NOTES ON EXTENSIONS OF HOPF ALGEBRAS

Nicolás Andruskiewitsch

FAMAF
Medina Allende y Haya de la Torre
5000 Ciudad Universitaria
Córdoba

Argentina

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

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NICOLÁS ANDRUSKIEWITSCH

FAMAF. Medina Allende y Haya de la Torre.
5000 Ciudad Universitaria. Córdoba. Argentina
e-mail: andrus@mpim-bonn.mpg.de,
atina!smimaf!andrus@uunet.UU.NET

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§0. INTRODUCTION

This paper, a sequel to [AD], contains examples, applications and further developments of the notion of exact sequences of Hopf algebras. In §1, we recall notations and facts on quantum groups and on the definition of exact sequences. In §2, we prove that any Hopf algebra has a maximal central Hopf subalgebra, which shall be called the Hopf center. §3.1 is devoted to recall the construction of extensions from cohomological data ([Mj], [AD]) and to the notion of cleftness ([AD], [By], [Sch2]); in §3.2 we discuss cleft extensions of $*$-Hopf algebras. §3.3 contains several basic results on quotient theory of Hopf algebras. The Frobenius morphism defined by Lusztig from the quantized enveloping algebra at an odd root of unity to the usual enveloping algebra and its dual version give rise to exact sequences of Hopf algebras. This was asserted by some authors but the rigourous verification, contained in §3.4, follows from the present definition and a result of Schneider on faithful flatness of the inclusion of a finite dimensional normal Hopf algebra. It follows from previous work of Schneider (for extensions of algebras by a Hopf algebras) that any extension of finite dimensional Hopf algebras is cleft. Therefore, one sees that to classify the Hopf algebras up to certain finite dimension, one needs to classify the simple ones (in the sense of Hopf algebras, i.e. without normal Hopf subalgebras) and then to construct inductively the non-simple ones by the extension method. As for the first task, we prove in §4 that Taft's finite dimensional Hopf algebras [Tf] are simple (these Hopf algebras also appear as the $+$–part of the Frobenius–Lusztig kernels for $SL(2)$; I do not know if the $+$–part of a general Frobenius–Lusztig kernel is simple). I further conjecture that the Frobenius–Lusztig kernels are simple. (This was proved in [T4] for type A). This would imply, in particular, that the inclusion of the algebra of regular functions on a semisimple algebraic group in its quantum analogue at a root of 1 as the Hopf center of the later.

Key words and phrases. Quantum Groups, Hopf Algebras.

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As for the second task, I begin to analize the cohomological meaning of the construction above. If A is commutative and B cocommutative, one has a nice cohomology theory by taking the total complex associated to a certain double complex. This idea goes back to Singer who worked out the graded case; the translation to our setting offers no difficulty (§5.1, see [Hf]). The double complex arises because one has a cocycle for the algebra structure, another for the coalgebra structure and a compatibility condition for them. By a spectral sequence argument, we prove in §5.2 that Hopf algebras of dimension pq, with p and q primes, $p \leq q$, p and $q-1$ coprimes, are commutative and cocommutative, provided they are not simple (compare with [H, p. 57]). For A and B general, the interpretation of the extensions in cohomological terms is a "double" version of the non-abelian group cohomology problem.

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§1. Preliminaries

§1.0 Notations and conventions. Our main reference for the general theory of Hopf algebras is [Sw]. Let us fix a commutative field $\urcorner$; then "Hopf algebra" will mean Hopf algebra over $\urcorner$, unless an explicit mention. We shall use the following notation: m, Δ (or δ), $\varepsilon, \mathcal{S}$ denote respectively the multiplication, comultiplication, counit, antipode of a Hopf algebra (or an algebra or a coalgebra), specified with a subscript if necessary. The opposite multiplication or comultiplication are indicated by a superscript "op"., resp. "cop". We shall also use the following convention: if c is an element of a tensor product $A \otimes B$, then we write $c = c_i \otimes c^i$, omitting the summation symbol. An exception is the case $c = \Delta(x)$, where we use Sweedler's "sigma" notation but dropping again the summatory. We also denote by Δ^n the n-iteration of the comultiplication, e.g. $\Delta^2 = (\Delta \otimes \mathrm{id})\Delta$. The kernel of the counit of a Hopf algebra A is denoted by A^+.

We shall abbreviate "finite Hopf algebra" for a finite dimensional one.

Let A be an algebra, B a coalgebra. We shall always consider the algebra structure in $\mathrm{Hom}(B, A)$ given by the convolution product $f * g(b) = f(b_{(1)})g(b_{(2)})$ [Sw], unless explicitly stated. (An exception: $\mathcal{S}^{-1}$ will denote the inverse of the antipode—always assumed bijective—for the composition). The group of invertible elements will be denoted by $\mathrm{Reg}(B, A)$; its unit will be sometimes denoted by 0. Suppose in addition that A and B are Hopf algebras; then we denote $\mathrm{Reg}_1(B, A) = \{\phi \in \mathrm{Reg}(B, A) : \phi(1) = 1\}$, $\mathrm{Reg}_\varepsilon(B, A) = \{\phi \in \mathrm{Reg}(B, A) : \varepsilon\phi = \varepsilon\}$, $\mathrm{Reg}_{1,\varepsilon}(B, A) = \mathrm{Reg}_1(B, A) \cap \mathrm{Reg}_\varepsilon(B, A)$; these are subgroups of $\mathrm{Reg}(B, A)$.

The left (resp., right) adjoint action of a Hopf algebra on itself is $\mathrm{Ad}(b)a = b_{(1)}a\mathcal{S}(b_{(2)})$ (resp., $\mathrm{Ad}_r(b)a = \mathcal{S}(b_{(1)})ab_{(2)}$); the right (resp., left) adjoint coaction is $\mathrm{ad}(a) = a_{(2)} \otimes \mathcal{S}(a_{(1)})a_{(3)}$ (resp., $\mathrm{ad}_l(a) = a_{(2)} \otimes a_{(1)}\mathcal{S}(a_{(3)})$).

§1.1 Quantized enveloping algebras. In this subsection, we recall the construction of some algebras related to quantum groups at roots of 1.

Let q be an indeterminate. Given $r, s, d \in \mathbb{N}_0$, we denote (as usual)

$$[r]_d = \frac{q^{dr} - q^{-dr}}{q^d - q^{-d}}, \qquad [r]!_d = \prod_{1 \leq h \leq r} [h]_d, \qquad \begin{bmatrix} r+s \\ r \end{bmatrix}_d = \frac{[r+s]!_d}{[r]!_d [s]!_d}.$$

Let g be a complex simple finite dimensional Lie algebra of rank n; let $A = (a_{ij})$ be the corresponding Cartan matrix. There exists a diagonal matrix $D = (d_1, \ldots, d_n) \in \mathbb{Z}^{n\times n}$ such that $DA = AD$ and $\det D \neq 0$. We shall further assume that the d_i's are positive and relatively prime.

Let $\overline{\mathcal{U}}_{\mathbb{Q}}$ be the universal enveloping algebra of the split $\mathbb{Q}$-form of g. It is well-known [K] that $\overline{\mathcal{U}}_{\mathbb{Q}}$ has a $\mathbb{Z}$-form $\overline{\mathcal{U}}_{\mathbb{Z}}$ (of the Hopf algebra structure). One can therefore consider $\overline{\mathcal{U}}_R = \overline{\mathcal{U}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$, for any ring R. Let e_i, f_i, h_i, be the usual generators of g; let $e_i^{(r)} = \frac{e_i^r}{r!}$, $f_i^{(r)} = \frac{f_i^r}{r!}$, $\binom{h_i}{r} = \frac{h_i(h_i-1)\ldots(h_i-r+1)}{r!}$, $r \in \mathbb{N}_0$. These will be also the notations for their images in $\overline{\mathcal{U}}_R$. The subalgebra spanned by the $e_i^{(r)}$ (resp., $f_i^{(r)}$, $\binom{h_i}{r}$) will be denoted $\overline{\mathcal{U}}_{R,+}$ (resp., $\overline{\mathcal{U}}_{R,-}$, $\overline{\mathcal{U}}_{R,0}$). It is known [K] that the following elements form a basis of $\overline{\mathcal{U}}_R$:

$$e_{\beta_1}^{(a_1)} \ldots e_{\beta_N}^{(a_N)} \prod_{1\le i\le n} \binom{h_i}{b_i} f_{\beta_N}^{(c_N)} \ldots f_{\beta_1}^{(c_1)},$$

where the a_i's, b_j's and c_k's are non-negative integers. Here $\{e_{\alpha_1}, \ldots e_{\alpha_N}\} \cup \{h_i\} \cup \{f_{\alpha_1}, \ldots f_{\alpha_N}\}$ is a Chevalley basis of g.

Let P (resp. $Q^\vee$) be the free abelian group with basis ω_i (resp. $\alpha_i^\vee$), $1 \le i \le n$. Let $\langle , \rangle : P \times Q^\vee \to \mathbb{Z}$ be the bilinear pairing defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let $\alpha_j \in P$ be defined by $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ and let Q (resp. Q^+) be the subgroup (resp. the subsemigroup) of P generated by $\alpha_1, \ldots, \alpha_n$.

Let s_i be the linear automorphism of P defined by $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$; let $\mathcal{W}$ be the subgroup of $GL(P)$ generated by the s_i's. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $R = \mathcal{W}\Pi$ (the set of roots) and $R^+ = R \cap Q^+$ (the positive roots).

For $i \neq j$, let $m_{ij} = 2, 3, 4, 6$ whenever $a_{ij}a_{ji} = 0, 1, 2, 3$. Let $\mathcal{B}$ be the group presented by generators T_i and relations

$$T_iT_jT_i \cdots = T_jT_iT_j \ldots, \qquad i \neq j$$

with m_{ij} factors in each member of the equality. It is well-known that $T_i \mapsto s_i$ defines an epimorphism from the braid group $\mathcal{B}$ onto the Weyl group $\mathcal{W}$.

Let $(|) : P \times Q \to \mathbb{Z}$ be the symmetric bilinear non-degenerate form defined by $(\omega_i|\alpha_j) = d_i\delta_{ij}$; we have $(\alpha_i|\alpha_j) = d_ia_{ij} = d_ja_{ji}$; $(|)$ is $\mathcal{W}$-invariant.

Definition 1.1.1. ([Dr], [J], see also [L5], [dCKP]). The *simply connected* quantized enveloping algebra U_P is the associative $\mathbb{Q}(q)$-algebra given by generators E_i, F_i, L_i, $L_i{}^{-1}$ and relations

$$L_iL_i{}^{-1} = L_i{}^{-1}L_i = 1, \quad L_iL_j = L_jL_i,$$
$$L_iE_j = q^{d_i\delta_{ij}}E_jL_i, \quad L_iF_j = q^{-d_i\delta_{ij}}F_jL_i,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},$$

and if $i \neq j$

$$\sum_{h+\ell=1-a_{ij}} (-1)^h E_i^{(\ell)}E_jE_i^{(h)} = 0, \qquad \sum_{h+\ell=1-a_{ij}} (-1)^h F_i^{(\ell)}F_jF_i^{(h)} = 0.$$

Here $E_i^{(h)}$ denotes E_i^h divided by $[h]!_{d_i}$ (idem for $F_i^{(h)}$); furthermore, for $\alpha = \sum m_i\omega_i \in P$, one denotes $K_\beta = \prod_i L_i{}^{m_i}$, and $K_i = K_{\alpha_i}$.

For any lattice M, $P \supseteq M \supseteq Q$, U_M denotes the $\mathbb{Q}(q)$-subalgebra of U_P generated by E_i, F_i and K_β ($1 \leq i \leq n$, $\beta \in M$). Thus $U = U_Q$ is the adjoint quantized enveloping algebra, as first defined by Drinfeld and Jimbo. Let U_+ (resp., U_-, $U_{M,0}$) be the $\mathbb{Q}(q)$-subalgebra of U_P generated by E_i, $1 \leq i \leq n$ (resp., by the F_i's, by the K_β for $\beta \in M$).

It is well-known that U_P (and *a fortiori* any U_M) carries a Hopf algebra structure, with comultiplication Δ, antipode S and counit ϵ defined by

$$\begin{array}{lll} \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i & \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i & \Delta L_i = L_i \otimes L_i \\ S(E_i) = -K_i^{-1}E_i & S(F_i) = -F_iK_i & S(L_i) = L_i{}^{-1} \\ \epsilon(E_i) = 0 & \epsilon(F_i) = 0 & \epsilon(L_i) = 1. \end{array}$$

Now we recall the action of $\mathcal{B}$ by algebra automorphisms on U_M defined in [L1]. Denoting still by T_i its image in $\mathrm{Aut}(U_M)$, one has

$$T_i(K_\beta) = K_{s_i(\beta)}, \qquad T_i(E_i) = -F_iK_i, \qquad T_i(F_i) = -K_i{}^{-1}E_i,$$

and if $i \neq j$

$$T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r q^{-d_is}E_i^{(r)}E_jE_i^{(s)}, \quad T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r q^{d_is}F_i^{(s)}F_jF_i^{(r)}.$$

Fix a reduced expression $w_0 = s_{i_1} \dots s_{i_N}$ of the longest element of $\mathcal{W}$; thus N is the number of positive roots. Then one has an ordering $\{\beta_1, \dots, \beta_N\}$ of R^+ by setting $\beta_t = s_{i_1} \dots s_{i_{t-1}}(\alpha_{i_t})$. Let $E_\beta = T_{i_1} \dots T_{i_{t-1}}(E_{i_t})$, $F_\beta = T_{i_1} \dots T_{i_{t-1}}(F_{i_t})$. Let us also denote $E_\beta^{(h)} = T_{i_1} \dots T_{i_{t-1}}(E_{i_t}^{(h)})$.

Now let us introduce some $\mathbb{Z}[q,q^{-1}]$-forms of U_M. First we present a form due to Lusztig. Let $\begin{bmatrix} K_i \\ h \end{bmatrix} = \begin{bmatrix} K_i;0 \\ h \end{bmatrix}$, where

$$\begin{bmatrix} K_i;c \\ h \end{bmatrix} = \prod_{1\le s\le h} \frac{q^{d_i(c+1-s)}K_i - q^{-d_i(c+1-s)}K_i^{-1}}{q^{d_is}-q^{-d_is}}.$$

Definition 1.1.2. (cf. [L1]). Let $\mathcal{U}$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U = U_Q$ generated by $E_i^{(h)}$, $F_i^{(h)}$, $K_i^{\pm 1}$, $1\le i\le n$, $h\in\mathbb{N}_0$. It is known that

(1) $\mathcal{U}$ is a Hopf subalgebra of U. Precisely (cf. [L5], [DeCL]),

$$\Delta(E_i^{(h)}) = \sum_{j=0}^{h} q^{d_ij(h-j)}E_i^{(h-j)}K_i^j\otimes E_i^{(j)},$$

$$\Delta(F_i) = \sum_{j=0}^{h} q^{-d_ij(h-j)}F_i^{(j)}\otimes K_i^{-j}F_i^{(h-j)}, \tag{1.1.3}$$

$$\Delta\left(\begin{bmatrix} K_i \\ h \end{bmatrix}\right) = \sum_{j=0}^{h} K_i^{-j}\begin{bmatrix} K_i \\ h-j \end{bmatrix}\otimes K_i^{h-j}\begin{bmatrix} K_i \\ j \end{bmatrix}.$$

(2) $\mathcal{U}$ is a form of U, i.e. $\mathcal{U}\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Q}(q)\simeq U$.

(3) The elements

$$E_{\beta_1}^{(h_1)}\dots E_{\beta_N}^{(h_N)}\prod_{1\le i\le n}(K_i^{\delta_i}\begin{bmatrix} K_i \\ t_i \end{bmatrix})\, F_{\beta_N}^{(\ell_N)}\dots F_{\beta_1}^{(\ell_1)},$$

with $h_1,\dots,h_N,t_1,\dots,t_n,\ell_1,\dots,\ell_N\in\mathbb{N}_0$, $\delta_i = 0$ or 1, form a basis of $\mathcal{U}$.

We recall now some more notation, this time from [DeCP]:

$$\overline{E}_i = (q^{(\alpha_i|\alpha_i)/2} - q^{-(\alpha_i|\alpha_i)/2})E_i;\quad \overline{F}_i = (q^{(\alpha_i|\alpha_i)/2} - q^{-(\alpha_i|\alpha_i)/2})F_i;$$

and more generally,

$$\overline{E}_{\beta_t} = (q^{(\beta_t|\beta_t)/2} - q^{-(\beta_t|\beta_t)/2})E_{\beta_t};\quad \overline{F}_{\beta_t} = (q^{(\beta_t|\beta_t)/2} - q^{-(\beta_t|\beta_t)/2})F_{\beta_t}.$$

Definition 1.1.4. (cf. [DeCP]). Let $\mathcal{A}_M$ be the smallest B-stable $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_M generated by $\overline{E}_i$, $\overline{F}_i$ and K_β, $1\le i\le n$, $\beta\in M$. It is known that

(1) $\mathcal{A}_M$ is a Hopf subalgebra of U_M.

(2) $\mathcal{A}_M$ is a form of U_M, i.e. $\mathcal{U}_M\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Q}(q)\simeq U_M$.

(3) The elements

$$\overline{E}_{\beta_1}^{h_1}\dots\overline{E}_{\beta_N}^{h_N}K_\beta\overline{F}_{\beta_N}^{j_N}\dots\overline{F}_{\beta_1}^{j_1},$$

with $h_1,\dots,h_N,j_1,\dots,j_N\in\mathbb{N}_0$, $\beta\in M$, form a basis of $\mathcal{A}_M$.

We shall abbreviate $\mathcal{A} = \mathcal{A}_P$.

Let R be a ring, $v \in R$ an invertible element, and $\mathbb{Z}[q, q^{-1}] \to R$ the ring homomorphism sending $q \mapsto v$. The Hopf algebra over R obtained by extensions of scalars $\mathcal{U} \otimes_{\mathbb{Z}[q,q^{-1}]} R$ is denoted by $\mathcal{U}_R$. In the same vein, $\mathcal{A}_R$ denotes $\mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} R$. The main example we are interested in is the cyclotomic field $\mathbf{B} = \mathbb{Q}(v)$, where v is a primitive ℓ-root of unity (ℓ is odd and greather than 3). We shall denote by the same letters the images of E, etc. in the respective specializations. $\mathcal{A}_{\mathbf{B}}$ has a central Hopf subalgebra Z_0 generated by the monomials $\overline{E}_{\beta_1}^{\ell h_1} \dots \overline{E}_{\beta_N}^{\ell h_N} K_{\ell\beta} \overline{F}_{\beta_N}^{\ell j_N} \dots \overline{F}_{\beta_1}^{\ell j_1}$ (see [DeCP]), which is in fact isomorphic to the algebra of functions on the group dual (in the sense of Drinfeld) to the group corresponding to g. This algebra plays an important role in representation theory of quantum groups at roots of 1, see [DeCP], [DeCKP].

