,v)\}, 2z+1). \tag{5}$$

with arbitrary integers u and v. This means, that the intersections $B_i \cap L_{2z+1}$ have the form $\alpha^z(B_i)$, $i = 3,4$. In particular, these four classes cover G and $R(\phi) = 4$.

To obtain a complete description of B_i let us remark that directly from the definition of α

$$\alpha(x,y) = (2x+y, x+y)$$

one has the following properties:

- α maps the set of elements with an even (resp., odd) sum of coordinates onto the set of elements with an even (resp., odd) second coordinate,
- α maps the set of elements with an even (resp., odd) second coordinate onto the set of elements with an even (resp., odd) first coordinate,
- α maps the set of elements with an even (resp., odd) first coordinate onto the set of elements with an even (resp., odd) sum of coordinates.

Hence, the elements $((m,k),n)$ in intersections $B_i \cap L_j$ are of the form

i	1	2	3	4
$j \equiv 0 \mod 6$	$m+k$ is even	$m+k$ is odd	$\emptyset$	$\emptyset$
$j \equiv 1 \mod 6$	$\emptyset$	$\emptyset$	m is even	m is odd
$j \equiv 2 \mod 6$	k is even	k is odd	$\emptyset$	$\emptyset$
$j \equiv 3 \mod 6$	$\emptyset$	$\emptyset$	$m+k$ is even	$m+k$ is odd
$j \equiv 4 \mod 6$	m is even	m is odd	$\emptyset$	$\emptyset$
$j \equiv 5 \mod 6$	$\emptyset$	$\emptyset$	k is even	k is odd

Now we want to study the fixed points of the homeomorphism $\widehat{\phi} : \widehat{G} \to \widehat{G}$, $[\rho] \mapsto [\rho\phi]$ of the unitary dual. Let us start from the finite-dimensional representations. As it was shown in [5] in this case there exists exactly one twisted-invariant functional on $L^1(G)$, or ϕ-central L^∞ function, coming from a twisted-invariant functional on $\rho(L^1(G)) \cong M(\dim \rho, \mathbb{C})$ (up to scaling), namely

$$\varphi_\rho : g \mapsto \mathrm{Tr}(S\rho(g)), \tag{6}$$

where S is the intertwining operator between ρ and $\rho\phi$.

First, we have to find μ-invariant finite α-orbits on $\mathbb{T}^2$. One can notice that

$$\det(A^n - M) = \det A^n + 1 = 2$$

for any n. Hence, the mentioned orbits are formed by points with coordinates 0 and 1/2. We have 2 orbits: one of them consists of 1 point $(0,0)$ and gives rise to 1-dimensional trivial representation ρ_1, and the other consists of $A_1 = (0,1/2)$, $A_2 = (1/2,0)$ and $A_3 = (1/2,1/2)$ and gives rise to a 3-dimensional (irreducible) representation ρ_2. Also, one has the following 1-dimensional representation π:

$$\pi((m,k),2n) = 1, \qquad \pi((m,k),2n+1) = -1.$$

So, we have 4 representations

$$\rho_1, \quad \rho_2, \quad \pi, \quad \rho_2 \otimes \pi.$$

We claim that they give rise via (6) to 4 linear independent twisted-invariant functionals. In particular, there is no more finite-dimensional fixed points of $\widehat{\phi}$.

Clearly,

$$\varphi_{\rho_1} \equiv 1, \quad \varphi_\pi = \begin{cases} 1, & \text{on } \cup_n L_{2n} = B_1 \cup B_2, \\ -1, & \text{on } \cup_n L_{2n+1} = B_3 \cup B_4, \end{cases} \qquad \varphi_{\rho_2 \otimes \pi} = \varphi_{\rho_2} \cdot \varphi_\pi. \tag{7}$$

Let us find φ_{ρ_2}. In the space $L^2(A_1, A_2, A_3)$ we take the base ε_1, ε_2, ε_3 of characteristic functions in these points. One has

$$\alpha(\varepsilon_1) = \varepsilon_3, \qquad \alpha(\varepsilon_3) = \varepsilon_2, \qquad \alpha(\varepsilon_2) = \varepsilon_1,$$
$$\mu(\varepsilon_1) = \varepsilon_2, \qquad \mu(\varepsilon_2) = \varepsilon_1, \qquad \mu(\varepsilon_3) = \varepsilon_3.$$

