Sectional curvatures of Kähler moduli

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Abstract

If X is a compact Kähler manifold of dimension n, we let $\mathcal{K} \subset H^{1,1}(X, \mathbf{R})$ denote the cone of Kähler classes, and $\mathcal{K}_1$ the level set given by classes D with $D^n = 1$. This space is naturally a Riemannian manifold and is isometric to the manifold $\tilde{\mathcal{K}}_1$ of Kähler *forms* ω with ω^n some fixed volume form, equipped with the Hodge metric, as studied previously by Huybrechts. We study these spaces further, in particular their geodesics and sectional curvatures. Conjecturally, at least for Calabi–Yau manifolds and probably rather more generally, these sectional curvatures should be bounded between $-\frac{1}{2}n(n-1)$ and zero. We find a simple formula for the sectional curvatures, and prove both the bounds hold for various classes of varieties, developing along the way a mirror to the Weil–Petersson theory of complex moduli. In the case of threefolds with $h^{1,1} = 3$, we produce an explicit formula for this curvature in terms of the invariants of the cubic form. This enables us to check the bounds by computer for a wide range of examples. Finally, we explore the implications of non-positivity.

0. Introduction

Let X denote a compact Kähler manifold of dimension n. The cup product on $H^2(X, \mathbf{Z})$ determines a degree n form on $H^{1,1}(X, \mathbf{R})$, and we define the *positive cone* to be $\{D \in H^{1,1}(X, \mathbf{R}) : D^n > 0\}$. Cup product also determines an *index cone* (cf. [14]), which will be denoted by W, consisting of elements D in the positive cone for which the quadratic form on $H^{1,1}(X, \mathbf{R})$ given by $L \mapsto D^{n-2} \cup L^2$ has signature $(1, h^{1,1} - 1)$. By the Hodge index theorem, we have an inclusion of open cones in $H^{1,1}(X, \mathbf{R})$ from the Kähler cone $\mathcal{K}$ into W. We denote by W_1 the level set $\{D \in W : D^n = 1\}$ in the index cone. It is clear that W_1 is a smooth manifold, usually non-connected, whose tangent space at a point D is identified as $\{L \in H^{1,1}(X, \mathbf{R}) : D^{n-1} \cup L = 0\}$. Moreover, there is a natural Riemannian metric on W_1 given by the pairing, for tangent vectors L_1, L_2 at $D \in W_1$,

$$(L_1, L_2) \mapsto -D^{n-2} \cup L_1 \cup L_2.$$

We note that for $n = 2$, the cup product on $H^{1,1}(X, \mathbf{R})$ is just a Lorentzian real quadratic form, and W coincides with the positive cone. Moreover, on each connected component of W_1, our construction reduces to the standard construction of real hyperbolic space (Example 10.2 in Chapter XI of [9], or page 189 of [6]); in particular it has constant negative curvature -1.

For a given complex structure on X, the Kähler structure is determined by the Kähler form ω, a closed real $(1,1)$-form. For a fixed complex structure and volume form $\omega_0^n/n!$, Huybrechts introduced what he called the *curved Kähler cone* $\tilde{\mathcal{K}}$ consisting of all Kähler forms ω with $\omega^n = c\,\omega_0^n$ for some $c > 0$ [5]. The Aubin–Calabi–Yau theorem [1] then implies that the projection map from $\tilde{\mathcal{K}}$ to $\mathcal{K}$ is a bijection. We normalise the volume form so that $\int_X \omega_0^n = 1$, and set $\tilde{\mathcal{K}}_1$ to consist of the Kähler forms ω with $\omega^n = \omega_0^n$. Thus, setting $\mathcal{K}_1 = \mathcal{K} \cap W_1$, the projection $\tilde{\mathcal{K}}_1 \to \mathcal{K}_1$ is a bijection. Moreover, Huybrechts observes that $\tilde{\mathcal{K}}_1$ is a smooth manifold, with the tangent space at $\omega \in \tilde{\mathcal{K}}_1$ consisting of the *primitive* closed real $(1,1)$-forms α; essentially this is saying that to first order $(\omega + \epsilon\,\alpha)^n = \omega^n$. He remarks that the Hodge identities imply that such forms are harmonic with respect to ω, and so the tangent space is identified as the $(h^{1,1} - 1)$-dimensional space of primitive harmonic $(1,1)$-forms. There is now a natural Riemannian metric on $\tilde{\mathcal{K}}_1$ given by the Hodge metric on harmonic forms. Given elements α_1, α_2 in the tangent space to $\tilde{\mathcal{K}}_1$ at ω, the metric is specified by the pairing

$$(\alpha_1, \alpha_2) \mapsto -\int_X \omega^{n-2} \wedge \alpha_1 \wedge \alpha_2.$$

Thus the projection map $\tilde{\mathcal{K}}_1 \to \mathcal{K}_1$ is in fact an isometry of Riemannian manifolds, enabling us to identify $\tilde{\mathcal{K}}_1$ with $\mathcal{K}_1$. We shall call this Riemannian manifold the *normalised Kähler moduli space*, and it is now clearly independent of the choice of normalised volume form.

In the case when X is a Calabi–Yau n-fold, not necessarily with $h^{2,0} = 0$, we have a nowhere vanishing holomorphic n-form Ω on X, and we can take the volume form to be a suitable multiple of $(i/2)^n\, \Omega \wedge \bar{\Omega}$. In this case, one is struck by the similar properties enjoyed by $\mathcal{K}_1$ and the space of complex structures on a Calabi–Yau n-fold. For instance, $\mathcal{K}_1$ has degenerations at *finite distance*, which correspond to singular Calabi–Yau n-folds with canonical singularities (cf [22] in dimension 3), and also degenerations at *infinite distance.* This may be compared with the properties of the Weil–Petersson metric for the complex moduli space [21]. In Section 1, we shall use Mirror Symmetry to suggest an explanation for such similarities. The sectional curvatures of the Weil–Petersson metric on the complex moduli space of the mirror are non-positive by results of Todorov [18]; we argue why this might indicate that the sectional curvatures on $\mathcal{K}_1$ are also non-positive. In general, we shall say that the *Kähler moduli curvature is semi-negative* if the sectional curvatures of $\mathcal{K}_1$ are non-positive.

The argument from Section 1 suggests that a mirror version of Weil–Petersson theory should be developed, to hold on the normalised Kähler moduli space; the basics of such a theory are developed in Sections 2 to 4. In Section 2, a mirror version of the Tian–Todorov Lemma is proved, and a DGLA structure on $\bigoplus_{p\geq 0} A^{p,1}$ is defined (throughout the paper, $A^{p,q}(X)$ denotes the space of (p,q)-forms on X). In Section 3, we investigate geodesics on the normalised Kähler moduli space, finding the all important quadratic term (3.6), and in Section 4, we obtain a simple formula (4.1) for the sectional curvatures, which is relevant for the question of semi-negativity, and another formula (4.3), which leads us naturally to the question of a *lower* bound. The following question turns out to be the natural one:

Question. *For which compact Kähler manifolds X are the sectional curvatures of $\mathcal{K}_1$ non-positive, and for which X are they bounded below by $-\frac{1}{2}n(n-1)$?*

We conjecture that both these bounds should hold for Calabi–Yau manifolds, but in fact they likely to be true rather more widely than that. The author knows of no examples of compact Kähler manifolds where the bounds fail. The question of when the formulae (4.1) and (4.3) do yield the suspected bounds, and also a strategy for proving them, are discussed in (4.4).

An instructive example where the bounds hold is provided (4.2) when X is a complex torus. We observe that the normalised Kähler moduli space $\mathcal{K}_1$ may be identified as the space of positive definite hermitian matrices of determinant 1, that is, the symmetric space $SL(n, \mathbf{C})/SU(n)$, and that the Hodge metric on $\mathcal{K}_1$ corresponds to some multiple of the symmetric space metric. The standard formula for the sectional curvatures of $SL(n, \mathbf{C})/SU(n)$ then corresponds to the formula we derive in (4.1).

If we could prove the conjectured semi-negativity for Calabi–Yau manifolds, it would provide new information on the possible location of the Kähler cone $\mathcal{K}$ in cohomology, and potentially useful information concerning which differentiable manifolds may support Calabi–Yau structures. For $h^{1,1}(X) \leq 2$, the conjecture gives no information, but one should expect increasingly more information as $h^{1,1}$ increases.

The case of Kähler manifolds with $h^{1,1} = 3$ does have the advantage that the curvature is easier to calculate, and for threefolds we derive a very explicit formula (5.1) for it in terms of the invariants of the ternary cubic form given by cup product (this formula has recently been shown to have a natural extension, in a similar shape, for arbitrary degrees > 2 [19]). Using the formula, the author has checked a large number of Kähler threefolds with $b_2 = 3$, and has verified the bounds in each case. In the case for instance of complete intersections in the product of three projective spaces, there is persuasive numerical evidence for the bounds, and for the case of complete intersections in $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$ even a computer proof of the result.

These examples also show that, in the Calabi–Yau case, the semi-negativity condition usually places stronger restrictions concerning the location of the Kähler cone than are provided by just the index cone. On the other hand, we show that, for Kähler threefolds with $b_2(X) = 3$, even the stronger condition that the Kähler moduli curvature lies between -3 and zero does not rule out further cases for the cubic form than are excluded by the standard results from Kähler geometry. I conjectured in an earlier version of this paper that this would not be the case for higher dimensions or higher second Betti number, and this expectation has recently been confirmed in [19]. The case of threefolds is of particular interest because, by results of Wall and Jupp, there is a very simple criterion for a given cubic form to be realisable as the cup product on $H^2(X, \mathbf{Z})$, for some smooth simply connected 6-manifold X with torsion free homology, and moreover such manifolds are classified by means of their invariants [14]. Note that all simply connected 6-manifolds are formal [12], and so in dimension 6 we cannot use this to distinguish those manifolds not supporting a Kähler structure. If the conjectured semi-negativity is true, then we can give examples of simply connected compact differentiable 6-manifolds with torsion free homology (specified by invariants including the cubic form, the first Pontryagin class and the third betti number) which do not support any Calabi–Yau structures. The non-existence of Calabi–Yau structures on these 6-manifolds would not appear to follow from existing criteria.

Finally, I mention two cases where the bounds on the Kähler moduli sectional curvatures are easy to check.

Example 1. Suppose X is a compact irreducible complex symplectic (hence hyperkähler) manifold of dimension $2n$. Letting q_X denote the Beauville–Bogomolov quadratic form, this has the property that for any $\alpha \in H^2(X, \mathbf{R})$, $q_X(\alpha)^n = \lambda \int_X \alpha^n$, for some fixed positive constant λ. Moreover, q_X is positive on any Kähler class ω and negative definite on the corresponding primitive classes $H^{1,1}(X, \mathbf{R})_\omega$. Setting $2\, b(\alpha, \beta) = q_X(\alpha+\beta) - q_X(\alpha) - q_X(\beta)$, we observe that for ω a Kähler class and α a primitive $(1,1)$-class, $q_X(\omega + t\alpha) = q_X(\omega) + 2t\, b(\omega, \alpha) + t^2 q_X(\alpha)$, whilst $\int_X (\omega + t\,\alpha)^n = \int_X \omega^n + n(2n-1)t^2\, \omega^{2n-2} \cup \alpha^2 + O(t^3)$; thus $b(\omega, \alpha) = 0$. Assuming that $\int \omega^n = 1$ and that q_X is normalised so that $\lambda = 1$, we have $q_X(\omega) = 1$, and we deduce that $q_X(\alpha) = (2n-1)\, \omega^{2n-2} \cup \alpha^2$ for all $\alpha \in H^{1,1}(X, \mathbf{R})_\omega$. Since q_X is Lorentzian on $H^{1,1}(X, \mathbf{R})$, this implies that W_1 is hyperbolic, as in the surface case, with constant sectional curvatures of value $-(2n-1)$. Note that this is consistent with the fact that, for a general irreducible complex symplectic manifold, the Kähler cone is a connected component of what we've called the positive cone.