§1.2 Exact sequences. We recall here the definition of short exact sequences given in [AD], [Sch]. Consider a sequence of morphisms of Hopf algebras

$$(\mathcal{C}) \qquad 0 \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow 0$$

where 0 denotes the trivial Hopf algebra $\mathbb{k}$. We shall say that $(\mathcal{C})$ is exact if and only if the following conditions hold

(1.2.1) ι is injective. (Identify in such case A with its image.)
(1.2.2) π is surjective.
(1.2.3) $\ker \pi = C\iota(A)^+$.
(1.2.4) $\iota(A) = \mathrm{LKer}(\pi) = \{x \in C : (\pi \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}$.

Either (1.2.3) or (1.2.4) imply $\pi\iota = \varepsilon_A 1_B$ (the trivial morphism of Hopf algebras). Moreover, if $A \xrightarrow{\iota} C$ (resp., $C \xrightarrow{\pi} B$) is faithfully flat and $\iota(A)$ is stable by the adjoint actions (resp., faithfully coflat and B is stable by the adjoint coactions), then (1.2.1), (1.2.2), (1.2.3) imply (1.2.4) (resp., (1.2.1), (1.2.2), (1.2.4) imply (1.2.3)) and π is faithfully coflat (resp., ι is faithfully flat) (see [AD], [Sch]). Notice that $\iota(A) = \mathrm{LKer}(\pi)$ implies $\iota(A) = \mathrm{RKer}(\pi)$ and hence A is stable for both adjoint actions; (1.2.3) implies the dual statement.

One says that a Hopf subalgebra $A \hookrightarrow C$ (resp., a quotient Hopf algebra $C \twoheadrightarrow B$) is *strongly normal* (resp., *strongly conormal*) if A is stable for both adjoint actions (resp., B is a quotient comodule for the left and right adjoint coactions).

The preceding notion of short exact sequence is supported by the following more general definition. A sequence $A \xrightarrow{\iota} C \xrightarrow{\pi} B$ is exact if and only if (1.2.3), (1.2.4), (1.2.5) and (1.2.6) hold, where

(1.2.5) $\ker \iota \subseteq A(\mathrm{HKer}\, \iota)^+ A$.
(1.2.6) $\mathrm{HKer}(\mathrm{HCoker}\, \pi) \subseteq \pi(C)$.

Here, for a morphism of Hopf algebras $X \xrightarrow{f} Y$, one denotes

$$\mathrm{HKer}(f) = \{x \in X : (\mathrm{id} \otimes f \otimes \mathrm{id})\Delta^2(x) = x_{(1)} \otimes 1 \otimes x_{(2)}\},$$
$$\mathrm{HCoker}(f) = Y/Yf(X^+)Y, \qquad \mathrm{LCoker}(f) = Y/Yf(X^+).$$

This definition is stronger than the given in [AD] because it also requires π to have a Hopf image. A long sequence $\dots \to A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \to \dots$ is exact if and only if each "piece" $A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2}$ is. Thus for infinite sequences exactness means the same with respect to the preceding definition or to [AD]. Observe also that $\dots \to A_i \xrightarrow{f_i} A_{i+1} \to \dots$ is exact if and only if both $\dots \to A_i \to V \to 0$ and $0 \to V \to A_{i+1} \to \dots$ are exact, where $V = \operatorname{Im} f_i = \operatorname{HKer} f_{i+1}$.

Lemma 1.2.7. *(i) The sequence*

$$(\mathcal{C}_{left}) \qquad 0 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B$$

is exact if and only if (1.2.1), (1.2.3), (1.2.4), (1.2.6) hold.

(ii) The sequence

$$(\mathcal{C}_{right}) \qquad A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 0$$

is exact if and only if (1.2.2), (1.2.3), (1.2.4), (1.2.5) hold.

Proof. *(i)* is left to the reader.

(ii). $C \xrightarrow{\pi} B \to 0$ is exact if and only if if and only if π is surjective and $\ker \pi \subseteq C(\operatorname{HKer} \pi)^+ C$. Assume that $(\mathcal{C}_{right})$ is exact; then $\ker \pi \subseteq C\iota(A)^+ C \subseteq C\iota(A)^+$ and the other inclusion holds because $\iota(A)^+ \subseteq \ker \pi$. Conversely, assume (1.2.2, 3, 4, 5). As $\ker \pi$ is a two-sided ideal, $A \xrightarrow{\iota} C \xrightarrow{\pi} B$ is exact; by (1.2.3, 5) $C \xrightarrow{\pi} B \to 0$ also is. $\square$

We collect a number of results about faithful (co)flatness due to Nichols-Zoeller and Schneider.

Theorem 1.2.8. *(i) ([NZ]) Let $B \hookrightarrow H$ be an inclusion of finite Hopf algebras. Then every left (H, B)-Hopf module is free as left module over B. In particular, H is free over B.*

(Recall that a (H, B)-Hopf module M is a left B-module and a left H-comodule such that the coaction $M \to H \otimes M$ is a morphism of B-modules.)

(ii) ([Sch, 3.3]) Noetherian Hopf algebras are faithfully flat over its central Hopf subalgebras.

(iii) ([Sch, 2.1(2)]) Hopf algebras are free over finite strongly normal Hopf subalgebras.

(iv) ([Sch, 2.1(1)]) Hopf algebras are faithfully coflat over its finite strongly conormal quotient Hopf algebras.

§2. Invariants

§2.1. Let A be an algebra, B a Hopf algebra. Recall that a weak action of B on A is a morphism of vector spaces $\rightharpoonup: B \otimes A \to A$, $b \otimes a \mapsto h \rightharpoonup a$, satisfying

(2.1.1) $b \rightharpoonup a\tilde{a} = (b_{(1)} \rightharpoonup a)(b_{(2)} \rightharpoonup \tilde{a})$,
(2.1.2) $b \rightharpoonup 1 = \varepsilon(b)1$,
(2.1.3) $1 \rightharpoonup a = a$.

We shall say that it is an action if in addition it satisfies the module axioms (no confusion should arise between actions on modules and actions on algebras, the latter including in addition the axioms (2.1.1,2)).

The significance of (2.1.1), (2.1.2) is the following [Sw]: Let $\Theta : A \to \operatorname{Hom}(B, A)$ be defined by

$$\Theta(a)(b) = b \rightharpoonup a. \tag{2.1.4}$$

Then Θ is a morphism of unital algebras if and only if (2.1.1), (2.1.2) hold.

Now suppose that A is also a Hopf algebra. One is naturally led to consider some Hopf algebra structure on $\operatorname{Hom}(B, A)$. Let $\Delta : \operatorname{Hom}(B, A) \to \operatorname{Hom}(B \otimes B, A \otimes A)$ be defined by

$$\Delta(f)(b \otimes \tilde{b}) = \Delta\left(f(b\tilde{b})\right). \tag{2.1.5}$$

$\operatorname{Hom}(B, A)$ is a complete topological algebra with respect to the topology defined by the annihilators of finite dimensional subspaces of B [T1]. This topology will be called the finite topology. Moreover, $\operatorname{Hom}(B \otimes B, A \otimes A)$ is the completion of $\operatorname{Hom}(B, A) \otimes \operatorname{Hom}(B, A)$ with respect to the product topology. It is not difficult to see that (2.1.5) provides $\operatorname{Hom}(B, A)$ a structure of topological Hopf algebra; the counit is given by

$$\langle \varepsilon_{\operatorname{Hom}(B,A)}, T \rangle = \langle \varepsilon_A, T(1_B) \rangle \tag{2.1.6}$$

and the antipode by

$$\mathcal{S}_{\operatorname{Hom}(B,A)}(T) = \mathcal{S}_A T \mathcal{S}_B. \tag{2.1.7}$$

However, Θ, defined by (2.1.4), is not, in general, a coalgebra morphism. This is, however, true if $\rightharpoonup$ is the trivial action $b \rightharpoonup a = \varepsilon(b)a$; the morphism $A \to \operatorname{Hom}(B, A)$ for the trivial action will be henceforth denoted by Υ.

In some circumstances, it is possible to twist the comultiplication of $\operatorname{Hom}(B, A)$ in order to have a morphism of Hopf algebras. Assume further the existence of an algebra C containing A as a subalgebra, and $\chi \in \operatorname{Reg}(B, C)$ such that

$$b \rightharpoonup a = \chi(b_{(1)}) a \chi^{-1}(b_{(2)}).$$

Define $\Delta_\chi : \operatorname{Hom}(B, A) \to \operatorname{Hom}(B \otimes B, C \otimes C)$ by

$$\begin{aligned} \Delta_\chi(T)(b \otimes d) &= \chi(b_{(1)}) \otimes \chi(d_{(1)}) \\ &\quad \Delta\left(\chi^{-1}(b_{(2)}d_{(2)}) f(b_{(3)}d_{(3)}) \chi(b_{(4)}d_{(4)})\right) \chi^{-1}(b_{(5)}) \otimes \chi^{-1}(d_{(5)}). \end{aligned}$$

Lemma 2.1.8. *Assume that* $\operatorname{Im}\Delta_\chi \subset \operatorname{Hom}(B \otimes B, A \otimes A)$. *Then* $(\operatorname{Hom}(B; A), \Delta_\chi)$ *is a Hopf algebra, with counit given by (2.1.6) and antipode by*

$$\mathcal{S}_\chi(T)(b) = \chi(b_{(1)}) \mathcal{S}\left(\chi^{-1}(\mathcal{S}b_{(4)}) T(\mathcal{S}b_{(3)}) \chi(\mathcal{S}b_{(2)})\right) \chi^{-1}(b_{(5)}).$$

Furthermore, Θ *is a morphism of Hopf algebras.*

Proof. Left to the reader. □

§2.2 The Hopf algebra of invariants. We want to attach, to a Hopf algebra B acting on a Hopf algebra A as in the preceding subsection, a Hopf subalgebra A^B of A such that $b \rightharpoonup a = \varepsilon(b)a$ for all $b \in B$, $a \in A^B$, and maximal with this property. We can not take directly profit of Hopf equalizers (cf. [AD], [Sch]) because the Hopf algebra structures on $\operatorname{Hom}(B, A)$ making Θ and Υ morphisms of Hopf algebras are in general different.

Let A be a Hopf algebra, C an algebra, $f, g : A \to C$ two algebra morphisms. Define

$$\mathrm{Equal}(f,g) = \{a \in A : a_{(1)} \otimes f(a_{(2)}) \otimes a_{(3)} = a_{(1)} \otimes g(a_{(2)}) \otimes a_{(3)} \in A \otimes C \otimes A\};$$

it is not difficult to show that $\mathrm{Equal}(f,g)$ is a sub-bialgebra of A (since our base ring is a field). To obtain a Hopf subalgebra we use the following recipe (compare with [Ma], [A]).

Lemma 2.2.1. *Let A be a Hopf algebra, $E \subset A$ a sub-bialgebra. Then $\mathcal{S}^i E$ is again a sub-bialgebra, for any $i \in \mathbb{Z}$, and*

$$H(E) = \bigcap_{i \in \mathbb{Z}} \mathcal{S}^i E$$

is a Hopf subalgebra of A. If H is any Hopf subalgebra of A contained in E, then $H \subseteq H(E)$.

Proof. $\mathcal{S}^i E$ is a sub-bialgebra because $\mathcal{S}$ is antimultiplicative and anticomultiplicative. Therefore $H(E)$ is a bialgebra (cf. [Sw, p. 45]) and clearly $\mathcal{S}(H(E)) = H(E)$; thus it is a Hopf subalgebra. The rest is obvious. □

Here one takes negative powers of the antipode because of the convention on the bijectivity of the antipode.

Corollary 2.2.2. *Let* $\mathrm{HEqual}(f,g) = H(\mathrm{Equal}(f,g))$, *a Hopf subalgebra of A. If H is any Hopf subalgebra of A such that $f(x) = g(x)$ for any $x \in H$, then $H \subseteq$* $\mathrm{HEqual}(f,g)$ *(and clearly* $\mathrm{HEqual}(f,g)$ *satisfies this property).* □

Let B act weakly on A as above and denote

$$A^{(B)} = \mathrm{Equal}(\Theta, \Upsilon), \qquad A^B = \mathrm{HEqual}(\Theta, \Upsilon).$$

Then $A^{(B)}$ (resp., A^B) is the maximal sub-bialgebra (resp., Hopf subalgebra) of A among those whose elements a satisfy $b \rightharpoonup a = \varepsilon(b)a$ for all $b \in B$. We shall say that A^B is the Hopf algebra of invariants (of the weak action of B on A).

Definition 2.2.3. Let $B = A$ act on itself by the adjoint (cf §1.0). In this case, $A^{(A)} = A^A$ will be called the Hopf center of A. It is the maximal central Hopf subalgebra of A.

Proof of the equality. Observe first that $A^{(A)}$ contains any coalgebra consisting of central elements. For, if H is such a coalgebra and $a \in H$ then

$$a_{(1)} \otimes \Theta(a_{(2)})(h) \otimes a_{(3)} = a_{(1)} \otimes h_{(1)} a_{(2)} \mathcal{S}(h_{(2)}) \otimes a_{(3)} = a_{(1)} \otimes \varepsilon(h) a_{(2)} \otimes a_{(3)}.$$

The elements of $A^{(A)}$ are central, and *a fortiori* those of $\mathcal{S}(A^{(A)})$: if $a \in A^{(A)}$ and $h \in A$ then $ha = h_{(1)} a \mathcal{S}(h_{(2)}) h_{(3)} = ah$. Thus $A^{(A)} = A^A$. □

Example. Let A be either the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ or the group algebra of a group G, and suppose that char $\urcorner = 0$. Then from the fundamental theorem of cocommutative Hopf algebras, we infer that its Hopf center is either the universal enveloping algebra of the center of $\mathfrak{g}$, or respectively the group algebra of the center of G.

§2.3 The Hopf algebra of covariants. Now we pass to the dual version of the material presented above.

Let A be a Hopf algebra, B a coalgebra. A linear map $\rho : B \to B \otimes A$ is a weak coaction if

(2.3.1) $(\delta \otimes \mathrm{id})\rho = m^{24}(\rho \otimes \rho)\delta$, where $m^{24} : B \otimes A \otimes B \otimes A \to B \otimes B \otimes A$ is the map $c \otimes h \otimes d \otimes k \mapsto c \otimes d \otimes hk$.

(2.3.2) $(\varepsilon_B \otimes \mathrm{id})\rho = \varepsilon_B \otimes 1$.

(2.3.3) $(\mathrm{id} \otimes \varepsilon_A)\rho = \mathrm{id}_B$.

Again, we shall say that it is a coaction if in addition it satisfies the comodule axioms.

Let us consider $\mathrm{Hom}(A, B)$ with the coalgebra structure given by (2.1.5). Let $\mathrm{Hom}_{fin}(A, B)$ be the subspace of maps with finite rank; $\mathrm{Hom}_{fin}(A, B) \simeq A^* \otimes B$. Then $\Delta(\mathrm{Hom}_{fin}(A, B)) \subseteq \mathrm{Hom}_{fin}(A \otimes A, B \otimes B)$. Indeed, the image of $\Delta(f)$ is contained in $C \otimes C$ where C is the coalgebra generated by $\mathrm{Im}\, f$; and C is finite dimensional if $\mathrm{Im}\, f$ is [Sw, Cor. 2.2.2, p.47]. (Observe that in fact $\Delta(\alpha \otimes b)(x \otimes y) = \langle \alpha, xy \rangle b_{(1)} \otimes b_{(2)}$).

Now for any $\rho : B \to B \otimes A$, let $\Xi : \mathrm{Hom}_{fin}(A, B) \simeq A^* \otimes B \to B$ be the map defined by

$$\Xi(\alpha \otimes b) = \langle \mathrm{id} \otimes \alpha, \rho(b) \rangle.$$

Notice first of all that Ξ can not be extended in general to the whole of $\mathrm{Hom}(A, B)$. Take for example $A = B = \Bbbk\langle G \rangle$, the group algebra of an infinite group G, and $\rho : A \to A \otimes A$, $\rho(e_g) = e_g \otimes e_g$ (the usual comultiplication of A). Let $\mathrm{id} \in \mathrm{Hom}(A, A)$; then $\Xi(\mathrm{id})$ should be the formal sum $\sum_g e_g \notin \Bbbk\langle G \rangle$.

Lemma 2.3.4. *Ξ is comultiplicative if and only if ρ satisfies (2.3.1), and preserves the counit if and only if ρ satisfies (2.3.2).*

Proof. Left to the reader. □

When ρ is the trivial coaction ($\rho(b) = 1 \otimes b$), we shall denote η instead of Ξ. Then $\eta(\alpha \otimes b) = \langle \alpha, 1 \rangle b$.

Lemma 2.3.5. *Let B be a bialgebra, C a coalgebra, $f, g : C \to B$ two coalgebra maps. Let $\mathrm{Coeq}(f, g) = B/BJB$, where J is the image of $f - g$. Then $\mathrm{Coeq}(f, g)$ is a quotient bialgebra of B. Moreover, if $q : B \to D$ is any morphism of bialgebras such that $qf = qg$, then it factorizes through $\mathrm{Coeq}(f, g)$.*

Proof. Left to the reader. □

Lemma 2.3.6. *Let B be a Hopf algebra, $p : B \to C$ a quotient bialgebra and let I denote the kernel of p. Let $J = \sum_{n \in \mathbb{Z}} S^n I$ and $H(C) = B/J$. Then $H(C)$ is a quotient Hopf algebra of B. If $q : C \to H$ is a morphism of bialgebras to an arbitrary Hopf algebra H, then q factorizes through $H(C)$.*

Proof. Left to the reader (use [Sw, 4.0.4]). □

In particular, we denote $\mathrm{HCoeq}(f, g) = H(\mathrm{Coeq}(f, g))$.

Now let $\rho : B \to B \otimes A$ be a weak coaction of a Hopf algebra A on a Hopf algebra B and set

$$B_{(A)} = \mathrm{Coeq}(\Xi, \eta), \qquad B_A = \mathrm{HCoeq}(\Xi, \eta).$$

Let D be a quotient coalgebra of B. We say that D trivializes the coaction ρ if the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\rho} & B \otimes A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q\otimes \mathrm{id}} \\ D & \xrightarrow{\text{trivial}} & D \otimes A. \end{array}$$

Lemma 2.3.7. *$B_{(A)}$ (resp., B_A) is the minimal quotient bialgebra (resp., Hopf algebra) among those trivializing ρ.*

Proof. Let $b \in B$, $x = \rho(b)_i \otimes \rho(b)^i - 1 \otimes b$. Let (a_j) be a basis of A, (α^j) its dual basis; then $x = \sum_j x_j \otimes a_j$, where $x_j = \langle \mathrm{id} \otimes \alpha^j, x\rangle$. With this notation, D trivializes ρ if and only if $x \in \ker q \otimes A$ (for any $b \in B$), if and only if $x_j \in \ker q$ for all j, if and only if $\ker q \supseteq (\Xi - \eta)(\alpha^j \otimes b)$, for any j, $b \in B$. □

Definition 2.3.8. Let $\mathrm{ad} : A \to A \otimes A$ be the right adjoint coaction. Then $A_{(A)} = A_A$ will be called the Hopf cocenter of A.

