PIERI FOR ISOTROPIC GRASSMANNIANS;

THE OPERATOR APPROACH

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Piotr Pragacz[1] & **Jan Ratajski**

INTRODUCTION

The goal of this paper is to give a simple, transparent proof of a Pieri-type formula for the multiplication in the Cohomology/Chow ring of the Grassmannian of (maximal) isotropic spaces. Originally, this formula was given by Boe and Hiller in [H-B], with a proof being a very complicated, inductive application of the Chevalley formula for multiplication in the Cohomology/Chow ring of isotropic flag variety. In contrast to [H-B] our proof makes no use of the Chevalley formula; two main tools used here are: a Leibnitz-type formula for the [B-G-G]&[D] - operators, and a choice of special reduced decompositions of elements apearing in the Pieri formula. The present approach determines in an efficient and fast way both the shapes of Schubert cycles in the Pieri formula, as well as their multiplicities (which are powers of 2). This Pieri formula together with a Giambelli-type formula from [P] and the Basis theorem give us a symplectic and orthogonal Schubert Calculus - see Section 7 where a new simple proof of the Basis Theorem is also given.

[1] Visiting the Max-Planck Institut during the preparation of this paper.

1. PRELIMINARIES, NOTATION [2] AND CONVENTIONS

Let G denote the Grassmannian of n-dimensional isotropic subspaces in $\mathbb{C}^{2n}$ with respect to a non-degenerate symplectic form on $\mathbb{C}^{2n}$. Let F denote the flag variety of (total) isotropic flags in $\mathbb{C}^{2n}$ (with respect to the same symplectic form). By ρ we will denote the partition $(n,\dots,2,1)$. Let $\lambda\subset\rho$ be a strict partition $\lambda = (\lambda_1>\lambda_2>\dots>\lambda_k>0)$. We associate to λ the element w_λ of the symplectic Weyl group W :

$$w_\lambda = s_{n-\lambda_k+1}\, s_{n-\lambda_k+2} \cdots s_{n-1}\, s_n \cdots s_{n-\lambda_1+1}\, s_{n-\lambda_1+2} \cdots s_{n-1}\, s_n \quad {}^{3}$$

(see [H-B] for details about W). Note that w_λ has a form

$$(y_1,\dots,y_{n-k};\overline{n+1-\lambda_k},\overline{n+1-\lambda_{k-1}},\dots,\overline{n+1-\lambda_1}) \quad ,$$

where $y_1< \dots <y_{n-k}$ and $\{y_1,\dots,y_{n-k};n+1-\lambda_k,n+1-\lambda_{k-1},\dots,n+1-\lambda_1\} = \{1,\dots,n\}$, in the standard "barred-permutation notation" (see loc.cit.). Then, denoting by α the right end root, the subgroup W_α of W generated by $\{s_i,\ i<n\}$ is the symmetric group S_n, and w_λ belongs to the set of minimal left coset representatives of W_α in W.

From the theory in [B-G-G] and [D] we have a Schubert cycle $X_{w_\lambda} \in A^{|\lambda|}(F)$ which in fact belongs to $A^{|\lambda|}(G) \subset A^{|\lambda|}(F)$, where $|\lambda|=\sum \lambda_i$. Denote this element in $A^{|\lambda|}(G)$ by $\sigma(\lambda)$, for short.

Define the numbers $z_i := n+1-\lambda_i$. Then

$$w_\lambda = (y_1,\dots,y_{n-k};\bar{z}_k,\overline{z_{k-1}},\dots,\bar{z}_1).$$

As usual, we will associate to a partition λ a diagram D_λ. The elements of the D_λ will be boxes (and not dots). This will allow us to speak about "connected components" of differences between diagrams without misunderstandings.

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[2] Our notation here is a combination of notations from [H-B] and [P].

[3] Writing here and in the sequel $s_{i_1}\, s_{i_2} \dots s_{i_k}$ we mean that we perform first s_{i_1}, then - s_{i_2} etc.

The following result was proved originally in [H-B].

<u>Theorem 1.1</u> Let $\lambda=(\lambda_1,\dots,\lambda_k) \subset \rho$ be a strict partition. The following equality holds in $A^*(G)$ $(p=1,\dots,n)$:

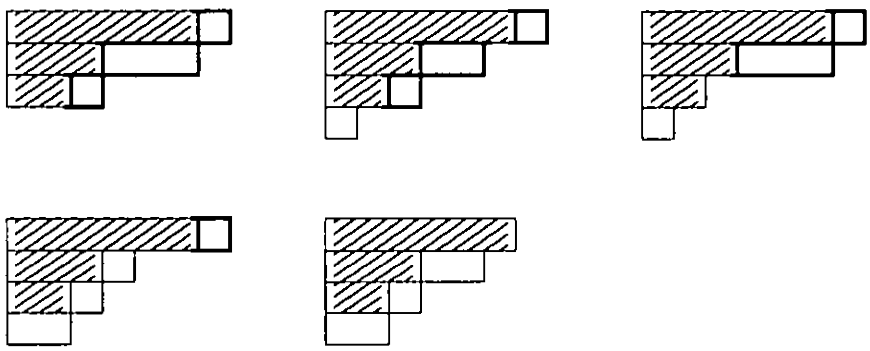
$$\sigma(\lambda)\ \sigma(p) = \sum 2^{m(\lambda,\mu)}\ \sigma(\mu)\ ,$$

where the sum is over strict partitions μ such that $\lambda_{i-1} \geq \mu_i \geq \lambda_i$ $(\lambda_0=n,$ $\lambda_{k+1}=0)$, $|\mu|=|\lambda|+p$ and $m(\lambda,\mu)$ is the number of connected components of $D_\mu \backslash D_\lambda$ not meeting the first column.

<u>Example 1.2</u> n=7

$\sigma(632)\ \sigma(5) = 2\ \sigma(763) + 2^2\ \sigma(7531) + 2\ \sigma(7621) + 2\ \sigma(7432) + \sigma(6532).$

Fig. 1



(If we adopt a name "the characteristic box of a component" for the lowest box to the right of a component, then the original Hiller-Boe's formulation used the cardinality of the set of the characteristic boxes.)

For a given $w \in W$, we denote by $R(w)$ the set of reduced decompositions of w.

2. [B-G-G] & [D] - OPERATORS

Let $x=(x_1,\dots,x_n)$ be independent variables. It follows from [B-G-G] and [D] that $A^*(F)$ is identified with $\mathbb{Z}[x]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by symmetric polynomials in $x_1^2,\dots,x_n^2$ without constant term. Also, $A^*(G)$ is identified with $(\mathbb{Z}[x]/\mathcal{I})^{S_n}$ i.e. with the quotient of the symmetric polynomials modulo $\mathcal{I}$ restricted to the ring of symmetric polynomials.