Example 2. Suppose $h^{1,1}(X) = m+1$, and there are m independent divisorial contractions of irreducible divisors E_i on X to (distinct) points. We have a basis of $H^{1,1}(X, \mathbf{R}) \subset H^2(X, \mathbf{R})$ given by $H, -s_1 E_1, \ldots, -s_m E_m$, with H nef and the $s_i > 0$, with respect to which the degree n form on $H^{1,1}(X, \mathbf{R})$ is diagonal of the form

$$x_0^n - x_1^n - \ldots - x_m^n.$$

Let W_1^+ denote the open subset of W_1 given by intersecting with the positive orthant (with all coordinates strictly positive); when $n > 2$, this is just the connected component of W_1 that contains $\mathcal{K}_1$. The metric on W_1^+ is the restriction of

$$-x_0^{n-2} dx_0^2 + x_1^{n-2} dx_1^2 + \ldots + x_m^{n-2} dx_m^2.$$

Let U_1 denote the upper sheet of the hypersurface $y_0^2 - y_1^2 - \ldots - y_m^2 = 1$ in $\mathbf{R}^{m+1}$, equipped with the metric given by restricting $-dy_0^2 + dy_1^2 + \ldots + dy_m^2$; as commented above, this is a standard construction for m-dimensional hyperbolic space. We let U_1^+ denote the intersection of U_1 with the positive orthant, and consider the diffeomorphism from W_1^+ onto U_1^+ given by $y_i = x_i^{n/2}$. Since $dy_i = \frac{n}{2}\, x_i^{(n-2)/2}\, dx_i$, the pullback of the hyperbolic metric to W_1^+ is simply the given metric scaled by a factor of $(\frac{n}{2})^2$. So scaling distances on W_1^+ by a factor of $\frac{2}{n}$, we obtain a manifold isometric to U_1^+. Since hyperbolic space has constant sectional curvatures of value -1, our manifold W_1^+ has constant sectional curvatures of value $-(\frac{n}{2})^2$. For $n = 3$, this is the reason for the term $-9/4$ in (5.1).

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1. Motivation from Mirror Symmetry

One possible interpretation of Mirror Symmetry is in terms of the SYZ Conjecture. The simplest case here is that of a (non-compact) Calabi–Yau n-fold X fibered by special lagrangian tori for which the Ricci flat metric is semiflat (that is invariant under the torus action). This case was studied by Hitchin, Gross and Leung [4,3,10]. If the holomorphic n-form on X is denoted by Ω and the Kähler form by ω, recall that the condition of Ricci flatness may be written as

$$\omega^n/n! = c\,(\tfrac{i}{2})^n\,\Omega \wedge \bar{\Omega}$$

for some positive real constant c. The solutions sought correspond to solutions (with appropriate boundary conditions) of the real Monge–Ampère equation on the base of the fibration.

In [10], Leung produces an explicit holomorphic map (defined in terms of the fibration) between the Kähler moduli space of X and the complex moduli space of the mirror. In Leung's notation, fixing a standard complex structure on X, the Kähler moduli space consists of T^n-invariant complex 2-forms $\omega + i\beta$, with ω as above, and furthermore satisfying

$$\mathrm{Im}\,(e^{i\theta}(\omega + i\beta)^n) = 0$$

for some angle θ. At a given point $\omega + i\beta$ of this space, Leung remarks that the tangent space consists of complex harmonic $(1,1)$-forms, and he argues that therefore the Hodge L^2-metric on the harmonic forms (defined at a given point by the real part ω) determines a Riemannian metric on the whole space. On the corresponding space of complex structures on the mirror, one can take the Weil–Petersson metric (also defined as a Hodge L^2-metric on harmonic forms), and he argues formally that the explicit correspondence that he has defined is an isometry (see [10], Section 2).

For compact Calabi–Yau n-folds, we cannot hope for the metrics to be semi-flat and there will be instanton corrections to consider. We can however argue as follows: by standard theory, a Calabi–Yau manifold in the sense used in this paper has a finite unramified cover which is the product of a complex torus, irreducible complex symplectic manifolds, and simply connected Calabi–Yau manifolds with $h^{2,0} = 0$. As we know the conjectured bounds hold for the first two types of manifold, we may reduce the question to the third type; we do not prove this assertion here, but it follows for instance from the formula derived in (4.1), the key point here being that the *mixed* sectional curvatures of $\tilde{\mathcal{K}}_1$ for the product are necessarily zero. For simply connected Calabi–Yau manifolds in dimension 3 with $h^{2,0} = 0$, the Main Theorem from [22] says that the Kähler cone $\mathcal{K}$ is essentially invariant under deformations of the complex structure; to be precise, there are some complex codimension one loci in moduli, corresponding to the existence of elliptic quasi-ruled surfaces, and here the Kähler cone jumps down. A similar result is true for general n, without the explicit description of the loci where the Kähler cone jumps down. The normalised Kähler moduli space $\mathcal{K}_1$, considered as a subset of $H^2(X, \mathbf{R})$, can then be considered as an invariant of the complex structure. When addressing the conjecture on the sectional curvatures of $\mathcal{K}_1$, we are therefore at liberty to take arbitrary general points in the complex moduli space, for instance degenerating to a large complex structure limit point on the boundary, if such a limit exists; here the SYZ Conjecture will be relevant. One hopes that Leung's results are then indicative of what we might expect in the large Kähler structure limit. In particular, in view of Todorov's result that on the complex moduli space of Calabi–Yau n-folds, the Weil–Petersson metric has non-positive sectional curvatures, one is led to conjecture that the same is true of the Hodge metric on the complex Kähler moduli space for large Kähler structure limit. On Leung's complex Kähler moduli space, complex conjugation defines an isometry. The moduli space of real Kähler forms is then just the fixed locus of this isometry, and is therefore a totally geodesic submanifold (see [8], page 59). Thus the real Kähler moduli space should also have non-positive sectional curvatures, at least in the large Kähler structure limit.

Suppose now $\omega \in \mathcal{K}_1$, and let g denote the Hodge metric on the tangent space to $\mathcal{K}_1$ at ω, namely the primitive real harmonic $(1,1)$-forms. Clearly then, for $r > 0$, the metric at $r\omega$ is $r(dr^2 + g)$. For $\lambda > 0$, there is a scaling map θ_λ from $\mathcal{K} = \mathbf{R}_+ \times \mathcal{K}_1$ to itself, given by $\theta_\lambda(r, x) = (r', x)$, where $r' = \lambda r$. Moreover, we check that

$$\lambda^{-1} \theta_\lambda^* \, r'((dr')^2 + g) = r(\lambda^2 dr^2 + g).$$

Thus the scaling map sends the level set $r = 1$ to the level set $r' = \lambda$, and the normalised pullback of the metric from points on the level set $r' = \lambda$ is the same as the metric at the level set $r = 1$, except that the distances in the radial direction have been stretched by a factor λ. Hence, in the limit, only the sectional curvatures along the level set $\mathcal{K}_1$ will survive. In particular, the Hodge metric on $\mathcal{K}_1$ should also have non-positive sectional curvatures.

All this is of course only a plausability argument. To prove that the Kähler moduli curvature is semi-negative in the Calabi–Yau case, the author anticipates that a theory describing Kähler moduli which is mirror to the Weil-Petersson theory on complex moduli will be needed, and the basics of such a theory are developed in the next three sections.

2. Bracket operation on $(1,1)$-forms

Let X denote a compact Kähler n-fold, equipped with a fixed complex structure and a Kähler structure, determined by a closed real $(1,1)$-form ω. We shall assume that ω is normalised so that $\int_X \omega^n = 1$.

Motivated by the theory on the mirror side [17,18], we define a bracket operation on the $(1,1)$-forms $A^{1,1}$. Given $\alpha \in A^{1,1}$, there exists a unique element $\theta \in A^{1,0}(\Theta)$ such that $\theta \lrcorner \omega = \alpha$, where Θ denotes the sheaf of holomorphic vector fields on X and $\lrcorner$ denotes interior product. At any given point, we can find local coordinates $z_1, \ldots, z_n$, for which $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$ and $\alpha = \frac{i}{2} \sum a_{ij} dz_i \wedge d\bar{z}_j$, and then θ can be written at that point as

$$\theta = \sum a_{ij}\ \partial/\partial z_j \otimes dz_i.$$

Note that α is real if and only if $a_{ji} = \bar{a}_{ij}$ for all i, j — mostly we shall only consider real $(1,1)$-forms.

If α, β real $(1,1)$-forms, then with coordinates as above such that $\alpha = \frac{i}{2} \sum a_{ij}\ dz_i \wedge d\bar{z}_j$ and $\beta = \frac{i}{2} \sum b_{kl}\ dz_k \wedge d\bar{z}_l$, we see at the given point that

$$\theta \lrcorner \beta = \frac{i}{2} \sum a_{ij} b_{jl}\ dz_i \wedge d\bar{z}_l.$$

We observe for future use the identity $\langle \alpha, \beta \rangle = \Lambda(\theta \lrcorner \beta)$, where $\langle\, ,\, \rangle$ denotes the induced (pointwise) metric on forms and Λ denotes the adjoint of the Lefshetz operator L, given in coordinates on page 114 of [2]. We also note that if $\beta = \phi \lrcorner \omega$, then

$$\overline{\theta \lrcorner \beta} = \bar{\theta} \lrcorner \beta = \phi \lrcorner \alpha,$$

where $\bar{\theta}$ is an element of $A^{0,1}(\bar{\Theta})$.

The Kähler condition ensures that we can define a bracket operation

$$[\ ,\ ] \quad : \quad A^{1,0}(\Theta) \times A^{1,0}(\Theta) \to A^{2,0}(\Theta),$$

or equivalently, via $[\alpha, \beta] := [\theta, \phi] \lrcorner \omega$, a (symmetric) bracket

$$[\ ,\ ] \quad : A^{1,1} \times A^{1,1} \to A^{2,1}.$$

If at a given point, local coordinates $z_1, \ldots, z_n$ are chosen such that

$$\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i \ + O(\,|z|^2)$$

(using here the metric is Kähler), then we write $[\alpha, \beta]$ at that point as

$$[\alpha, \beta] = \frac{i}{2} \sum_{i,j,k,l} \left(a_{ij} \frac{\partial b_{kl}}{\partial z_j} + b_{ij} \frac{\partial a_{kl}}{\partial z_j} \right) d\bar{z}_l \wedge dz_i \wedge dz_k;$$

the corresponding bilinear form on $A^{1,0}(\Theta)$ wedges the dz_i and takes the Lie bracket of vector fields. Since only one derivative is being taken, this is independent of our choice of local coordinates of the specified type, a fact also confirmed by (2.2) below, which provides a visibly coordinate-independent characterization of the pairing.

There is an alternative way to proceed here, where we define the real bracket of real forms α and β to be $[\alpha, \beta]_{\mathbf{R}} := \frac{1}{2}([\alpha, \beta] + \overline{[\alpha, \beta]})$. Observing that $\frac{1}{2}(\theta + \bar{\theta}) \lrcorner \omega = \alpha$ and $\frac{1}{2}(\phi + \bar{\phi}) \lrcorner \omega = \beta$, the reader will easily check that $[\alpha, \beta]_{\mathbf{R}} = [\frac{1}{2}(\theta + \bar{\theta}), \frac{1}{2}(\phi + \bar{\phi})] \lrcorner \omega$.

Lemma 2.1. *For α a real $(1,1)$-form, $[\omega, \alpha] = \partial\alpha$ and $[\omega, \alpha]_{\mathbf{R}} = \frac{1}{2} d\alpha$.*

Proof. Immediate from the local formula given above.

Proposition 2.2. *With the notation as above,*

$$[\alpha, \beta] \, + \, \partial(\theta \lrcorner \beta \, + \, \phi \lrcorner \alpha) \; = \; (\#\partial\alpha) \lrcorner \beta \, + \, (\#\partial\beta) \lrcorner \alpha \, + \, \theta \lrcorner \partial\beta \, + \, \phi \lrcorner \partial\alpha,$$

where $\# : A^{2,1} \to A^{2,0}(\Theta)$ is the natural map given by $(\#\Gamma) \lrcorner \omega = \Gamma$.

Proof. This is the mirror of what is sometimes referred to as the Tian–Todorov Lemma, and as in that case, once one knows the correct statement of the result, the proof reduces to a (rather unenlightening) calculation in terms of local coordinates. If we wish to check the identity at a given point, we choose local coordinates $z_1, \ldots, z_n$ such that $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i \ + O(\,|z|^2)$. At the given point therefore

$$[\alpha, \beta] = \frac{i}{2} \sum_{i,j,k,l} \left(a_{ij} \frac{\partial b_{kl}}{\partial z_j} + b_{ij} \frac{\partial a_{kl}}{\partial z_j} \right) d\bar{z}_l \wedge dz_i \wedge dz_k$$

and

$$\partial(\theta \lrcorner \beta) = \frac{i}{2} \sum_{i,j,k,l} \left(a_{ij} \frac{\partial b_{jl}}{\partial z_k} + b_{jl} \frac{\partial a_{ij}}{\partial z_k} \right) dz_k \wedge dz_i \wedge d\bar{z}_l.$$

Also, since $\partial \alpha = \sum_{i,j,k} \frac{\partial a_{ij}}{\partial z_k} \, dz_k \wedge dz_i \wedge d\bar{z}_j$, we see that $\# \partial \alpha = \sum_{i,j,k} \frac{\partial a_{ij}}{\partial z_k} \, \partial/\partial z_j \otimes dz_k \wedge dz_i$ at the given point, from which it follows that

$$(\# \partial \alpha) \lrcorner \beta \ = \ \sum_{i,j,k} \frac{\partial a_{ij}}{\partial z_k} b_{jl} \ dz_k \wedge dz_i \wedge d\bar{z}_l.$$

Finally we have

$$\theta \lrcorner \partial \beta \ = \ \frac{i}{2} \sum_{i,j,k,l} \left(a_{ij} \frac{\partial b_{kl}}{\partial z_j} + a_{ij} \frac{\partial b_{jl}}{\partial z_k} \right) dz_i \wedge dz_k \wedge d\bar{z}_l.$$

From this it follows that, at the given point,

$$-\partial(\theta \lrcorner \beta) + (\# \partial \alpha) \lrcorner \beta \ + \ \theta \lrcorner \partial \beta \ = \ \frac{i}{2} \sum_{i,j,k,l} a_{ij} \frac{\partial b_{kl}}{\partial z_j} \ dz_i \wedge dz_k \wedge d\bar{z}_l.$$

The symmetry between α and β now yields the formula claimed.