The representation (see [1, Ch. 17, § 1]) is defined by:

$$\rho_2(m,k,0)(\varepsilon_i) = \chi_{A_i}(m,k) \cdot \varepsilon_i, \qquad \rho_2(0,0,n)(\varepsilon_i) = \alpha^{-n}(\varepsilon_i) = \varepsilon_{i+n \mod 3},$$
$$\rho_2(m,k,0)(\varepsilon_1) = e^{2\pi i(0\cdot m + 1/2\cdot k)} \cdot \varepsilon_1 = e^{\pi i k}\varepsilon_1,$$
$$\rho_2(m,k,0)(\varepsilon_2) = e^{2\pi i(1/2\cdot m + 0\cdot k)} \cdot \varepsilon_2 = e^{\pi i m}\varepsilon_2,$$
$$\rho_2(m,k,0)(\varepsilon_3) = e^{2\pi i(1/2\cdot m + 1/2\cdot k)} \cdot \varepsilon_3 = e^{\pi i (m+k)}\varepsilon_3.$$

The representation $\widehat{\phi}\rho_2$ is defined by

$$\widehat{\phi}\rho_2(0,0,n)(\varepsilon_i) = \rho_2(0,0,-n)(\varepsilon_i) = \varepsilon_{i-n \mod 3},$$
$$\widehat{\phi}\rho_2(m,k,0)(\varepsilon_1) = \rho_2(k,-m,0)(\varepsilon_1) = e^{-\pi i m}\varepsilon_1 = e^{\pi i m}\varepsilon_1,$$
$$\widehat{\phi}\rho_2(m,k,0)(\varepsilon_2) = \rho_2(k,-m,0)(\varepsilon_2) = e^{\pi i k}\varepsilon_2,$$
$$\widehat{\phi}\rho_2(m,k,0)(\varepsilon_3) = \rho_2(k,-m,0)(\varepsilon_3) = e^{\pi i (k-m)}\varepsilon_3 = e^{\pi i (m+k)}\varepsilon_3.$$

The intertwining operator is induced by ϕ and has the matrix $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence,

$$\varphi_{\rho_2}(m,k,n) = \operatorname{Tr}(S\rho_2(m,k,0)\rho_2(0,0,n)) =$$
$$= \operatorname{Tr}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\pi i k} & 0 & 0 \\ 0 & e^{\pi i m} & 0 \\ 0 & 0 & e^{\pi i (k+m)} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\right] =$$
$$= \operatorname{Tr}\left[\begin{pmatrix} 0 & e^{\pi i k} & 0 \\ e^{\pi i k} & 0 & 0 \\ 0 & 0 & e^{\pi i (k+m)} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\right].$$

For $n \equiv 0 \mod 3$

$$\varphi_{\rho_2}(m,k,n) = \operatorname{Tr}\left[\begin{pmatrix} 0 & e^{\pi i m} & 0 \\ e^{\pi i k} & 0 & 0 \\ 0 & 0 & e^{\pi i (k+m)} \end{pmatrix}\right] = e^{\pi i (k+m)} = \begin{cases} 1, & \text{if } m+k \text{ is even} \\ -1, & \text{if } m+k \text{ is odd} \end{cases},$$

for $n \equiv 1 \mod 3$

$$\varphi_{\rho_2}(m,k,n) = \operatorname{Tr}\left[\begin{pmatrix} 0 & e^{\pi i m} & 0 \\ e^{\pi i k} & 0 & 0 \\ 0 & 0 & e^{\pi i (k+m)} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right] =$$

$$= \mathrm{Tr}\begin{pmatrix} e^{\pi i m} & 0 & 0 \\ 0 & 0 & e^{\pi i k} \\ 0 & e^{\pi i(k+m)} & 0 \end{pmatrix} = e^{\pi i m} = \begin{cases} 1, & \text{if } m \text{ is even} \\ -1, & \text{if } m \text{ is odd} \end{cases},$$

for $n \equiv 2 \mod 3$

$$\varphi_{\rho_2}(m,k,n) = \mathrm{Tr}\left[\begin{pmatrix} 0 & e^{\pi i m} & 0 \\ e^{\pi i k} & 0 & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right] =$$

$$= \mathrm{Tr}\begin{pmatrix} 0 & 0 & e^{\pi i m} \\ 0 & e^{\pi i k} & 0 \\ e^{\pi i(k+m)} & 0 & 0 \end{pmatrix} = e^{\pi i k} = \begin{cases} 1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}.$$

