February 12, 1998

Parabolic subgroups with Abelian unipotent radical as a testing site for Invariant Theory

Dmitri I. Panyushev

Introduction

Let L be a simple algebraic group over an algebraically closed field k of characteristic zero and P a parabolic subgroup with Abelian unipotent radical (= with *aura*). In this case $\mathrm{Lie}\, L = \mathfrak{l}$ admits a $\mathbb{Z}$-grading with only three nonzero parts:

$$\mathfrak{l} = \mathfrak{l}(-1) \oplus \mathfrak{l}(0) \oplus \mathfrak{l}(1) \ .$$

Such a grading is said to be *short*. Here $\mathrm{Lie}\, P = \mathfrak{l}(0) \oplus \mathfrak{l}(1)$ and $exp\, \mathfrak{l}(1)$ is the Abelian unipotent radical of P.

In this paper we consider several invariant-theoretic problems related to the representation of $G := L(0)$ on $\mathfrak{l}(1)$. As is well known, G has finitely many orbits in $\mathfrak{l}(1)$. Let $\mathcal{O}_i$ be one of them, $\overline{\mathcal{O}}_i$ its closure, and $\mathfrak{C}_i$ the closure of the conormal bundle of $\mathcal{O}_i$. Since $\mathfrak{l}(-1)$ is G-equivariantly identified with the dual space $\mathfrak{l}(1)^*$, we have $\mathfrak{C}_i$ is a subvariety in $\mathfrak{l}(-1) \oplus \mathfrak{l}(1)$. Our principal results are the following:

- We give a unified construction of a G-equivariant resolution of singularities for $\overline{\mathcal{O}}_i$ and $\mathfrak{C}_i$. In all cases, the covering smooth variety is a homogeneous bundle $G *_R V$, where R is a parabolic subgroup of G and V is a completely reducible R-module.
- It is proved that $L \cdot \mathfrak{l}(1) \cap (\mathfrak{l}(1) \oplus \mathfrak{l}(-1)) = \cup_i \mathfrak{C}_i$ and it is the variety of pairs (x, y) $(x \in \mathfrak{l}(1), y \in \mathfrak{l}(-1))$ such that $[x, y] = 0$. A relationship between the double coset space $G \backslash L / P$ and the G-orbit structure of $L \cdot \mathfrak{l}(1) \cap (\mathfrak{l}(1) \oplus \mathfrak{l}(-1))$ is investigated. This provides some link to results in [11].
- Let U be a maximal unipotent subgroup of G. The (well-known) description of the algebra of covariants $k[\mathfrak{l}(1)]^U$ will be obtained as a consequence of "a restriction theorem for U-invariants" [7]. We prove that $k[\mathfrak{l}(1)]$ is a free $k[\mathfrak{l}(1)]^U$-module. In our context, it is equivalent to that the quotient map $\pi_{\mathfrak{l}(1)} : \mathfrak{l}(1) \to \mathfrak{l}(1)/\!\!/U$ is equidimensional.
- Actually, to derive the previous result, we prove a sufficient condition for the quotient map $\pi_X : X \to X/\!\!/U$ to be flat, where X is an affine G-variety. As a by-product, we classify the irreducible representations of simple algebraic groups with this property. The main part of the list consists of representations arising from parabolic subgroups with *aura*.

The proofs are based on properties of the partition of the root system Δ determined by the short grading. We work with two specific sequences of orthogonal long roots in $\Delta(1)$

(see notation below). We call them respectively lower and upper canonical strings. The construction of them is not new, it goes back to Harish Chandra. It seems however that a systematic utilization of their properties given by Lemma 1.2 constitutes some novelty.

It is quite typical for parabolic subgroups with *aura* that many problems for them can be solved in a case-by-case fashion. For instance, this is true for resolution of singularities of G-orbits in $\mathfrak{l}(1)$. Indeed, we meet as the orbit closures the determinantal varieties, their symmetric and skew-symmetric analogues, and quadrics, if $\mathfrak{l}$ is classical. Associated to $\mathfrak{l}=\mathbf{E}_7$, one obtains a 27-dimensional representation of $\mathbf{E}_6$ and in this case resolutions of singularities were constructed in [3, §2]. Finally, the $\mathbf{E}_6$-case is simple enough. Our approach to this problem gives a unified description of resolutions and then a proof working for all simple Lie algebras. The same can be said about all other proofs in the paper. We never use case-by-case arguments and do not distinguish between the 'tube' or 'non-tube' case.

Acknoledgements. I would like to thank T. Springer for drawing my attention to parabolic subgroups with *aura*. This work was done during my stay at the University of Poitiers and at the MPI (Bonn). I am grateful to both Institutions for hospitality and support. The research was supported in part by RFFI Grant No. 95–01–00783a.

Notation and conventions.