Corollary 2.3. *If $\partial \alpha = 0 = \partial \beta$, then $[\alpha, \beta] + \partial(\theta \lrcorner \beta \ + \ \phi \lrcorner \alpha) = 0$.*

The conditions of (2.3) are equivalent to $d\alpha = 0 = d\beta$. The conclusion may be written as

$$[\alpha, \beta]_{\mathbf{R}} + \tfrac{1}{2} \, d(\theta \lrcorner \beta \ + \ \phi \lrcorner \alpha) = 0.$$

Note that by a previous comment, $\theta \lrcorner \beta \ + \ \phi \lrcorner \alpha \ = 2 \, \mathrm{Re}(\theta \lrcorner \beta)$ is a real $(1,1)$-form.

The pairing we've defined may be extended in an obvious way to give a bilinear pairing

$$A^{p,1} \times A^{q,1} \rightarrow A^{p+q,1},$$

which is symmetric or antisymmetric according to the parity of $pq + 1$, where the corresponding pairing on $A^{p,0}(\Theta) \times A^{q,0}(\Theta)$ is again defined by wedging the dz_i and taking the Lie bracket of vector fields. We shall also assume here that the metric is real analytic.

Theorem 2.4. *The bracket defined above satisfies the Jacobi identity*

$$[\gamma, [\alpha, \beta]] + [\beta, [\gamma, \alpha]] + [\alpha, [\beta, \gamma]] = 0.$$

Proof. To see this at a given point, we need to be slightly more careful concerning our choice of local holomorphic coordinates, choosing *canonical* holomorphic normal coordinates. The metric being Kähler and real analytic, we can choose local holomorphic coordinates $z_1, \ldots, z_n$ in a neighbourhood of a given point so that the Kähler form may be written as $\frac{i}{2} \sum h_{ij}\ dz_i \wedge d\bar{z}_j$, where

$$h_{ij} = \delta_{ij}\ + \sum_{k,l} c_{ijkl} z_k \bar{z}_l\ +\ O(\,|z|^3),$$

with $c_{ijkl} = \frac{1}{2} R_{i\bar{j}k\bar{l}}$, where R denotes the Riemannian curvature tensor (see Appendix 1 from [11], or Exercise 9 on p. 188 of [23]). For small $|z|$, the inverse matrix $h^{ij} = \delta_{ij}\ - \sum_{k,l} c_{ijkl} z_k \bar{z}_l\ +\ O(\,|z|^3)$. Now, we use (2.2) to give a formula for $[\alpha, \beta]$ valid in a neighbourhood of our given point. Thus we can then use our local definition of the bracket to calculate $[\gamma, [\alpha, \beta]]$ at our given point — although this now involves taking two derivatives, both are with respect to the holomorphic rather than anti-holomorphic coordinates. Since the forms involved are

$$\theta \lrcorner \beta = \frac{i}{2} \sum a_{ij} h^{jk} b_{kl} dz_i \wedge d\bar{z}_l,$$

and similar terms, when we evaluate at the given point, the derivatives of h^{ij} do not contribute. The conclusion therefore is that, by using canonical holomorphic coordinates, we can calculate $[\gamma, [\alpha, \beta]]$ *at the given point* as if the metric were just the flat metric. The claimed result then reduces to the Jacobi identity on vector fields of type $(1, 0)$.

Corollary 2.5. *For* $\alpha, \beta \in A^{1,1}$*, we have* $\partial[\alpha, \beta] = [\partial\alpha, \beta] - [\alpha, \partial\beta]$.

Proof. Applying (2.4),

$$[\omega, [\alpha, \beta]] + [\beta, [\omega, \alpha]] + [\alpha, [\beta, \omega]] = 0,$$

which by (2.1) implies the claimed result.

Extending these results in a straightforward way, we can show that $(\bigoplus_{p \geq 0} A^{p,1}, [\ ,\ ], \partial)$ has the structure of a DGLA.

The results of this section are not used in any crucial way in the following sections, but they do indicate an underlying structure behind the calculations we perform.

3. Geodesics

Before proceeding further, we shall need various identities on $(1,1)$-forms involving interior and exterior products, where at least one of the forms is assumed primitive.

Lemma 3.1. *If $\alpha, \beta \in A^{1,1}$, with $\alpha = \theta \lrcorner \omega$ and α primitive, then*

$$(\theta \lrcorner \beta) \wedge \omega^{n-1} = -(n-1)\ \alpha \wedge \beta \wedge \omega^{n-2}.$$

In particular, $(\theta \lrcorner \alpha) \wedge \omega^{n-1} = -(n-1)\, \alpha^2 \wedge \omega^{n-2}$.

Proof. We check the identity at a given point P by simultaneously diagonalising the forms ω and α, that is choosing coordinates $z_1, \ldots, z_n$ at P so that $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i\ + O(1)$ and $\alpha = \frac{i}{2} \sum a_{ii}\, dz_i \wedge d\bar{z}_i\ + O(1)$. Since α assumed primitive, we have $\sum_i a_{ii}\ = 0$ at P. Thus, recalling that $(\frac{i}{2})^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n = \omega^n/n!$, at the given point P,

$$\begin{aligned}(\theta \lrcorner \beta) \wedge \omega^{n-1} &= \tfrac{i}{2} \sum_{i,j} a_{ii} b_{ij}\, dz_i \wedge d\bar{z}_j \wedge \omega^{n-1} \\ &= \Big(\sum_i a_{ii} b_{ii}\Big)\ \omega^n/n \qquad\qquad . \\ &= -(n-1)\, \alpha \wedge \beta \wedge \omega^{n-2}\end{aligned}$$

Lemma 3.2. *(a) Suppose $\alpha, \beta \in A^{1,1}$, with $\alpha = \theta \lrcorner \omega$. If β is primitive, then*

$$2(\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2} = -(n-2)\, \alpha^2 \wedge \beta \wedge \omega^{n-3}.$$

If α is primitive, then

$$(\theta \lrcorner \alpha) \wedge \beta \wedge \omega^{n-2} + (\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2} = -(n-2)\, \alpha^2 \wedge \beta \wedge \omega^{n-3}.$$

In particular, if both α and β are primitive, then

$$(\theta \lrcorner \alpha) \wedge \beta \wedge \omega^{n-2} = (\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2} = -\tfrac{1}{2}(n-2)\, \alpha^2 \wedge \beta \wedge \omega^{n-3}.$$

Furthermore, if $\alpha_1, \alpha_2, \alpha_3$ are all primitive, then

$$(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \alpha_3 \wedge \omega^{n-2} = -(n-2)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \omega^{n-3}.$$

(b) If α is primitive, then

$$\beta^2 \wedge (\theta \lrcorner \alpha) \wedge \omega^{n-3} + 2\alpha \wedge \beta \wedge (\theta \lrcorner \beta) \wedge \omega^{n-3} = -(n-3)\, \alpha^2 \wedge \beta^2 \wedge \omega^{n-4},$$

where the right-hand side is zero for $n \leq 3$. In particular,

$$3\alpha^2 \wedge (\theta \lrcorner \alpha) \wedge \omega^{n-3} = -(n-3)\, \alpha^4 \wedge \omega^{n-4}.$$

Proof. (a) With coordinates at P chosen as in (3.1), observe that

$$\alpha^2 \wedge \beta \wedge \omega^{n-3} = \Big(\sum_{\substack{l,k,i \\ \text{distinct}}} a_{ll} a_{kk} b_{ii} \Big)\, \omega^n / n(n-1)(n-2).$$

If now β is primitive,

$$\begin{aligned}
(\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2} &= \Big(\sum_{k \neq i} a_{ii} b_{ii} a_{kk} \Big)\, \omega^n / n(n-1) \\
&= \Big(- \sum_{\substack{i,j,k \\ \text{distinct}}} a_{ii} b_{jj} a_{kk} \; - \sum_{k \neq i} a_{ii} b_{kk} a_{kk} \Big)\, \omega^n / n(n-1), \\
&= -(n-2)\, \alpha^2 \wedge \beta \wedge \omega^{n-3} - (\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2},
\end{aligned}$$

and hence the claim. If instead α is primitive, then

$$\begin{aligned}
(\theta \lrcorner \beta) \wedge \alpha \wedge \omega^{n-2} &= \Big(\sum_{k \neq i} a_{ii} b_{ii} a_{kk} \Big)\, \omega^n / n(n-1) \\
&= \Big(- \sum_{\substack{l,i,k \\ \text{distinct}}} a_{ll} b_{ii} a_{kk} \; - \sum_{k \neq i} a_{kk}^2 b_{ii} \Big)\, \omega^n / n(n-1) \\
&= -(n-2)\, \alpha^2 \wedge \beta \wedge \omega^{n-3} - (\theta \lrcorner \alpha) \wedge \beta \wedge \omega^{n-2}.
\end{aligned}$$

If both α and β are primitive, the next statement follows immediately.

For the final part, we observe from the previous statement that

$$(\theta_1 \pm \theta_2) \lrcorner (\alpha_1 \pm \alpha_2) \wedge \alpha_3 \wedge \omega^{n-2} = -\tfrac{1}{2}(n-2)(\alpha_1 \pm \alpha_2)^2 \wedge \alpha_3 \wedge \omega^{n-3},$$

from which the claim follows.

(b) With notation as in (a),

$$\beta^2 \wedge (\theta \lrcorner \alpha) \wedge \omega^{n-3} = \Bigg(\sum_{\substack{i,j,k \\ \text{distinct}}} (-a_{ii}^2 b_{jk} b_{kj} + a_{ii}^2 b_{jj} b_{kk}) \Bigg) \omega^n / n(n-1)(n-2),$$

and

$$2\alpha \wedge \beta \wedge (\alpha \lrcorner \beta) \wedge \omega^{n-3} =$$
$$\Big(\sum_{\substack{i,j,k \\ \text{distinct}}} (-a_{ii}a_{jj}b_{jk}b_{kj} - a_{ii}a_{kk}b_{jk}b_{kj} + a_{ii}a_{jj}b_{jj}b_{kk} + a_{ii}a_{kk}b_{jj}b_{kk}) \Big) \omega^n / n(n-1)(n-2).$$

The sum is therefore

$$\Big(\sum_{\substack{i,j,k,l \\ \text{distinct}}} (a_{ii}a_{ll}b_{jk}b_{kj} - a_{ii}a_{ll}b_{jj}b_{kk}) \Big) \omega^n / n(n-1)(n-2) = -(n-3)\alpha^2 \wedge \beta^2 \wedge \omega^{n-4}.$$

Remark 3.3. For α a primitive $(1,1)$-form, one can show similarly, using the formula for the adjoint Lefshetz operator Λ on page 114 of [2], that $\theta \lrcorner \alpha = -\frac{1}{2}\Lambda(\alpha \wedge \alpha)$; if both α and β are primitive, then $\theta \lrcorner \beta + \phi \lrcorner \alpha = -\Lambda(\alpha \wedge \beta)$. When α is also closed (and hence harmonic), it follows from the Hodge identity $i\bar{\partial}^* = [\Lambda, \partial]$ that

$$-i\bar{\partial}^*(\alpha \wedge \alpha) = -2\partial(\theta \lrcorner \alpha) = [\alpha, \alpha].$$

So if a primitive harmonic form α satisfies $[\alpha, \alpha] = 0$, or equivalently by (2.3) that $\theta \lrcorner \alpha$ is closed, then $i\bar{\partial}^*(\alpha \wedge \alpha) = 0$. The Hodge decomposition corresponding to $\bar{\partial}$ then implies that the closed form $\alpha \wedge \alpha$ is harmonic, and hence so too is $\theta \lrcorner \alpha$. In the general case, given a primitive harmonic $(1,1)$-form α, we prove below that $[\alpha, \alpha]$ is primitive, and it is ∂-exact by (2.3); thus $[\alpha, \alpha]$ is closed (equivalently, $\bar{\partial}$-closed) if and only if it is zero, which we have just seen happens if and only if $\theta \lrcorner \alpha$ is harmonic.