We have "divided differences":

$$\partial_i : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x] \quad \text{(of degree } -1\text{)}, \ i=1,\dots,n, \quad \text{defined by}$$

$$\partial_i(f)=(f - s_i f)/(x_i - x_{i+1}) \qquad i=1,\dots,n-1,$$

$$\partial_n(f) = (f - s_n f)/2x_n \quad .$$

The key tool for our purposes is a Leibnitz-type formula:

(2.1) $$\partial_i(f\cdot g) = (\partial_i f)\cdot g + (s_i f)\cdot(\partial_i g).$$

We will need in the sequel the following formulas for generating functions. Let $\mathbf{a}=(a_1,\dots,a_n)\in \{-1,0,1\}^n$. Define

$$E_{\mathbf{a}} := E_{\mathbf{a}}(t) := \prod_{i=1}^{n} (1+a_i x_i t)$$

(small bold letters will be reserved only for sequences of this form). For example for $\mathbf{o}=(0,\dots,0)$, $E_{\mathbf{o}} = 1$. For $\mathbf{1}=(1,\dots,1)$, $E_{\mathbf{1}}=(1+x_1 t)\dots$ $\dots(1+x_n t)=E$, say, is the generating function for the elementary symmetric polynomials.

In formulas below, 0 (usual zero) will denote the zero function.

<u>Lemma 2.2</u> :

a) $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$, where $\mathbf{a}' = \begin{cases} (a_1,\dots,a_{i-1},a_{i+1},a_i,a_{i+2},\dots,a_n) & i<n \\ (a_1,\dots,a_{n-1},-a_n) & i=n \end{cases}$

b) For $i=1,2,\dots,n-1$

$$\partial_i(E_{\mathbf{a}}) = \begin{cases} 0 & a_i=a_{i+1} \\ tE_{\mathbf{a}'} & a_i=a_{i+1}+1 \\ -tE_{\mathbf{a}'} & a_i=a_{i+1}-1 \\ 2tE_{\mathbf{a}'} & a_i=a_{i+1}+2 \\ -2tE_{\mathbf{a}'} & a_i=a_{i+1}-2 \end{cases} ,$$

where $\mathbf{a}'=(a_1,\dots,0,0,\dots,a_n)$ is $\mathbf{a}$ with a_i, a_{i+1} replaced by zeros.

c) $$\partial_n(E_{\mathbf{a}}) = a_n \; t \; E_{(a_1,\dots,a_{n-1},0)} .$$

Proof. – a straightforward verification. We check, for instance, (b4). In this case $(a_i, a_{i+1}) = (1,-1)$. Then

$$\partial_i(E_{\mathbf{a}}) = \prod_{j\neq i,i+1} (1+a_j x_j t)\ \partial_i\big((1+x_i t)(1-x_{i+1})\big) = \prod_{j\neq i,i+1} (1+a_j x_j t)\cdot 2t =$$

$$= 2t\cdot E_{\mathbf{a}'}\ . \quad \square$$

Note that the effect of applying the ∂_i to $E_{\mathbf{a}}$ (if nonzero) is $E_{\mathbf{a}'}$, where $a'_i = a'_{i+1} = 0$.

3. PIERI'S FORMULA AND THE LEIBNITZ RULE

First, we summarize the theory from [B-G-G] and [D]. For every reduced decomposition $w = s_{i_1} \dots s_{i_k}$ one can define $\partial_w = \partial_{i_1} \circ \dots \circ \partial_{i_k}$ – an operator on $\mathbb{Z}[x]$ of degree $-\ell(w)$. In fact ∂_w does not depend on the reduced decomposition chosen. There exists a ring homomorphism

$$c:\ \mathbb{Z}[x] \longrightarrow A^*(F)\quad ,$$

(called the characteristic map) defined for a homogeneous $f \in \mathbb{Z}[x]$ by

$$c(f) = \sum_{\ell(w)=\deg f} \partial_w(f)\ X_w\ .$$

For instance, denoting by e_p the p-th elementary symmetric polynomial in x, we have

$$c(e_p) = \sigma(p) = X_{s_{n-p+1}\dots s_{n-1}s_n} \in A^p(G)$$

([H-B, Lemma 2.13']).

The operators ∂_w give rise to operators on $A^*(F)$ (denoted by the same letters) and these two families of operators commute with c. Moreover for w,v, $\partial_w(X_v) = 1$ iff $w=v$.

Let f_λ be such that $c(f_\lambda) = \sigma(\lambda)$. Our goal is to find coefficients m_μ appearing in

$$c(\ f_\lambda\cdot e_p) = \sum m_\mu\ \sigma(\mu)\ .$$

Consider $D \subset D_\mu$. The boxes in D_μ which belong to D will be called D-boxes; the boxes in $D_\mu \setminus D$ will be called non D-boxes. We associate with D the following operators $\bar{\partial}^D_\mu$ and $\partial^D_{-\mu}$. For technical reasons we will use, from now on, the following coordinates for indexing boxes in $\mu\subset\rho$:

$$\begin{array}{c|cccc} & n & n-1 & \dots & 2 \quad 1 \\ \hline 1 & & & & \\ 2 & & & & \\ \vdots & & & & \\ n & & & & \end{array}$$

(i.e. the first column has the number n).

In Definitions (3.1),(3.2) we read D_μ row by row from left to right staring from the first row.

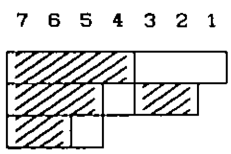
(3.1) Definition of $\partial^D_{\underline{\mu}}$: Read D_μ. Every D-box in the i-th column gives us the s_i. Every non D-box in the i-th column gives the ∂_i. Then $\partial^D_{\underline{\mu}}$ is the composition of the so obtained s_i's and ∂_i's (the composition written from the right to left).

(3.2) Definition of r_D : Read D_μ. Every D-box in the i-th column gives us the s_i. Non D-boxes have no influence on r_D. Then r_D is the word obtained by writing the so obtained s_i's from right to left.

(3.3) Definition of $\bar{\partial}^D_\mu$: $\bar{\partial}^D_\mu := \partial_{r_D}$.

Example 3.4 $\mu=(763)$, $n=7$.

Fig. 2



(D-boxes are "dark" here)

$$\partial^D_{\underline{\mu}} = \partial_5\circ s_6\circ s_7\circ s_2\circ s_3\circ\partial_4\circ s_5\circ s_6\circ s_7\circ\partial_1\circ\partial_2\circ\partial_3\circ s_4\circ s_5\circ s_6\circ s_7 ,$$

$$r_D = s_6 s_7 s_2 s_3 s_5 s_6 s_7 s_4 s_5 s_6 s_7 ,$$

$$\bar{\partial}^D_\mu = \partial_6\circ\partial_7\circ\partial_2\circ\partial_3\circ\partial_5\circ\partial_6\circ\partial_7\circ\partial_4\circ\partial_5\circ\partial_6\circ\partial_7 .$$

Proposition 3.5 In the above notation,

$$m_\mu = \sum \bar{\partial}^D_\mu(f_\lambda)\cdot\partial^D_{-\mu}(e_p) \quad ,$$

where the sum is over all $D\subset D_\mu$ such that $r_D\in R(w_\lambda)$ and $\partial^D_{-\mu}(e_p)\neq 0$.