Lie algebras of algebraic groups are denoted by the corresponding small Gothik letters.
$\mathfrak{l}$ is a simple Lie algebra equipped with a short grading $\mathfrak{l}=\mathfrak{l}(-1)\oplus\mathfrak{l}(0)\oplus\mathfrak{l}(1)$; $\mathfrak{g}:=\mathfrak{l}(0)$.
T is a maximal torus in G and hence in L.
Δ is the root system of $(\mathfrak{l},\mathfrak{t})$.
$\Delta=\Delta(-1)\cup\Delta(0)\cup\Delta(1)$—the partition corresponding to the short grading.
We fix a Borel subgroup B in G, containing T. This choice determines a set of positive roots $\Delta(0)^+$ in $\Delta(0)$ and also in Δ: $\Delta^+=\Delta(0)^+\cup\Delta(1)$; γ is the highest root in Δ^+.
Π is the set of simple roots in Δ^+ and $\Pi(0)=\Pi\cap\Delta(0)$.
W (resp. $W(0)$) is the Weyl group of $\mathfrak{l}$ (resp. $\mathfrak{g}$) with respect to $\mathfrak{t}$.
For $\alpha\in\Delta$, we let w_α denote the corresponding reflection in W, $\alpha^\vee=2\alpha/(\alpha,\alpha)$ the corresponding coroot, and e_α a nonzero element in the root space $\mathfrak{l}_\alpha\subset\mathfrak{l}$.
For $r\in\mathbb{N}$, let $[1,r]:=\{1,2,\dots,r\}$.
$\#M$ is the cardinality of a finite set M.

1 Canonical strings of roots and their properties

We begin with recalling some properties of short gradings. Since $\mathfrak{l}$ is simple, the representation of $\mathfrak{g}$ on $\mathfrak{l}(1)$ is faithful and irreducible. Therefore the centre of $\mathfrak{g}$ is one-dimensional, P is a maximal parabolic subgroup, and $\#\Pi(0)=\#\Pi-1$. Thus, there is a unique simple root in $\Delta(1)$. Call it β. Then β is the unique lowest weight of $\mathfrak{g}$-module $\mathfrak{l}(1)$. The longest element in $W(0)$ takes β to the highest weight of $\mathfrak{l}(1)$, i.e., to γ. Hence β is long. This also proves that β-height of γ, i.e., the coefficient n_β in the sum

$\gamma = n_\beta\beta + \sum_{\alpha\in\Pi(0)} n_\alpha\alpha$, is equal to 1. Therefore β-height of any root in $\Delta(1)$ is equal to 1.

We shall consider two strings of pairwise orthogonal roots in $\Delta(1)$. Both strings are defined inductively. The *lower canonical string* (l.c.s.) is the cascade up from $\beta_1 = \beta$ within $\Delta(1)$: at each stage, β_{i+1} is *the* minimal root in $\Delta_i(1) = \{\alpha \in \Delta(1) \mid (\alpha,\beta_1) = \ldots = (\alpha,\beta_i) = 0\}$. The process terminates when $\Delta_i(1)$ becomes empty. The construction is originally due to Harish Chandra and is well known nowadays. The following lemma is included for completeness and convenience of the reader.

1.1 Lemma.
The above procedure is well-defined, i.e., at each stage $\Delta_i(1)$ contains a unique minimal element. All the roots in the string are long.

Proof. It goes by induction on i. Consider $\Delta_i = \{\alpha \in \Delta \mid (\alpha,\beta_1) = \ldots = (\alpha,\beta_i) = 0\}$. Then $\Delta_i = \Delta_i(-1) \cup \Delta_i(0) \cup \Delta_i(1)$. This partition corresponds to a short grading of a reductive subalgebra $\mathfrak{l}_i \subset \mathfrak{l}$. It is easy to see that $\Delta_i(1) \neq \varnothing$ if and only if $\gamma \in \Delta_i(1)$ and in this case γ is the unique maximal element in $\Delta_i(1)$ (as an element of Δ_i). Therefore the unique minimal element in $\Delta_i(1)$, which is $W(0)$-conjugate to γ, is long. □

The *upper canonical string* (u.c.s.) is the cascade down from $\gamma_1 = \gamma$ within $\Delta(1)$: at each stage, γ_{i+1} is *the* maximal element in $\Delta^i(1)$, where $\Delta^i = \{\alpha \in \Delta \mid (\alpha,\gamma_1) = \ldots = (\alpha,\gamma_i) = 0\}$. The process terminates when $\Delta^i(1)$ becomes empty. Obviously, the longest element in $W(0)$ takes the lower canonical string to the upper canonical one. Hence, the latter is well-defined and both strings have the same cardinality, say r. ¿From now on, $\beta = \beta_1, \beta_2, \ldots, \beta_r$ and $\gamma = \gamma_1, \ldots, \gamma_r$ are respectively the l.c.s and the u.c.s.

Remarks. 1. It is easily seen that the l.c.s. and u.c.s. are sequences of orthogonal long roots of *maximal* length. Moreover, one can prove that if a sequence of strongly orthogonal roots has the maximal length, then all the roots in it are long.
2. By [11, 2.8], $W(0)$ is transitive on the set of sequences, of a fixed length, that consist of orthogonal long roots in $\Delta(1)$.
3. The number r is an important invariant of the short grading. For instance, it is the rank of the symmetric variety L/G or Krull dimension of the algebra $k[\mathfrak{l}(1)]^U$ or the number of nonzero G-orbits in $\mathfrak{l}(1)$.