Lemma 3.4. *If α is a primitive and closed $(1,1)$-form (and hence also harmonic), then*

$$\bar{\partial}^*(\theta \lrcorner \alpha) = i\partial\langle \alpha, \alpha \rangle,$$

where $\langle\, , \rangle$ here denotes the pointwise inner-product of forms. Equivalently, this says that $[\alpha, \alpha]$ is primitive.

Proof. We check this at a given point. Choose complex normal coordinates as in Section 2 (canonical coordinates are not needed here). Given α, we write $\alpha = \frac{i}{2}\sum a_{ij} dz_i \wedge d\bar{z}_j$, and observe that

$$\langle \alpha, \alpha \rangle = \sum_{i,j} |a_{ij}|^2 + O(|z|^2).$$

We now write as before

$$\theta \lrcorner \alpha = \frac{i}{2} \sum_{i,j,k} \tilde{a}_{ik} a_{kj} \, dz_i \wedge d\bar{z}_j = \frac{i}{2} \sum_{i,j,k} a_{ik} a_{kj} \, dz_i \wedge d\bar{z}_j + O(|z|^2).$$

The formula for $\bar{\partial}^*$ on page 113 of [2] shows that

$$\bar{\partial}^* (\theta \lrcorner \alpha) = - * \bar{\partial} * (\theta \lrcorner \alpha) = i \sum_{i,j,k} \frac{\partial (a_{ik} a_{kj})}{\partial z_j} dz_i + O(|z|).$$

Expanding out,

$$\bar{\partial}^* (\theta \lrcorner \alpha) = i \big(\sum_{i,j,k} \frac{\partial a_{ik}}{\partial z_j} a_{kj} \, dz_i + \sum_{i,j,k} a_{ik} \frac{\partial a_{kj}}{\partial z_j} \, dz_i \big) + O(|z|).$$

The second term in the bracket is $\sum_{i,j,k} a_{ik} \frac{\partial a_{jj}}{\partial z_k} dz_i$ since α is closed, and this is zero at the point since $\sum_j a_{jj} = O(|z|^2)$ from the primitivity of α. The first term can however be rewritten as $\sum_{i,j,k} \frac{\partial a_{jk}}{\partial z_i} a_{kj} \, dz_i$ since α is closed, which in turn can be written as $\partial (\sum_{j,k} |a_{jk}|^2)$. Thus at the point we have $\bar{\partial}^* (\theta \lrcorner \alpha) = i \partial \langle \alpha, \alpha \rangle$, and hence this identity holds everywhere. This however is equivalent to the condition that $[\alpha, \alpha]$ is primitive, from the fact that $\langle \alpha, \alpha \rangle = \Lambda (\theta \lrcorner \alpha)$, the Hodge identity $i \bar{\partial}^* = [\Lambda, \partial]$, and (2.3).

Let us now apply (3.4) to give information concerning the Hodge decomposition of $\theta \lrcorner \alpha$. We write

$$(\theta \lrcorner \alpha) = (\theta \lrcorner \alpha)^h + \bar{\partial} \gamma_1 + \bar{\partial}^* \gamma_2.$$

Hence

$$i \partial \langle \alpha, \alpha \rangle = \bar{\partial}^* (\theta \lrcorner \alpha) = \bar{\partial}^* \bar{\partial} \gamma_1$$

for some $(1,0)$-form γ_1. This latter term may be written as

$$\Delta_{\bar{\partial}} \gamma_1 = \Delta_{\partial} \gamma_1 = \partial \partial^* \gamma_1 + \partial^* \partial \gamma_1 .$$

Hodge decomposition for ∂ now implies that $\partial^* \partial \gamma_1 = 0$, and hence $\partial \gamma_1 = 0$. Thus $\bar{\partial} \gamma_1 = -i \bar{\partial} \partial f$ for some real function f. The Hodge decomposition therefore reads

$$(\theta \lrcorner \alpha) = (\theta \lrcorner \alpha)^h + i \partial \bar{\partial} f + \bar{\partial}^* \gamma_2, \qquad (\dagger)$$

where $\bar{\partial}^* \gamma_2 = \partial^* \bar{\gamma}_2$, and so

$$(\theta \lrcorner \alpha) = (\theta \lrcorner \alpha)^h + i \partial \bar{\partial} f + i \partial^* \bar{\partial}^* \Gamma,$$

for some $(2,2)$-form Γ.

This decomposition has another special property; observe that

$$i\partial\langle\alpha,\alpha\rangle = \bar\partial^*(\theta\lrcorner\alpha) = i\bar\partial^*\partial\bar\partial f = -i\partial\bar\partial^*\bar\partial f = -i\partial\Delta_{\bar\partial} f.$$

Thus $\partial(\Delta_{\bar\partial} f + \langle\alpha,\alpha\rangle) = 0$, and so $\Delta_{\bar\partial} f + \langle\alpha,\alpha\rangle$ is constant on the manifold. The constant may be found by integrating over the manifold. Clearly $(\Delta_{\bar\partial} f)\omega^n$ has integral zero, and the calculation in (3.1) shows that

$$\int_X \langle\alpha,\alpha\rangle\,\omega^n = n\int_X (\theta\lrcorner\alpha)\wedge\omega^{n-1} = -n(n-1)\int_X \alpha^2\wedge\omega^{n-2} = n(n-1)A,$$

where $A = -\alpha^2\cup\omega^{n-2} > 0$. Thus

$$-\Delta_{\bar\partial} f = \langle\alpha,\alpha\rangle - n(n-1)A.$$

We now wedge equation (†) with ω^{n-1}. We have seen that $(\theta\lrcorner\alpha)\wedge\omega^{n-1} = \langle\alpha,\alpha\rangle\,\omega^n/n$; since $(\theta\lrcorner\alpha)^h\wedge\omega^{n-1}$ is harmonic, it is a constant multiple of ω^n, where the constant is clearly $(n-1)A$. A standard calculation shows that $i\partial\bar\partial f\wedge\omega^{n-1} = -(\Delta_{\bar\partial} f)\,\omega^n/n$, which is therefore $(\frac{1}{n}\langle\alpha,\alpha\rangle - (n-1)A)\,\omega^n$. We deduce therefore:

Proposition 3.5. *In the above decomposition (†), the form $(\theta\lrcorner\alpha)-(\theta\lrcorner\alpha)^h - i\partial\bar\partial f = \bar\partial^*\gamma_2$ is primitive.*

Notation. For any form η, we shall denote by η^{cl} the closed part of the Hodge decomposition of η, dependent of course on the metric. Thus, for α a primitive harmonic $(1,1)$-form, equation (†) says that $(\theta\lrcorner\alpha)^{cl} = (\theta\lrcorner\alpha)^h + i\partial\bar\partial f$, and then (3.4) says that $(\theta\lrcorner\alpha)^{cl}\wedge\omega^{n-1} = (\theta\lrcorner\alpha)\wedge\omega^{n-1}$. Moreover, if we write $(\theta\lrcorner\alpha)^{cl} = (\theta\lrcorner\alpha)+\gamma$, then γ is a primitive element of d^*A^3.

We now apply these results to get information about curves on the manifold $\tilde{\mathcal{K}}_1$. Suppose now that $\omega + \alpha_1 t + \frac12\alpha_{11}t^2 + \frac16\alpha_{111}t^3 + \dots$ is a deformation of ω such that

$$\begin{aligned}\omega^n &= (\omega + \alpha_1 t + \tfrac12\alpha_{11}t^2 + \dots)^n\\ &= \omega^n + (n\omega^{n-1}\wedge\alpha_1)\,t + \tfrac12\big(n\omega^{n-1}\wedge\alpha_{11} + n(n-1)\,\omega^{n-2}\wedge\alpha_1^2\big)\,t^2 + O(t^3).\end{aligned}$$

The reason for the slightly odd subscripts on the coefficients will become clear in the next section. Considering the degree one term, we have seen in the Introduction that this implies that α_1 is primitive and closed, and hence harmonic. If we set

$$\alpha_{11} = (\theta_1\lrcorner\alpha_1)^{cl} + \xi,$$

then ξ is a real and closed $(1,1)$-form. The equation in t^2 implies that

$$n\,(\theta_1 \lrcorner \alpha_1)^{cl} \wedge \omega^{n-1} + n\,\xi \wedge \omega^{n-1} + n(n-1)\,\omega^{n-2} \wedge \alpha_1^2 = 0,$$

and so by (3.1) and (3.5), ξ is primitive, and hence also harmonic.

Theorem 3.6. *Given a 1-parameter family on the manifold $\tilde{\mathcal{K}}_1$ through ω in the direction α_1, with α_1 primitive harmonic and $-\int \omega^{n-2} \wedge \alpha_1^2 = 1$, we can write it as*

$$\omega(t) = \omega + \alpha_1 t + \tfrac{1}{2}((\theta_1 \lrcorner \alpha_1)^{cl} \; + \xi)\, t^2 + O(t^3),$$

where ξ is a primitive harmonic form. If $\omega(t)$ is parametrised by arclength, then $\int_X \alpha_1 \wedge \xi \wedge \omega^{n-2} = 0$. If, furthermore, the 1-parameter family is a geodesic, then $\xi = 0$.

Proof. The metric on $\tilde{\mathcal{K}}_1$ is just the Hodge metric. Since

$$\frac{d\omega}{dt} = \alpha_1 + \alpha_{11} t + \tfrac{1}{2}\alpha_{111} t^2 + \ldots,$$

the condition that this has unit norm is

$$-\int_X (\omega + \alpha_1 t + \tfrac{1}{2}\alpha_{11} t^2 + \ldots)^{n-2} \wedge (\alpha_1 + \alpha_{11} t + \tfrac{1}{2}\alpha_{111} t^2 + \ldots)^2 = 1.$$

Equating terms in t to zero gives

$$\int_X 2\alpha_1 \wedge \alpha_{11} \wedge \omega^{n-2} + (n-2) \int_X \alpha_1^3 \wedge \omega^{n-3} = 0.$$

Recall that $\alpha_{11} = (\theta_1 \lrcorner \alpha_1)^{cl} \; + \xi = (\theta_1 \lrcorner \alpha_1) - i d^* \bar{\partial}^* \Gamma \; + \xi$. Observe that $\alpha_1 \wedge \omega^{n-2}$ is harmonic (and hence the image of a harmonic form under the Hodge $*$-operator), and so $\int_X \alpha_1 \wedge \omega^{n-2} \wedge d^* \bar{\partial}^* \Gamma = 0$. From (3.2)(a),

$$2\alpha_1 \wedge (\theta_1 \lrcorner \alpha_1) \wedge \omega^{n-2} = -(n-2)\, \alpha_1^3 \wedge \omega^{n-3},$$

and thus the condition obtained from equating terms in t to zero is

$$\int_X \alpha_1 \wedge \xi \wedge \omega^{n-2} = 0.$$

Consider now $\omega(t)$ for $0 \le t \le \delta$, with δ small, and suppose that it is geodesic; we claim that $\xi = 0$. If $\xi \ne 0$, we can choose a family $\xi(t) = \xi + t\xi_1 + \ldots$ of tangent vectors of the manifold $\tilde{\mathcal{K}}_1$ along the curve $\omega(t)$. We let $p(t) = t(\delta - t)$, although the argument also works replacing $t(\delta - t)$ by other polynomials $p(t)$ vanishing at 0 and δ, and deform the curve $\omega(t)$ using the tangent field $p(t)\xi(t)$. This yields a family of nearby curves $\omega_s(t)$ on $\tilde{\mathcal{K}}_1$, with the same initial and end points, of the form

$$\omega_s(t) = \omega(t) + s\, p(t)\, \xi(t) + O(s^2).$$

Note that

$$\dot{\omega}_s(t) = \dot{\omega}(t) + s\, p'(t)\, \xi(t) + s\, p(t)\, \dot{\xi}(t) + O(s^2).$$

We use the fact that $\omega(t)$ minimizes the energy.