Proof. This is a consequence of consecutive applications of the Leibnitz rule (2.1) used in this way: we apply only the ∂_i's (and the identity operators) to f_λ ; and both the s_i's and ∂_i's to e_p. □

4. WHAT ARE THE $D \subset D_\mu$ FOR WHICH $r_D\in R(w_\lambda)$?

We will treat a reduced decomposition of w_λ as a composition $s_{i_1} s_{i_2} \ldots s_{i_m}$ of "simple transposition"-operations (we will call them "s_{i_k} - operations", $k=1,\ldots,m$) such that

$$\{\ldots\Big(\big((1,\ldots,n)\circ s_{i_1}\big)\circ s_{i_2}\Big)\circ\ldots\Big)\circ s_{i_m} = (y_1,\ldots,y_{n-k};\bar{z}_k,\overline{z_{k-1}},\ldots,\bar{z}_1)$$

$(1,\ldots,n)$ denoting the identity permutation. Recall that s_i, $i<n$, acting from the right on v, interchanges the value of v on the i-th and (i+1)-th place. The s_n supplies the last component of v (in the "barred notation") with a bar, if this component is bar-free.

Proposition 4.1 Exactly one (bar-free) z. is nontrivially involved in a "s_{i_k}-operation". More precisely,

a) If $i_k=n$, then the operation is:

$$\ldots\ z. \longrightarrow \ldots\bar{z}. \quad .$$

b) If $i_k<n$, then the operation is:

$$\ldots\ z.\ x\ \ldots \longrightarrow \ldots x\ z.\ \ldots \quad ,$$

where $x \neq z_i$ $(i=1,\ldots,k)$.

Proof. Since $s_{i_1} \ldots s_{i_m} \in R(w_\lambda)$, we have for $k=1,\ldots,m$,

$$\ell(s_{i_1} \ldots s_{i_k}) = \ell(s_{i_1} \ldots s_{i_{k-1}}) + 1 .$$

It is well known (and easy to check) that the length of a "barred permutation" in W is a sum of the length of the same permutation without bars (in S_n) plus the sum of the numbers $2d_i+1$, each coming from a "barred place": to a given "barred place" i, say, we associate

$$d_i = \text{card} \{ j : j>i \ \& \ w(j)>w(i) \} .$$

It follows from this formula that no "s_{i_k}-operation" interchanges $y.,y_*$ and $\bar{z}.,\bar{z}_*$ (after they have received bars). Consequently $z_1,\ldots,z_k$ receive their bars in the order $k,\ldots,2,1$. This information and the above length-formula imply that no "s_{i_k}-operation" can interchange $z_*,z.$ (before they are supplied with bars). Thus, at most one $z.$ is nontrivially involved in every "s_{i_k}-operation". Every z_i needs $\geq n-z_i$ "s_{i_k}-operations" to pass from the z_i-th place the n-th place (where it receives its bar). This, in sum, requires $\geq n-z_i+1=\lambda_i$ "s_{i_k}-operations". Since $|\lambda|=m$, we conclude that there is exactly one (bar-free) $z.$ involved nontrivially in any "s_{i_k}-operation". This proves the Proposition. □

<u>Corollary 4.2</u> No "s_{i_k}-operation" as above can interchange $z.$ and z_* .

Now, following definitions and notation of Section 2, we introduce a notion of a *mark of a D-box*. Assume that a D-box appears in the i-th column. Its mark is defined to be the integer m such that the "s_i-operation" supplies z_m with a bar if $i=n$; or it acts on the i-th , $(i+1)$-th places as:

$$\ldots z_m \, x \ldots \quad \longrightarrow \quad \ldots x \, z_m \ldots$$

(here, $x\neq z_k$ $k=1,\ldots,n$; $i<n$).

<u>Lemma 4.3</u> :

a) The D-boxes with a fixed mark in one row form a connected set.

b) In a fixed row, the two sets of D-boxes labelled by different marks are disconnected (i.e. there is at least one non D-box between them).

c) The sequence of boxes with mark i is of the form

$$(t_n, n) \ , \ (t_{n-1}, n-1) \ , \ \dots \ , \ (t_{z_i}, z_i) \quad ,$$

where $t_n \le t_{n-1} \le \dots \le t_{z_i}$. (Recall that $n-z_i+1=\lambda_i$.)

d) The marks of boxes in fixed column increase from the top to the bottom.

Proof. The assertions a) and b) are obvious. As for c) the fact that the columns of the (mark i)-boxes are $n, n-1, \dots, z_i$ is clear as z_i is transformed from the n-th to the z_i-th place by a sequence of succesive transpositions. assume that the second assertion of c) is not valid. This means that the following configuration of (mark i)-boxes appears

Fig. 3

(the picture presents three consecutive rows; d is in the p-th column, d' is in the q-th column; q>p+1).

But d and d' cannot have the same mark! Indeed, the sequence of "s.-operations" here is: $\dots s_p \dots s_q \dots$ and z_i is not involved in any "s.-operation" from the interval between s_p and s_q. Moreover, a fixed z_i cannot be nontrivially involved succesively in s_p and s_q for q>p+1. This contradiction proves c).

d) Two marks appear in a fixed column in a not asserted order only if some "barred-free" $z_{..}z_{*}$ have changed their order during "s.-operations". This contradicts Corollary 4.2. □

Corollary 4.4 Every configuration of boxes $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$ can be obtained from $D_\lambda \subset D_\mu$ by the following operation applied consecutively to the rows of D_λ with numbers $i=\ell(\lambda), \ell(\lambda)-1, \dots, 2, 1$: the boxes

$$(i,n), \ (i,n-1), \ \dots \ ,(i,z_i)$$

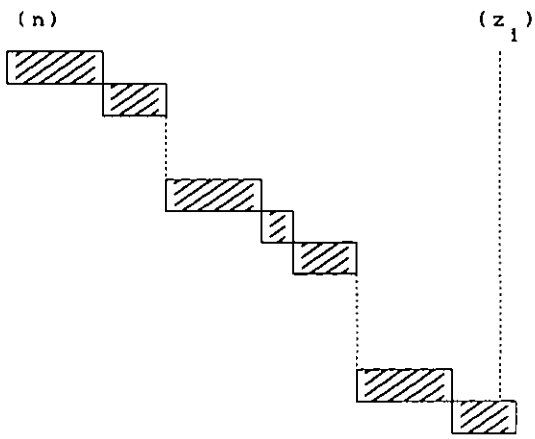
are transformed to

$$(t_n, n) \ , \ (t_{n-1}, n-1) \ , \ \dots \ , \ (t_{z_i}, z_i) \quad ,$$

where $i \le t_n \le t_{n-1} \le \dots \le t_{z_i}$. Note that Lemma 4.3 b) d) gives two (necessity) conditions of such configurations.