Proof of the equality. (Compare with [AD, before Prop. 2.16]). Let us say that a quotient bialgebra $q : A \to C$ is cocentral if $q(a_{(1)}) \otimes a_{(2)} = q(a_{(2)}) \otimes a_{(1)}$ for any $a \in A$. We claim that $q : A \to C$ is cocentral if and only if trivializes the adjoint coaction, i.e. if and only if

$$q(a_{(2)}) \otimes \mathcal{S}(a_{(1)})a_{(3)} = q(a) \otimes 1.$$

For, let $x, y : A \to C \otimes A$ be the applications $x(c) = q(c) \otimes 1$, $y(c) = 1 \otimes c$; y is invertible with respect to the convolution product, and in fact $y^{-1}(c) = 1 \otimes \mathcal{S}c$. But "$D$ cocentral" is equivalent to $x * y = y * x$, whereas D trivializes the adjoint coaction if and only if $y^{-1} * x * y = x$. Thus, in particular, $A \to A_{(A)}$ is the minimal cocentral quotient bialgebra. Let $I = \ker A \to A_{(A)}$; then $A \to A/\mathcal{S}(I)$ is also a quotient bialgebra. Thus $\mathcal{S}(I) = I$ and $A_{(A)} = A_A$. □

Remark. More generally, the Hopf centralizer of a quotient Hopf algebra $p : A \to C$ is A_C, where $\rho = (1 \otimes p)\,\mathrm{ad} : A \to A \otimes C$. One still has $A_C = A_{(C)}$, with the same proof as above.

§**2.4 Hopf systems.** We introduce here a formalism inspired by the approach of [DeCKP], [DeCL], [DeCP] to representations of quantum groups at roots of 1. The contents of this subsection will be not used in the rest of the paper.

A Hopf system is a family of unital k-algebras $(A_g)_{g\in G}$, where G is a group with identity e, provided with morphisms of k-algebras

$$\begin{aligned} \delta_{g,h} &= A_{gh} \to A_g \otimes A_h, & g, h &\in G, \\ \varepsilon_e &= A_e \to k, & S_g &= A_g \to A^{op}_{g^{-1}}, \end{aligned}$$

subject to the axioms wich can be expresse by the commutativity of the following diagrams, for any $g, h, \ell \in G$

(2.4.1)
$$\begin{tikzcd} A_{gh\ell} \arrow[r, "\delta_{g,h\ell}"] \arrow[d, "\delta_{gh,\ell}"'] & A_g \otimes A_{h\ell} \arrow[d, "id\otimes\delta_{h,\ell}"] \\ A_{gh} \otimes A_\ell \arrow[r, "\delta_{g,h}\otimes id"] & A_g \otimes A_h \otimes A_\ell \end{tikzcd}$$

(2.4.2)
$$\begin{tikzcd} A_g \arrow[r, "\delta_{g,e}"] \arrow[d, equal] & A_g \otimes A_e \arrow[d, "id\otimes\varepsilon_e"] \\ A_g \arrow[r, "\sim"] & A_g \otimes k \end{tikzcd} \qquad \begin{tikzcd} A_g \arrow[r, "\delta_{e,g}"] \arrow[d, equal] & A_e \otimes A_g \arrow[d, "\varepsilon_e\otimes id"] \\ A_g \arrow[r, "\sim"] & k \otimes A_g \end{tikzcd}$$

(2.4.3)
$$\begin{tikzcd} A_e \arrow[r, "\varepsilon_e 1_g"] \arrow[d, "\delta_{g,g^{-1}}"'] & A_g \\ A_g \otimes A_{g^{-1}} \arrow[r, "id\otimes S_{g^{-1}}"] & A_g \otimes A_g \arrow[u, "m_g"'] \end{tikzcd} \qquad \begin{tikzcd} A_e \arrow[r, "\varepsilon_e 1_g"] \arrow[d, "\delta_{g^{-1},g}"'] & A_g \\ A_{g^{-1}} \otimes A_g \arrow[r, "S_{g^{-1}}\otimes id"] & A_g \otimes A_g \arrow[u, "m_g"'] \end{tikzcd}$$

Here, 1_g and m_g are, respectively, the unit and the multiplication of A_g. Remark that in particular A_e is a Hopf algebra and each A_g is an A_e-bi-comodule. On the other hand, a Hopf algebra is the same thing as a Hopf system over the trivial group. We will always assume that S_g is bijective, for every g.

Let us denote, for $p \in \mathbb{N}$,

$$\Gamma_p = \{\alpha : \underbrace{G \times \cdots \times G}_{p-times} \to \prod_{g_1,\dots g_p} A_{g_1} \otimes \cdots \otimes A_{g_p} : \alpha(g_1, \dots, g_p) \in A_{g_1} \otimes \cdots \otimes A_{g_p}\}$$

and $\Gamma = \Gamma_1$. When necessary, we will write $\Gamma_p(G, A_g)$ instead of Γ_p. Each Γ_p is a k-algebra with pointwise operations. The elements of Γ_p will be called sections. The support of a section α is, as always,the set $\{g \in G : \alpha(g) \neq 0\}$. Let $\Delta : \Gamma \to \Gamma_2, \varepsilon : \Gamma \to k, S : \Gamma \to \Gamma^{op}$, be the morphisms defined by

$$(\Delta\alpha)(g,h) = \delta_{g,h}(\alpha(gh)), \quad \varepsilon(\alpha) = \varepsilon_e(\alpha(e)), \quad S(\alpha)(g) = S_{g^{-1}}(\alpha(g^{-1})).$$

and let also Δ^{12}, $\Delta^{23} = \Gamma_2 \to \Gamma_3$ be given by

$$\Delta^{12}(\alpha)(g,h,\ell) = (\delta_{g,h} \otimes id)\alpha(gh,\ell), \quad \Delta^{23}(\alpha)(g,h,\ell) = (id \otimes \delta h, \ell\alpha(g, h\ell)$$

Then the axioms (2.4.1–3) imply

(2.4.4) $$\Delta^{12}\Delta = \Delta^{23}\Delta.$$

(2.4.5) $$(\mathrm{id} \otimes \varepsilon)\Gamma = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}_\Gamma,$$

(2.4.6)
$$\begin{tikzcd} \Gamma_2 \arrow[r, "id\otimes S"] & \Gamma_2 \arrow[d, "m"] & \Gamma_2 \arrow[r, "\S\otimes id"] & \Gamma_2 \arrow[d, "m"] \\ \Gamma \arrow[u, "\Delta"] \arrow[r, "\varepsilon 1"] & \Gamma & \Gamma \arrow[u, "\Delta"] \arrow[r, "\varepsilon 1"] & \Gamma \end{tikzcd}$$

Here $\varepsilon \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varepsilon$ denote the morphisms from Γ_2 to Γ given by $(\varepsilon \otimes \mathrm{id})(\alpha)(g) = (\varepsilon_e \otimes \mathrm{id})(\alpha(e, g))$, $(\mathrm{id} \otimes \varepsilon)(\alpha)(g) = (\mathrm{id} \otimes \varepsilon_e)(\alpha(g, e))$; 1, m are defined by $1(\lambda)(g) = \lambda 1g$, where $1g$ is the unity of A_g; $m\alpha(g) = m_g \alpha(g, g)$, where m_g is the multiplication of A_g; and $S \otimes \mathrm{id}$ (resp.,$\mathrm{id} \otimes S$) denotes the morphism given by $(S \otimes id)\alpha(g, h) = (S_{g^{-1}} \otimes id)\alpha(g^{-1}, h)$, (resp. $(id \otimes S)\alpha(g, h) = (id \otimes S_{h^{-1}}\alpha(g, h^{-1}))$.

Let $(A_g)_{g\in G}, (B_g)_{g\in G}$ be two Hopf systems. A morphism of Hopf systems $(A_g) \to (B_g)$ is a collection of morphisms of k-algebras $\psi_g : A_g \to B_g$ satisfying the natural compatibility requirements. Such morphism gives rise to algebra maps $\psi_p : \Gamma_p(G, A_g) \to \Gamma_p(G, B_g)$. Let $\psi_1 = \psi$; ψ verifies $\Delta'\psi = \psi_2\Delta, S'\psi = \psi S, \varepsilon'\psi = \psi\varepsilon$, with the same conventions as above.

On the other hand, $(A_g \otimes B_g)g \in G$ is also a Hopf system, and the category of Hopf systems over a fixed group G is monoidal.

Identify $\Gamma^{\otimes p}$ with its image in Γ_p under the monomorphism wich sends $\alpha_1 \otimes \cdots \otimes \alpha_p$ in the funtion $G \times \cdots \times G \to \prod A_{g_1} \otimes \cdots \otimes A_{g_p}, (g_1, \ldots, g_p) \longmapsto \alpha_1(g_1) \otimes \cdots \otimes \alpha_p(g_p)$. Sometimes it si possible to find subalgebras Γ_f of Γ such that $\Delta\Gamma_f \subseteq \Gamma_f \otimes \Gamma_f$, $s(\Gamma_f) = \Gamma_f$; axioms $(4), \ldots, (6)$ unguarantee that they are actually Hopf algebras.

Let X be a set, $(V_x)_{x\in X}$ a family of k vector spaces, and denote $\Gamma(X, V_x) = \{s : X \to \prod_x V_x, s(x) \in V_x\}$. Let $\delta_x v (v \in V_x, x \in X)$ denote the element of $\Gamma(X, (V_x))$ defined by $(\delta_x v)(y) = \delta_{x,y} v$. Let $(v_i)_{i\in I_x}$ be a basis of V_x. Then the family $(v_i \delta_x)_{x\in X, i\in I_x}$ is linealy independent and if X is finite, is a basis of the vector space $\Gamma = \Gamma(X, V_x)$. In particular, the natural application $\Gamma \otimes \Gamma \to \Gamma(X \times X, V_x \otimes V_y)$ is a bijection. It follows that for a Hopf system$(A_g)_{g\in G}$ over a finite group G, $\Gamma(G, A_g)$ is a Hopf algebra.

Conversely, let C be a Hopf algebra and let A be its Hopf center. Assume that A is the algebra of regular functions on an algebraic group G (this will be always the case under certain "finiteness" assumptions). Let $g \in G$, $\mathcal{M}_g$ the corresponding maximal ideal of A, I_g the two-sided ideal of C generated by $\phi(\mathcal{M}_g)$, and $A_g = C/I_g$. If h also belongs to G, there exists a morphism $\delta_{g,h} : A_{g,h} \to A_g \otimes A_h$ making commutative the following diagram:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d] & C \otimes C \arrow[d] \\ C/I_{gh} \arrow[r, "\delta_{g,h}"] & C/I_g \otimes C/I_h \end{tikzcd}$$

Let $B = A_e$; one has an exact sequence of Hopf algebras $0 \to A \to C \to B \to 0$ (use Schneider's theorem 1.2.8 (ii); so in the preceding one should assume C noetherian). Finally let $S_g : A_g \to A^{op}_{g^{-1}}$ be the morphism in the bottom horizontal arrow of the following commutative diagram, whose top horizontal arrow is the antipode of C:

$$\begin{array}{ccc} C & \longrightarrow & C \\ \downarrow & & \downarrow \\ C/I_g & \longrightarrow & C/I_{g^{-1}} \end{array}$$

The introduced morphisms of algebras $\delta_{g,h}$, ε_e, S_g satisfy the axiomas expressed by (2.4.1–3). There exists an algebra morphism $\sigma : C \to \Gamma(G, A_g)$ given by $\sigma(c)(g) =$ class of c in A_g.

Now assume further that C is a Poisson-Hopf algebra. Then A inherits the Poisson structure and therefore, G is a Poisson algebraic group. Indeed, if $x \in C$ and $z \in Z :=$ the center of C, then

$$\{x, z\}y = \{x, zy\} - z\{x, y\} = \{x, yz\} - \{x, y\}z = y\{x, z\}.$$

Consider the intersection T of all the Poisson subalgebras of Z containing A; this possible by the preceding computation. Then T is a Hopf subalgebra, by the following argument (taken from [DeP]): the algebra $U = \{t \in T : \Delta(t) \in T \otimes T\}$ contains A and is closed by the Poisson bracket, so it equals T; thus $T = U$. But by definition of Hopf center, $T = A$.

§3. Extensions of Hopf algebras

In this section, we pursue the study of extensions of Hopf algebras begun in [AD]; cf. §**1.2**.

§3.1 Construction of cleft extensions. Let A, B be two Hopf algebras. Let also be given a weak action $\rightharpoonup: B \otimes A \to A$ (cf. §*2.1*) and a weak coaction $\rho : B \to B \otimes A$ (cf. §*2.3*). Let $\sigma : B \times B \to A$ be a bilinear map; assume that

(unitary condition)

(3.1.1) $$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1;$$

(cocycle condition)

(3.1.2) $$[h_{(1)} \rightharpoonup \sigma(l_{(1)}, m_{(1)})]\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m);$$

(twisted module condition)

(3.1.3) $$(h_{(1)} \rightharpoonup (l_{(1)} \rightharpoonup a))\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \rightharpoonup a),$$

for any $h, l, m \in B$ and $a \in A$.

Furthermore, let $\tau : B \to A \otimes A$; assume that

(counitary condition)

(3.1.4) $$\varepsilon_B(c)1_A = (\varepsilon_A \otimes \mathrm{id})\tau(c) = (\mathrm{id} \otimes \varepsilon_A)\tau(c);$$

(co-cocycle condition)

(3.1.5) $$m_{A^{\otimes 3}}(\Delta \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id})(\tau \otimes \rho)\Delta = (\mathrm{id} \otimes m_{A^{\otimes 2}})(\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id})(\tau \otimes \tau)\Delta;$$

(twisted comodule condition)

(3.1.6) $$(\mathrm{id} \otimes m_{A^{\otimes 2}})(\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id})(\rho \otimes \tau)\Delta = m^{13}_{A^{\otimes 2}}(\mathrm{id} \otimes \mathrm{id} \otimes \rho \otimes \mathrm{id})(\tau \otimes \rho)\Delta,$$

where $m^{13}_{A^{\otimes 2}} : A \otimes A \otimes B \otimes A \otimes A \to B \otimes A \otimes A$ sends $h \otimes k \otimes c \otimes \tilde{h} \otimes \tilde{k} \mapsto c \otimes h\tilde{h} \otimes k\tilde{k}$.

Further, assume that the following compatibility conditions hold:

$$(3.1.7) \qquad \rho(1)=\tau(1)=1\otimes 1, \qquad \varepsilon\circ\sigma=\varepsilon\otimes\varepsilon, \qquad \varepsilon(a\rightharpoonup b)=\varepsilon(a)\varepsilon(b),$$

(Parts of this axiom are redundant, see [T2]).

$$(3.1.8) \qquad \Delta(b_{(1)}\rightharpoonup a)\tau(b_{(2)})=\tau(b_{(1)})\left((\rho(b_{(2)})_i\rightharpoonup a_{(1)})\otimes\rho(b_{(2)})^i(b_{(3)}\rightharpoonup a_{(2)})\right).$$

$$(3.1.9) \qquad \left(1\otimes\sigma(b_{(1)}\otimes\tilde b_{(1)})\right)\rho(b_{(2)}\tilde b_{(2)}) = \rho(b_1)\left(\rho(\tilde b_1)_k\otimes(b_{(2)}\rightharpoonup\rho(\tilde b_1)^k)\right)\left(1\otimes\sigma(b_{(3)}\otimes\tilde b_{(2)})\right).$$

$$(3.1.10) \qquad \left(1\otimes b_{(1)}\rightharpoonup a\right)\rho(b_{(2)})=\rho(b_{(1)})\left(1\otimes b_{(2)}\rightharpoonup a\right)$$

$$(3.1.11) \qquad \Delta\left(\sigma(b_{(1)}\otimes\tilde b_{(1)})\right)\tau(b_{(2)}\tilde b_{(2)}) = \tau(b_{(1)})\left(\rho(b_{(2)})_i\rightharpoonup\tau(\tilde b_{(1)})_p\otimes\rho(b_{(2)})^i(b_{(3)}\rightharpoonup\tau(\tilde b_{(1)})^p\right) \left(\sigma(\rho(b_{(4)})_j\otimes\rho(\tilde b_{(2)})_q)\otimes\rho(b_{(4)})^j(b_{(5)}\rightharpoonup\rho(\tilde b_{(2)})^q)\right)\left(1\otimes\sigma(b_{(6)}\otimes\tilde b_{(3)})\right).$$

(In all the preceding formulas, we use implicitly the usual tensor product multiplication in $A\otimes A$.)

Let $C=A\,{}^\tau\#_\sigma B$ denote the vector space $A\otimes B$ provided with the multiplication

$$(a\otimes b)(\tilde a\otimes\tilde b)=a(b_{(1)}\rightharpoonup\tilde a)\sigma(b_{(2)},\tilde b_{(1)})\otimes b_{(3)}\tilde b_{(2)}$$

and the comultiplication

$$\Delta(a\otimes b)=a_{(1)}\tau(b_{(1)})_j\otimes\rho(b_{(2)})_i\otimes a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i\otimes b_{(3)}.$$

Let $\iota:A\to C$ and $\pi:C\to B$ be given by $\iota(a)=a\otimes 1$, $\pi(a\otimes b)=\varepsilon(a)b$. $A\#_\sigma B$ (resp., $A\,{}^\tau\#B$) denotes the same space considered merely as an algebra (resp., as a coalgebra).

Proposition 3.1.12 ([Mj], [AD]). *$C=A\,{}^\tau\#_\sigma B$ is a bialgebra. Moreover, if σ and τ are invertible with respect to the convolution product, then C is a Hopf algebra and its antipode is given by*

$$\mathcal S(a\#b)=\left[(\sigma^{-1}\left(\mathcal S\rho(b_{(2)})_h\otimes\rho(b_{(3)})_j\right)\otimes\mathcal S\rho(b_{(1)})_i\right] \left[\tau^{-1}(b_{(4)})_k\mathcal S\left(a\rho(b_{(1)})^i\rho(b_{(2)})^h\rho(b_{(3)})^j\tau^{-1}(b_{(4)})^k\right)\otimes 1\right].$$

In this case,

$$(\mathcal C) \qquad \urcorner\to A\xrightarrow{\iota}C\xrightarrow{\pi}B\to\urcorner.$$

is an exact sequence of Hopf algebras.