The energy of ω_s is $-\int_0^\delta \omega_s(t)^{n-2} \cup \dot{\omega}_s(t)^2\, dt$, which is then of the form

$$-\int_0^\delta \big(\omega(t) + s\, p(t)\, \xi(t) + O(s^2)\big)^{n-2} \cup \big(\dot{\omega}(t) + s\, p'(t)\, \xi(t) + s\, p(t)\, \dot{\xi}(t) + O(s^2)\big)^2\, dt,$$

where as before $\cup$ denotes the cup product of cohomology classes represented by closed forms. If $\omega(t) = \omega_0(t)$ is geodesic, then

$$\begin{aligned}(n-2) \int_0^\delta p(t)\ \omega(t)^{n-3} \cup \xi(t) \cup \dot{\omega}(t)^2\, dt \ +\ 2 \int_0^\delta p'(t)\ \omega(t)^{n-2} \cup \dot{\omega}(t) \cup \xi(t)\, dt \\ +\ 2 \int_0^\delta p(t)\ \omega(t)^{n-2} \cup \dot{\omega}(t) \cup \dot{\xi}(t)\, dt \ =\ 0.\end{aligned}$$

The term in δ^2 is $2\, \omega^{n-2} \cup \alpha_1 \cup \xi \displaystyle\int_0^\delta p'(t)\, dt$, which is zero as required. Expanding out, the terms in δ^3 are as follows :

$$\begin{aligned}&(n-2)\, \omega^{n-3} \cup \alpha_1^2 \cup \xi \int_0^\delta \{p(t) + 2tp'(t)\}\, dt \\ &+\ 2\, \omega^{n-2} \cup \alpha_{11} \cup \xi \int_0^\delta tp'(t)\, dt \ +\ 2\, \omega^{n-2} \cup \alpha_1 \cup \xi_1 \int_0^\delta \{tp'(t) + p(t)\}\, dt.\end{aligned}$$

The third term here is clearly zero. Note that by definition of α_{11},

$$\omega^{n-2} \wedge \alpha_{11} \wedge \xi = \omega^{n-2} \wedge (\theta_1 \lrcorner \alpha_1) \wedge \xi\ - \omega^{n-2} \wedge \xi \wedge d^* \bar{\partial}^* \Gamma + \omega^{n-2} \wedge \xi^2,$$

where by (3.2)(a)

$$2\, \omega^{n-2} \wedge (\theta_1 \lrcorner \alpha_1) \wedge \xi \ = \ -(n-2)\, \omega^{n-3} \wedge \alpha_1^2 \wedge \xi.$$

Observe also that $\int_X \omega^{n-2} \wedge \xi \wedge d^* \bar{\partial}^* \Gamma = 0$, and so

$$2\, \alpha_{11} \cup \xi \cup \omega^{n-2} = -(n-2)\, \omega^{n-3} \cup \alpha_1^2 \cup \xi + 2\, \omega^{n-2} \cup \xi^2.$$

Collecting then the terms in $\omega^{n-3}\cup\alpha_1^2\cup\xi$, we obtain a coefficient which is

$$\int_0^\delta \{p(t)+2tp'(t)-tp'(t)\}\,dt \;=\; \int_0^\delta \{p(t)+tp'(t)\}\,dt \;=\; 0.$$

Thus, only one term of order δ^3 survives, and this is

$$2\omega^{n-2}\cup\xi^2 \int_0^\delta tp'(t)\,dt \;=\; -\tfrac{1}{3}\delta^3\,\omega^{n-2}\cup\xi^2.$$

As this has to be zero, we deduce that ξ is zero (recalling the Hodge Index theorem and that ξ is primitive), and the Theorem is proved.

Remarks 3.7. Note that there is an equality on harmonic parts $\alpha_{11}^h = (\theta_1 \lrcorner \alpha_1)^h$, and so the expansion for the geodesic $\bar\omega(t)$ on $\mathcal{K}_1$, considered as a submanifold of the harmonic forms $\mathcal{H}^{1,1}(\omega)$, is just

$$\bar\omega(t) = \omega + \alpha_1 t + \tfrac{1}{2}(\theta_1 \lrcorner \alpha_1)^h\, t^2 + O(t^3).$$

There is no reason why in general α_{11} should be harmonic. We note that $\alpha_{11} = (\theta_1 \lrcorner \alpha_1)^{cl}$ being harmonic is equivalent to $\bar\partial^*(\theta_1 \lrcorner \alpha_1) = 0$, which by (3.4) is equivalent to $\langle \alpha_1, \alpha_1\rangle$ being constant over the manifold. Thus, if for instance $h^{1,1} > n^2$, at any point of the manifold the harmonic $(1,1)$-forms are linearly dependent, and so some non-zero harmonic form α vanishes there. Since no primitive form is a multiple of ω at the point, it follows that α is primitive. Since $\langle\alpha,\alpha\rangle$ cannot be constant on the manifold, we deduce that $(\theta \lrcorner \alpha)^{cl}$ is not harmonic. For example, we can deduce for $h^{1,1} > n^2$ that the curved Kähler cone $\tilde{\mathcal{K}}$ is not linear, and in particular that there exists a primitive harmonic form α with α^2 not harmonic; these facts may be compared with the results from [5]. In the example of ω representing the flat metric on a complex torus, the harmonic forms α do however have $\langle\alpha,\alpha\rangle$ constant, with the forms $\theta \lrcorner \alpha$ all being harmonic.

4. Curvature calculations

In this section, we let $\alpha_1, \dots, \alpha_r$ denote an orthonormal basis for the tangent space to the manifold $\tilde{\mathcal{K}}_1$ at ω, or equivalently a basis of the primitive harmonic forms such that

$$-\omega^{n-2}\cup\alpha_i\cup\alpha_j = \delta_{ij}$$

for all i, j. This will determine a local normal coordinate system $(t_1, \ldots, t_r)$; locally the corresponding points of $\tilde{\mathcal{K}}_1$ will be of the form

$$\omega(t_1, \ldots, t_r) = \omega + \sum_1^r \alpha_i t_i + \frac{1}{2} \sum_{i,j} \alpha_{ij} t_i t_j + \frac{1}{6} \sum_{i,j,k} \alpha_{ijk} t_i t_j t_k + O(t^4).$$

The sectional curvature corresponding to the plane spanned by α_i and α_j in the tangent space of $\tilde{\mathcal{K}}_1$ at ω may be determined by expressing the metric in powers of the normal coordinates t_i and t_j (all the other t_k being held to be zero). For simplicity of notation, we shall do this for α_1 and α_2, and suppress all the coordinates t_k for $k > 2$. If the metric is written as $\sum g_{ij} dt_i \otimes dt_j$, then g_{12} has an expansion

$$g_{12}(t_1, t_2) = \tfrac{1}{3} R_{1212}\, t_1 t_2 + O(|t|^3)$$

by for instance [15], page 41, where R_{1212} is the sectional curvature we seek. Note here that the convention we adopt concerning the last two indices of the Riemannian curvature tensor differs from that of [15], but coincides with that of [9]. Writing $\omega(t_1, t_2)$ as

$$\omega + \alpha_1 t_1 + \alpha_2 t_2 + \tfrac{1}{2}\alpha_{11} t_1^2 + \alpha_{12} t_1 t_2 + \tfrac{1}{2}\alpha_{22} t_2^2 + \tfrac{1}{6}\alpha_{111} t_1^3 + \tfrac{1}{2}\alpha_{112} t_1^2 t_2 + \tfrac{1}{2}\alpha_{122} t_1 t_2^2 + \tfrac{1}{6}\alpha_{222} t_2^3 + \ldots,$$

we have

$$\partial\omega(t_1, t_2)/\partial t_1 = \alpha_1 + \alpha_{11} t_1 + \alpha_{12} t_2 + \tfrac{1}{2}\alpha_{111} t_1^2 + \alpha_{112} t_1 t_2 + \tfrac{1}{2}\alpha_{122} t_2^2 + \ldots$$

and

$$\partial\omega(t_1, t_2)/\partial t_2 = \alpha_2 + \alpha_{12} t_1 + \alpha_{22} t_2 + \tfrac{1}{2}\alpha_{112} t_1^2 + \alpha_{122} t_1 t_2 + \tfrac{1}{2}\alpha_{222} t_2^2 + \ldots.$$

To calculate $g_{12}(t_1, t_2)$, we need

$$-\int_X \omega(t_1, t_2)^{n-2} \wedge \partial\omega(t_1, t_2)/\partial t_1 \wedge \partial\omega(t_1, t_2)/\partial t_2.$$

Picking out the terms in $t_1 t_2$, we find that

$$\begin{aligned} -\tfrac{1}{3} R_{1212} &= \omega^{n-2} \cup \alpha_1 \cup \alpha_{122} + \omega^{n-2} \cup \alpha_2 \cup \alpha_{112} + \omega^{n-2} \cup \alpha_{11} \cup \alpha_{22} \\ &\quad + \omega^{n-2} \cup \alpha_{12}^2 + (n-2)\omega^{n-3} \cup \alpha_1^2 \cup \alpha_{22} + (n-2)\omega^{n-3} \cup \alpha_2^2 \cup \alpha_{11} \\ &\quad + 3(n-2)\omega^{n-3} \cup \alpha_1 \cup \alpha_2 \cup \alpha_{12} + (n-2)(n-3)\omega^{n-4} \cup \alpha_1^2 \cup \alpha_2^2. \end{aligned}$$

We can also calculate the curvature from the fact that g_{11} has an expansion

$$g_{11}(t_1, t_2) = 1 + \tfrac{1}{3} R_{1221}\, t_2^2 + O(|t|^3).$$

To calculate $g_{11}(t_1,t_2)$, we need

$$-\int_X \omega(t_1,t_2)^{n-2}\wedge \partial\omega(t_1,t_2)/\partial t_1 \wedge \partial\omega(t_1,t_2)/\partial t_1.$$

Picking out the terms in t_2^2, we find that

$$\begin{aligned}\tfrac{1}{3}R_{1212} &= \omega^{n-2}\cup\alpha_1\cup\alpha_{122} + \omega^{n-2}\cup\alpha_{12}^2\\ &+ \tfrac{1}{2}(n-2)\omega^{n-3}\cup\alpha_1^2\cup\alpha_{22} + 2(n-2)\omega^{n-3}\cup\alpha_1\cup\alpha_2\cup\alpha_{12}\\ &+ \tfrac{1}{2}(n-2)(n-3)\omega^{n-4}\cup\alpha_1^2\cup\alpha_2^2.\end{aligned}$$

By symmetry therefore,

$$\begin{aligned}\tfrac{2}{3}R_{1212} &= \omega^{n-2}\cup\alpha_1\cup\alpha_{122} + \omega^{n-2}\cup\alpha_2\cup\alpha_{112}\\ &+ 2\omega^{n-2}\cup\alpha_{12}^2 + \tfrac{1}{2}(n-2)\omega^{n-3}\cup\alpha_1^2\cup\alpha_{22} + \tfrac{1}{2}(n-2)\omega^{n-3}\cup\alpha_2^2\cup\alpha_{11}\\ &+ 4(n-2)\omega^{n-3}\cup\alpha_1\cup\alpha_2\cup\alpha_{12} + (n-2)(n-3)\omega^{n-4}\cup\alpha_1^2\cup\alpha_2^2.\end{aligned}$$

Subtracting our previous expression for $-\frac{1}{3}R_{1212}$ from this, we obtain

$$\begin{aligned}R_{1212} &= \omega^{n-2}\cup\alpha_{12}^2 - \omega^{n-2}\cup\alpha_{11}\cup\alpha_{22}\\ &- \tfrac{1}{2}(n-2)\omega^{n-3}\cup\alpha_1^2\cup\alpha_{22} - \tfrac{1}{2}(n-2)\omega^{n-3}\cup\alpha_2^2\cup\alpha_{11}\\ &+ (n-2)\omega^{n-3}\cup\alpha_1\cup\alpha_2\cup\alpha_{12}.\end{aligned}$$

Thus far, we have yet to use the information gleaned from the previous section. Extending our previous notation in an obvious way, we know that $\alpha_{ii} = (\theta_i \lrcorner \alpha_i)^{cl} = \theta_i \lrcorner \alpha_i + \gamma_{ii}$, where $\gamma_{ii}\in d^*A^3$ and is primitive, and as usual $\theta_i\lrcorner\omega = \alpha_i$. Moreover, γ_{ii} may be recovered from $\alpha_i\wedge\alpha_i$ as follows: There exists a (unique) ∂^*-coexact $(1,2)$-form η_i such that $\alpha_i\wedge\alpha_i = (\alpha_i\wedge\alpha_i)^h + \partial\eta_i$; then $\theta_i\lrcorner\alpha_i = -\frac{1}{2}\Lambda(\alpha_i\wedge\alpha_i)^h - \frac{1}{2}\Lambda\partial\eta_i$. Using the Hodge identity $[\Lambda,\partial] = i\bar\partial^*$, we deduce that $\gamma_{ii} = -\frac{i}{2}\bar\partial^*\eta_i \in \partial^*\bar\partial^*A^{2,2}$.