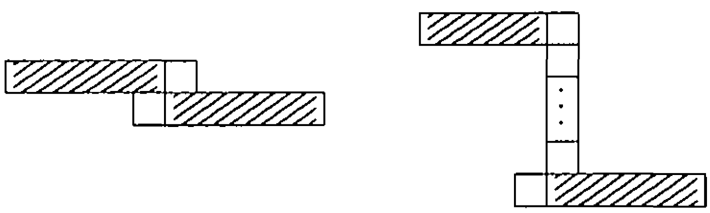
<u>Remark 4.5</u> Assume $r_D \in R(w_\lambda)$. Then the set of boxes with mark i can be visualized as follows

Fig. 4



(Here, $z_i = n+1-\lambda_i$ and the number of boxes is λ_i). Two sorts of "steps" can appear:

Fig. 5



<u>Corollary 4.6</u> If λ is not contained in μ, then there is no $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$.

5. WHEN $r_D \in R(w_\lambda)$ AND $\partial^D_{\underline{\mu}}(e_p) \neq 0$?

Observe that $E = \sum e_i t^i$ implies $\partial^D_{\underline{\mu}}(E) = \sum \partial^D_{\underline{\mu}}(e_i)t^i$. Hence $\partial^D_{\underline{\mu}}(e_p)$ equals the coefficient of t^p in $\partial^D_{\underline{\mu}}(E)$.

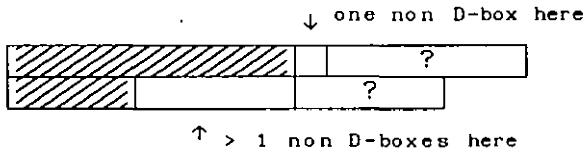
We start with a list of (some) cases when $\partial^D_{\underline{\mu}}(E)=0$.

Lemma 5.1 The equality $\partial^D_{\underline{\mu}}(E)=0$ holds true in the following cases (D-boxes are marked as "dark area" and non D-boxes are white, in the figures below) :

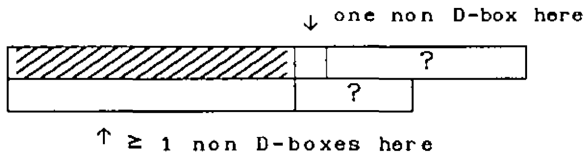
a) Assume that in the i-th and (i+1)-th row of D_μ the following configuration of boxes appears

Fig. 6

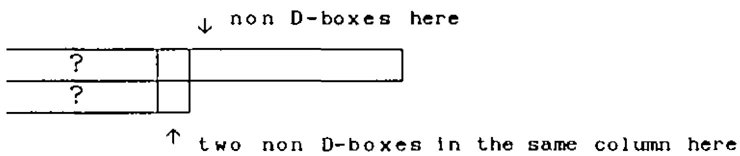
a 1)



a 2)

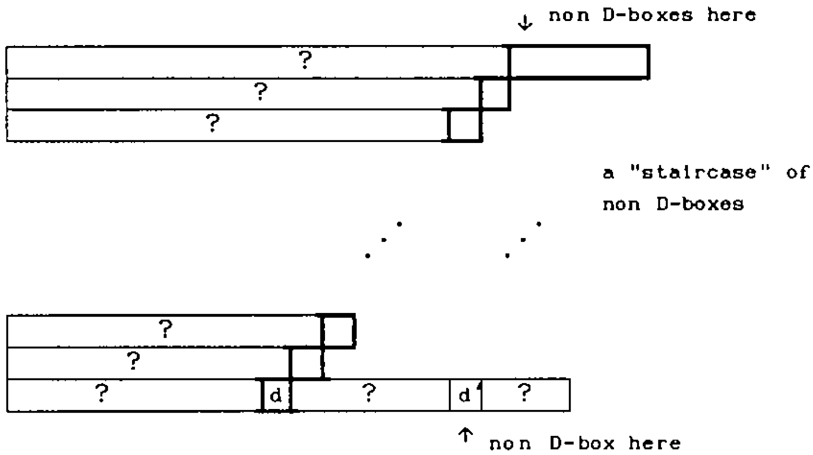


a 3)



b) Assume that in a sequence of consecutive rows the following configuration of non D-boxes appears:

Fig. 7



Proof. - a straightforward application of Lemma 2.2. We check, for instance (a1) and b). In the first case, let us assume that the marked non D-box in the i-th row appears in the j-th column. Then, after applying ∂_j (coming from the i-th row) we get $E_{\mathbf{a}}$ where $a_j=a_{j+1}=0$. The value $a_{j+1}=0$ will be not affected by "later" operators coming from the i-th row. If we will not reach 0 before by applying all the operators up to the ∂_{j+2} (coming from the (i+1)-th row), an application of ∂_{j+2} will give us $a_{j+2}=0$. Finally,

$$\partial_{j+1}(E_{(\ldots,0,0,\ldots)}) = 0 \quad ,$$

where zeros are on the (j+1)-th and (j+2)- th place.

In the b)-case we see that an application of operators coming from pictured rows up to the box d in the j-th column, will give us 0 or $E_{\mathbf{a}}$ with $a_j=a_{j-1}=\ldots=a_1=0$. But then. the operator ∂ associated with d' will anihilate the function in question (if not zero before). □

Proposition 5.2 Assume that $r_D\in R(w_\lambda)$ and $\partial^D_{-\mu}(E) \neq 0$. If $(i,n)\notin D$ then

a) the i-th row of D_μ consists entirely of non D-boxes.

b) every j-th row, with j>i, consists entirely of D-boxes.

Proof. a) follows from Lemma 5.1 (a2).

b) We know from a) that the operator $\partial^D_{-\mu}$ has the following contribution coming from the i-th row

$$\ldots \circ \partial_{n-\mu_i+1} \circ \ldots \circ \partial_{n-1} \circ \partial_n \circ \ldots \quad .$$

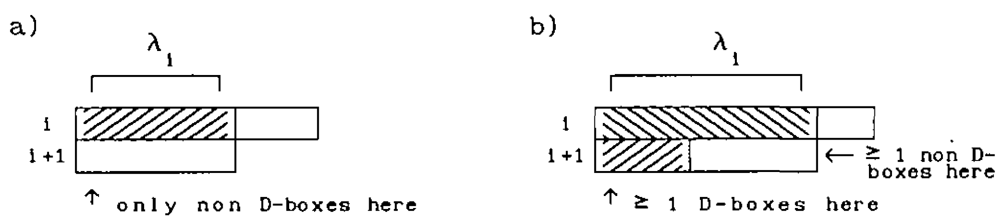
After applying all the ∂'s above we get $E_{\mathbf{a}}$, where $a_n = a_{n-1} = \ldots$ $\ldots = a_{n-\mu_i+1} = 0$. Since $\mu_j < \mu_i$ for $j>i$, an appearance of non D-boxes in the j-th row, j>i, gives rise to an operator $\partial_.$ which will anihilate the function $E_{\ldots}$ in question (if not zero before). □

Corollary 5.3 If $\partial^D_{\underline{\mu}}(E)\neq 0$ and $r_D \in R(w_\lambda)$ then $\ell(\mu) \le \ell(\lambda) + 1$.