φ_{ρ_2} is 3-periodical in n, while the characteristic functions of B_i are 6-periodical. For $j = 0, \dots, 5$ one has

$$\begin{array}{llll} \varphi_{\rho_2}|_{B_1\cap L_0} \equiv 1, & \varphi_{\rho_2}|_{B_2\cap L_0} \equiv -1, & \varphi_{\rho_2}|_{B_3\cap L_1} \equiv 1, & \varphi_{\rho_2}|_{B_4\cap L_1} \equiv -1, \\ \varphi_{\rho_2}|_{B_1\cap L_2} \equiv 1, & \varphi_{\rho_2}|_{B_2\cap L_2} \equiv -1, & \varphi_{\rho_2}|_{B_3\cap L_3} \equiv 1, & \varphi_{\rho_2}|_{B_4\cap L_3} \equiv -1, \\ \varphi_{\rho_2}|_{B_1\cap L_4} \equiv 1, & \varphi_{\rho_2}|_{B_2\cap L_4} \equiv -1, & \varphi_{\rho_2}|_{B_3\cap L_5} \equiv 1, & \varphi_{\rho_2}|_{B_4\cap L_5} \equiv -1, \end{array}$$

so $\varphi_{\rho_2}|_{B_1\cup B_3} \equiv 1$, $\varphi_{\rho_2}|_{B_2\cup B_4} \equiv -1$. the determinant of the values of the functions φ_{ρ_1}, φ_π, φ_{ρ_2}, $\varphi_{\rho_2\otimes\pi}$ on B_1, B_2, B_3, B_4 is

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = -8 \neq 0.$$

Hence, they are linearly independent.

Nevertheless, there are infinite-dimensional irreducible $\widehat{\phi}$-invariant representations. E.g. we have a representation ρ of G on $L^2(\mathbb{T}^2)$ with the respect to the Lebesgue measure, with $\rho(m,k,0)$ be the multiplier by characters in the appropriate points and $\rho(0,0,1)$ is α, in the same manner as for ρ_2.

This disproves the conjecture of Fel'shtyn and Hill [4], who supposed that the Reidemeister number equals to the number of fixed points of $\widehat{\phi}$ on $\widehat{G}$

This representation is not traceable, but one can nevertheless try to calculate (6). Let us choose an orthonormal base of $L^2(\mathbb{T}^2)$ formed by $\varepsilon_{st}(x,y) = e^{2\pi i(sx+ty)}$, $x, y \in [0,1]$. The intertwining operator S is generated by μ. Then

$$\mathrm{Tr}(S\rho(m,k,n)) = \sum_{s,t}\int_0^1\int_0^1 (\rho(m,k,0)\rho(0,0,n)\varepsilon_{s,t})(\mu(x,y))\overline{\varepsilon_{s,t}}(x,y)\,dx\,dy =$$

$$= \sum_{s,t}\int_0^1\int_0^1 e^{2\pi i\langle (m,k),\mu(x,y)\rangle}(\rho(0,0,n)\varepsilon_{s,t})(\mu(x,y))\varepsilon_{-s,-t}(x,y)\,dx\,dy =$$

$$= \sum_{s,t}\int_0^1\int_0^1 e^{2\pi i\langle (m,k),\mu(x,y)\rangle}\varepsilon_{s,t}(\alpha^n\mu(x,y))\varepsilon_{-s,-t}(x,y)\,dx\,dy =$$

$$= \sum_{s,t}\int_0^1\int_0^1 e^{2\pi i\langle (m,k)+(\alpha^n-\mu)(s,t),\mu(x,y)\rangle}\,dx\,dy = \sum_{(m,k)=(\mu-\alpha^n)(s,t)} 1.$$

For $n=0$ this equals 1 for $m+k$ even and 0 for $m+k$ odd. Hence, $\varphi_\rho|_{B_1\cap L_0}=1$, $\varphi_\rho|_{B_2\cap L_0}=0$. For $n=1$ the equality takes the form $(m,k)=(-2s-2t,-t)$. Hence, $\varphi_\rho|_{B_3\cap L_1}=1$, $\varphi_\rho|_{B_4\cap L_1}=0$. If $n=2r$, the equality takes form

$$(m,k)=(\mu-\alpha^{2r})(s,t),\quad (m,k)=(\alpha^r\mu\alpha^r-\alpha^{2r})(s,t),$$
$$\alpha^{-r}(m,k)=(\mu-\alpha^0)\alpha^r(s,t).$$

Since α is an automorphism of $\mathbb{Z}\oplus\mathbb{Z}$ and by the description of B_i via the action of α, we obtain that $\varphi_\rho|_{B_1}=1$, $\varphi_\rho|_{B_2}=0$. Similarly, for n odd. So, φ_ρ is well defined and

$$\varphi_\rho=\chi_{B_1}+\chi_{B_3}=\frac{1}{2}(\varphi_{\rho_1}+\varphi_{\rho_2}).$$

Of course, there are also $\widehat{\phi}$-invariant traceable factor representations of this group G, e.g. the regular representation. Since its kernel is trivial, evidently all twisted-invariant functionals can be pushed back from it.

This example shows that the conjecture of [4] for general groups can survive only after eliminating badly separated points in $\widehat{G}$.

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