Given $\alpha \in \Delta$, consider two sequences of rational numbers:

$$(*) \qquad (\alpha,\beta_1),\ldots,(\alpha,\beta_r)\ ,$$

$$(**) \qquad (\alpha,\gamma_1),\ldots,(\alpha,\gamma_r)\ .$$

1.2 Lemma.
(i) *Let $\alpha \in \Delta(0)$. Then both $(*)$ and $(**)$ contain at most one positive and one negative term. Moreover, $\alpha \in \Delta(0)^+$ if and only if the first nonzero term (if any) is negative in $(*)$ and positive in $(**)$;*

(ii) *Let* $\alpha \in \Delta(1)$. *Then* $(*)$ *and* $(**)$ *contain at most two nonzero terms, which are necessarily positive;*
(iii) *The nonzero terms in* $(*)$ *and* $(**)$ *have the same absolute value* $\frac{1}{2}(\gamma,\gamma)$.

Proof. 1. Since the proofs for $(*)$ and $(**)$ are similar, we consider only the second sequence. If $(\alpha,\gamma_i) > 0$ and $(\alpha,\gamma_j) > 0$ for $i \neq j$, then $\gamma_i - \alpha \in \Delta(1)$ and $(\gamma_i - \alpha, \gamma_j) < 0$. But this is impossible, because the sum of two elements in $\Delta(1)$ is never a root. For $\alpha \in \Delta(0)^+$, the first nonzero term is positive by the very definition of the u.c.s. As each root is either positive or negative, the converse is also true.
2 & 3. Left to the reader. $\square$

1.3 Corollary. *For any* $\alpha \in \Delta(0)^+$ *and* $m \in [1,r]$, *we have* $(\alpha, \beta_1 + \ldots + \beta_m) \le 0$ *and* $(\alpha, \gamma_1 + \ldots + \gamma_m) \ge 0$. *Moreover,* $\gamma_1 + \ldots + \gamma_m$ *is also dominant with respect to* Δ^+.
$\square$

Take an $\alpha \in \Delta(0)^+$ such that $(\alpha,\beta_i) < 0$ for some i. By Lemma 1.2, there is at most one $j > i$ such that $(\alpha,\beta_j) > 0$. The question arises whether such a j exists and is there a relationship between i and j? The next result says that compensation in case of simple roots occurs as soon as possible. This plays a crucial rôle in sect. 4, in the proof of flatness of the quotient map $\pi_{\mathfrak{l}(1)}$.

1.4 Theorem. *1. Let* $(\alpha,\beta_i) < 0$ *for some* $\alpha \in \Pi(0)$ *and* $i < r$. *Then* $(\alpha,\beta_{i+1}) > 0$.
2. Dually, if $(\alpha,\gamma_j) > 0$ *for some* $j < r$, *then* $(\alpha,\gamma_{j+1}) < 0$.

Proof. The second claim follows from the first one by the application of the longest element in $W(0)$. So, we consider only the l.c.s.
Replacing Δ by Δ_{i-1}, we may assume that $i = 1$. If $r = 1$, then there is nothing to prove. Suppose $r \ge 2$. By definition, $(\beta_2,\beta_1) = 0$ and β_2 is the minimal root in $\Delta(1)$ with this property. Since β_2 is not simple, there is $\mu \in \Pi(0)$ such that $\beta_2 - \mu \in \Delta(1)$. Then $(\beta_2 - \mu, \beta_1) > 0$. Hence $(\mu,\beta_1) < 0$ and μ,β_1 are adjacent simple roots. Now, assume that $(\alpha,\beta_1) < 0$ and $(\alpha,\beta_2) = 0$. Since $(\beta_2,\mu) > 0$, we have $\alpha \ne \mu$. Thus, we have detected a fragment of the Dynkin diagram:

$$\cdots\!-\!\!-\underset{\alpha}{\circ}\!-\!\!-\underset{\beta_1}{\circ}\!-\!\!-\underset{\mu}{\circ}\!-\!\!-\cdots$$

(At the moment, nothing is claimed about length of α and μ.) In particular, $(\alpha,\mu) = 0$. Consider the support[1] of $\beta_2 - \mu - \beta_1 \in \Delta(0)^+$. Since $(\beta_2 - \mu - \beta_1, \alpha) > 0$, we have $\{\alpha\} \in \operatorname{supp}(\beta_2 - \mu - \beta_1)$. The support of any root is connected, hence $\{\mu\} \notin \operatorname{supp}(\beta_2 - \mu - \beta_1)$ and even $(\mu, \beta_2 - \mu - \beta_1) = 0$. The latter equality implies $(\mu^\vee, \beta_2) = -(\mu^\vee, \beta_1) = 1$, i.e., μ is long and $\beta_2 - \mu - \beta_1$ is also long. Next, $\beta_2 - \mu - \beta_1 - \alpha \in \Delta(0)^+ \cup \{0\}$ and $(\beta_2 - \mu - \beta_1 - \alpha, \beta_1^\vee) = 1 - 2 + 1 = 0$. Hence $\{\alpha\} \notin \operatorname{supp}(\beta_2 - \mu - \beta_1 - \alpha)$. There are then two possibilities and we shall see that either of them leads to a contradiction.