Now, starting from $D_\lambda \subset D_\mu$ and using the operations of "deforming rows" of D_λ as described in Corollary 4.4, we will try to construct $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$ and $\partial^D_{\underline{\mu}}(E)\neq 0$. The next facts give some necessity conditions for that.

Lemma 5.4 Assume that $\lambda_i = \mu_{i+1}$ for some i. Suppose that the operations described in Corollary 4.4 have been applied to the rows with numbers $\ell(\lambda)$, $\ell(\lambda)-1, \ldots, i+1$,but the i-th row has not been affected yet. Then we have two possibilities for the configuration of D-boxes in the i-th and (i+1)-th rows (dark boxes visualize D-boxes)

Fig. 8



Then, to avoid $\partial^D_{\underline{\mu}}(E)=0$, the following operation applied to the i-th row is necessary in both, corresponding cases:

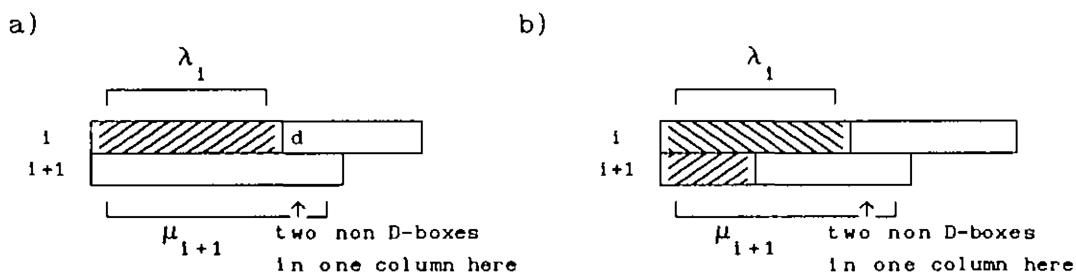
Fig. 9

Proof. The fact that after performing operations to the rows with numbers $\ell(\lambda),\dots,i+1$ we can have only two possibilities pictured in a) and b) is a consequence of Lemma 4.3. Then, a necessity of a') follows from Lemma 5.1(a2). Since by Proposition 5.2b) every j-th row, $j>i+1$, consists entirely of D-boxes , no further change of (mark i)-boxes is possible. A necessity of b') follows from Lemma 5.1(a1). Note that the place of a unique non D-box in the (i+1)-row is uniquely determined: just after the right-most D-box in the (i+1)-th row in b). Moreover, since $\mu_{i+2} < \mu_{i+1}$ no further change of (mark i)-boxes via pushing them down in columns is possible. □

Proposition 5.5 If $\mu_{i+1} > \lambda_i$ for some i , then $\partial^{D}_{-\mu}(E)=0$ for every $D\subset D_\mu$ such that $r_D \in R(w_\lambda)$.

Proof. Assume that i is the smallest number such that $\mu_{i+1} > \lambda_i$. We start from $D_\lambda \subset D_\mu$ and perform operations described in Corollary 4.4. Suppose that we have performed them for the rows with numbers $\ell(\lambda)$, $\ell(\lambda)-1,\dots,i+1$. Then we have two possibilities for the configuration of D-boxes in the i-th and (i+1)-th row:

Fig. 10



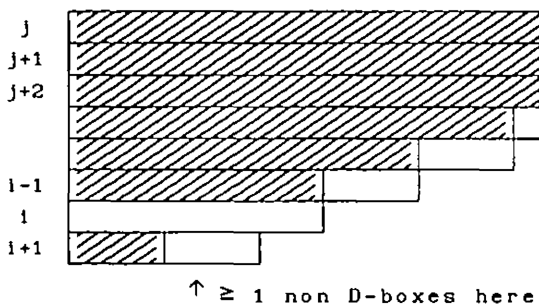
(This is a consequence of Lemma 4.3.)

In the case a), to avoid $\partial^{D}_{-\mu}(E)=0$, we must push down all the D-boxes from the i-th row to the (i+1)-th row (use Lemma 5.1(a2) remarking that if the configuration of D-boxes in the i-th row will not change; then the box d cannot be filled up with a D-box coming from higher rows). We can assume that for some $j<i$,

$$\mu_i=\lambda_{i-1}\ ,\ \mu_{i-1}=\lambda_{i-2}\ ,\ \dots\ ,\ \mu_{j+2}=\lambda_{j+1} \quad \text{but} \quad \lambda_j>\mu_{j+1} \quad .$$

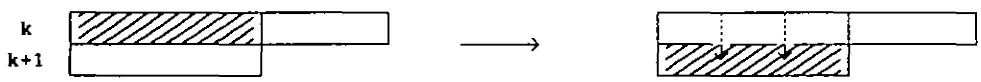
(If no such j exists, we put $j=0$). Pictorially

Fig. 11



Now, Lemma 5.1(a2) forces us - if we want to avoid $\partial^{D}_{-\mu}(E)=0$ - to perform the operations

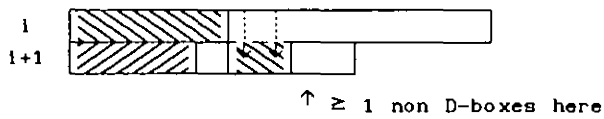
Fig. 12



in the rows with numbers $k=i-1,\dots,j+1$ succesively. We obtain a diagram where the (j+1)-th consists entirely of non D-boxes. According to Proposition 5.2b), no non D-box can appear in lower rows. Since there is a non D-box in the (i+1)-th row, we have $\partial^{D}_{-\mu}(E)=0$.