Conversely, let $(\mathcal C)$ be an exact sequence of Hopf algebras and assume that in addition it is cleft (see below). Then there exist $\rightharpoonup,\sigma,\rho,\tau$ satisfying the conditions above, such that $C\simeq A\,{}^\tau\#_\sigma B$.

In addition, the description of exactly which data produce isomorphic extensions is given in [AD, Thm. 3.2.14] (previous work under abelian restrictions was also done in [Si], [By], [Hf]).

The following definition was independiently found by the authors of [AD], [By]; the author of the second paper was inspired by the given in [Sch2], [Sch3] for algebraic groups.

Definition 3.1.13. The extension $(\mathcal{C})$ is cleft if

(a) there exists $\chi \in \operatorname{Reg}_1(B, C)$ such that $(\mathrm{id} \otimes \pi)\Delta\chi = (\chi \otimes \mathrm{id})\Delta$ (such χ is called a section);

(b) there exists $\xi \in \operatorname{Reg}_\varepsilon(C, A)$ such that $\xi(ac) = a\xi(c), \quad \forall a \in A, c \in C$ (ξ is then called a retraction);

(c) $\xi\chi = \varepsilon_B 1_A$.

One deduces from (c) that $\xi(1) = 1$, $\varepsilon\chi = \varepsilon$, and henceforth $\pi\chi = \mathrm{id}_B$ and $\xi\iota = \mathrm{id}_A$.

The following Lemma was first proved by Byott; the author rediscovered independiently part of it before the publication of [By]. See also [Sch3, 2.1].

Lemma 3.1.14. *Let $(\mathcal{C})$ be an exact sequence of Hopf algebras. The following statements are equivalent:*

(i) *$(\mathcal{C})$ is cleft.*

(ii) *there exists $\chi \in \operatorname{Reg}_{1,\varepsilon}(B, C)$ satisfying (3.1.13) (a).*

(iii) *there exists $\xi \in \operatorname{Reg}_{1,\varepsilon}(C, A)$ satisfying (3.1.13) (b).*

(iv) *there exist a morphism of A-modules $\xi : C \to A$ and a morphism of B-comodules $\chi : B \to C$ such that $\xi\chi = \varepsilon_B 1_A$ and $(\iota\xi) * (\chi\pi) = \mathrm{id}_C$.*

Proof. By definition, *(i)* implies both *(ii)* and *(iii)*. We shall show that *(iii)* $\Longrightarrow$ *(i)*; *(ii)* $\Longrightarrow$ *(i)* is similar and will be left to the reader.

Assume *(iii)*. Let $\chi : B \to C$ be defined by $\chi(\pi c) = \xi^{-1}(c_{(1)})c_{(2)}$. χ is actually well-defined: if $c \in \ker\pi$, then $c = \sum a_i c_i$ for some $a_i \in A^+$ and hence $\xi^{-1}(c_{(1)})c_{(2)} = \xi^{-1}(c_{i(1)})\mathcal{S}(a_{i(1)})a_{i(2)}c_{i(2)} = 0$ by the formula $\xi^{-1}(ac) = \xi^{-1}(c)\mathcal{S}(a)$ [AD, 3.2]. Clearly, $\chi(1) = 1$ and $\varepsilon\chi = \varepsilon$; moreover χ is invertible and $\chi^{-1}(c) = \mathcal{S}(c_{(1)})\xi(c_{(2)})$. Finally, $(\mathrm{id} \otimes \pi)\Delta(\chi\pi c) = \xi^{-1}(c_{(1)})_{(1)}c_{(2)} \otimes \pi(\xi^{-1}(c_{(1)})_{(2)})\pi(c_{(3)}) = \xi^{-1}(c_{(1)})c_{(2)} \otimes \pi(c_{(3)}) = (\chi \otimes \mathrm{id})\Delta(\pi c)$ because $\xi^{-1}(c) \in A = \operatorname{LKer}\pi$.

We refer to [By, Lemma 4.5] for a proof of the equivalence between *(i)* and *(iv)*. □

It follows from *(iv)* in the preceding Lemma that, in the setting of Proposition 3.1.12, $C = A^\tau\#_\sigma B$ has an antipode if and only if σ and τ are invertible. If the last holds, then C has an antipode [AD], see Proposition 3.1.12. For the converse, let $\xi : C \to A$, $\xi(a \otimes b) = a\varepsilon(b)$, $\chi : B \to C$, $\chi(b) = 1 \otimes b$. Clearly, χ (resp., ξ) is a morphism of B-comodules (resp., of A-modules). It is known that σ (resp., τ) is invertible if and only if χ (resp., ξ) is, see [BM, Prop. 1.8] (resp., its dual [AD, 3.2.5]).

In any case let $\tilde{\chi} : B \to C$ be another morphism of right B-comodules. Then, using the formula $c = \xi(c_{(1)}) \otimes \pi(c_{(2)})$, one sees that $\tilde{\chi}(b) = f(b_{(1)}) \otimes b_{(2)}$ for some $f : B \to A$ (explicitly, $f(b) = \xi\tilde{\chi}(b)$). Conversely, any linear map $f : B \to A$ induces a morphism of comodules $\tilde{\chi} : B \to C$ by that recipe; and $\tilde{\chi}(1) = 1$ if and only if $f(1) = 1$. Assume that χ is invertible. Then $\tilde{\chi}$ is invertible if and only if f is (observe that $\tilde{\chi} = (\iota f) * \chi$). Thus the set of sections is in bijective correspondance with $\operatorname{Reg}_1(B, A)$.

Back to the general situation, let $\tilde{\xi} : C \to A$ be a morphism of left A-modules and set $g(b) = \tilde{\xi}(1 \otimes b)$. One has $\tilde{\xi} = \xi * g\pi$ and therefore, if the extension is cleft, the set of retractions is parametrized by $\operatorname{Reg}_\varepsilon(B, A)$.

Assume now that σ is trivial, i.e. that χ is a morphism of algebras (such extensions are called in the literature–under commutativity assumptions–Hochschild extensions [DG]). Then $\tilde{\chi}$ is a morphism of algebras if and only if

$$f(b\tilde{b}) = (b_{(1)} \rightharpoonup f(\tilde{b}))f(b_{(2)}), \tag{3.1.15}$$

and $f(1) = 1$. Dually, if τ is trivial (that is, if ξ is a coalgebra map), then $\tilde{\xi}$ is a coalgebra map if and only if

$$\Delta(f(b)) = \left(1 \otimes f(b_{(1)})\right)\left(f(\rho(b_{(2)})_j \otimes \rho(b_{(2)})^j\right), \tag{3.1.16}$$

and $\varepsilon g = \varepsilon$.

In a cleft extension like $(\mathcal{C})$, C is free as a left module over A.[1] There are examples of commutative, cocommutative Hopf algebras which are not free over some Hopf subalgebra [OS] (this example was rediscovered in [T3]); thus there are extensions which are not cleft. On the other hand, there are some important positive results, for example the following is a consequence of [Sch4, Thm. 2.2], whose proof is based on a result by Kramer and Takeuchi (the commutative case was first treated in [OS]):

Theorem 3.1.17. *An extension of finite dimensional Hopf algebras is always cleft.*

Other useful criteria are stated in [Sch4, Thm. 4.3]. These criteria apply in our setting because in any exact sequence $(\mathcal{C})$, C is an B-algebra extension of A (and an A-coalgebra extension of B). Moreover, the Hopf algebra extension is cleft if and only if the algebra extension is (this is the content of Lemma 3.1.14). Now an extension of algebras is cleft if and only if

(1) it is Galois,
(2) it has a normal basis.

(See [DT], [BCM], [BM]). In the case of our interest (exact sequences like $(\mathcal{C})$), one does not need to wonder about the Galois property: if ι is faithfully flat, then C is a Galois B-extension of A by [T3]. Now "normal basis" means that C is simultaneously isomorphic to $A \otimes B$ as A-module and B-comodule. Thus the notion of extension managed by several authors, beginning by Singer [Si], coincides with that of cleft extensions, as in this paper. The interested reader could find more examples of extensions which are not cleft arising from the theory of algebraic groups in [Sch3] (even of Hopf algebras which are free, but not cleft, over a suitable Hopf subalgebra). A nice survey on what is known about Hopf Galois extensions is [Sch5]. Finally, we quote a result which follows from [BM] and will be useful later.

Proposition 3.1.18. *Let $(\mathcal{C})$ be a cleft extension of finite Hopf algebras. If A and B are semisimple, then C also is.*

[1] The following remarks are well-known to specialists in Hopf algebra theory.

§3.2 Extensions of $*$-Hopf algebras. In this subsection, we shall use results quoted in the previous section to construct extensions of $*$-Hopf algebras. We will asume that the base field is $\mathbb{C}$.

Let A be a $*$-algebra and C a $*$-coalgebra (i. e., C has an antilinear involution $x \mapsto x^\circ$ such that $x_{(1)}{}^\circ \otimes x_{(2)}{}^\circ = x^\circ{}_{(2)} \otimes x^\circ{}_{(1)}$). Then one endows $\mathrm{Hom}(C, A)$ with an involution $*$ defined by

$$f^*(c) = f(c^\circ)^*, \quad f \in \mathrm{Hom}(C, A).$$

If g also belongs to $\mathrm{Hom}(C, A)$, then

$$(f * g)^*(c) = (f * g)(c^\circ)^* = g(c_{(1)}{}^\circ)^* f(c_{(2)}{}^\circ)^* = (g^* * f^*)(c).$$

That is, $\mathrm{Hom}(C, A)$ is a $*$-algebra. In the same vein, $\mathrm{Hom}(A, C)$ is a (topological) $*$-coalgebra–notice that $\Delta^{op}(f)(x \otimes y) = \Delta^{op}(f(y \otimes x))$.

Recall now that a $*$-Hopf algebra is a pair $(A, *)$, where A is a Hopf algebra, $*$ an antilinear involution making it a $*$-algebra, and Δ is a morphism of $*$-algebras (here $(x \otimes y)^* = x^* \otimes y^*$). Given a Hopf algebra A, it is equivalent to specify a $*$-Hopf algebra involution, or an antilinear involution $x \mapsto x^\circ$ such that $(A, \circ)$ is a $*$-coalgebra and m is a morphism of $*$-coalgebras. Indeed, the correspondance is given by

$$x^* = (\mathcal{S}^{-1}x)^\circ.$$

On the other hand, $A \xrightarrow{*} A^{op}$ is a morphism of (real) bialgebras, hence preserves the antipode [Sw, 4.0.4]. That is, $\mathcal{S}(x)^* = \mathcal{S}^{-1}(x^*)$ (compare with [W]).

Observe also that if A, B are $*$-Hopf algebras, then $\mathrm{Hom}(A, B)$ is a (topological) $*$-Hopf algebra, by $f^*(c) = f(c^\circ)^*$ as above.

It is clear what a morphism of $*$-Hopf algebras is; for example, the counit and the unit are. Also, the various kernels and cokernels of morphisms of $*$-Hopf algebras inherit the $*$-structure. We shall say that a sequence of morphisms of $*$-Hopf algebras

$$(\mathcal{C}) \qquad 0 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 0$$

is exact if it is exact as a sequence of the underlying Hopf algebras. Let us fix two $*$-Hopf algebras A, B and seek for conditions on a data $\rightharpoonup$, σ, ρ, τ as in Proposition 3.1.12, in order to get an extension of $*$-Hopf algebras.

First, we look the algebra case. It seems reasonable to impose Θ (cf. 2.1.4) to be a $*$-morphism. This translates into the following condition:

$$(3.2.1) \qquad (b \rightharpoonup a)^* = b^\circ \rightharpoonup a^*.$$

Equivalently, $(b \rightharpoonup a)^* = \mathcal{S}(b)^* \rightharpoonup a^*$, or $b \rightharpoonup a^* = (\mathcal{S}(b)^* \rightharpoonup a)^*$, or ... Let $\sigma : B \otimes B \to A$ satisfy (3.1.1-3) and assume in addition that σ is invertible. Analyzing the corresponding section χ, one sees it is plausible to ask

$$(3.2.2) \qquad \sigma(b \otimes \tilde{b})^* = \sigma^{-1}(\mathcal{S}(b)^* \otimes \mathcal{S}(\tilde{b})^*).$$

Lemma 3.2.3. *Let B be a $*$-Hopf algebra, A a $*$-algebra, $\rightharpoonup: B \otimes A \to A$ a weak action satisfying (3.2.1), σ a cocycle satisfying (3.2.2). Then $C = A \otimes_\sigma B$ is a $*$-algebra with involution*

$$(a \otimes b)^* = \sigma^{-1}(b^*_{(2)} \otimes \mathcal{S}^{-1}(b^*_{(1)})(b_{(3)} \rightharpoonup a^*) \otimes b^*_{(4)}. \tag{3.2.4}$$

Proof. The lenghty but straightforward verfication is left to the reader, as well as that of the following dual statement. □

Lemma 3.2.5. *Let B be a $*$-coalgebra, A a $*$-Hopf algebra, $\rho : B \to B \otimes A$ a weak coaction satisfying*

$$\rho(b^\circ) = \rho(b_i)^\circ \otimes \rho(b^i)^*. \tag{3.2.6}$$

Let $\tau : B \to A \otimes A$ be an invertible co-cocycle (i.e., it satisfies (3.1.4-6)), satisfying

$$\tau(b^\circ) = \tau^{-1}(b)^*. \tag{3.2.7}$$

Then $C = A^\tau \otimes B$ is a $*$-coalgebra with involution

$$(a \otimes b)^\circ = \mathcal{S}\left(a\rho(b_{(1)})^i \tau^{-1}(b_{(2)})_k\right)^* \tau^{-1}(b_{(2)})^{k*} \otimes \rho(b_{(1)})_i{}^\circ. \quad \square \tag{3.2.8}$$

Remark. (3.2.6) means that Ξ is a morphism of $*$-coalgebras.

Proposition 3.2.9. *Let A, B be $*$-Hopf algebras, and $\rightharpoonup$, σ, ρ, τ as in §3.1. Assume in addition they satisfy (3.2.1, 2, 6, 7). Then $C = A^\tau\#_\sigma B$ is a $*$-Hopf algebra and $(\mathcal{C})$ is an extension of $*$-Hopf algebras.*

Proof. We need only to check that the convolutions given by (3.2.4) and (3.2.8) agree, i. e. that $(a \otimes b)^* = \mathcal{S}\left((a \otimes b)^\circ\right)$. Again, this is a lenghty computation which will be omitted. (use the formula for the antipode given in [AD, 3.2.17]). □

§**3.3 Basic properties of extensions.** We collect in this subsection a number of facts about extensions of Hopf algebras. For brevity, we shall refer to an exact sequence $(\mathcal{C})$ as in §1.2. If $(\mathcal{C})$ is cleft, χ and ξ will denote respectively a section and a retraction satisfying (3.1.13) (c).

(3.3.1). *Let $(\mathcal{C})$ be a sequence with ι injective and π surjective. If C is finite, then the following are equivalent*

(a) *$(\mathcal{C})$ is exact.*
(b) $\ker \pi = C\iota(A)^+$.
(c) $\iota(A) = \operatorname{LKer}(\pi)$.
(d) *$(\mathcal{C}^*)$ is exact, where*

$$\daleth \to B^* \xrightarrow{\pi^*} C^* \xrightarrow{\iota^*} A^* \to \daleth. \tag{$\mathcal{C}^*$}$$

Proof. It can be found in [By, 4.1]; the non-trivial implication between (a) and (b) follows from [Sw, 16.0.2]. We sketch however, for further use, the proof of (a) $\implies$ (d).

We check first (1.2.4), i.e. that $B^* = \operatorname{LKer} \iota^*$. Let $\beta \in B^*$, $a \in A$, $c \in C$. Then

$$\langle(\iota^* \otimes \mathrm{id})\Delta(\beta), a \otimes c\rangle = \langle\beta, \pi(ac)\rangle = \langle 1 \otimes \beta, a \otimes c\rangle.$$

On the other hand, if $\gamma \in \operatorname{LKer} \iota^*$ then $\gamma(A^+C) = 0$ and hence $\gamma = \beta\pi$ for some $\beta \in B^*$. To prove (1.2.3) notice that $(\ker \iota^*)^\perp = A$ and $(\pi^*(B^*)^+C^*)^\perp = \operatorname{LKer} \pi$. □

(3.3.2). *(i). Let D be another Hopf algebra. Then*

$$(\mathcal{C}') \qquad \daleth \to D\otimes A \xrightarrow{\iota'} D\otimes C \xrightarrow{\pi'} B \to \daleth$$

is exact, where $\iota' = \mathrm{id}\otimes\iota$, $\pi' = \varepsilon\otimes\pi$.

(ii). The following sequence is also exact:

$$(\mathcal{C}'') \qquad \daleth \to A \xrightarrow{\iota''} D\otimes C \xrightarrow{\pi''} D\otimes B \to \daleth,$$

where $\iota'' = 1\otimes\iota$, $\pi'' = \mathrm{id}\otimes\pi$.

Proof. (i). Clearly, π' is surjective and ι' is injective. Now $\ker\pi' = D^+\otimes C + D\otimes\ker\pi = (D\otimes A)^+ D\otimes C$. Finally, let a_i be a basis of D and let $x = \sum_i a_i\otimes c_i \in \mathrm{LKer}\,\pi'$. Applying $\varepsilon_D\otimes\mathrm{id}$ to both sides of the equality defining LKer, one sees that $c_i \in \mathrm{LKer}\,\pi = A$.

(ii). Similar to the preceding. □

(3.3.3). *If $(\tilde{\mathcal{C}})$ is another exact sequence, then $\daleth \to A\otimes\tilde{A} \to C\otimes\tilde{C} \to B\otimes\tilde{B} \to \daleth$ is also exact.* □

Now we give a generalization of (3.3.1). Let H be a Hopf algebra and $\mathcal{F}$ a family of finite dimensional representations of H closed by finite direct sums, tensor products and taking the contragredient; we shall say that $\mathcal{F}$ is tensorial. Then the linear span $\Gamma_{\mathcal{F}}$ of the matrix coefficients of representations in $\mathcal{F}$ is a Hopf algebra contained in the dual of H. Sometimes we will emphasize $\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{F}}(H)$. If $f : H' \to H$ is a morphism of Hopf algebras, then $f^*\mathcal{F}$ is the family of representations of H' obtained composing with f.

Consider tensorial families of representations $\mathcal{F}'$, $\mathcal{F}$" of B, C as in $(\mathcal{C})$, respectively, and set $\mathcal{F} = \iota^*\mathcal{F}$".

(3.3.4). *Assume that*

(a) $\pi^*\mathcal{F}' \subseteq \mathcal{F}$"

(b) $\pi^*(\Gamma_{\mathcal{F}'}(B)) = \Gamma_{\mathcal{F}"}(C)\cap\pi^*(B^*)$.