[1]If $\nu = \sum_{\alpha\in\Pi} n_\alpha \alpha$, then $\operatorname{supp}(\nu) = \{\alpha \mid n_\alpha \ne 0\}$.

(a) $\beta_2 - \mu - \beta_1 = \alpha$. Then α is long and $0 = (\alpha^\vee, \beta_2) = (\alpha^\vee, \alpha + \mu + \beta_1) = 2 - 1$, a contradiction.

(b) $\beta_2 - \mu - \beta_1 - \alpha \neq 0$. The support of this positive root must then contain a simple root adjacent to α, i.e., $(\beta_2 - \mu - \beta_1 - \alpha, \alpha) < 0$. That is, $(\alpha, \alpha) + (\beta_1, \alpha) > 0$ and hence α is long. Therefore $\alpha + \beta_1 + \mu$ is a long root in $\Delta(1)$ and $(\alpha + \beta_1 + \mu, \beta_1) = 0$. Thus, $\alpha + \beta_1 + \mu$ must be equal to β_2, which is again a contradiction. $\square$

Let φ be the fundamental weight of L corresponding to β_1, i.e., $(\varphi, \beta_1) = \frac{1}{2}(\beta_1, \beta_1)$ and $(\varphi, \alpha) = 0$ for all $\alpha \in \Delta(0)$.

1.5 Proposition. *The following conditions are equivalent:*

1. $\beta_{r+1-i} = \gamma_i$ *for* $i = 1, \dots, r$;
2. $\beta_r = \gamma_1$;
3. $2\varphi = \beta_1 + \dots + \beta_r$;
4. *The element* $\tilde{w} := \prod_{i=1}^r w_{\beta_i} \in W$ *takes* $\Delta(1)$ *to* $\Delta(-1)$.

Proof. $1 \Rightarrow 2$. Obvious.

$2 \Rightarrow 3$. We have $(\alpha, \beta_r) \geq 0$ for all $\alpha \in \Pi(0)$. It then follows from (1.2) and (1.4) that the number of β_i's that are not orthogonal to such an α is either 0 or 2. Therefore $(\alpha, \beta_1 + \dots + \beta_r) = 0$ and hence $\beta_1 + \dots + \beta_r = c\varphi$. Then clearly $c = 2$.

$3 \Rightarrow 4$. It follows from (3) that $\tilde{w}.\varphi = -\varphi$. Therefore $\tilde{w}$ takes $\Delta(0)$ to $\Delta(0)$. Hence $\tilde{w}.\alpha$ belongs to either $\Delta(-1)$ or $\Delta(1)$ for any $\alpha \in \Delta(1)$. The assumption $\tilde{w}.\alpha \in \Delta(1)$ clearly implies that $\tilde{w}.\alpha = \alpha$, i.e., α is orthogonal to all the β_i's. But this contradicts to maximality of the l.c.s.

$4 \Rightarrow 2$. If $\beta_r \neq \gamma_1$, then $\tilde{w}.\gamma_1 = \gamma_1 - \beta_r \in \Delta(0)$.

$3 \Rightarrow 1$. We have $(\alpha, \beta_1 + \dots + \beta_r) = 0$ for all $\alpha \in \Delta(0)^+$. Then Lemma 1.2 implies that the number of β_i's that are not orthogonal to α is either 0 or 2. Therefore the first nonzero value (if any) in the sequence $(\alpha, \beta_r), \dots, (\alpha, \beta_1)$ is positive. Since also $(\beta_i, \mu) \geq 0$ for all $\mu \in \Delta(1)$, we have $\beta_r = \gamma_1$ and each β_i is the highest root in the $+1$-part of the set of roots that are orthogonal to $\beta_r, \dots, \beta_{i+1}$. It remains to note that this is just the definition of u.c.s. $\square$

Remarks. 1. The equivalent conditions of 1.5 characterize the case, where the Hermitian symmetric space G/P is of tube type (for $k = \mathbb{C}$) and/or the prehomogeneous vector space $(G, \mathfrak{l}(1))$ is regular.
2. The list of equivalent conditions can easily be extended. If one sticks to those that concern only the root system, the condition $\varphi = -w^o.\varphi$ should be mentioned (here $w^o \in W$ is the longest element). It is not hard to deduce it from, say, condition 4. But I do not know a *direct* proof for the converse.

2 Resolution of singularities of the closures of G-orbits in $\mathfrak{l}(1)$

It follows from a result of Vinberg [14, §2] that the number of G-orbits in $\mathfrak{l}(1)$ is finite. An explicit classification has been obtained by Muller et al. [5] (see also [11]). Their description can be stated as follows.