In case b), it follows from Lemma 5.1(a1) that to avoid $\partial^{D}_{-\mu}(E)=0$ we are forced to change the configuration to:

Fig. 13

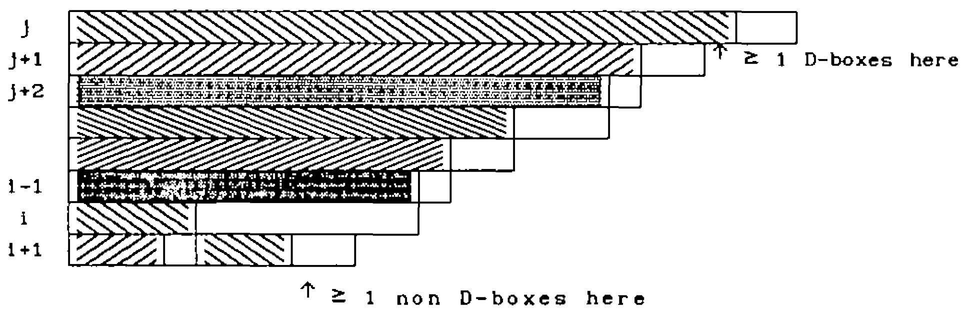


(some of (mark i)-boxes can be moved even to lower rows). We can assume

$$\mu_i=\lambda_{i-1}\ ,\ \mu_{i-1}=\lambda_{i-2}\ ,\ \dots\ ,\ \mu_{j+2}=\lambda_{j+1} \quad \text{but} \quad \lambda_j>\mu_{j+1} \quad .$$

(If no such j exists, we put $j=0$). Pictorially

Fig. 14

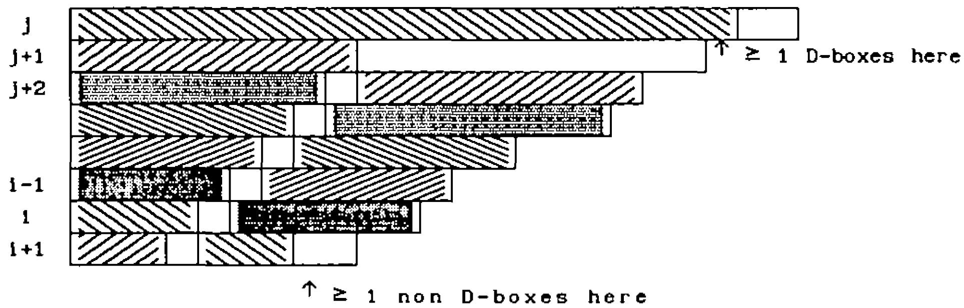


Now, Lemma 5.1(a1) forces us - if we want to avoid $\partial^D_{-\mu}(E)=0$ - to perform the operations

Fig. 15

in the rows with numbers $k=i-1,\dots,j+1$ succesively. We obtain the followong configuration od D-boxes:

Fig. 16



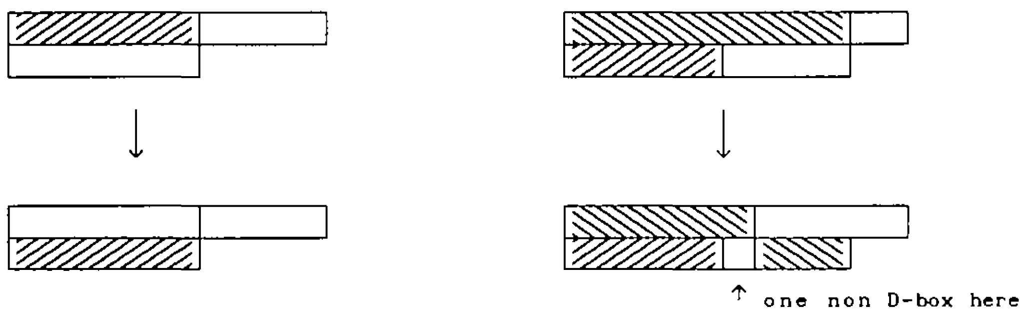
But then, Lemma 5.1 b) implies $\partial_{-\mu}^{D}(E)=0$. □

<u>Proposition 5.6</u> Fix a strict partition $\lambda\subset\rho$. Let μ be a strict partition such that $\lambda\subset\mu\subset\rho$, $\ell(\mu)\le\ell(\lambda)+1$, $\mu_{i+1}\le\lambda_i$ for every i. Then there exists at most one $D\subset D_\mu$ such that $r_D\in R(w_\lambda)$ and $\partial_{-\mu}^{D}(E)\ne 0$.

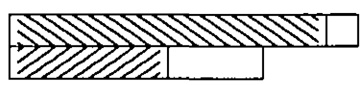
<u>Proof.</u> To find such a D we use operations described in Corollary 4.4. We start with $D_\lambda\subset D_\mu$. The operations in question are performed succesively in rows with numbers $\ell(\lambda)$, $\ell(\lambda)-1,\dots,2,1$. At each stage the operation is uniquely determined and is:

Fig. 17

a) or b)



or c) no change



□

This the *only way* to avoid $\partial_{-\mu}^{D}(E)=0$ (see Lemma 5.1). The assumption

$\mu_{i+1} \le \lambda_i$ for every i, allows us to continue this procedure up to reaching the first row. In this way we obtain a uniquely determined subset $D^{\lambda,\mu} \subset D_\mu$ (card $D^{\lambda,\mu} = |\lambda|$). □

6. CALCULATION OF $\underline{\partial}_\mu^D(e_p)$ FOR $D=D^{\lambda,\mu}$

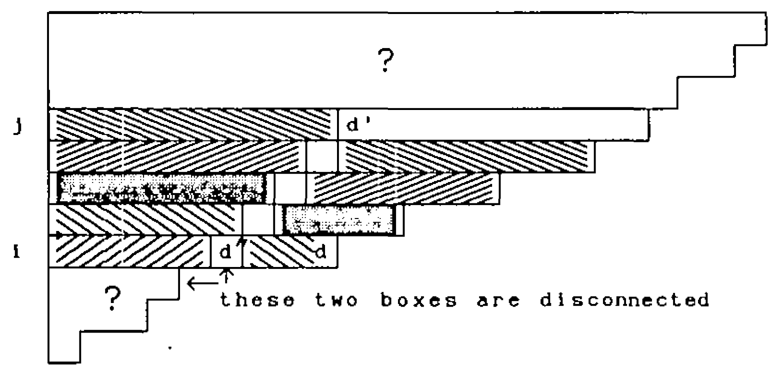
Fix a strict partition $\lambda\subset\rho$ and a number p = 1,...,n. Let μ be a strict partition such that $\lambda\subset\mu\subset\rho$, $|\mu|=|\lambda|+p$, $\ell(\mu)\le\ell(\lambda)+1$, $\mu_{i+1}\le\lambda_i$ for every i. Let $D=D^{\lambda,\mu}\subset D_\mu$ be a collection of boxes from Proposition 5.6. Every ∂_i involved in $\underline{\partial}_\mu^D$ is associated to a box in $D_\mu \setminus D$. We will analyze the connected components of $D_\mu\setminus D$.