If A is finite then

$$(\mathcal{C}^*) \qquad \daleth \to \Gamma_{\mathcal{F}'}(B) \xrightarrow{\pi^*} \Gamma_{\mathcal{F}"}(C) \xrightarrow{\iota^*} \Gamma_{\mathcal{F}}(A) \to \daleth.$$

is exact.

Proof. By definition of $\mathcal{F}$, ι^* is well defined and surjective. By (a), π^* is well defined, hence injective since is the restriction of an injection. One sees, as in the proof of (3.3.1), that $\mathrm{LKer}\,\iota^* = \Gamma_{\mathcal{F}'}(C)\cap\pi^*(B^*)$. To prove the assertion, we show finally that $\Gamma_{\mathcal{F}}(A)$ is a quotient comodule for the adjoint coactions. But this follows from the following general observation:

Let $\iota : A \to C$ be an inclusion of Hopf algebras and let $A' \subseteq A^*$, $C' \subseteq C^*$ be dual Hopf algebras such that ι^* induces an epimorphism $C' \to A'$. If A is stable by the adjoint actions, then A' is a quotient comodule for the adjoint coactions. Indeed, let $\alpha \in \ker\iota^* = C'\cap A^{\perp}$. Then $\langle(\iota^*\otimes\mathrm{id})\,\mathrm{ad}\,\alpha, a\otimes c\rangle = \langle\alpha, \mathrm{Ad}_r(c)a\rangle = 0$ and hence $(\iota^*\otimes\mathrm{id})\,\mathrm{ad}\,\alpha = 0$, which is our claim. □

Remarks. (i). The hypothesis on A are in order to apply (1.2.8) (iv) and can be replaced by any requirement insuring that ι^* is faithfully coflat.

(ii). By (a), one always has $\pi^*(\Gamma_{\mathcal{F}'}(B)) \subseteq \Gamma_{\mathcal{F}''}(C) \cap \pi^*(B^*)$. Let $\phi \in \Gamma_{\mathcal{F}''}(C) \cap \pi^*(B^*)$. Then $\phi \in \pi^*(B^\circ)$ (the image of the restricted dual of B). Indeed, if R denotes the right action of a Hopf algebra on its dual, then clearly $\pi^*\left(R_{\pi(c)}(\beta)\right) = R_c(\pi^*(\beta))$ and the claim follows from [Sw, 6.0.3].

For a Hopf algebra H, let H^{op} (resp., H^{cop}) be the Hopf algebra obtained by taking the opposite multiplication (resp., comultiplication); let $H^{\mathrm{bop}} = (H^{\mathrm{op}})^{\mathrm{cop}}$. Define $(\mathcal{C}^{\mathrm{op}})$, $(\mathcal{C}^{\mathrm{cop}})$, $(\mathcal{C}^{\mathrm{bop}})$ in a similar way. One proves easily that the three are exact if $(\mathcal{C})$ is. Moreover,

(3.3.5). *If $(\mathcal{C})$ is cleft, then $(\mathcal{C}^{\mathrm{op}})$, $(\mathcal{C}^{\mathrm{cop}})$, $(\mathcal{C}^{\mathrm{bop}})$ are.*

Proof. A section for $(\mathcal{C}^{\mathrm{op}})$ is $\xi^{\mathrm{op}}(c) = \xi^{-1}(\mathcal{S}^{-1}c)$; a retraction for $(\mathcal{C}^{\mathrm{cop}})$ is $\chi^{\mathrm{cop}}(b) = \mathcal{S}^{-1}(\chi^{-1}b)$. □

(3.3.6). *Retain the notations and hypothesis of (3.3.4). If $(\mathcal{C})$ is cleft, $\chi^*\left(\Gamma_{\mathcal{F}''}(C)\right) \subseteq \Gamma_{\mathcal{F}'}(B)$ or $\xi^*\left(\Gamma_{\mathcal{F}}(A)\right) \subseteq \Gamma_{\mathcal{F}''}(C)$, then $(\mathcal{C}^{*\mathrm{bop}})$ is also cleft.*

Proof. The candidates for retraction and section are respectively χ^* and ξ^*. Let $\beta \in \Gamma_{\mathcal{F}'}(B)$, $\gamma \in \Gamma_{\mathcal{F}''}(C)$. Then $\langle \chi^*(\pi^*\beta \cdot_{\mathrm{op}} \gamma), b\rangle = \langle \gamma \otimes \beta, (\mathrm{id} \otimes \pi)\Delta(\chi b)\rangle = \langle \gamma \otimes \beta, (\chi \otimes \mathrm{id})\Delta(b)\rangle = \langle \beta \cdot_{\mathrm{op}} \chi^*(\gamma), b\rangle$ for all $b \in B$. In a similar way, one proves that ξ^* is a section. □

(3.3.7). *Let $J \subseteq A$ be a Hopf subalgebra stable by the left adjoint action of C. Assume that*

(a) *there exists $\xi : C \to A$ such that $\xi(1) = 1$, $\xi(ac) = a\xi(c)$ for all $a \in A$, $c \in C$.*

Assume further that A is finite. Then the sequence

$$(\mathcal{C}_J) \qquad 0 \to A/J^+A \xrightarrow{\bar{\iota}} C/J^+C \xrightarrow{\bar{\pi}} B \to 0$$

is exact; moreover it is cleft if $(\mathcal{C})$ is.

Proof. First, $A \cap J^+C = J^+A$. Indeed, let $x = \sum j_i c_i \in A$, with $j_i \in J^+$, $c_i \in C$. Then $x = \xi(x) = \sum j_i \xi(c_i) \in J^+A$. Therefore $\bar{\iota}$ is well-defined and injective. It is clear that $\ker \bar{\pi} = \bar{\iota}(A^+/J^+A)$. Thus $(\mathcal{C}_J)$ is exact because $\bar{\iota}$ is faithfully flat (1.2.8) (iii). Finally, if $\xi \in \mathrm{Reg}_\varepsilon(C, A)$, then it defines $\bar{\xi} : C/J^+C \to A/J^+A$ and this a retraction of $(\mathcal{C}_J)$, which is then cleft by (3.1.14). □

The condition (3.3.7) (a) holds of course if $(\mathcal{C})$ is cleft, e.g. if C is finite dimensional. But also holds if B has a functional $\mu \in B^*$ such that $b_{(1)}\mu(b_{(2)}) = \mu(b)1$ (i.e. if B^* has a left integral) and $\mu(1) = 1$. For, let $\xi : C \to A$ be given by $\xi(c) = (\mathrm{id} \otimes \mu\pi)\Delta(c)$; it is easy to see that it satisfies (a). This admits a generalization to non necessarily normal Hopf subalgebras and integrals in their quotient coalgebras, whose explicit formulation we leave to the reader.

The following statement should be certainly improved; it should be useful to prove a sort of Jordan-Hölder theorem for finite quantum groups.

3.3.8. *Let A, D be finite dimensional Hopf subalgebras of a Hopf algebra, such that* $\mathrm{Ad}(A)D \subseteq D$. *Assume that there exists* $\xi : AD \to A$ *such that* $\xi(ad) = a\xi(d)$, $\xi(1) = 1$, $\xi(D^+) \subseteq A(A \cap D)^+$. *Then* $A/A(A \cap D)^+ \simeq DA/DAD^+$.

Proof. By the bijectivity of the antipode, $DA = AD$ and this is a Hopf subalgebra. Clearly, $(DA)D^+ = D^+A = AD^+$. The map $q : A \to DA/(DA)D^+$ is surjective; let $x \in \ker q = A \cap AD^+$. Then $x = \sum a_i d_i = \xi(x) = \sum a_i \xi(d_i) \in A(A \cap D)^+$. □

3.3.9. *If A is central in C and B has trivial Hopf center, then A is the Hopf center of C. (Dually, if B is cocentral and A has trivial Hopf cocenter, then B is the Hopf cocenter of C.)*

Proof. Let $A' \supseteq A$ be the Hopf center of C; then $\pi(A') = \urcorner$ and hence $A' \subseteq \mathrm{LKer}\,\pi = A$. □

(3.3.9) applies when B is simple (as Hopf algebra) and noncommutative.

§3.4 The Frobenius morphism. Lusztig [L4, Thm. 8.10, 8.16] has shown the existence (and uniqueness) of a Hopf algebra homomorphism $Fr : \mathcal{U}_{\mathbf{B}} \to \overline{\mathcal{U}}_{\mathbf{B}}$ such that

$$Fr(E_i^{(N)}) = \begin{cases} e_i^{(N/\ell)}, & \text{if } \ell \text{ divides } N \\ 0, & \text{if not;} \end{cases}$$
$$Fr(F_i^{(N)}) = \begin{cases} f_i^{(N/\ell)}, & \text{if } \ell \text{ divides } N \\ 0, & \text{if not;} \end{cases}$$
$$Fr\left(\begin{bmatrix} K_i \\ N \end{bmatrix}\right) = \begin{cases} \begin{pmatrix} h_i \\ N/\ell \end{pmatrix}, & \text{if } \ell \text{ divides } N \\ 0, & \text{if not;} \end{cases}$$
$$Fr(K_i) = 1.$$

It is known that $\mathcal{U}_{\mathbf{B}}$ is generated by E_i, F_i, K_i, $E_i^{(\ell)}$, $F_i^{(\ell)}$. Let $\mathbf{u}$ be the Hopf subalgebra of $\mathcal{U}_{\mathbf{B}}$ generated by E_i, F_i, K_i [L4]; it is known that $\dim \mathbf{u} = 2^n \ell^{\dim \mathfrak{g}}$.

Lemma 3.4.1. *The sequence*

$$(\mathcal{FR}) \qquad 0 \to \mathbf{u} \hookrightarrow \mathcal{U}_{\mathbf{B}} \xrightarrow{Fr} \overline{\mathcal{U}}_{\mathbf{B}} \to 0$$

is exact.

Proof. (1.2.1,2) being clear, we proceed with (1.2.3). It is known that $\ker Fr = \mathcal{U}_{\mathbf{B}}\mathbf{u}^+\mathcal{U}_{\mathbf{B}}$ [L4, 8.16]. Thus, from the formulas [L4, 5.3 and 5.4] (see also [L2, 4.1]) (1.2.3) follows. However, and in order to use Schneider's theorem (1.2.8) (iii) we prove the following Lemma (proved independently but previously in [Li]); it will imply also (1.2.4). □

Lemma 3.4.2. *The finite dimensional Hopf subalgebra* $\mathbf{u}$ *is stable by the right and left adjoint actions.*

Proof. We give the proof of the stability by the left adjoint and leave that of the right one (which is very similar) to the reader. Clearly, it suffices to show that $\mathrm{Ad}(E_i^{(\ell)})\mathbf{u} \subseteq \mathbf{u}$. Let $x \in \mathcal{U}_{\mathbf{B}}$ be such that $K_i x K_i^{-1} = q^{d_i m_i}$ for some integers m_i. It follows from (1.1.3) that

$$\mathrm{Ad}(E_i^{(\ell)})(x) = \sum_{j=0}^{\ell}(-1)^j q^{d_i j(\ell-1)} E_i^{(\ell-j)} K_i^j x K_i^{-j} E_i^{(j)}$$
$$= \sum_{j=0}^{\ell}(-1)^j q^{d_i j(\ell-1+m_i)} E_i^{(\ell-j)} x E_i^{(j)}.$$

Let x be either E_h with $h=i$ or $a_{ih}=0$, or F_h with $h \neq i$, or any K_h. Then, as

$$E_i^{(\ell-j)} E_i^{(j)} = \begin{bmatrix} \ell \\ j \end{bmatrix}_{d_i} E_i^{(\ell)} = 0,$$

we conclude that $\mathrm{Ad}(E_i^{(\ell)})(x) = 0$. Now let $x = E_h$ with $a_{ih} \neq 0$. Applying either the commutation relation [L2, 4.1(g)], or [L4, 5.3(i)], or [L4, 5.4(a6)] (depending on whether $-a_{ih} = 1, 2$ or 3) we see that $\mathrm{Ad}(E_i^{(\ell)})(x) \in \mathbf{u}$; here one uses that $\mathbf{u}$ is preserved by the action of the braid group [L4, 8.12]. For $x = F_i$, one uses instead [L2, 4.1(a)]. So we have proved that $\mathrm{Ad}(E_i^{(\ell)})(x) \in \mathbf{u}$ for x in a family of generators of $\mathbf{u}$. But

$$\mathrm{Ad}(E_i^{(\ell)})(xz) = \sum_{j=0}^{\ell} q^{d_i j(\ell-j)}\, \mathrm{Ad}(E_i^{(\ell-j)} K_i^j)(x)\, \mathrm{Ad}(E_i^{(j)})(z)$$

and we are done. $\square$

Lemma 3.4.3. *Consider* $\mathcal{U}_{\mathbf{B}}$ *as a* $\overline{\mathcal{U}}_{\mathbf{B}}$*-comodule algebra via* $\gamma = (\mathrm{id} \otimes Fr)\Delta$. *Then* $\mathcal{U}_{\mathbf{B}}$ *is an algebra cleft extension of* $\mathbf{u}$ *by* $\overline{\mathcal{U}}_{\mathbf{B}}$.

Proof. We need to define $\chi \in \mathrm{Reg}_1(\overline{\mathcal{U}}_{\mathbf{B}}, \mathcal{U}_{\mathbf{B}})$ such that $\gamma\chi = (\chi \otimes \mathrm{id})\Delta$. We will proceed by steps.

(i). There exists an algebra homomorphism $\chi_+ : \overline{\mathcal{U}}_{\mathbf{B},+} \to \mathcal{U}_{\mathbf{B}}$ uniquely defined by $\chi_+(e_i) = K_i^{-\ell} E_i^{(\ell)}$. This follows from [L4, 8.6] since $K_i^{-\ell}$ is central. Now

$$\gamma\chi_+(e_i) = (\mathrm{id} \otimes Fr)\left(\sum_{j=0}^{\ell} q^{d_i j(\ell-j)} E_i^{(\ell-j)} K_i^j \otimes E_i^{(j)}\right)$$
$$= K_i^{-\ell} E_i^{(\ell)} \otimes 1 + 1 \otimes e_i = (\chi_+ \otimes \mathrm{id})\Delta(e_i).$$

As χ_+ is multiplicative, $\gamma\chi_+ = (\chi_+ \otimes \mathrm{id})\Delta$.

(ii). There exists an algebra homomorphism $\chi_- : \overline{\mathcal{U}}_{\mathbf{B},-} \to \mathcal{U}_{\mathbf{B}}$ uniquely defined by $\chi_-(f_i) = F_i^{(\ell)}$. (Same proof as for [L4, 8.6].) Again, $\gamma\chi_- = (\chi_- \otimes \mathrm{id})\Delta$.

(iii). Let $\chi_0 : \overline{\mathcal{U}}_{\mathbf{B},0} \to \mathcal{U}_{\mathbf{B}}$ be the algebra homomorphism such that $\chi_0(h_i) = K_i^\ell \begin{bmatrix} K_i \\ \ell \end{bmatrix}$. Using (1.1.3), one has again $\gamma\chi_0 = (\chi_0 \otimes \mathrm{id})\Delta$.

(iv). Let $\chi : \overline{\mathcal{U}}_{\mathbf{B}} \to \mathcal{U}_{\mathbf{B}}$ be defined by $\chi(x_+x_0x_-) = \chi_+(x_+)\chi_0(x_0)\chi_-(x_-)$, for $x_I \in \overline{\mathcal{U}}_{\mathbf{B},I}$, $I = +,0,-$. Then

$$\begin{aligned}\gamma\chi(x_+x_0x_-) &= \gamma\left(\chi_+(x_+)\right)\gamma\left(\chi_0(x_0)\right)\gamma\left(\chi_-(x_-)\right)\\ &= (\chi_+ \otimes \mathrm{id})\Delta(x_+)(\chi_0 \otimes \mathrm{id})\Delta(x_0)(\chi_- \otimes \mathrm{id})\Delta(x_-) = (\chi \otimes \mathrm{id})\Delta(x).\end{aligned}$$

(v). Being a morphism of algebras, χ_+ is invertible and $\chi_+^{-1} = \chi_+\mathcal{S}$. Thus χ is invertible, and $\chi^{-1}(x_+x_0x_-) = \chi_-^{-1}(x_-)\chi_0^{-1}(x_0)\chi_+^{-1}(x_+)$. □

The last result also follows from a general result in [Sch4, 4.3]. From Lemma 3.1.14 one deduces immediately (compare with [Li, 5.5]):

Proposition 3.4.4. *The exact sequence ($\mathcal{FR}$) given by the Frobenius morphism is cleft.* □

It is interesting to see the failure of χ to be a morphism of algebras. First, one deduces from [L4, 6.5(a2)]

$$[\chi(e_i), \chi(f_j)] = \delta_{ij}\left(\chi(h_i) - K_i^{-\ell}\sum_{1\le t\le \ell-1} F_i^{(\ell-t)}\begin{bmatrix} K_i; 2(t-\ell) \\ t \end{bmatrix} E_i^{(\ell-t)}\right).$$

Next,

$$[\chi(h_i), \chi(e_j)] = E_j^{(\ell)}\left(\begin{bmatrix} K_i; \ell a_{ij} \\ \ell \end{bmatrix} - \begin{bmatrix} K_i \\ \ell \end{bmatrix}\right)$$

$$= E_j^{(\ell)}\left(\sum_{0\le t\le \ell}(-1)^t q^{-\ell a_{ij}(t-\ell)}\begin{bmatrix} -\ell a_{ij} + t - 1 \\ t \end{bmatrix} K_i^{-t}\begin{bmatrix} K_i \\ \ell - t \end{bmatrix} - \begin{bmatrix} K_i \\ \ell \end{bmatrix}\right) = a_{ij}\chi(e_j).$$

Here, the first equality follows from [L2, 4.1(c)]; the second, from $\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \sum_{1\le j\le t}(-1)^j q^{c(j-t)}\begin{bmatrix} c + j - 1 \\ j \end{bmatrix} K_i^{-j}\begin{bmatrix} K_i \\ t - j \end{bmatrix}$, for $c \le -1$, $t \ge 0$ (this is [L3, 1.3 (g9)]); in the third, one uses that $\begin{bmatrix} -\ell a_{ij} + t - 1 \\ t \end{bmatrix} = 0$ if $1 < t < \ell$, that $\begin{bmatrix} -\ell a_{ij} + \ell - 1 \\ \ell \end{bmatrix} = -a_{ij}$ [L2, 3.3] and that ℓ is odd.