2.1 Theorem. *Let $\mu_1, \ldots, \mu_r$ be an arbitrary maximal sequence of orthogonal long roots in $\Delta(1)$. Then $\{\sum_{j=1}^{i} e_{\mu_j} \mid 1 \le i \le r\} \cup \{0\}$ is a system of representatives of the G-orbits in $\mathfrak{l}(1)$. In particular, the total number of orbits is $r+1$.* □

We let $\mathcal{O}_i$ denote the G-orbit which contains the elements of the form $\sum_{j=1}^{i} e_{\mu_j}$. Because $\mu_1, \ldots, \mu_r$ are linearly independent, it immediately follows that $\mathcal{O}_i \subset \overline{\mathcal{O}_j}$ if and only if $i \le j$. (This is also a result in [5].) To construct an equivariant resolution of the $\overline{\mathcal{O}}_i$'s, it will be convenient to deal with the representatives given in terms of the u.c.s. Define the point $x_i \in \mathcal{O}_i$ by $x_i = \sum_{j=1}^{i} e_{\gamma_j}$. Recall that $\Delta(0)$ is the root system of G and $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ that determines $\Delta(0)^+$.

2.2 Proposition. *Define the subspace $V_i \subset \mathfrak{l}(1)$ by $V_i := [\mathfrak{b}, x_i]$. Then*

(i) $V_i = \bigoplus_{\alpha \in \Gamma_i} \mathfrak{l}(1)_\alpha$, *where* $\Gamma_i = \{\alpha \in \Delta(1) \mid (\alpha, \gamma_1 + \ldots + \gamma_i) = (\gamma, \gamma)\}$;

(ii) V_i *is a B-stable subspace of* $\mathfrak{l}(1)$.

Proof. (i) Let Γ_i' be the subset of $\Delta(1)$ consisting of all α such that $\alpha = \mu + \gamma_j$ for some $\mu \in \Delta(0)^+ \cup \{0\}$ and $1 \le j \le i$. Obviously, $V_i \subset \bigoplus_{\alpha \in \Gamma_i'} \mathfrak{l}(1)_\alpha$. Since $\gamma_1, \ldots, \gamma_i$ are linearly independent, $[\mathfrak{t}, x_i] = \mathfrak{l}(1)_{\gamma_1} \oplus \ldots \oplus \mathfrak{l}(1)_{\gamma_i} \subset V_i$. Suppose $\alpha \notin \{\gamma_1, \ldots, \gamma_i\}$ and $\alpha - \gamma_j =: \mu \in \Delta(0)^+$ for some $j \in [1, i]$. Then $(\alpha, \gamma_j) > 0$ and $(\mu, \gamma_j) < 0$. By Lemma 1.2, we have $(\mu, \gamma_s) \ge 0$, if $s \ne j$. It follows that $[e_\mu, x_i] = [e_\mu, e_{\gamma_j}] = e_\alpha \in V_i$. Thus, $V_i = \bigoplus_{\alpha \in \Gamma_i'} \mathfrak{l}(1)_\alpha$. It remains to prove that $\Gamma_i' = \Gamma_i$.

Clearly, γ_j $(j \le i)$ belongs to both Γ_i' and Γ_i. Next, assume that $\alpha = \mu + \gamma_j$ with $\mu \in \Delta(0)^+$. By Lemma 1.2(i), the condition $(\mu, \gamma_j) > 0$ implies that there exists a unique $m < j$ such that $(\mu, \gamma_m) > 0$. Therefore $(\alpha, \gamma_m) = (\mu + \gamma_j, \gamma_m) > 0$. From lemma 1.2(ii) it then follows that $(\alpha, \gamma_1 + \ldots + \gamma_i) = (\alpha, \gamma_j) + (\alpha, \gamma_m) = \frac{1}{2}(\gamma, \gamma) + \frac{1}{2}(\gamma, \gamma) = (\gamma, \gamma)$. Thus, $\Gamma_i' \subset \Gamma_i$.

On the other side, suppose $\alpha \in \Delta(1)$, $\alpha \notin \{\gamma_1, \ldots, \gamma_i\}$, and $(\alpha, \gamma_1 + \ldots + \gamma_i) = (\gamma, \gamma)$. Since all the γ_j's are long and $(\alpha, \gamma_j) \ge 0$, there exist exactly two indices $m, m' \in [1, i]$ such that $(\alpha, \gamma_m) = (\alpha, \gamma_{m'}) > 0$. Assume that $m > m'$ and consider $\alpha - \gamma_m \in \Delta(0)$. We have $(\alpha - \gamma_m, \gamma_m) < 0$, $(\alpha - \gamma_m, \gamma_{m'}) > 0$, and $(\alpha - \gamma_m, \gamma_s) = 0$ for $s \ne m, m'$. Then Lemma 1.2(i) forces $\alpha - \gamma_m$ is positive. Hence $\alpha = (\alpha - \gamma_m) + \gamma_m \in \Gamma_i'$.