Lemma 6.1

a) Assume that a connected component of $D_\mu\setminus D$ meets the n-th column. Then this is a unique component with this property and is a row in D_μ. Moreover, all the rows with bigger numbers than the component consist entirely of D-boxes.

b) Assume that a connected component of $D_\mu\setminus D$ does not meet the n-th column. Then this component can be pictured as follows:

Fig. 18



More precisely, assume that the lowest row meeting this component is the i-th one, and the highest - the j-th one. We claim that

1) this component consists of a "staircase of boxes" having one box $d''=(i,n-\lambda_i+1)$ in the i-th row , one box $(i-1,n-\lambda_i)$ in the (i-1)-th row, ..., one box in the (j+1)-th row and a connected set of boxes

in the j-th row going up to the right border of D_μ.

2) denoting by d the the last box in the i-th row, an assignment: (component)$\longrightarrow$ (the box d), gives a bijection between the set of connected components of $D_\mu \setminus D$ and the set of "characteristic boxes" (see Section 1).

Proof. a) Since $\ell(\mu)\le\ell(\lambda)+1$, there is at most one non D-box in the n-th column (precisely when $\ell(\mu)=\ell(\lambda)+1$). Thus there is at most one connected component meeting the n-th column and this component equals to, say, the i-th row by Proposition 5.2 a). Then every j-th row with $j>i$, consists entirely of D-boxes by Proposition 5.2 b).

b) Taking into account the operations which give us D starting from $D_\lambda \subset D_\mu$ (see Corollary 4.4 and for more precise description - the proof of Proposition 5.4) we see that $\mu_{i+1}<\lambda_i$, $\mu_j<\lambda_{j-1}$ and $\mu_i=\lambda_{i-1}$, $\mu_{i-1}=\lambda_{i-2}$, ... , $\mu_{j-1}=\lambda_{j-2}$. Thus the area of D lying between the j-th and the i-th row is obtained using the following operations. At the starting point the D-boxes in the i-th row are the same as in D_λ. Then for the rows with numbers $i-1, i-2, \ldots, j+1, j$ we apply the operation b) from the proof of Proposition 5.4. The D-boxes in the (j-1)-th row are the same as in D_λ. This discussion implies 1) and 2). □

Let us divide now the set of rows of D_μ into disjoint subsets I_1, $I_2, \ldots, I_q$, each I_k consisting of the rows meeting a fixed connected component, or consisting of a (single) row built of D-boxes only. The operator $\partial^D_{\underline{\mu}}$ is the composition $\Delta_q \circ \Delta_{q-1} \circ \ldots \circ \Delta_2 \circ \Delta_1$ where each Δ_k is defined in the same way as $\partial^D_{\underline{\mu}}$ but instead of D_μ and D we take their intersections with the sum of the rows in I_k. Since $\partial^D_{\underline{\mu}}(e_p)$ is the coefficient of t^p on $\partial^D_{\underline{\mu}}(E)$, we have

$$\Delta_1(E) = c_1 t^{p_1} E_{a_1}$$

$$\Delta_2 \circ \Delta_1(E) = c_2 c_1 t^{p_1+p_2} E_{a_2}$$

$$\cdots$$

$$\Delta_q \circ \Delta_{q-1} \circ \ldots \circ \Delta_1(E) = c_q c_{q-1} \ldots c_1 \, t^{p_1+p_2+\ldots+p_q} E_{a_q} .$$

It follows from Lemma 2.2 that $p_1+p_2+\ldots+p_q = \mathrm{card}(D_\mu \setminus D) = p$. Since the constant term in E_{a_q} is 1, we have $\underline{\partial}^D_{\mu}(e_p) = c_1\ldots c_q$.

Let ℓ_k denote the biggest length of a row in I_k, $k=1,\ldots,q$.

Lemma 6.2

a) If I_k does not equal the connected component of $D_\mu \setminus D$ meeting the n-th column then $a_k = (*,\ldots,*,1,\ldots,1)$ (1 appears ℓ_{k+1} times) , $k=1,\ldots,q$.

b) If I_k equals the connected component of $D_\mu \setminus D$ meeting the n-th column then $a_k = (*,\ldots,*,0,\ldots,0)$ (0 appears ℓ_{k+1} times), $k=1,\ldots,q$.

Proof. Notice first that Δ_k can change only last ℓ_q components of a_{k-1} (i.e. the components $a_n, a_{n-1},\ldots,a_{n-\ell_q+1}$).

a) We use induction on k. Assume that $a_{k-1}=(*,\ldots,*,1,\ldots,1)$ (where 1 appears ℓ_k times). Suppose that Fig. 18 presents the k-th component. Then it is is clear that the last ℓ_{k+1} 1's in a_{k-1} are not affected by an operator ∂_i from Δ_k and the operators s_i from Δ_k can only transpose these units. Therefore $a_k=(*,\ldots,*,1,\ldots,1)$, where 1 appears ℓ_{k+1} times.

b) It follows from a) and Lemma 2.2 that after applying Δ_k to a_{k-1} the last ℓ_{k+1} components of a_k become zero. □

Combining the above Lemma with Lemma 2.2 we get

Lemma 6.3 a) If I_k is a row consisting of D-boxes only, then $c_k=1$.

b) If the sum of the rows in I_k contains a connected component of $D_\mu \setminus D$ not meeting the n-th column then $c_k=2$.

c) If I_k equals the connected component of $D_\mu \setminus D$ meeting the n-th column then $c_k=1$.

(Note that the multiplicity 2 in b) comes from ∂_r (r is the number of the box d' on Fig. 18) applied to

$$c_1\ldots c_{k-1} t^{p_1+\ldots+p_{k-1}} s_{r+1}\circ\ldots\circ s_{n-1}\circ s_n (E_{a_{k-1}})$$

$$= c_1 \ldots c_{k-1} t^{p_1+\ldots+p_{k-1}} E_{(*,\ldots,*,1,-1,1,\ldots,1)} ,$$

the pair (1,-1) occupying the r-th and (r+1)-th places.) □

Summing up we have proved

Proposition 6.4 In the above notation

$$\underline{\partial}_{\mu}^{D}(e_p) = 2^{m(\lambda,\mu)} ,$$

where $m(\lambda,\mu)$ is the number of connected components of $D \setminus D_\mu$ not meeting the n-th component.