We want now to present another exact sequence, dual to $(\mathcal{FR})$. Let V be a $\mathcal{U}$-module. Let $\lambda \in P$. Set

$$V_\lambda = \{v \in V : K_i v = q^{d_i\langle\lambda,\alpha_i^\vee\rangle}, \quad \begin{bmatrix} K_i \\ r \end{bmatrix} v = \begin{bmatrix} \langle\lambda,\alpha_i^\vee\rangle \\ r \end{bmatrix}_{d_i} v, \quad 1 \le i \le n\}.$$

For any lattice M, $P \supseteq M \supseteq Q$, let $\tilde{\mathcal{F}}_M$ be the category of free $\mathcal{U}$-modules of finite rank V such that $V = \oplus_{\lambda\in M} V_\lambda$. Let $\mathcal{F}" = \mathcal{F}"_M$ be the category of $\mathcal{U}_{\mathbf{B}}$-modules obtained from $\tilde{\mathcal{F}}_M$ by extension of scalars. Let $\mathcal{F}' = \mathcal{F}'_M$ be the category of finite dimensional $\overline{\mathcal{U}}_{\mathbf{B}}$-modules whose weights belong to M. Let $\mathcal{F} = \iota^*\mathcal{F}"$. Let us denote $\mathbf{B}[G] = \Gamma_{\mathcal{F}'}(\overline{\mathcal{U}}_{\mathbf{B}})$, $\mathbf{B}_v[G] = \Gamma_{\mathcal{F}"}(\mathcal{U}_{\mathbf{B}})$, $\mathbf{v} = \Gamma_{\mathcal{F}}(\mathbf{u})$. (That is: G is the connected semisimple algebraic group with Lie algebra g whose "π_1" equals P/M).

Proposition 3.4.5. *Assume that $M = P$. The sequence*

$$(\mathcal{FR}^*) \qquad 0 \to \mathbf{B}[G] \to \mathbf{B}_v[G] \to \mathbf{v} \to 0$$

is exact.

Proof. This follows from (3.4.4) thanks to (3.3.4). Indeed $Fr^*(\mathcal{F}') \subset \mathcal{F}"$ is a consequence of complete reducibility of simple Lie algebras and [L2, 7.2]. On the other hand, $\mathbf{B}[G] = (\overline{\mathcal{U}}_{\mathbf{B}})^\circ$ and we can apply the second remark after (3.3.4). □

Remark. It was proved in [DeCL] that $\mathbf{B}_v[G]$ is projective over $\mathbf{B}[G]$ of rank $\ell^{\dim g}$.

Let $\mathbf{j}$ be the subalgebra of $\mathbf{u}$ generated by K_i^ℓ; it is a central Hopf subalgebra, isomorphic to the group algebra of $(\mathbb{Z}/2\mathbb{Z})^\ell$. It follows from (3.3.7) that the sequence

$$0 \to \mathbf{u}/\mathbf{u}\mathbf{j}^+ \hookrightarrow \mathcal{U}_{\mathbf{B}}/\mathcal{U}_{\mathbf{B}}\mathbf{j}^+ \xrightarrow{Fr} \overline{\mathcal{U}}_{\mathbf{B}} \to 0$$

is also exact and cleft. One can also deduce (3.4.5) from this fact.

Recall the notation from 1.1.4 and the subsequent lines.

Proposition 3.4.6. *The sequence $0 \to Z_0 \to \mathcal{A}_{\mathbf{B}} \to \mathbf{u}/\mathbf{u}\mathbf{j}^+ \to 0$ is exact.*

Proof. Left to the reader; use (1.2.8) (ii). □

§4. Some simple finite quantum groups

Let us a say that a finite Hopf algebra is *quantum simple* (q-simple for short) if it has no strongly normal Hopf subalgebra or, equivalently, it has no strongly conormal quotient Hopf algebra; see [AD], [Sch], [By]. Thus the dual of a finite q-simple Hopf algebra is again q-simple. Suppose that a finite Hopf algebra C is not q-simple and pick a non-trivial strongly normal Hopf subalgebra A. Then by (3.3.1) C fits into an exact sequence like (C); by (3.1.17) this exact sequence is cleft and hence it is possible to reconstruct C from A and B and some data. Therefore, to classify all the possible finite Hopf algebras of order (that is, dimension) $\le N$, we need first to classify the q-simple ones, and then glue them via that data. We will discuss some basic features of this second step in the next section; here we shall give examples of q-simple Hopf algebras. Let us work in this section over an algebraically closed field $\daleth$. If G is a finite simple group, then clearly both the algebra of functions on G, $\daleth[G]$, and the group algebra of G, $\daleth\langle G\rangle$, are simple. On the other hand, any Hopf algebra of prime order is simple, thanks to the Nichols-Zoeller theorem [NZ]. More precisely

Theorem 4.1 [Z]. *If the characteristic of $\Bbbk$ is 0, then any Hopf algebra of prime order is the group algebra of a cyclic group, that is, it is commutative, cocommutative and semisimple.*

This was conjectured by Kaplansky. The commutative Hopf algebras of prime order over an arbitrary field are well-known; see e.g. [TO]. In particular, they are all cocommutative. It is now easy to deduce the following criteria of q-simplicity:

Proposition 4.2. *If a finite Hopf algebra of order pq is not q-simple, where p and q are primes, then it is semisimple.*

Proof. This follows at once from (4.1) and (3.1.17, 18). □

Some results on the classification of semisimple Hopf algebras of low order can be found in [LR], [Ms]. A result more precise than Proposition 4.2, intersecting also [LR], is given by Theorem 5.2.7 below.

Let us consider the algebra generated by three elements x and $g^{\pm 1}$ with relations $gg^{-1} = g^{-1}g = 1$ and $gxg^{-1} = q^2x$, for some $q \in \Bbbk$. It has a Hopf algebra structure given by

$$\begin{aligned} \Delta(g) &= g \otimes g, \quad \mathcal{S}(g) = g^{-1}, \quad \varepsilon(g) = 1, \\ \Delta(x) &= x \otimes g + 1 \otimes x, \quad \mathcal{S}(x) = -xg^{-1}, \quad \varepsilon(x) = 0. \end{aligned} \tag{4.3}$$

Then by the quantum binomial formula one has

$$\Delta(x^n) = \sum_{t=0}^{n} q^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix} x^{n-t} \otimes g^{n-t}x^t. \tag{4.4}$$

Assume further that q^2 is a primitive n-root of 1; then $\Delta(x^n) = x^n \otimes g^n + 1 \otimes x^n$. We want to study the Hopf algebra $\mathfrak{D}_{n,m}$ generated by x and g, with the preceding relations and structure plus $x^n = 0$, $g^{nm} = 1$, for some $m \in \mathbb{Z}$. This algebra has dimension n^2m and it is isomorphic to the dual of the quiver Hopf algebra constructed in [Ci] (use [Ci, 3.8] to prove the isomorphism). The Hopf algebra $\mathfrak{D}_n := \mathfrak{D}_{n,1}$ ($= \mathfrak{D}$ if n is fixed) was introduced in [Tf] generalyzing an example of Sweedler (namely, the case $n = 2$). It is isomorphic to the $+$-part of the Lusztig kernel corresponding to $s\ell(2)$ discussed in §3.4. The Hopf subalgebra of $\mathfrak{D}_{n,m}$ generated by g^n is central (indeed it is the Hopf center by 4.7 below) and one has an exact sequence

$$0 \to \Bbbk[g^n] \to \mathfrak{D}_{n,m} \to \mathfrak{D}_n \to 0.$$

We concentrate now on $\mathfrak{D}_n$. Notice that $\mathrm{Ad}_g = \mathcal{S}^2$. It follows that the order of the antipode is $2n$ [Tf]. Let $\mathfrak{D}^j$ denote the span of $\{g^ix^j : 0 \le i \le n-1\}$.

Lemma 4.5. *Let H be a Hopf subalgebra of $\mathfrak{D}$ of dimension greather than 1. Then there exists an integer s, $1 \le s \le n-1$ such that $g^s \in H$. Moreover, if $x^j \in H$ for some $j \ge 1$, then $x \in H$.*

Proof. As H is $\mathcal{S}^2$-stable, $H = \oplus_j H^j$, where $H^j = H \cap \mathfrak{D}^j$. Assume that $H^j \neq 0$ for some $j \neq 0$ and fix $z = f(g)x^j \in H$, where f is a polynomial of degree $\le n-1$, $f \neq 0$, say

$f(g) = \sum_{i=0}^{n-1} a_i g^i$. Now, as $H \otimes H = \oplus_{i,j} H^i \otimes H^j$ and $H^i \otimes H^j = (H \otimes H) \cap (\mathfrak{D}^i \otimes \mathfrak{D}^j)$, we conclude from (4.4) that

$$f(g \otimes g) x^{j-t} \otimes g^{j-t} x^t \in H. \tag{4.6}$$

Take $t = j$; then $f(g \otimes g)(1 \otimes x^j) \in H \otimes H$. Applying $m^{k+1}(\Delta^k \otimes \mathrm{id})$ to this element, we see that $f(g^k)x^j \in H$, and in particular $f(1)x^j \in H$. We claim that it is always possible to choose f such that $f(1) \neq 0$. Indeed, applying m to (4.6) for $t = j - 1$, we see that, if $h(g)x^j \in H$, then $gh(q^{-2}g^2)x^j \in H$, and therefore $gf(q^{-2k}g^{2k})x^j \in H$ for all k. Set $f_k(g) = gf(q^{-2k}g^{2k})$; if $f_k(1) = 0$ for all k, then $f(q^{-2k}) = 0$ for all k; as degree of f is $\leq n-1$, f should be 0.

So we can assume that $x^j \in H$ and by (4.6) again, x^{j-t} and g^j belong to H. □

Now it is easy to determine the Hopf subalgebras of $\mathfrak{D}$. Indeed, if $x \in H$ then $g \in H$ and hence $H = \mathfrak{D}$. Otherwise, H is a Hopf subalgebra of $\urcorner[g] \simeq \urcorner(\mathbb{Z}/n\mathbb{Z})$.

Proposition 4.7. *$\mathfrak{D}$ is q-simple.*

Proof. Let H be a non-trivial normal Hopf subalgebra of $\mathfrak{D}$; then for some s, $1 \leq s \leq n-1$, $g^s \in H$. But then $\mathrm{Ad}(x)(g^s) = q^2(q^{-2s} - 1)g^{s-1}x \in H$. Thus $H = D$. □

For n prime, the last result follows from (4.2).

We discuss now briefly the (well-known) representation theory of $\mathfrak{D}$. Let us fix j, m, $0 \leq j, m \leq n-1$. Let $V_{j,m}$ be the following $\mathfrak{D}$-module: it has a basis $\{v_h : 0 \leq h \leq j\}$ such that $x.v_h = v_{h+1}$, where by convention $v_{j+1} = 0$, and g acts by the scalar q^{2m}. Then the $V_{j,m}$'s constitute a classification of the indecomposable $\mathfrak{D}$-modules.

The following questions arise naturally from the preceding discussion:

(a) Is the +-part of a Lusztig kernel always a simple finite quantum group?

(b) Has it always has only a finite number of decomposables?

On another direction, it is likely that Lusztig kernels are also q-simple. (For type A_n this was proved in [T4]). If true, we shall deduce from (3.3.9) that $\mathbf{B}[G]$ is the Hopf center of $\mathbf{B}_v[G]$ (resp., that Z_0 is the Hopf center of $\mathcal{A}_{\mathbf{B}}$).

§5. Cohomology of Hopf algebras

§5.1 Singer's cohomology. Singer [Si] defined a cohomology theory for a pair of (graded, connected) Hopf algebras A, B, with B cocommutative and acting on A, A commutative and coacting on B, subject to two compatibility conditions. He also showed that the 2-cohomology group classifies the extensions of A by B. Singer's cohomology can be also defined without the "graded and connected" assumption and again the 2-cohomology group classifies the (isomorphy classes of) extensions of A by B. Details can be found in [Hf]; the classification theorem is also a particular case of [AD, Th. 3.2.14].

We begin by reviewing Singer's cohomology in the setting of our interest; for the moment, it is convenient to do not suppose that A is commutative and B, cocommutative; we shall latter do so. Let us fix two Hopf algebras A and B together with an action $\rightharpoonup: B \otimes A \to A$ and a coaction $\rho : B \to B \otimes A$, such that $\rho(1) = 1$, $\varepsilon(b \rightharpoonup a) = \varepsilon(a)\varepsilon(b)$, and

$$\Delta(b \rightharpoonup a) = \left(\rho(b_{(1)})_i \rightharpoonup a_{(1)}\right) \otimes \rho(b_{(1)})^i \left(b_{(2)} \rightharpoonup a_{(2)}\right), \tag{5.1.1}$$

$$\rho(b\tilde{b}) = \rho(b_{(1)}) \left(\rho(\tilde{b})_k \otimes b_{(2)} \rightharpoonup \rho(\tilde{b})^k\right), \tag{5.1.2}$$

$$\left(1 \otimes b_{(1)} \rightharpoonup a\right) \rho(b_{(2)}) = \rho(b_{(1)}) \left(1 \otimes b_{(2)} \rightharpoonup a\right). \tag{5.1.3}$$

Clearly, (5.1.3) is superflous if A is commutative and B cocommutative. For brevity, one says that the pair (A, B) is *compatible.*

Let N be a left B-module and define an action of B on $N \otimes A$ by

$$b(n \otimes a) = \rho(b_{(1)})_i n \otimes \rho(b_{(1)})^i (b_{(2)} \rightharpoonup a). \tag{5.1.4}$$

This is a left B-module action thanks to (5.1.2) and $N \otimes A$ with this action will be denoted by $N \tilde{\otimes} A$. Notice that $\Delta : A \to A \tilde{\otimes} A$ is a morphism of B-modules by (5.1.1). Let X be a right A-comodule (with structural morphism c) and define a coaction $B \otimes X \to B \otimes X \otimes A$ (still called c by abuse of notation) by

$$c(b \otimes x) = \rho(b_{(1)})_i \otimes c(x)_j \otimes \rho(b_{(1)})^i (b_{(2)} \rightharpoonup c(x)^j). \tag{5.1.5}$$

This a right A-comodule (denoted by $B \tilde{\otimes} X$) by (5.1.1); (5.1.2) implies now that the multiplication $B \tilde{\otimes} B \to B$ is a morphism of comodules.

Now we consider the category $\mathfrak{C}(A, B)$. An object M of $\mathfrak{C}(A, B)$ is simultaneously a left B-module and a right A-comodule, such that the action $B \tilde{\otimes} M \to M$ is a morphism of A-comodules, and the coaction $M \to M \tilde{\otimes} A$ is a morphism of B-modules. Both conditions are expressed by one equality:

$$c(bm) = \rho(b_{(1)})_i c(m)_j \otimes \rho(b_{(1)})^i (b_{(2)} \rightharpoonup c(m)^j). \tag{5.1.6}$$

(The arrows in this category are those linear morphisms which preserve all the structures involved; we shall use the notation Hom_B^A for them).

If N is a left B-module, then $N \tilde{\otimes} A$ belongs to $\mathfrak{C}(A, B)$ with coaction $n \otimes a \mapsto a \otimes a_{(1)} \otimes a_{(2)}$: this is again a consequence of (5.1.1). If X is a right A-comodule then it follows from (5.1.2) that $B \tilde{\otimes} X$ belongs to $\mathfrak{C}(A, B)$ by letting B act on the first factor. Thus, we have in particular two translation functors $S, T : \mathfrak{C}(A, B) \to \mathfrak{C}(A, B)$, $S(M) = M \tilde{\otimes} A$, $T(M) = B \tilde{\otimes} M$. Observe that $M \simeq \{z \in S(M) : c(z) = z \otimes 1\} \simeq T(M)/B^+ T(M)$.

On the other hand, if M, P belong to $\mathfrak{C}(A, B)$ then $M \otimes P$ (considered as A-comodule via the multiplication of A and as B-module via the comultiplication of B) also belongs to $\mathfrak{C}(A, B)$. Here one use for the first time the hypothesis (5.1.3).

Next, one says that M in $\mathfrak{C}(A, B)$ is an (A, B)-algebra (resp., coalgebra) if the multiplication $m : M \otimes M \to M$ (resp., the comultiplication $\delta : M \to M \otimes M$) is a morphism in $\mathfrak{C}(A, B)$. (In such case, we shall denote ρ instead of c for the coaction and $\rightharpoonup$ for the action). If X is an (A, B)-algebra then $S(X) = X \tilde{\otimes} A$ also is, with the tensor product algebra structure; similarly, if Y is an (A, B)-coalgebra then $T(Y) = B \tilde{\otimes} Y$ also is. Notice that in this case, there is an isomorphism

$$\mathrm{Reg}(Y, X) \xrightarrow{\theta} \mathrm{Reg}_B^A(T(Y), S(X)). \tag{5.1.7}$$

Here, $\mathrm{Reg}_B^A(T(Y), S(X)) := \mathrm{Reg}(T(Y), S(X)) \cap \mathrm{Hom}_B^A(T(Y), S(X))$. Explicitly, $\theta(f)(b \otimes x) = b \rightharpoonup ((f \otimes \mathrm{id})\rho(x))$.

Remark. The monoidal category $\mathfrak{C}(A,B)$ is in fact the category of representations of a Hopf algebra. Indeed, assume for simplicity that A is finite dimensional (otherwise one should consider a structure of topological Hopf algebra on $\mathrm{Hom}(A,B)$, see [T1]). Let $H = B \otimes A^*$; following the recipe of [Ma, Prop. 3.13], one considers the matched Hopf algebra structure on H induced by the left action $\rightharpoonup: A^* \otimes B \to B$ and the right action $\leftharpoonup: A^* \otimes B \to A^*$ (see also [T2]). These actions are explicitly

$$\alpha \rightharpoonup b = \sum_i \rho(b)_i \langle \alpha, \rho(b)^i \rangle, \qquad \langle \alpha \leftharpoonup b, a \rangle = \langle \alpha, b \rightharpoonup a \rangle,$$

and the corresponding multiplication is

$$(b \otimes \alpha)(d \otimes \gamma) = b(\alpha_{(1)} \rightharpoonup d_{(1)}) \otimes (\alpha_{(2)} \leftharpoonup d_{(2)})\gamma.$$

Thus if $M \in \mathfrak{C}(A,B)$, one defines an action of H on M by $(b \otimes \alpha).m = b.(\langle \alpha, c(m)^i \rangle c(m)_i$; (5.1.6) guarantees that this is effectively an action. Conversely, any H-module gives rise, by reversing the procedure just described, to an object of $\mathfrak{C}(A,B)$, and the tensor product in $\mathfrak{C}(A,B)$ corresponds to the comultiplication of H.