(ii) One has to prove that if $\alpha \in \Gamma_i$ and $\alpha + \mu \in \Delta$ for some $\mu \in \Delta(0)^+$, then $\alpha + \mu \in \Gamma_i$. Recall from (1.3) that $(\mu, \gamma_1 + \ldots + \gamma_i) \ge 0$ for each $\mu \in \Delta(0)^+$.

(a) If $(\mu, \gamma_1 + \ldots + \gamma_i) = 0$ and $\alpha + \mu$ is a root, then $(\alpha + \mu, \gamma_1 + \ldots + \gamma_i) = (\gamma, \gamma)$. That is, $\alpha + \gamma \subset \Gamma_i$ as well.

(b) If $(\mu, \gamma_1 + \ldots + \gamma_i) > 0$, then $(\alpha + \mu, \gamma_1 + \ldots + \gamma_i) > (\gamma, \gamma)$. However, it follows from Lemma 1.2(ii) that $(\nu, \gamma_1 + \ldots + \gamma_i) \le (\gamma, \gamma)$ for any $\nu \in \Delta(1)$. Hence $\alpha + \mu$ is not a root in this case. $\square$

2.3 Corollary. *The orbit Bx_i is open and dense in V_i.*

Proof. One has $Bx_i \subset V_i$ and $\dim Bx_i = \dim V_i$. $\square$

Since V_i is B-stable, its normalizer in G is a parabolic subgroup. Call it R_i. Let $R_i = G_i N_i$ be the standard Levi decomposition and $\Delta(G_i)$ the root system of G_i. Define $\Delta(N_i)$ to be the set of weights of T-module N_i.

2.4 Proposition. *If $\mu \in \Delta(0)$, then $\mu \in \Delta(G_i)$ if and only if $(\mu, \gamma_1 + \ldots + \gamma_i) = 0$.*

Proof. An equivalent formulation is that $\mu \in \Delta(G_i)$ if and only if either all the numbers $(\mu, \gamma_1), \ldots, (\mu, \gamma_i)$ are equal to zero or precisely two of them are nonzero (of different signs).

(a) Let us prove that if $\mu \in \Delta(0)^+$ and there is a single nonzero value among $\{(\mu, \gamma_j) \mid j = 1, \ldots, i\}$, then $e_{-\mu}$ does not preserve V_i.
Suppose $(\mu, \gamma_m) > 0$, while all other products are equal to zero. Then $\gamma_m - \mu \in \Delta(1)$ and $(\gamma_m - \mu, \gamma_j) = 0$ for $j \ne m$. Since $(\gamma_m - \mu) - \gamma_j \notin \Delta$ and $(\gamma_m - \mu) - \gamma_m$ is negative, $\gamma_m - \mu \notin \Gamma_i$. Thus, $e_{\gamma_m} \in V_i$ and $[e_{-\mu}, e_{\gamma_m}] \notin V_i$.

(b) It remains to prove that if $\mu \in \Delta(0)^+$ and $(\mu, \gamma_1 + \ldots + \gamma_i) = 0$, then $e_{-\mu} V_i \subset V_i$. Essentially, it was already shown at the end of the proof of (2.2): If $\alpha \in \Gamma_i$ and $\alpha - \mu$ is a root, then $(\alpha - \mu, \gamma_1 + \ldots + \gamma_i) = (\gamma, \gamma)$ and hence $\alpha - \mu \in \Gamma_i$. $\square$

2.5 Corollary.
1. $\Delta(N_i) = \{\mu \in \Delta(0)^+ \mid (\mu, \gamma_1 + \ldots + \gamma_i) > 0\}$*;*
2. The representation of R_i on V_i is completely reducible.

Proof. 1. Compare (1.3) and (2.4).
2. One has to prove that N_i acts trivially on V_i. Invoking the first assertion, we see that it was done in the proof of Proposition 2.2, part (ii)b. $\square$

The following is the principal result of this section.

2.6 Theorem. *The natural mapping $\tau_i : G *_{R_i} V_i \to G{\cdot}V_i = \overline{\mathcal{O}}_i \subset \mathfrak{l}(1)$ is a G-equivariant resolution of singularities of $\overline{\mathcal{O}}_i$.*

Proof. It is well-known that τ_i is proper and $G{\cdot}V_i$ is closed (see e.g. [3]). The equality $G{\cdot}V_i = \overline{\mathcal{O}}_i$ follows from (2.3). The only assertion that still has to be proved is that τ_i is birational. In our situation, it is equivalent to that $G_{x_i} \subset R_i$.