We can however also identify the cross-terms α_{12}, as we know that the curve given by $t_1 = t_2 = t/\sqrt{2}$ and $t_k = 0$ otherwise, will also be a geodesic, corresponding to the unit tangent vector $(\alpha_1+\alpha_2)/\sqrt{2}$. From this it follows that

$$\tfrac{1}{2}\alpha_{11} + \tfrac{1}{2}\alpha_{22} + \alpha_{12} = \tfrac{1}{2}((\theta_1+\theta_2)\lrcorner(\alpha_1+\alpha_2))^{cl}.$$

Thus

$$\alpha_{12} = \tfrac{1}{2}\big(\theta_1\lrcorner\alpha_2 + \theta_2\lrcorner\alpha_1\big) + \gamma_{12} = \mathrm{Re}\,\big(\theta_1\lrcorner\alpha_2\big) + \gamma_{12} = \mathrm{Re}\,\big(\theta_i\lrcorner\alpha_j\big)^{cl},$$

with $\gamma_{12}\in d^*A^3$ and primitive. In particular,

$$\omega^{n-1}\cup\alpha_{12} = \int_X \omega^{n-1}\wedge\mathrm{Re}\,(\theta_1\lrcorner\alpha_2) = -(n-1)\,\alpha_1\cup\alpha_2\cup\omega^{n-2},$$

using (3.1). By assumption $-\omega^{n-2}\cup\alpha_1\cup\alpha_2 = 0$, and thus $\omega^{n-1}\cup\alpha_{12} = 0$.

Theorem 4.1. *Given primitive harmonic forms α_1, α_2 as above on the compact Kähler manifold X, the corresponding sectional curvature of $\mathcal{K}_1$ is given by the formula*

$$R_{1212} = \int_X (\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \int_X \gamma_{12}^2 \wedge \omega^{n-2} - \int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2},$$

where the first two terms are non-positive.

Proof. From (3.2)(a), we know that for any primitive form α_3,

$$(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \alpha_3 \wedge \omega^{n-2} = -(n-2)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \omega^{n-3}.$$

We have

$$\alpha_{12} = \tfrac{1}{2}(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) + \gamma_{12},$$

with γ_{12} primitive. Thus,

$$\begin{aligned}\alpha_{12}^2 \wedge \omega^{n-2} &= \tfrac{1}{4}(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} + (\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \gamma_{12} \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} \\ &= \tfrac{1}{4}(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} - (n-2)\alpha_1 \wedge \alpha_2 \wedge \gamma_{12} \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2}.\end{aligned}$$

Therefore, the sum

$$\begin{aligned}&\alpha_{12}^2 \wedge \omega^{n-2} + (n-2)\alpha_1 \wedge \alpha_2 \wedge \alpha_{12} \wedge \omega^{n-3} = \\ &\tfrac{1}{4}(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} + (n-2)\alpha_1 \wedge \alpha_2 \wedge (\alpha_{12} - \gamma_{12}) \wedge \omega^{n-3} = \\ &\tfrac{1}{4}(\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} + \tfrac{1}{2}(n-2)\alpha_1 \wedge \alpha_2 \wedge (\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \omega^{n-3}.\end{aligned}$$

This may be written as

$$\begin{aligned}&- \tfrac{1}{4}(\theta_1 \lrcorner \alpha_2 - \theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} + \tfrac{1}{2}(\theta_1 \lrcorner \alpha_2)^2 \wedge \omega^{n-2} + \tfrac{1}{2}(\theta_2 \lrcorner \alpha_1)^2 \wedge \omega^{n-2} \\ &+ \tfrac{1}{2}(n-2)\alpha_1 \wedge \alpha_2 \wedge (\theta_1 \lrcorner \alpha_2) \wedge \omega^{n-3} + \tfrac{1}{2}(n-2)\alpha_1 \wedge \alpha_2 \wedge (\theta_2 \lrcorner \alpha_1) \wedge \omega^{n-3},\end{aligned}$$

which may be rewritten, using (3.2)(a), as

$$(\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} - \tfrac{1}{2}(\theta_1 \lrcorner \theta_1 \lrcorner \alpha_2) \wedge \alpha_2 \wedge \omega^{n-2} - \tfrac{1}{2}(\theta_2 \lrcorner \theta_2 \lrcorner \alpha_1) \wedge \alpha_1 \wedge \omega^{n-2}.$$

A straightforward local calculation verifies that

$$(\theta_1 \lrcorner \theta_1 \lrcorner \alpha_2) \wedge \alpha_2 \wedge \omega^{n-2} = (\theta_2 \lrcorner \theta_1 \lrcorner \alpha_1) \wedge \alpha_2 \wedge \omega^{n-2},$$

and hence also the corresponding formula with indices exchanged. Hence

$$\begin{aligned}\alpha_{12}^2 \wedge \omega^{n-2} + (n-2)\alpha_1 \wedge \alpha_2 \wedge \alpha_{12} \wedge \omega^{n-3} &= (\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} \\ &- \tfrac{1}{2}(\theta_2 \lrcorner \theta_1 \lrcorner \alpha_1) \wedge \alpha_2 \wedge \omega^{n-2} - \tfrac{1}{2}(\theta_1 \lrcorner \theta_2 \lrcorner \alpha_2) \wedge \alpha_1 \wedge \omega^{n-2},\end{aligned}$$

which using (3.2)(a) again may be written as

$$(\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} + (\theta_1 \lrcorner \alpha_1) \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-2}$$
$$+ \tfrac{1}{2}(n-2)\alpha_1^2 \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-3} + \tfrac{1}{2}(n-2)\alpha_2^2 \wedge (\theta_1 \lrcorner \alpha_1) \wedge \omega^{n-3}.$$

Substituting this into our formula for R_{1212}, we get that R_{1212} is the integral of

$$(\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} - \gamma_{11} \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-2} - \gamma_{22} \wedge (\theta_1 \lrcorner \alpha_1) \wedge \omega^{n-2}$$
$$- \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2} - \tfrac{1}{2}(n-2)\alpha_1^2 \wedge \gamma_{22} \wedge \omega^{n-2} - \tfrac{1}{2}(n-2)\alpha_2^2 \wedge \gamma_{11} \wedge \omega^{n-2},$$

which, using (3.2)(a) yet again, simplifies to

$$(\mathrm{Im}(\theta_1 \lrcorner \alpha_2))^2 \wedge \omega^{n-2} + \gamma_{12}^2 \wedge \omega^{n-2} - \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2}.$$

Finally, we observe that $\mathrm{Im}(\theta_1 \lrcorner \alpha_2) = -\frac{i}{2}(\theta_1 \lrcorner \alpha_2 - \theta_2 \lrcorner \alpha_1)$ is primitive by (3.1), and γ_{12} is primitive by (3.5), and so the first two terms in the above formula are non-positive.

Remark. Our formula (4.1) for the sectional curvature may be rewritten as

$$n(n-1)R_{1212} = -(\mathrm{Im}(\theta_1 \lrcorner \alpha_2), \mathrm{Im}(\theta_1 \lrcorner \alpha_2)) - (\gamma_{12}, \gamma_{12}) + (\gamma_{11}, \gamma_{22}),$$

where the bracket (,) here denotes the global inner-product. Compare this formula with the Gauss formula for the curvature of a submanifold, page 48 of [23], relevant because we can consider $\tilde{\mathcal{K}}_1$ as a submanifold of the infinite dimensional Riemannian manifold of all $(1,1)$-forms with top power being ω^n. The γ_{ij} are analogous to the values of an appropriate multiple of the second fundamental form.

Example 4.2. An instructive example is provided when X is an n-dimensional complex torus. After choosing a basis for the global holomorphic 1-forms, $H^{1,1}(X, \mathbf{R})$ may be identified as the hermitian $n \times n$ matrices. The Kähler cone corresponds to the positive definite hermitian matrices, and the form of degree n coming from cup product is some positive multiple of the determinant. Thus the normalised Kähler moduli space $\mathcal{K}_1$ may be identified as the space of positive definite hermitian matrices of determinant 1, that is, the symmetric space $SL(n, \mathbf{C})/SU(n)$. The Hodge metric on $\mathcal{K}_1$ is invariant under the action of $SL(n, \mathbf{C})$, corresponding to different choices of bases for the holomorphic 1-forms (not changing the volume form). Since $SL(n, \mathbf{C})/SU(n)$ is an irreducible symmetric space, the metric we have defined must be a constant multiple of the symmetric space metric; thus in this case $\mathcal{K}_1$ is also complete. A very concrete description of the symmetric space $SL(n, \mathbf{R})/SO(n)$ is given in Section 5.4 of [6]; the theory for $SL(n, \mathbf{C})/SU(n)$ is entirely analogous, just replacing the transpose of a matrix by its hermitian conjugate, and with the properties symmetric, respectively antisymmetric, being replaced by hermitian, respectively skew-hemitian. The tangent space at I_n is given by the trace-free hermitian matrices; given two such matrices A and B representing orthonormal tangent vectors, the sectional curvature of the corresponding tangent plane is $\mathcal{B}([A,B],[A,B])$ (cf. [6], Corollary 5.4.1), where the bracket operation $[\,,\,]$ is just the commutator of the matrices and $\mathcal{B}$ denotes the standard Killing bilinear form. This then corresponds to the formula we derived above, modulo an expected constant factor, once we have lifted the Kähler class to the form representing the flat metric — here of course γ_{11}, γ_{22} and γ_{12} are all zero.

For a trace-free skew-hermitian matrix $C = (c_{ij})$, we have that

$$\mathcal{B}(C, C) = 2n \ \mathrm{tr}(C^2) = -2n \ \mathrm{tr}(CC^*) = -2n \sum_{i,j} |c_{ij}|^2.$$

By choosing a suitable basis for the 1-forms, we may assume that the Kähler form corresponds to the identity matrix and that A is diagonal, and then a routine check verifies that $\mathcal{B}([A,B],[A,B]) = -2n \sum_{i,j}(a_{ii} - a_{jj})^2 |b_{ij}|^2$. We observe in passing that, when $n \geq 3$, the curvature in this example is not strictly negative.

There is however a second expression for the sectional curvature, which is relevant when considering the possible existence of a *lower* bound for the sectional curvatures.

Theorem 4.3. *With the same notation as in (4.1), the sectional curvature R_{1212} is also given by the formula*

$$-3\alpha_{12}^2 \cup \omega^{n-2} + \tfrac{1}{2}(n-2)(n-3)\alpha_1^2 \cup \alpha_2^2 \cup \omega^{n-4} - 2\int_X \gamma_{12}^2 \wedge \omega^{n-2} - \int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2} - \tfrac{1}{2}n(n-1),$$

where here the first three terms are non-negative.

Proof. We observe first that

$$\alpha_{12}^2 \cup \omega^{n-2} = -\tfrac{1}{2}(n-2)\, \alpha_1 \cup \alpha_2 \cup \alpha_{12} \cup \omega^{n-3}.$$

To prove this, note that

$$\alpha_{12}^2 \cup \omega^{n-2} = \alpha_{12} \cup \alpha_{12}^h \cup \omega^{n-2} = \tfrac{1}{2} \int_X (\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \alpha_{12}^h \wedge \omega^{n-2},$$

and that α_{12}^h is also primitive, and so the claim follows from (3.2)(a).