Consider now the category $\mathfrak{C}_1$ of (A,B)-algebras. The coaction $\rho : N \to N \otimes A$ gives rise to a natural transformation η from the identity functor to S. Moreover, there is a natural transformation $\mu : S^2 \to S$ given by $\mu_N : N\tilde{\otimes}A\tilde{\otimes}A \to N\tilde{\otimes}A$, $\mu_N = \mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$, and (S, η, μ) is a monad (or triple) in $\mathfrak{C}_1$ [McL, p. 133]. Therefore, for each $X \in \mathfrak{C}_1$ one can form the corresponding simplicial object [McL, p.171]. For the benefit of the reader, let us write down the formulas explicitly. Let $F^q = S^{q+1}(X)$, for $q \geq 0$; the face operators $\underline{\delta}_q^i : F^q \to F^{q+1}$, are

$$\underline{\delta}_q^0(x \otimes a_0 \otimes \cdots \otimes a_q) = \rho(x) \otimes a_0 \otimes \cdots \otimes a_q,$$
$$\underline{\delta}_q^i(x \otimes a_0 \otimes \cdots \otimes a_q) = x \otimes a_0 \otimes \cdots \otimes \Delta(a_{i-1}) \cdots \otimes a_q, \quad 1 \leq i \leq q+1;$$

the degeneracy operators $\underline{\sigma}_q^i : F^{q+1} \to F^q$ are

$$\underline{\sigma}_q^i(x \otimes a_0 \otimes \cdots \otimes a_{q+1}) = x \otimes a_0 \otimes \ldots \varepsilon(a_i) \cdots \otimes a_{q+1}, \quad 0 \leq i \leq q.$$

Similarly, one has a comonad (or cotriple) (T, δ, χ) in the category $\mathfrak{C}_2$ of (A,B)-coalgebras by setting $\delta : T \to \mathrm{id}$, $\delta_Y : B\tilde{\otimes}Y \to Y$ the action, $\chi : T \to T^2$, $\chi_Y : B\tilde{\otimes}Y \to B\tilde{\otimes}B\tilde{\otimes}Y$, $\chi_Y(b \otimes y) = b \otimes 1 \otimes y$. Let $G^p(Y) = T^{p+1}(Y)$. The explicit expressions for the coface operators $\underline{d}_j^p : G^{p+1}(Y) \to G^p(Y)$ and codegeneracy operators $\underline{s}_j^p : G^p(Y) \to G^{p+1}(Y)$ are as follows:

$$\underline{d}_j^p(b_0 \otimes \cdots \otimes b_{p+1} \otimes y) = b_0 \otimes \ldots b_j b_{j+1} \otimes \cdots \otimes b_{p+1} \otimes y, \quad 0 \leq j \leq p,$$
$$\underline{d}_{p+1}^p(b_0 \otimes \cdots \otimes b_{p+1} \otimes y) = b_0 \otimes \cdots \otimes \cdots \otimes b_{p+1} \rightharpoonup y,$$
$$\underline{s}_j^p(b_0 \otimes \cdots \otimes b_p \otimes y) = b_0 \otimes \ldots b_j \otimes 1 \otimes b_{j+1} \cdots \otimes b_{p+1} \otimes y, \quad 0 \leq j \leq p.$$

For X in $\mathfrak{C}_1$, Y in $\mathfrak{C}_2$, set $C^{p,q} = \mathrm{Reg}_B^A(G^p(Y), F^q(X)) = \mathrm{Reg}_B^A(T^{p+1}(Y), S^{q+1}(X))$.

The family of groups $C^{p,q}$ together with the group homomorphisms $\underline{\delta}_q^i$, $\underline{\sigma}_q^i$, $\underline{d}_j^p$ and $\underline{s}_j^p$ is a "double" simplicial group; more precisely we have a bifunctor from $\mathfrak{C}_2 \times \mathfrak{C}_1$ to the category of "double" simplicial groups. In particular, the group $\mathrm{Aut}_{\mathrm{coalg}}{}^A_B(Y)$ acts on the right on $C^{p,q}$ by group homomorphisms. It follows that $C^{p,q}$ has a $\daleth$-action if $Y = \daleth[T]$, with the usual comultiplication, and trivial action and coaction.

Assume from now on, till the end of this section, that A is commutative, B is cocommutative. The preceding constructions apply in the (full) subcategories $\mathfrak{C}_1^{ab}$ (resp., $\mathfrak{C}_2^{ab}$) of commutative (resp. cocommutative) algebras (coalgebras).

Let $\underline{\partial}_1^{p,q} : C^{p,q} \to C^{p+1,q}$ and $\underline{\partial}_2^{p,q} : C^{p,q} \to C^{p,q+1}$ be the differentiation operators

$$\begin{aligned} \underline{\partial}_1^{p,q} &= (f \circ \underline{d}_0^p) * (f^{-1} \circ \underline{d}_1^p) * \dots * (f^{\pm 1} \circ \underline{d}_{p+1}^p), \\ \underline{\partial}_2^{p,q} &= \left[(\underline{\delta}_p^0 \circ f) * (\underline{\delta}_p^1 \circ f^{-1}) * \dots * (\underline{\delta}_p^{p+1} \circ f^{\pm 1})\right]^{(-1)^p}, \end{aligned} \tag{5.1.8}$$

The $(-1)^p$ guarantees that $\underline{\partial}_2\underline{\partial}_1 + \underline{\partial}_1\underline{\partial}_2 = 0$ (cf. [CE, p.63]) and hence $C^{p,q}$ is a double complex. The use of monads avoids various tedious computations; however, it is convenient to note that, by (5.1.7), this double complex is isomorphic to $D^{p,q} = \mathrm{Reg}(T^p(Y), S^q(X))$, that is, to

$$\begin{array}{ccccc} \dots & & \dots & & \\ \uparrow & & \uparrow & & \\ \mathrm{Reg}(Y, X \otimes A^{\otimes 2}) & \longrightarrow & \mathrm{Reg}(B \otimes Y, X \otimes A^{\otimes 2}) & \longrightarrow & \dots \\ \uparrow & & \uparrow & & \\ \mathrm{Reg}(Y, X \otimes A) & \longrightarrow & \mathrm{Reg}(B \otimes Y, X \otimes A) & \longrightarrow & \dots \\ \partial_2^{0,0}\uparrow & & \partial_2^{1,0}\uparrow & & \\ \mathrm{Reg}(Y, X) & \xrightarrow{\partial_1^{0,0}} & \mathrm{Reg}(B \otimes Y, X) & \xrightarrow{\partial_1^{1,0}} & \dots \end{array} \tag{5.1.9}$$

In this presentation, the face, coface, degeneracy and codegeneracy operators are given by

$$\begin{aligned} \delta_q^0 f(b_1 \otimes \dots b_p \otimes y) &= (\rho_X \otimes \mathrm{id})\left(f(b_1 \otimes \dots b_p \otimes y)\right), \\ \delta_q^i f(b_1 \otimes \dots b_p \otimes y) &= (\mathrm{id}_{X \otimes A^{\otimes i-1}} \otimes \Delta \otimes \mathrm{id})\left(f(b_1 \otimes \dots b_p \otimes y)\right), \quad 1 \le i \le q, \\ \delta_q^{q+1} f(b_1 \otimes \dots b_p \otimes y) &= f\left(\rho(b_1 \otimes \dots b_p \otimes y)_k\right) \otimes \rho(b_1 \otimes \dots b_p \otimes y)^k; \\ \sigma_q^i f(b_1 \otimes \dots b_p \otimes y) &= (\mathrm{id}_{X \otimes A^{\otimes i}} \otimes \varepsilon \otimes \mathrm{id})\left(f(b_1 \otimes \dots b_p \otimes y)\right), \quad 0 \le i \le q; \\ d_0^p f(b_1 \otimes \dots b_{p+1} \otimes y) &= b_1 \rightharpoonup f(b_2 \otimes \dots b_{p+1} \otimes y), \\ d_i^p f(b_1 \otimes \dots b_{p+1} \otimes y) &= f(b_1 \otimes \dots b_i b_{i+1} \otimes \dots b_{p+1} \otimes y), \quad 1 \le i \le p, \\ d_{p+1}^p f(b_1 \otimes \dots b_{p+1} \otimes y) &= f(b_1 \otimes \dots b_p \otimes b_{p+1} \rightharpoonup y); \\ s_i^p f(b_1 \otimes \dots b_p \otimes y) &= f(b_1 \otimes \dots b_i \otimes 1 \otimes b_{i+1} \dots b_p \otimes y). \end{aligned}$$

The differential ∂_1, ∂_2 are given by formulas similar to (5.1.8). Let $\mathrm{Reg}_+(B^{\otimes p}\otimes Y, X\otimes A^{\otimes q})$ be the subgroup of those $f \in \mathrm{Reg}(B^{\otimes p}\otimes Y, X\otimes A^{\otimes q})$ such that $f(b_1\otimes\cdots\otimes b_p\otimes y) = \varepsilon(y)\prod_j \varepsilon(b_j)$ if one of the b_j's is equal to 1, and also $(\mathrm{id}_{X\otimes A^{\otimes i}}\otimes\varepsilon\otimes\mathrm{id})\left(f(b_1\otimes\dots b_p\otimes y)\right) = \varepsilon(y)\prod_j\varepsilon(b_j)$. Adding a subscript $+$ throughout in (5.1.9) (with standard conventions if $p=0$ or $q=0$), one obtains a double complex $(5.1.9)_+$ whose total cohomology is the same as that of (5.1.9). This cohomology will be denoted by $H^*(A,B;X,Y)$. Observe that if A is the trivial Hopf algebra (and hence several actions and coactions are uniquely determined) and also Y is trivial, then the cohomology of complex in the lowest row is exactly Sweedler's cohomology [Sw2]; it will be denoted H^*_{Sw}.

Take now $X = Y = \urcorner$ and define $H^*(B,A)$ as the cohomology of the total complex $E^m = \oplus_{p+q=m-2,p\geq1,q\geq1}D^{p,q}$. That is, we drop the first column from the left and the lowest row in (5.1.9), decrease both the vertical and the horizontal index by 1 and take the cohomology of the total complex of the resulting double complex.

Here is a description of the low index cohomology groups. $Z^0(B,A)$ is the subgroup of $\mathrm{Reg}_{1,\varepsilon}(B,A)$ of those maps f satisfying

$$f(b\tilde{b}) = (b_{(1)} \rightharpoonup f(\tilde{b}))f(b_{(2)}), \tag{5.1.10a}$$

$$\Delta(f(b)) = (1\otimes f(b_{(1)})\left(f(\rho(b_{(2)})_j)\otimes\rho(b_{(2)})^j\right). \tag{5.1.10b}$$

By (3.1.15), (3.1.16), f provides simultaneously an algebra map $\tilde{\chi}: B\to C$ and a coalgebra map $\tilde{\xi}: C\to A$, where $C = A\#B$ with trivial cocycle and co-cocycle. Denote $\tilde{\chi} = \chi_f$, $\tilde{\xi} = \xi_f$ to emphasize the dependence on f. Then the multiplication in $\mathrm{Reg}(B,A)$ translates into $\xi_f\circ\chi_g = g*f$.

Next, $Z^1(B,A)$ is the subgroup of those pairs $(\sigma,\tau)\in\mathrm{Reg}_+(B^{\otimes2},A)\times\mathrm{Reg}_+(B,A^{\otimes2})$ such that σ satisfies (3.1.2), τ satisfies (3.1.5), and

$$\begin{aligned}\Delta\left(\sigma(b_{(1)}\otimes\tilde{b}_{(1)})\right)\tau(b_{(2)}\tilde{b}_{(2)}) &= (b_{(1)}\rightharpoonup\tau(\tilde{b}_{(1)}))\\ &\tau(b_{(2)})\left(1\otimes\sigma(b_{(3)}\otimes\tilde{b}_{(2)})\right)\left(\sigma(\rho(b_{(4)})_i\otimes\rho(\tilde{b}_{(3)})_j)\otimes\rho(b_{(4)})^i(b_{(5)}\rightharpoonup\rho(\tilde{b}_{(3)})^j)\right).\end{aligned}\tag{5.1.11}$$

So, let $0\to A\to C\to B\to 0$ be a cleft extension. Then there exist $\rightharpoonup,\rho,\sigma,\tau$ satisfying (3.1.1-9). As A is commutative, B cocommutative and σ and τ invertible, one deduces from (3.1.3, ... , 6) that $\rightharpoonup$ is an action and ρ a coaction; from (3.1.8, ... , 10) that $\rightharpoonup,\rho$ satisfy (5.1.1, 2, 3) and from (3.1.11) that σ,τ fulfill (5.1.11). Thus (σ,τ) is an element of $Z^1(B,A)$. Conversely, given $(\sigma,\tau)\in Z^1(B,A)$, $C = A\,{}^\tau\#_\sigma B$ is a cleft extension of A by B; indeed, (3.1.8) and (3.1.9) follow from (5.1.1, 2, 10) because A is commutative and B cocommutative. Moreover, it follows from [AD, 3.2.14] (see also [Hf]) that two extensions (with the same action and coaction) are equivalent if and only if the corresponding pairs (σ,τ), (σ',τ') are congruent modulo $B^1(B,A)$. Thus $H^1(B,A)$ classifies extensions with given action and coaction. (It is also possible to define a Baer sum translating the group operation of $H^1(B,A)$, see [Hf]).

We record the identities defining $H^2(B,A)$. It would be interesting to interpret these group in terms of the so-called Whitehead modules (cf. [Ho], [Lo]; the denomination varies greatly from author to author). An element of $Z^2(B,A)$ is a triple (ϕ,ψ,γ), where $\phi \in \operatorname{Reg}_+(B, A^{\otimes 3})$, $\psi \in \operatorname{Reg}_+(B^{\otimes 2}, A^{\otimes 2})$, $\gamma \in \operatorname{Reg}_+(B^{\otimes 3}, A)$, such that

(5.1.12)
$$\begin{aligned}&(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\phi(b_{(1)})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\phi(b_{(2)})\\ &\qquad =(1\otimes\phi(b_{(1)}))(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\phi(b_{(2)})\left(\phi(\rho(b_{(3)})_j)\otimes\rho(b_{(3)})^j\right);\end{aligned}$$

(5.1.13)
$$\begin{aligned}&\left(b_{(1)}\rightharpoonup\gamma(h_{(1)}\otimes k_{(1)}\otimes\ell_{(1)})\right)\gamma(b_{(2)}\otimes h_{(2)}k_{(2)}\otimes\ell_{(2)})\gamma(b_{(3)}\otimes h_{(3)}\otimes k_{(3)})\\ &\qquad =\gamma(b_{(1)}h_{(1)}\otimes k_{(1)}\otimes\ell_{(1)})\gamma(b_{(2)}\otimes h_{(2)}\otimes k_{(2)}\ell_{(2)});\end{aligned}$$

(5.1.14)
$$\begin{aligned}&\left(b_{(1)}\rightharpoonup\phi(h_{(1)})\right)\phi(b_{(2)})(1\otimes\psi(b_{(3)}\otimes h_{(2)}))(\mathrm{id}\otimes\Delta)\left(\psi(b_{(4)}\otimes h_{(3)})\right)\\ &\qquad =\phi(b_{(1)}h_{(1)})(\Delta\otimes\mathrm{id})\left(\psi(b_{(2)}\otimes h_{(2)})\right)\\ &\qquad\qquad \left(\psi(\rho(b_{(3)})_i\otimes\rho(h_{(2)})_j)\otimes\rho(b_{(3)})^i(b_{(4)}\rightharpoonup\rho(h_{(2)})^j)\right);\end{aligned}$$

(5.1.15)
$$\begin{aligned}&\left(b_{(1)}\rightharpoonup\psi(h_{(1)}\otimes k_{(1)})\right)\psi(b_{(2)}\otimes h_{(2)}k_{(2)})\Delta\gamma(b_{(3)}\otimes h_{(3)}\otimes k_{(3)})\\ &\qquad =\psi(b_{(1)}h_{(1)}\otimes k_{(1)})\psi(b_{(2)}\otimes h_{(2)})(1\otimes\gamma(b_{(3)}\otimes h_{(3)}\otimes k_{(2)}))\\ &\left(\gamma(\rho(b_{(4)})_i\otimes\rho(h_{(4)})_j\otimes\rho(k_{(3)})_t)\otimes\rho(b_{(4)})^i(b_{(5)}\rightharpoonup\rho(h_{(4)})^j)(b_{(6)}h_{(5)}\rightharpoonup\rho(k_{(3)})^t)\right).\end{aligned}$$

§5.2 Examples. In order to classify all possible cleft extensions between $\Bbbk[N]$ and $\Bbbk\langle G\rangle$, for some groups G and N, one should first determine all the posible compatible actions as in (5.1.1,2), and then compute the corresponding cohomology group. We discuss now some concrete examples to get a flavour of how difficult this task could be. All the groups in this subsection will be denoted multiplicatively, unless explicitly stated.

Assume first that A is a commutative Hopf algebra and $B = \Bbbk\langle G\rangle$ is the group algebra of a group G. Then, as is known, it is equivalent to give an action $\rightharpoonup: B\otimes A \to A$ or a representation π of G by algebra automorphisms on A. Moreover, $\rightharpoonup$ is compatible with the trivial coaction if and only if G acts (via π) by Hopf algebra automorphisms. Now assume that $A = \Bbbk[N]$ is the algebra of functions on a finite group N. Then a representation π by algebra automorphisms is uniquely determined by a homomorphism $\psi : G \to \mathbb{S}(N)$ (where $\mathbb{S}(X)$ denotes the group of bijections of a set X), and G acts, in such case, by Hopf algebra automorphisms if and only if $\psi(G) \subseteq \operatorname{Aut}(N)$. (Explicitly, $\pi(g)f = f\circ\psi(g^{-1})$.)

Dually, if B is a cocommutative Hopf algebra and $A = \Bbbk[N]$, then coactions $\rho : B \to B\otimes A$ are in bijective correspondance with representations of N on B by coalgebra automorphisms (explicitly, $\mu(h)(b) = (\mathrm{id}\otimes e_h)\rho(b)$, where e_h denotes the character of $\Bbbk[N]$ corresponding to $h\in N$); ρ is compatible with the trivial coaction if and only if $\mu(N)\subseteq \operatorname{Aut}_{\text{Hopf alg}}(B)$. If $B = \Bbbk\langle G\rangle$, the former condition amounts to an action of N on the set G, namely, $e_{\theta_h(g)} = \mu(h)(e_g)$, and such ρ is compatible with the trivial action if and only if $\theta(N)\subseteq \operatorname{Aut}(G)$.