Recall that $x_i = e_{\gamma_1} + \ldots + e_{\gamma_i}$. For each $j \in [1, i]$, one may choose $e_{-\gamma_j} \in \mathfrak{l}(-1)$ so that $\{e_{\gamma_j}, h_{\gamma_j}, e_{-\gamma_j}\}$ is an $\mathfrak{sl}_2$-triple, where $h_{\gamma_j} = [e_{\gamma_j}, e_{-\gamma_j}]$. Then $\alpha(h_{\gamma_j}) = (\alpha, \gamma_j^\vee)$ for any $\alpha \in \Delta$. Set $y_i = e_{-\gamma_1} + \ldots + e_{-\gamma_i}$ and $h_i = h_{\gamma_1} + \ldots + h_{\gamma_i}$. Since the γ_j's are pairwise orthogonal and long, $[x_i, y_i] = h_i$ and $\{x_i, h_i, y_i\}$ is again an $\mathfrak{sl}_2$-triple. As is well-known, to any nilpotent element $x_i \in \mathfrak{l}$ one associates the parabolic subgroup $P(x_i) \subset L$ such that $L_{x_i} \subset P(x_i)$. Although $P(x_i)$ depends only on x_i, the most simple description of it uses an $\mathfrak{sl}_2$-triple containing x_i (see [13, III, §4]): The eigenvalues of $\operatorname{ad} h_i$ are integral ($[h_i, x_i] = 2x_i$) and $\operatorname{Lie} P(x_i)$ is the sum of the eigenspaces corresponding to nonnegative eigenvalues. In our case, it easily follows from the definition of h_i that $\operatorname{Lie} P(x_i) = \oplus_\mu \mathfrak{l}_\mu$, where the sum is taken over $\mu \in \Delta$ such that $(\mu, \gamma_1 + \ldots + \gamma_i) \geq 0$. Comparing with the description of R_i, we see that $P(x_i) \cap G = R_i$. Thus, $G_{x_i} = L_{x_i} \cap G \subset P(x_i) \cap G = R_i$ and the mapping τ_i is birational. □

Since the representation of R_i on V_i is completely reducible, it follows from [3, Thm. 0] that $\overline{\mathcal{O}}_i$ is normal and has rational singularities. That is, our construction of a resolution of singularities yields another proof of this well-known result.

In Proposition 2.2, the tangent space of Bx_i at x_i was described. For future use, we compute now the tangent space of Gx_i at x_i.

2.7 Proposition. *Set* $\tilde\Gamma_i = \{\alpha \in \Delta(1) \mid (\alpha, \gamma_1 + \ldots + \gamma_i) > 0\}$. *Then* $[\mathfrak{g}, x_i] = \bigoplus_{\alpha \in \tilde\Gamma_i} \mathfrak{l}(1)_\alpha$ *and this space is* B*-stable.*

Proof. The next observations are obvious consequences of Lemma 1.2 :
- if $\mu \in \Delta(0)^+$, then the function $\mu \mapsto (\mu, \gamma_1 + \ldots + \gamma_i)$ may take only two values 0, $\frac{1}{2}(\gamma, \gamma)$.
- if $\alpha \in \Delta(1)$, then the function $\alpha \mapsto (\alpha, \gamma_1 + \ldots + \gamma_i)$ may take only values 0, $\frac{1}{2}(\gamma, \gamma), (\gamma, \gamma)$.

Whence $\tilde\Gamma_i$ is the subset of $\Delta(1)$ corresponding to the values $\frac{1}{2}(\gamma, \gamma)$ and (γ, γ). Since x_i belongs to the sum of weight spaces corresponding to the value (γ, γ), we see that $[\mathfrak{g}, x_i] \subset \bigoplus_{\alpha \in \tilde\Gamma_i} \mathfrak{l}(1)_\alpha$. On the other side, $[\mathfrak{g}, x_i] \supset [\mathfrak{b}, x_i] = \bigoplus_{\alpha \in \Gamma_i} \mathfrak{l}(1)_\alpha$. Next, let $\nu \in \Delta(1)$ and $(\nu, \gamma_1 + \ldots + \gamma_i) = \frac{1}{2}(\gamma, \gamma)$. Take the unique $m \in [1, i]$ such that $(\nu, \gamma_m) \neq 0$ (i.e., $(\nu, \gamma_m) > 0$). Then $\mu := \nu - \gamma_m \in \Delta(0)$ and $[e_\mu, x_i] = [e_\mu, e_{\gamma_m}] = e_\nu \in [\mathfrak{g}, x_i]$.
The second assertion easily follows from the definition of $\tilde\Gamma_i$. □

For a subspace $M \subset \mathfrak{l}(1)$, we let $M^\perp$ denote the orthogonal complement to M in $\mathfrak{l}(-1)$.