A further identity is that

$$\alpha_{11} \cup \alpha_{22} \cup \omega^{n-2} = -\tfrac{1}{2}(n-2)\alpha_2^2 \cup \alpha_{11} \cup \omega^{n-3} + \tfrac{1}{2}n(n-1).$$

This is proved by considering the harmonic form $\alpha_{11}^h - (n-1)\omega$, which is primitive by (3.1), and applying (3.2)(a). Thus for instance one can in general write

$$\begin{aligned} R_{1212} &= -\alpha_{12}^2 \cup \omega^{n-2} + \alpha_{11} \cup \alpha_{22} \cup \omega^{n-2} - n(n-1) \\ &= -\alpha_{12}^2 \cup \omega^{n-2} - \tfrac{1}{2}(n-2)\alpha_1^2 \cup \alpha_{22} \cup \omega^{n-3} - \tfrac{1}{2}n(n-1). \end{aligned}$$

From (3.2)(b), we have

$$\alpha_1^2 \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-3} + 2\alpha_1 \wedge \alpha_2 \wedge (\theta_2 \lrcorner \alpha_1) \wedge \omega^{n-3} = -(n-3)\, \alpha_1^2 \wedge \alpha_2^2 \wedge \omega^{n-4}.$$

An easy local calculation verifies that

$$\alpha_1 \wedge \alpha_2 \wedge (\theta_2 \lrcorner \alpha_1) \wedge \omega^{n-3} = \alpha_1 \wedge \alpha_2 \wedge (\theta_1 \lrcorner \alpha_2) \wedge \omega^{n-3},$$

and so

$$\alpha_1^2 \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-3} + \alpha_1 \wedge \alpha_2 \wedge (\theta_1 \lrcorner \alpha_2 + \theta_2 \lrcorner \alpha_1) \wedge \omega^{n-3} = -(n-3)\, \alpha_1^2 \wedge \alpha_2^2 \wedge \omega^{n-4}.$$

Thus

$$\begin{aligned} -\tfrac{1}{2}(n-2)\, \alpha_1^2 \cup \alpha_{22} \cup \omega^{n-3} + \tfrac{1}{2}(n-2) \int_X \alpha_1^2 \wedge \gamma_{22} \wedge \omega^{n-3} - (n-2)\, \alpha_1 \cup \alpha_2 \cup \alpha_{12} \cup \omega^{n-3} \\ + (n-2) \int_X \alpha_1 \wedge \alpha_2 \wedge \gamma_{12} \wedge \omega^{n-3} = \tfrac{1}{2}(n-2)(n-3)\, \alpha_1^2 \cup \alpha_2^2 \cup \omega^{n-4}. \end{aligned}$$

Using the fact that for any closed form α and primitive form $\gamma \in dA^3$, we have $\int_X \alpha \wedge \gamma \wedge \omega^{n-2} = 0$, we deduce, using (3.2)(a) several times and also the first identity of this proof, that

$$\begin{aligned} &\tfrac{1}{2}(n-2)(n-3)\, \alpha_1^2 \cup \alpha_2^2 \cup \omega^{n-4} = \\ &\quad -\tfrac{1}{2}(n-2)\, \alpha_1^2 \cup \alpha_{22} \cup \omega^{n-3} + \int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2} \, + \, 2\, \alpha_{12}^2 \cup \omega^{n-2} + 2 \int_X \gamma_{12}^2 \wedge \omega^{n-2}. \end{aligned}$$

Using this to substitute for $-\frac{1}{2}(n-2)\, \alpha_1^2 \cup \alpha_{22} \cup \omega^{n-3}$ into the second expression given above for R_{1212}, we obtain the formula claimed.

The first and the third terms are non-negative from the primitivity of the forms α_{12}^h and γ_{12}, and the second term is non-negative from the primitivity of the 4-form $(\alpha_1 \wedge \alpha_2)^h$ and the Hodge–Riemann bilinear relations.

Discussion 4.4. We have seen that it is natural therefore extend the question concerning semi-negativity of Kähler moduli curvature to one which asks for which Kähler manifolds the sectional curvatures of $\mathcal{K}_1$ lie between $-\frac{1}{2}n(n-1)$ and 0. The message of (4.1) and (4.2) is that for both bounds, the term $-\int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2}$ is crucial. If for instance α_1^2 is harmonic, then this term is zero and the sectional curvature R_{1212} has the conjectured bounds. We deduce immediately that if the squares of harmonic $(1,1)$-forms on X are harmonic, for instance if X is a hermitian symmetric space of compact type, then the bounds are as stated for all the sectional curvatures. The formulae (4.1) and (4.3) are however rather stronger than this.

In the Calabi–Yau case, we argued in Section 1 that we can reduce down to the case of simply connected Calabi–Yau manifolds with $h^{2,0} = 0$, and that in this case $\mathcal{K}_1$ was essentially independent of the complex structure. Using an argument via Yau's construction of the Ricci flat metric and Moser's Theorem, we may even take the symplectic form ω to be fixed as we vary the complex structure. In the general case, we can also fix ω, and then vary the complex structure among those compatible with ω and which induce the same Hodge structure on $H^2(X, \mathbf{R})$, this latter condition holding automatically when $h^{2,0} = 0$. In this more general case also, on the complement of countably many subvarieties in the complex structure moduli space, the normalised Kähler moduli space $\mathcal{K}_1$ is essentially independent of the complex structure. This then leads us to the following criterion:

Criterion. For a given Kähler class $\bar{\omega}$ and orthonormal primitive classes $\bar{\alpha}_1$ and $\bar{\alpha}_2$ in $H^{1,1}(X, \mathbf{R})$, we lift $\bar{\omega}$ to a Kähler form ω, which we now fix. We vary the complex structure as described above, and for each such compatible complex structure, we lift the classes $\bar{\alpha}_1$ and $\bar{\alpha}_2$ to their harmonic representatives. This then gives rise to the forms γ_{11} and γ_{22}, and hence the quantity $\int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2}$, which we may regard as a function of the complex structure. It follows from $\int_X (\theta_1 \lrcorner \alpha_1) \wedge (\theta_2 \lrcorner \alpha_2) \wedge \omega^{n-2} \geq 0$ that $\int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2}$ is bounded below by $-\alpha_{11} \cup \alpha_{22} \cup \omega^{n-2}$, independent of the complex structure. If now for some limit point of complex moduli, either in the interior or more likely maybe on the boundary, the above quantity tends to 0 as we approach the limit point suitably, then the corresponding sectional curvature R_{1212} of $\mathcal{K}_1$ satisfies $-\frac{1}{2}n(n-1) \leq R_{1212} \leq 0$.

If we write $\alpha_i \wedge \alpha_i = (\alpha_i \wedge \alpha_i)^h + \partial \eta_i$, with η_i assumed to be ∂^*-coexact, then as noted above $\gamma_{ii} = -\frac{1}{2} i \bar{\partial}^* \eta_i$. Since $\bar{\partial}\partial \eta_i = 0$, we have $\partial \bar{\partial} \eta_i = 0$. But $\partial^*(\bar{\partial}\eta_i) = -\bar{\partial}\partial^* \eta_i = 0$, and so $\bar{\partial}\eta_i$ is harmonic; in particular $\bar{\partial}^* \bar{\partial} \eta_i = 0$. Now since γ_{11} and γ_{22} are primitive,

$$n(n-1) \int_X \gamma_{11} \wedge \gamma_{22} \wedge \omega^{n-2} = -(\gamma_{11}, \gamma_{22}) = -\tfrac{1}{4}(\bar{\partial}^* \eta_1, \bar{\partial}^* \eta_2) = -\tfrac{1}{4}(\bar{\partial}\bar{\partial}^* \eta_1, \eta_2).$$

Since $\bar{\partial}^* \bar{\partial} \eta_1 = 0$, this may be written as

$$-\tfrac{1}{4}(\Delta_{\bar{\partial}} \eta_1, \eta_2) = -\tfrac{1}{4}(\Delta_{\partial} \eta_1, \eta_2) = -\tfrac{1}{4}(\partial^* \partial \eta_1, \eta_2) = -\tfrac{1}{4}(\partial \eta_1, \partial \eta_2).$$

Thus

$$(\alpha_1 \wedge \alpha_1, \alpha_2 \wedge \alpha_2) = ((\alpha_1 \wedge \alpha_1)^h, (\alpha_2 \wedge \alpha_2)^h) \; + \; (\partial \eta_1, \partial \eta_2),$$

the first term of which is invariant and the second term being (up to the constant) the quantity we wish to know about.

When dealing with Calabi–Yau manifolds with $h^{2,0} = 0$, it is tempting to believe that one should consider a large complex structure limit point for the above purposes. The author hopes to return to this question in a later paper

5. Kähler threefolds with $b_2 = 3$

Given a form of degree n on $\mathbf{R}^r$, we define the index cone W, as in the Introduction, to be the set of points at which the form is positive and the signature of the associated quadratic form is $(1, r-1)$. In principle, one can then calculate the sectional curvatures of the level set W_1. The easiest case will be that of cubic forms on $\mathbf{R}^3$, of relevance to Kähler threefolds with $h^{1,1} = 3$. Here we shall have only one sectional curvature to consider.

We denote the cubic form by F, and its Hessian by H, also a cubic form. Choosing a basis for the vector space, these may be considered as homogeneous cubics in three variables. We let S denote the degree 4 invariant of F, one of the two basic invariants S and T found by Aronhold, and written down explicitly in [16]. If the cubic is non-singular and in Weierstrass normal form, these are multiples of the better known invariants g_2, g_3 of the corresponding elliptic curve, with in particular $S = 4g_2/27$. For a general ternary cubic, S has 25 terms and T has 103 terms. Modulo a positive constant, the discriminant Δ of the cubic is $64S^3 - T^2$. Recall that the cubic is singular if and only if $\Delta = 0$, and in the smooth case the real curve has one or two components, dependent upon whether $\Delta < 0$ or $\Delta > 0$. Thus if $S < 0$, we can only have one component, whilst if $S > 0$ we may have one or two components. We let $H(X_0, X_1, X_2)$ denote the Hessian of F, also a cubic form.

Theorem 5.1. *If F is a ternary cubic form with Hessian H and basic degree 4 invariant S, then for any point of its index cone W, the curvature at the corresponding point of the level set W_1 is given by the formula*

$$R = -\frac{9}{4} + \frac{6^6 S\, F^2}{4H^2}.$$

Remarks. If in fact we are at a point of W_1, then by definition F takes value one. I prefer however to write the formula in the way given, since then all the required invariance under changes of coordinates and scalings hold good and are clear. Motivated by this formula, Totaro has recently demonstrated a natural extension of it, in a similar shape, for forms F of arbitrary degree $d > 2$.

Proof. Suppose the relevant point of W_1 is represented by an element L_0 of $\mathbf{R}^3$, at which the cubic form takes the value one. We can then find a real basis $\{L_0, L_1, L_2\}$, with $L_0^2 \cup L_i = 0$ for $i = 1, 2$, where $\cup$ is given by polarising the cubic form. We shall let $L_0 \cup L_1^2 = -p < 0$, $L_0 \cup L_1 \cup L_2 = -q$, $L_0 \cup L_2^2 = -r < 0$, and $L_1^i \cup L_2^j = a_{ij}$ for all non-negative $i + j = 3$. One could of course also choose L_1, L_2 so that $p = r = 1$ and $q = 0$, but that does not significantly simplify the problem and tends to obscure some of the structure. In these coordinates, the value of the cubic form at $x_0L_0 + x_1L_1 + x_2L_2$ is

$$(x_0L_0+x_1L_1+x_2L_2)^3 = x_0^3 - 3x_0(px_1^2+2qx_1x_2+rx_2^2) + a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3,$$

and so the level set W_1 has equation

$$F(x_0, x_1, x_2) = x_0^3 - 3x_0(px_1^2 + 2qx_1x_2 + rx_2^2) + a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3 = 1.$$

The Hodge metric at a point (x_0, x_1, x_2) then takes the form

$$\begin{aligned}
&- (x_0L_0 + x_1L_1 + x_2L_2) \cup (L_0dx_0 + L_1dx_1 + L_2dx_2)^2 \\
&= -dx_0^2 + x_0(pdx_1^2 + 2qdx_1dx_2 + rdx_2^2) + 2(px_1dx_1 + q(x_1dx_2 + x_2dx_1) + rx_2dx_2)dx_0 \\
&- (a_{30}x_1 + a_{21}x_2)dx_1^2 - 2(a_{21}x_1 + a_{12}x_2)dx_1dx_2 - (a_{12}x_1 + a_{03}x_2)dx_2^2.
\end{aligned}$$

Now set $x_0 = 1 + y_0$, so that

$$y_0 = (px_1^2 + 2qx_1x_2 + rx_2^2) + O(|x|^3),$$

where $|x|^2 = x_1^2 + x_2^2$. Therefore

$$dy_0 = dx_0 = 2(px_1dx_1 + qx_1dx_2 + qx_2dx_1 + rx_2dx_2) + O(|x|^2).$$

Substituting in, we find that the metric takes the form locally $g = \sum g_{ij} dx_i dx_j$, where

$$g_{11} = -a_{30}x_1 - a_{21}x_2 + p(1 + px_1^2 + 2qx_1x_2 + rx_2^2) + O(|x|^3),$$

$$g_{22} = -a_{12}x_1 - a_{03}x_2 + r(1 + px_1^2 + 2qx_1x_2 + rx_2^2) + O(|x|^3),$$

$$g_{12} = g_{21} = -a_{21}x_1 - a_{12}x_2 + q(1 + px_1^2 + 2qx_1x_2 + rx_2^2) + O(|x|^3).$$

Note also that at the point in question, $\det(g) = pr - q^2$. Running MAPLE, we obtain that the curvature tensor at this point has

$$R_{1212} = -2(pr - q^2) + \tfrac{1}{4}\big(p(a_{12}^2 - a_{21}a_{03}) + q(a_{30}a_{03} - a_{21}a_{12}) + r(a_{21}^2 - a_{30}a_{12})\big)/(pr - q^2),$$

and hence that the (sectional) curvature is

$$R = -2 + \frac{1}{4(pr - q^2)^2}\big(p(a_{12}^2 - a_{21}a_{03}) + q(a_{30}a_{03} - a_{21}a_{12}) + r(a_{21}^2 - a_{30}a_{12})\big).$$

If we started instead with a point L_0 of W, say $L_0^3 = A > 0$, then the curvature at the corresponding point of W_1 takes the form

$$R = -2 + \frac{A}{4(pr - q^2)^2}\big(p(a_{12}^2 - a_{21}a_{03}) + q(a_{30}a_{03} - a_{21}a_{12}) + r(a_{21}^2 - a_{30}a_{12})\big).$$

Turning now to the formula for the invariant S on page 167 of Sturmfels, we find that for the cubic

$$F(x_0, x_1, x_2) = Ax_0^3 - 3x_0(px_1^2 + 2qx_1x_2 + rx_2^2) + a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3,$$

the basic invariant S takes the form

$$S = (pr - q^2)^2 + A\big(p(a_{12}^2 - a_{21}a_{03}) + q(a_{30}a_{03} - a_{21}a_{12}) + r(a_{21}^2 - a_{30}a_{12})\big).$$

From this it follows that

$$R = -\frac{9}{4} + \frac{S}{4(pr - q^2)^2}.$$

An easy check verifies that at the point in question, the Hessian H takes the value $6^3 A(pr - q^2)$, where F takes the value A, and so

$$R = -\frac{9}{4} + \frac{6^6 S\, F(x_0, x_1, x_2)^2}{4H(x_0, x_1, x_2)^2}.$$

For an arbitrary point of W, we can always make a change of coordinates

$$x_0 \mapsto x_0 + \lambda x_1 + \mu x_2, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2,$$

so that the cubic takes the form assumed above. Since S is invariant under such a transformation in $SL(3, \mathbf{R})$, as is the value of the Hessian, the formula remains true in general.