Using the bijections (see Section 1 and Lemma 6.2 b)) :

{ connected components of $D_\mu \setminus D^{\lambda,\mu}$ not meeting the n-th column }

$\updownarrow$

{ characteristic boxes }

$\updownarrow$

{ connected components of $D_\mu \setminus D_\lambda$ not meeting the n-th column } ,

and changing the numbering of columns to the usual order, we infer

Corollary 6.5 In the above notation,

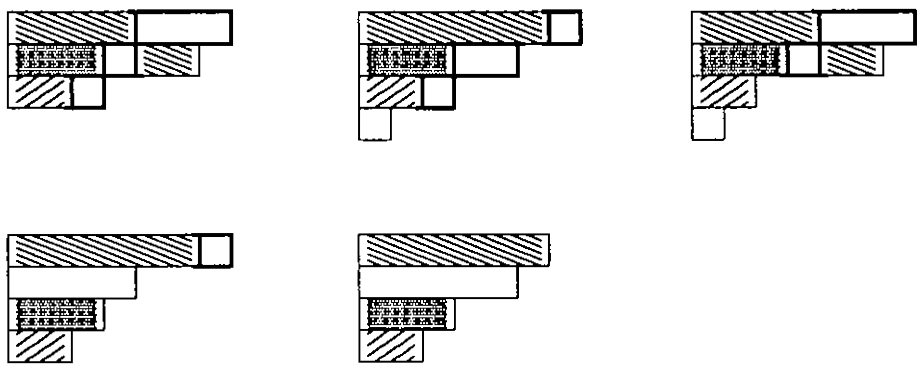
$$\underline{\partial}_{\mu}^{D}(e_p) = 2^{m(\lambda,\mu)} ,$$

where $m(\lambda,\mu)$ is the number of connected components of $D_\mu \setminus D_\lambda$ not meeting the first column.

By combining Propositions 3.5, 5.6 and Corollary 6.5 our proof of Theorem 1.1 is finished.

Example 6.6 The diagrams $D^{(632),\mu}$ for partitions μ appearing in the decomposition $\sigma(632)\ \sigma(5)$, are :

Fig. 19



7. CONCLUDING REMARKS

(7.1) A Giambelli - type formula

In [P, Sect.6] the first named author has deduced from Theorem 1.1 the following Giambelli-type formula. Let $\lambda=(\lambda_1,\ldots,\lambda_k) \subset \rho$ be a strict partition, k -even (we can always assume it by putting $\lambda_k=0$ if necessary). Then

$$\sigma(\lambda) = \text{Pfaffian} \left[\sigma(\lambda_i,\lambda_j) \right]_{1\le i<j\le k} ,$$

where $\sigma(\lambda_i,\lambda_j) = \sigma(\lambda_i)\sigma(\lambda_j) + 2\sum_{p=1}^{j} (-1)^p \sigma(\lambda_i+p)\,\sigma(\lambda_j-p)$, and where $\sigma(\lambda_i,0) = \sigma(\lambda_i)$.

(7.2) The orthogonal case

Using exactly the same method one can prove Pieri's formula for the Grassmannian of n-dimensional isotropic subspaces of (2n+1)-dimensional vector space endowed with an orthogonal nondegenerate form (for the precise Pieri-type formula in this case - see [H-B] ; and a Giambelli-type formula - see [P,Sect.6]. For analogous results in the case of Grassmannian of n-dimensional isotropic subspaces in an 2n-dimensional vector space endowed with an orthogonal nondegenerate form - see [P,Sect.6].

(7.3) Symplectic & orthogonal Schubert Calculus

We end with a geometric interpretation of the $\sigma(\lambda)$'s. For all notions which are used and not defined here, we refer to [P,Sect.6]. Let V be a 2n-dimensional vector space endowed with a symplectic nondegenerate form ϕ: $V\times V \longrightarrow \mathbb{C}$. Let $(v_1,\dots,v_n,w_n,\dots,w_1)$ be a symplectic basis of V. Let $V_1 \subset V_2 \subset \dots \subset V_n$ be a flag of isotropic subspaces spanned by the first i vectors in the sequence $(v_1,\dots,v_n)$. Then $\sigma(\lambda_1,\dots,\lambda_k)$ is the class in $A^{|\lambda|}(G)$ of the cycle of all isotropic n-subspaces L in V such that $\dim(L\cap V_{n+1-\lambda_i}) \geq i$, $i=1,\dots,k$.

The Schubert Calculus for usual Grassmannians is based on three main theorems: Pieri's formula, Giambelli's determinantal formula and the Basis theorem (see for example [L]). In the case of the isotropic Grassmannian G, a Pieri-type formula is described in Theorem 1.1 and a Giambelli -type formula is recalled in (7.1). A Basis-type theorem can be formulated as

$$A_*(G) = \oplus\, \mathbb{Z}\, \sigma(\lambda) ,$$

the sum over all strict partitions $\lambda\subset\rho$. This result can be deduced from a general theory of cellular Schubert/Bruhat decompositions of the spaces of the form G/P (see [B-G-G], [D]). The cellular decomposition in the case of G was described in details in [P,Sect.6]. Here, we use an opportunity to give a still another simple, conceptual proof of the Basis theorem.

The proof which we sketch is by induction on n and is inspired by the proof of the Basis theorem in [L]. Suppose that $V'\supset V$ is an (2n+2)-dimensional vector space endowed with a nondegenerate symplectic form ϕ': $V'\times V' \longrightarrow \mathbb{C}$ extending ϕ, and $(v_1,\dots,v_n,v,w,w_n,\dots,w_1)$ is a symplectic basis of V'. Let G' be the Grassmannian of (n+1)-dimensional subspaces of V', isotropic with respect to ϕ'. Let i: $G \subset G'$ be a closed imbedding defined by $L \longmapsto L\oplus \mathbb{C}v$. We have a map

$$p:\ G'\setminus i(G) \longrightarrow G''$$

where G" is the Grassmannian of (n+1)-dimensional subspaces in $V\oplus\mathbb{C}w$ (with respect to $\phi'_{|V\oplus\mathbb{C}w}$). The map p sends $L'\in G'$ to its image via the projection $V' \longrightarrow V\oplus\mathbb{C}w$. In fact, p is a vector bundle of rank n+1 over G". Moreover, G" is isomorphic to G. (Observe that if $L'\subset G'\setminus i(G)$ then $w\in L'$

because the maximum of dimension of an isotropic subspace in V is n.) An exact sequence

$$A_*(G) \xrightarrow{i_*} A_*(G') \xrightarrow{j^*} A_*(G'\setminus G) \longrightarrow 0$$

where $j: G'\setminus G \longrightarrow G'$ is the inclusion, can be rewritten as

$$A_*(G) \xrightarrow{i_*} A_*(G') \xrightarrow{(p\circ j)^*} A_*(G) \longrightarrow 0 . \qquad (\#)$$

Denoting by $\sigma'(\mu)$ the Schubert cycles in G' one can show (by using the above flags)

$$i_*\sigma(\lambda_1,\dots,\lambda_k) = \sigma'(n+1,\lambda_1,\dots,\lambda_k) ,$$

and

$$(p\circ j)^*\sigma'(\lambda_1,\dots,\lambda_k) = \sigma(\lambda_1,\dots,\lambda_k) \text{ if } \lambda_1\le n , \text{ - zero otherwise.}$$

The morphism i_* is in fact a monomorphism , the exact sequence (#) splits, and the Basis theorem follows by induction. Detailed arguments proving the above assertions are similar to the arguments in [L] and we omit them.

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