So assume that $A = \urcorner[N]$ and $B = \urcorner\langle G\rangle$ and fix $\psi : G \to \mathbb{S}(N)$, $\theta : N \to \mathbb{S}(G)$. Then (5.1.1), (5.1.2) take the following form

$$\psi_g(hy) = \psi_{\theta_y(g^{-1})^{-1}}(h)\psi_g(y), \tag{5.2.1}$$

$$\theta_h(gx) = \theta_h(g)\theta_{\psi_{g^{-1}}(h)}(x). \tag{5.2.2}$$

Here $g, x \in G$, $h, y \in N$. Thus, if in addition

$$\psi_g(1) = 1, \quad \theta_h(1) = 1 \tag{5.2.3}$$

for all g, h, then A, B is a compatible pair. Clearly, (5.2.1) (resp., 5.2.2) are always true for $h = 1$ or $y = 1$ (resp., for $g = 1$ or $x = 1$). (These conditions are equivalent to those in [Ma, Thm. 2.1], [T2]). Letting $\phi(g) = \psi(g)^{-1}$, one obtains the more readable formulas

$$\varphi_g(hy) = \varphi_{\theta_y(g)}(h)\varphi_g(y), \tag{5.2.1'}$$

$$\theta_h(gx) = \theta_h(g)\theta_{\varphi_g(h)}(x). \tag{5.2.2'}$$

In particular cases, one obtains from the above discussion some strong requirements; for instance, (5.2.3) implies the existence of a group morphism $G \to \mathbb{S}(N - \{1\})$. Assume for example that G has order p and $N = \mathbb{Z}/q\mathbb{Z}$, with $2 \le p \le q$. If θ is non-trivial, then some factor of q (different from 1) divides $(p-1)!$; thus if q is prime then θ is always trivial. In such case, ψ maps G onto a subgroup of $\mathrm{Aut}(N) \simeq \mathbb{Z}/(q-1)\mathbb{Z}$. If in addition p and $q-1$ are coprime, ψ is also trivial. See [Gr], [By2] for the classification in the case $p = q$ prime.

The conditions (5.2.1,2) have a cohomological interpretation. Let us consider the groups $\mathcal{C} = \{R : N \to G\}$ and $\mathcal{D} = \{T : G \to N\}$, with pointwise multiplication. We let N act on $\mathcal{D}$ and G act on $\mathcal{C}$ by $h.T = T \circ \theta(h^{-1})$, $g.R = R \circ \varphi(g)$. Let us define $E : N \to \mathcal{D}$, $F : G \to \mathcal{C}$, by

$$E_h(g) = \varphi_g(h^{-1})^{-1}, \qquad F_g(h) = \theta_h(g).$$

Lemma 5.2.4. *(5.2.1) (resp., (5.2.2)) holds if and only if E (resp., F) is a (non-commutative, see* [Se]*) 1-cocycle.*

Proof. F is a 1-cocycle if and only if $F_{gx} = F_g g.F_x$, if and only if $\theta_h(gx) = \theta_h(g)\theta_{\varphi_g(h)}(x)$, and this is (5.2.2'). The other is analogous. □

Assume for example that φ_g is a group homomorphism. Then (5.2.1) implies that $\varphi_g = \varphi_{\theta_y g}$, for all g, y. Assume further that F is a coboundary, i.e. that there exists $T : H \to G$ such that $\theta_h(g) = T(h)^{-1}T(\varphi_g(h))$. Now $\theta_{hy}(g) = \theta_h(\theta_y(g)) = T(h)^{-1}T(\varphi_{\theta_y(g)}(h)) = \theta_h(g)$; one concludes that $\theta_y = \mathrm{id}$, for all y. (Of course, the same holds if φ is injective).

Now we assume that the compatible actions are fixed. We discuss first the computation of the full cohomology and then that of the first cohomology group. The standard tools to deal with the total cohomology of a double complex are spectral sequences. Let

$${}_IE_2^{p,q} = H^q_{\mathrm{vert}}H^p_{\mathrm{hor}}(\tilde{D}^{\bullet,\bullet}), \qquad \left(\text{resp.,} \quad {}_{II}E_2^{p,q} = H^p_{\mathrm{hor}}H^q_{\mathrm{vert}}((\tilde{D}^{\bullet,\bullet})\right)$$

be the double complex obtained by taking first the cohomology of the rows–with respect to the horizontal differential–and then the cohomology with respect to the differential induced by the vertical one in the initial complex (resp., interchanging vertical by horizontal throughout). (Here ($\tilde{D}^{\bullet,\bullet}$ is the double complex arising from (5.1.9) by dropping a row and a column as explained above). In practice one needs to show that any of these E_2 degenerates sufficiently to allow to compute $H^\bullet(B, A)$. We include here some remarks of how this task could eventually be accomplished. Observe first that

$$H^{p-1}_{\text{hor}}(\tilde{D}^{\bullet,(q-1)}) = \begin{cases} H^p_{\text{Sw}}(B, A^{\otimes q}), & \text{if } p \geq 2, \\ Z^1_{\text{Sw}}(B, A^{\otimes q}), & \text{if } p = 1. \end{cases}$$

Moreover, if B is the group algebra of a finite group G this is isomorphic either to the group cohomology group $H^p(G, (A^{\otimes q})^\times)$, if $p \geq 2$, or to $Z^1(G, (A^{\otimes q})^\times)$, if $p = 1$, where the $\times$ indicates the group of units of the algebra in question, cf. [Sw2, Thm. 3.1]. Assume further that $A = \daleth[N]$, for some finite group N; then $A^{\otimes q} = \daleth[N^q]$, where N^q denotes the direct product of q copies of N, and the group of units of $A^{\otimes q}$ consists of the nowhere vanishing functions on N^q, i.e. $(\daleth^\times)^{N^q}$. Now G acts on N^q by bijections; let $\mathcal{O}_1, \dots \mathcal{O}_d$ be the orbits of this action. Then as G-module, $(\daleth^\times)^{N^q} \simeq \oplus_{1 \leq j \leq d} (\daleth^\times)^{\mathcal{O}_j}$ and the cohomology we are looking for can be deduced, as the cohomology functors are additive, from the cohomologies of all the G-modules $(\daleth^\times)^{\mathcal{O}}$, $\mathcal{O} = \mathcal{O}_j$ for some j. Let K be the isotropy group of some point in $\mathcal{O}$; that is $\mathcal{O} \simeq G/K$. Then by Shapiro's Lemma $H^\bullet(G, (\daleth^\times)^{\mathcal{O}}) \simeq H^\bullet(K, \daleth^\times)$, with K acting trivially on $\daleth^\times$ (see [Br, pp. 73 and 136]; here one uses that G is finite, and hence that $(\daleth^\times)^{\mathcal{O}} = \operatorname{Ind}_K^G(\daleth^\times) = \operatorname{Coind}_K^G(\daleth^\times)$, cf. [Br p. 70]). Finally one has the universal coefficient exact sequence, cf. [Br, p. 60]:

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbf{Z}}(H_{n-1}(K), \daleth^\times) \longrightarrow H^n(K, \daleth^\times) \longrightarrow \operatorname{Hom}_{\mathbf{Z}}(H_n(K), \daleth^\times) \longrightarrow 0.$$

A similar analysis holds for the other spectral sequence; one has to replace [Sw2, Thm. 3.1] by its dual version, which in turn follows from the following observation: if N is a finite group and B is a coalgebra, then $\operatorname{Reg}(B, \daleth[N])$ is naturally isomorphic to $\operatorname{Hom}_{\text{sets}}(N, (B^*)^\times)$, via $f \mapsto (h \mapsto e_h f)$, where for $h \in N$ e_h denotes the corresponding character of $\daleth[N]$. This isomorphism gives rise to an isomorphism of complexes, as in *loc. cit.*

Here is a more precise result.

Proposition 5.2.5. *Consider the compatible abelian pair $(\daleth[N], \daleth\langle G\rangle)$, where G and N are finite groups, with trivial action and coaction. Then*

$${}_I E_2^{p-1,q-1} = \begin{cases} H^q(N, H^p(G, \daleth^\times)), & \text{if } p \geq 2, q \geq 2, \\ H^q(N, Z^1(G, \daleth^\times)), & \text{if } p = 1, q \geq 2, \\ Z^1(N, H^p(G, \daleth^\times)), & \text{if } q = 1, p \geq 2, \\ Z^1(N, Z^1(G, \daleth^\times)), & \text{if } p = 1 = q. \end{cases}$$

Here G acts trivially on $\daleth^\times$ and the various groups appearing in the right-hand side refer to group cohomology.

Proof. Let X, Y be two sets. There is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{sets}}(X, \mathrm{Hom}_{\mathrm{sets}}(Y, ㄱ^{\times})) \simeq \mathrm{Hom}_{\mathrm{sets}}(Y, \mathrm{Hom}_{\mathrm{sets}}(X, ㄱ^{\times})),$$

and hence one has a natural isomorphism of double simplicial groups

$$\mathrm{Hom}_{\mathrm{sets}}(G^p, \mathrm{Hom}_{\mathrm{sets}}(N^q, ㄱ^{\times})) \simeq \mathrm{Hom}_{\mathrm{sets}}(N^q, \mathrm{Hom}_{\mathrm{sets}}(G^p, ㄱ^{\times})).$$

Taking cohomology with respect to G^p and as the functor $\mathrm{Hom}_{\mathrm{sets}}(N^q, \_)$ is exact, one gets

$$H^p(G, \mathrm{Hom}_{\mathrm{sets}}(N^q, ㄱ^{\times})) \simeq \mathrm{Hom}_{\mathrm{sets}}(N^q, H^p(G, ㄱ^{\times})).$$

(Recall the standard resolution of a trivial G-module [Br, p. 59]). Taking now cohomology with respect to N^q one has

$$H^q\left(H^p(G, \mathrm{Hom}_{\mathrm{sets}}(N^{\bullet}, ㄱ^{\times}))\right) \simeq H^q(N, H^p(G, ㄱ^{\times})).$$

But we have already observed above that ${}_IE_2^{p-1,q-1}$ coincides with the left hand side in the last isomorphism. □

Now if one is merely interested in extensions, the situation is neatly simpler [CE], since there are exact sequences

$$\begin{aligned} 0 \to {}_IE_2^{1,0} \to H^1 \to {}_IE_2^{0,1} \to {}_IE_2^{2,0} \to H^2, \\ 0 \to {}_{II}E_2^{0,1} \to H^1 \to {}_{II}E_2^{1,0} \to {}_{II}E_2^{0,2} \to H^2. \end{aligned}$$

Thus for a compatible abelian pair $(ㄱ[N], ㄱ\langle G\rangle)$, where G and N are finite groups, with trivial action and coaction, one has
(5.2.6)

$$0 \to \mathrm{Hom}_{\mathrm{gr}}(N, H^2(G, ㄱ^{\times})) \to H^1 \to H^2(N, \mathrm{Hom}_{\mathrm{gr}}(G, ㄱ^{\times})) \to H^2(N, H^2(G, ㄱ^{\times})).$$

As an application, we can now prove:

Theorem 5.2.7. *Assume that the characteristic of* $ㄱ$ *is 0. Let* C *be a finite Hopf algebra of order* pq *which is not* q*-simple, where* p *and* q *are primes,* $2 < p \leq q$, p *and* $q-1$ *coprime. Then* C *is commutative and cocommutative.*

Proof. By (4.1), (3.1.17), and passing to the dual if necessary, we can assume that C fits into a cleft extension $0 \to ㄱ[\mathbb{Z}/q\mathbb{Z}] \to C \to ㄱ\langle\mathbb{Z}/p\mathbb{Z}\rangle \to 0$. By the remarks before (5.2.4), we know that the action and the coaction are trivial, so we can use (5.2.5).

Assume first that $p < q$. On one hand, $H^2(\mathbb{Z}/q\mathbb{Z}, \mathrm{Hom}_{\mathrm{gr}}(\mathbb{Z}/p\mathbb{Z}, ㄱ^{\times})) = 0$ by [Br, 10.2]. On the other hand, we can also assume that $ㄱ$ is algebraically closed and hence $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{Z}/q\mathbb{Z}, H^2(\mathbb{Z}/p\mathbb{Z}, ㄱ^{\times})) = 0$ because $H^2(\mathbb{Z}/p\mathbb{Z}, ㄱ^{\times}) = ㄱ^{\times}/(ㄱ^{\times})^p$ [Br, p. 58]. Thus C is the trivial extension and the claim follows.

Assume now that $p = q$. Then the above argument shows this time that $H^1 \simeq H^2(\mathbb{Z}/p\mathbb{Z}, \mathrm{Hom}_{\mathrm{gr}}(\mathbb{Z}/p\mathbb{Z}, ㄱ^{\times})) = H^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, since (we assume that) $ㄱ$ is algebraically closed. The assertion follows now from group theory and (5.2.8) below. □

We still assume that $ㄱ$ is algebraically closed.

(5.2.8). *Let G, N be finite groups as above and let $A = \daleth[N]$, $B = \daleth\langle G\rangle$. Assume that N is abelian and denote its character group by $\hat{N}$. Fix an action of G on $\hat{N}$ by group automorphisms. Thus one has an action of G on N by group automorphisms and a fortiori a compatible pair structure on (A, B), with trivial coaction. Then there is a monomorphism $H^2(G, \hat{N}) \to H^1(A, B)$.*

Proof. Let

$$1 \to \hat{N} \to P \to G \to 1 \tag{5.2.9}$$

be an exact sequence of groups. Taking group algebras, one gets an exact sequence of Hopf algebras

$$\daleth \to \daleth\langle\hat{N}\rangle \to \daleth\langle P\rangle \to \daleth\langle G\rangle \to \daleth. \tag{5.2.10}$$

Clearly $\daleth\langle\hat{N}\rangle = \daleth[N]$. Moreover, two exact sequences like (5.2.9) are isomorphic if and only if the corresponding sequences like (5.2.10) are; here one uses that an isomorphism of Hopf algebras preserves group-like elements. □

§5.3 **Remarks on the general case.** Let (A, B) be a compatible pair as in the beginning of §5.1. We define $Z^0(B, A)$ as the subset of $\mathrm{Reg}_{1,\varepsilon}(B, A)$ of those maps f satisfying (5.1.10). Its elements could be called Hopf 0-cocycles. In the same vein, one can consider only the algebra (resp., coalgebra) structure on A (resp., on B), forgetting the compatibility conditions (5.1.1-3), and hence the algebra (resp. coalgebra) cocycles, which are the elements of $\mathrm{Reg}_1(B, A)$ (resp., $\mathrm{Reg}_\varepsilon(B, A)$) satisfying (5.1.10a) (resp., (b)); they are non-commutative versions of those in [Sw2].

Example. Let $C = B^{\mathrm{op}} \otimes A$ with the product Hopf algebra structure and define $\rightharpoonup : B \otimes C \to C$ and $\rho : B \to B \otimes C$ by $b \rightharpoonup (d \otimes a) = d \otimes b \rightharpoonup a$, $\rho(b) = \rho^{13}(b)$. Then (C, B) is a compatible pair. Moreover $\rho : B \to C$ is an algebra cocycle (whose inverse is $((\mathcal{S} \otimes \mathrm{id})\rho)$. There is an analogous dual statement. This generalizes (5.2.4).

Let $Z^1(B, A) = \{(\sigma, \tau) \in \mathrm{Reg}_{1,\varepsilon}(B \otimes B, A) \times \mathrm{Reg}_{1,\varepsilon}(B, A \otimes A) : \sigma$ satisfies (3.1.1,2), (5.3.1); τ satisfies (3.1.4,5), (5.3.2) and both satisfy (3.1.11), (5.3.3, 4)$\}$. Here (5.3.1) (resp., (5.3.2), (5.3.3), (5.3.4)) is the condition which follows from (3.1.3) (resp., from (3.1.6), (3.1.8), (3.1.9)) because $\rightharpoonup$ (resp., ρ, (A, B)) is an action (resp., a coaction, a compatible pair). Let $B^0(B, A)$ be the group of those $\phi \in \mathrm{Reg}_{1,\varepsilon}(B, A)$ such that $\phi(b_{(1)})b_{(2)} \rightharpoonup a = b_{(1)} \rightharpoonup a\phi(b_{(2)})$, $(1 \otimes \phi(b_{(1)}))\rho(b_{(2)}) = \rho(b_{(1)})(1 \otimes \phi(b_{(2)}))$. Let $H^1(B, A)$ be the quotient space of $Z^1(B, A)$ by the following action of $B^0(B, A)$ (see [AD]): $\phi(\sigma, \tau) = ({}^{\phi}\sigma, {}^{\phi}\tau)$, where

$$\begin{aligned} {}^{\phi}\sigma(b \otimes d) &= \phi(b_{(1)}) \left(b_{(2)} \rightharpoonup \phi(d_{(1)})\right) \sigma(b_{(3)} \otimes d_{(2)}) \phi^{-1}(b_{(4)} d_{(3)}), \\ {}^{\phi}\tau(b) &= \Delta\phi(b_{(1)})\tau(b_{(2)})(\phi^{-1} \otimes \mathrm{id})\rho(b_{(3)}) \left(1 \otimes \phi^{-1}(b_{(4)})\right). \end{aligned}$$

It is clear how a morphism of compatible pairs should be defined. Fix a Hopf algebra B and consider the category $\mathcal{H}_B$ whose objects are triples $(A, \rho, \rightharpoonup)$ giving rise to a compatible pair (A, B) and whose arrows are the Hopf algebra morphisms which preserve the action and the coaction. Defined as above, Z^0 gives rise to a functor from this category. Let

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

be an exact sequence of Hopf algebras in $H^1(B, A)$. In particular, it follows that the coaction $\rho_3 : B \to B \otimes A_3$ is trivial. One checks easily that the sequence

$$0 \longrightarrow Z^0(B, A_1) \xrightarrow{f_1^0} Z^0(B, A_2) \xrightarrow{f_2^0} Z^0(B, A_3)$$

is exact. One would be happy to extend this sequence to an exact sequence involving H^2. However, H^2, at least as defined here, is not functorial. Let $f : A \to A'$ be a map in $\mathcal{H}_B$ and let f^1 denote the map $\operatorname{Reg}(\mathrm{id}, f) \times \operatorname{Reg}(\mathrm{id}, f \otimes f) : \operatorname{Reg}_{1,\varepsilon}(B \otimes B, A) \times \operatorname{Reg}_{1,\varepsilon}(B, A \otimes A) \to \operatorname{Reg}_{1,\varepsilon}(B \otimes B, A') \times \operatorname{Reg}_{1,\varepsilon}(B, A' \otimes A')$. Then the conditions (3.1.1,4) are clearly preserved by f^1. The same is true for (3.1.2,5,11); this is more transparent when expressing this axioms in the following way:

(3.1.2) $$d_0\sigma * d_2\sigma = d_3\sigma * d_1\sigma,$$

(3.1.5) $$\delta^1\tau * \delta^3\tau = \delta^2\tau * \delta^0\tau,$$

(3.1.11) $$\delta^1\sigma * d_1\tau = d_2\tau * d_0\tau * \delta^2\sigma * \delta^0\sigma.$$

(5.3.2,4) also follow easily. But (5.3.1, 3) are apparently true only if f is surjective. These problems seem to be similar to those arising in the non-abelian group cohomology theory.

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