Examples. There are two cases above when the curvature is constant on W_1. The first is when $S = 0$, when the curvature is $-\frac{9}{4}$ everywhere. Note that $S = 0$ if and only if the cubic curve is isomorphic (over $\mathbf{C}$) to the unique elliptic curve with an automorphism of order three, namely the Fermat cubic $x^3 + y^3 + z^3 = 0$, or else some singular specialization of this. Compare this with the calculation from Example 2 described in the Introduction; in particular the $-9/4$ occurs because it is $-(\frac{n}{2})^2$. If $S \neq 0$ and the curvature is constant, then we must have that F and H are proportional. This says that every point of the curve is an inflexion point, and so the curve reduces to three lines. Assuming that W_1 is non-empty and so in particular the Hessian is not identically zero, the curve consists of three non-concurrent lines. An easy calculation then shows that $\frac{6^6 S\, F^2}{4H^2} = \frac{9}{4}$, and so R is identically zero. Both these cases do occur: if the threefold admits a birational contraction of a divisor down to a point, then it is an easy check from [16] that $S = 0$; if $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, then we are clearly in the second case [14].

Remarks. The discriminant Δ of the ternary cubic may be positive, negative or zero, and all these cases can be realised geometrically, as may be checked by various examples of Calabi–Yau complete intersections in a product of three projective spaces. For general Kähler threefolds with $b_2 = 3$, the invariant S may be positive, negative or zero. An example with $S < 0$ is for instance given by taking a line with normal bundle $(-1, -1)$ on a quintic hypersurface in $\mathbf{P}^4$ and blowing it up, and then blowing up a section of the resulting exceptional surface $\mathbf{P}^1 \times \mathbf{P}^1$.

The formula in (5.1) makes it possible to check whether a given Kähler threefold with $b_2 = 3$ does satisfy the suggested bounds on curvature for the Kähler moduli, and in all the examples calculated by the author the answer has been affirmative. In the Calabi–Yau cases, all the examples have $S \geq 0$, with strict inequality holding in most cases; when $S > 0$, the curvature condition has in all cases defined a proper subcone of the index cone.

In the case of Kähler threefolds with $b_2 = 3$ which are complete intersection in a product of three projective spaces, the author has made extensive computer calculations, and there is persuasive numerical evidence here for $S > 0$ and the Kähler moduli curvature always being semi-negative. When X is a complete intersection in $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$, this has in fact been laboriously proved by the author using MAPLE. For three general projective spaces, there is presumably a combinatorial proof for these assertions.

Example. To give the flavour of the calculations referred to above, let us consider the complete intersection Calabi–Yau threefold

$$\begin{pmatrix} \mathbf{P}^3 & \Big\| & 1 & 1 & 2 & 0 \\ \mathbf{P}^2 & \Big\| & 1 & 1 & 1 & 0 \\ \mathbf{P}^2 & \Big\| & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The configuration is such that the Kähler cone has generators corresponding to the three factors, and so the Kähler cone is given in coordinates (x, y, z) by all three coordinates being positive. Running MAPLE, one discovers that $S = 4624$ and the curvature of $\mathcal{K}_1$ is given by a formula whose denominator is a square and whose numerator consists of 13 monomials of degree 6 in x, y, z, all of whose coefficients are negative. This property of every term being negative is a feature of all the computer calculations performed (including various examples of Kähler 4-folds in the product of three projective spaces that the author has also calculated, using the generalised form [19] of (5.1) for quartics).

By results of Wall and Jupp [20,7], a simply connected compact differentiable 6-manifold with torsion-free cohomology is determined uniquely by a set of invariants, these invariants including the integral cubic form on second cohomology given by cup product, the integral linear form on second cohomology given by the first Pontryagin class p_1, and the third cohomology, the first two of these invariants being required to satisfy certain simple congruence relations (see [14] for a good summary). For the manifold to support a Kähler structure, the Hard Lefschetz Theorem says that the Hessian of the cubic form cannot be identically zero [14]. In the case when $b_2 = 3$, the only non-trivial cubic forms excluded by this condition are those where the corresponding curve consists of three concurrent lines (and degenerations of this). One might think that, even in the case $b_2 = 3$, the suggested bounds on the curvature of $\mathcal{K}_1$ would rule out some further cubic forms — this however is not the case.

Proposition 5.2. *For any real cubic form $F(X_0, X_1, X_2)$ with Hessian $H(X_0, X_1, X_2)$ not identically zero, there is a non-empty open subcone of the index cone W for which the curvature at the corresponding points of W_1 lies between -3 and zero.*

Proof. The reader is left as an exercise to check the cases when the corresponding complex projective curve is reducible. Suppose now the curve is irreducible (therefore either an elliptic curve, a nodal cubic, or a cuspidal cubic); it is then well-known that the curve contains at least one real inflexion point (in the case of a real elliptic curve, there will in fact be three). Such a real inflexion point is given by a transverse intersection of the real curves given by F and H. If we consider the complement of these two real curves in a small enough neighbourhood of the inflexion point in the affine plane, we obtain four regions, determined by the signs of F and H. On the region for which $F < 0$ and $H > 0$, the signature at the corresponding point of $\mathbf{R}^3$ has to be $(-,-,+)$; we use the convention here that the first entry indicates the sign of F. As one passes through the curve $F = 0$, the signature does not change, only the sign of F; we therefore have locally a region where the corresponding points of $\mathbf{R}^3$ have signature $(+,-,-)$, that is lie in the index cone. If however, we take points in this region away from the inflexion point but sufficiently close to the curve $F = 0$, then the formula in (5.1) ensures that the curvature condition is also satisfied, and the Proposition follows. Similarly, points in the region $F < 0$, $H < 0$, with F sufficiently small, will by taking negatives of the vectors in $\mathbf{R}^3$ also correspond to points in W of the type desired.

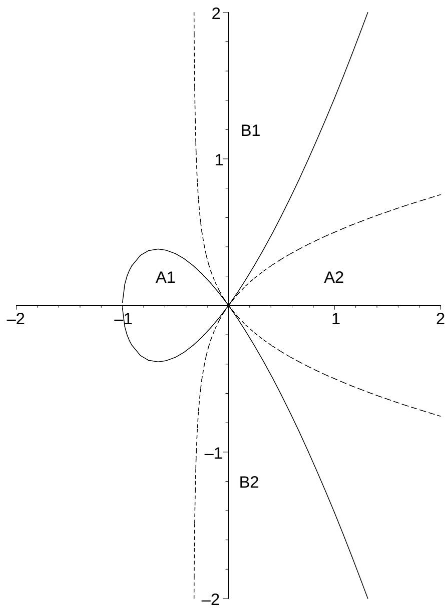
Let us take a specific cubic for illustrative purposes, namely the real nodal cubic $F(X_0, X_1, X_2) = X_1^3 + X_0X_1^2 - X_0X_2^2$, or in affine coordinates $x^3 + x^2 - y^2$. We remark that it is not known whether this nodal cubic does occur for a Kähler threefold [14]. By [16], we check that the invariant $S = 1/81$. The affine form of the Hessian is checked easily to be $8x^2 - 8y^2 - 24xy^2$, whose zero locus is given by the equation $y^2 = x^2/(3x+1)$. The two curves are illustrated in the Figure, the solid line denoting the nodal cubic and the broken line its Hessian curve.

We are looking for points for which the signature (with the above convention) is either $(+,-,-)$ or $(-,+,+)$, and these fall in the four regions of the plane shown (bounded by real branches of the two curves): for regions A1 and A2 the signature is $(+,-,-)$, and for regions B1 and B2 it is $(-,+,+)$.

We now impose the condition that the curvature is non-positive at the corresponding point of W_1; the criterion for this given by (5.1) is invariant on passing between a point and its negative in $\mathbf{R}^3$. Thus by (5.1), the curvature condition may be rephrased as the polynomial

$$(x^3 + x^2 - y^2)^2 - (x^2 - y^2 - 3xy^2)^2 = x\big(x^5 + 2(x^2 - y^2)(3y^2 + x^2) - 9xy^4\big)$$

being non-positive. An elementary calculation shows the real part of the zero locus of this polynomial consists of the y-axis, together with the curve $y^2 = x^2(x+2)/(2+3x)$. From



The nodal cubic and its Hessian

this, it is a routine check that the points (x, y) which satisfy both the index and curvature conditions fall in three regions, whose interiors are as follows:

(a) The bounded open set A1 of points with negative x-coordinate for which the cubic is positive.

(b) The two open subsets of B1 and B2 consisting of points (x, y) in the plane with positive x-coordinate and for which the cubic is negative.

We should however also take into account the linear form given by p_1; recall that for a complex manifold X, we have $p_1 = c_1^2 - 2c_2$, with c_1, c_2 denoting the Chern classes. Let us take the linear form p_1 on $\mathbf{R}^3$ to be a negative integral multiple of X_1; note that its zero locus corresponds to the y-axis in the affine plane, and therefore separates the open set in (a) from the open sets in (b). In the list of invariants given in (1.1) of [14], we may take $w_2 = 0$, $\tau = 0$, and b_3 an arbitrary (even) positive integer. By taking the cubic form to be a suitable positive integral multiple of $X_1^3 + X_0X_1^2 - X_0X_2^2$, and similarly with the linear form p_1, we may ensure that the congruency relations for these invariants to be represented by a simply connected smooth 6-manifold with torsion-free homology are automatically satisfied — for details of this, see [14]. The smooth manifold X is then uniquely determined. We can ask whether it admits any Calabi–Yau structures.

Proposition 5.3. *If the conjectured semi-negativity of the Kähler moduli curvature for*

Calabi–Yau threefolds is true, then the smooth 6-manifolds X determined by the above choices of invariants (with b_3 arbitrary) carry no Calabi–Yau structures.

Proof. Suppose X did admit such a Calabi–Yau structure. By Lemma 4 of [14], $h^{2,0} = 0$ (since otherwise the cubic curve consists of either a conic with a transversal line, or three non-concurrent lines), and so X is projective. Given that c_1 is trivial, the linear form on second cohomology defined by c_2 is the same as that defined by $-\frac{1}{2}p_1$. Since X is minimal, a standard result [13] ensures that c_2 is non-negative on any Kähler class, and hence p_1 will be non-positive. This contradicts our choice of p_1, which was chosen so as to be strictly positive on every possible Kähler class allowed by the conjectured semi-negativity of the Kähler moduli curvature.

Remark. We chose the nodal cubic for the above example because the equations turn out to be relatively simple. A slightly more complicated example would be to take the cubic form $x^3 - x + 1 - y^2$, with $S = 1/27$, corresponding to a smooth elliptic curve. Again, one finds that there are three open subsets in the plane that could correspond to points of $\mathcal{K}_1$ where the curvature was non-positive. The (rational) line through the three real inflexion points of the elliptic curve separates one of these subsets (on which both the cubic and its Hessian are positive) from the other two (on which both the cubic and its Hessian are negative). The same argument as above then goes through.

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