2.8 Corollary. $[\mathfrak{g}, x_i]^\perp = \bigoplus_{\alpha \in \Delta^i(-1)} \mathfrak{l}(-1)_\alpha$ *and this space is* B*-stable.*

Proof. Recall from section 1 that $\Delta^i = \{\alpha \in \Delta \mid (\alpha, \gamma_1) = \ldots = (\alpha, \gamma_i) = 0\}$. The above discussion about functions with at most three values implies that $[\mathfrak{g}, x_i]^\perp = \bigoplus_\alpha \mathfrak{l}(-1)_\alpha$, where α ranges over all roots in $\Delta(-1)$ such that $(\alpha, \gamma_1 + \ldots + \gamma_i) = 0$. But for $\alpha \in \Delta(-1)$ the last equality implies that $(\alpha, \gamma_1) = \ldots = (\alpha, \gamma_i) = 0$. □

3 The commuting variety and a double coset space

Define $\eta : \mathfrak{l}(1) \oplus \mathfrak{l}(-1) \to \mathfrak{l}(0) = \mathfrak{g}$ to be the restriction of the Lie bracket in $\mathfrak{l}$ to $\mathfrak{l}(1) \oplus \mathfrak{l}(-1)$, i.e., $\eta(x+y) = [x,y]$ for $x \in \mathfrak{l}(1)$, $y \in \mathfrak{l}(-1)$. In this section we study properties of the generalized *commuting variety* $\mathfrak{E} := \eta^{-1}(0)$.

It is well-known (and easy to prove) that $\mathfrak{l}(-1)_x$, the centralizer of $x \in \mathfrak{l}(1)$ in $\mathfrak{l}(-1)$, is nothing but $[\mathfrak{g}, x]^{\perp}$. Therefore $\mathfrak{E}$ is the union of the conormal bundles to the G-orbits in $\mathfrak{l}(1)$. Since any conormal bundle is of dimension $\dim \mathfrak{l}(1)$ and there are finitely many G-orbits in $\mathfrak{l}(1)$, $\mathfrak{E}$ is a variety of pure dimension $\dim \mathfrak{l}(1)$ and there is a bijection between the G-orbits in $\mathfrak{l}(1)$ and the irreducible components of $\mathfrak{E}$: each irreducible component is the closure of the conormal bundle of a unique G-orbit. By symmetry, the same is true G-orbits in $\mathfrak{l}(-1)$. Consequently, there is a natural bijection (duality) between the G-orbits in $\mathfrak{l}(1)$ and $\mathfrak{l}(-1)$. This is a particular case of Pyasetskii's theorem [10]. Given an orbit $\mathcal{O} \subset \mathfrak{l}(1)$, the dual orbit $\mathcal{O}^{\vee} \subset \mathfrak{l}(-1)$ is directly being described as follows. For an arbitrary $x \in \mathcal{O}$, take $[\mathfrak{g}, x]^{\perp}$ and then $G{\cdot}[\mathfrak{g}, x]^{\perp}$. The last set is irreducible and $\mathcal{O}^{\vee}$ is the dense orbit in it.

We use for G-orbits in $\mathfrak{l}(-1)$ the notation similar to that for $\mathfrak{l}(1)$. Namely, define $\mathcal{O}_i^*$ to be the orbit in $\mathfrak{l}(-1)$ containing representatives of the form $e_{\mu_1} + \ldots + e_{\mu_i}$, where $\mu_1, \ldots, \mu_i$ are orthogonal long roots in $\Delta(-1)$.

3.1 Lemma. $(\mathcal{O}_i)^{\vee} = \mathcal{O}_{r-i}^*$.

Proof. Take $x_i \in \mathcal{O}_i$. By Corollary 2.8, the set of weights of $[\mathfrak{g}, x_i]^{\perp}$ is $\Delta^i(-1)$. Since it contains a sequence of orthogonal long roots of length $r-i$ (e.g. $-\gamma_{i+1}, \ldots, -\gamma_r$), $\mathcal{O}_{r-i}^*$ has a nonempty intersection with $[\mathfrak{g}, x_i]^{\perp}$. Hence, denoting $(\mathcal{O}_i)^{\vee} = \mathcal{O}_{d(i)}^*$, we have $d(i) \geq r-i$. Because this holds for all i, one must have $d(i) = r-i$. □

3.2 Corollary. *Let $\mu_1, \ldots, \mu_i$ be an arbitrary sequence of orthogonal long roots in $\Delta(1)$. Let $\Psi = \{\alpha \in \Delta(1) \mid (\alpha, \mu_1) = \ldots = (\alpha, \mu_i) = 0\}$ and $\mathfrak{l}(1)_{\Psi} = \oplus_{\alpha \in \Psi} \mathfrak{l}(1)_{\alpha}$. Then $\mathcal{O}_{r-i}$ is the dense orbit in $G \cdot \mathfrak{l}(1)_{\Psi}$.*

Proof. The argument for a part of the u.c.s. is given in the lemma (up to switching between $\mathfrak{l}(-1)$ and $\mathfrak{l}(1)$). As is easily seen, it applies to arbitrary sequences of orthogonal long roots as well. □

Define $\mathfrak{E}_i$ to be the irreducible component of $\mathfrak{E}$ corresponding to $\mathcal{O}_i$. For instance, $\mathfrak{E}_0 = \mathfrak{l}(-1)$ and $\mathfrak{E}_r = \mathfrak{l}(1)$. The previous lemma says the orbit in $\mathfrak{l}(-1)$ that corresponds to $\mathfrak{E}_i$ is $\mathcal{O}_{r-i}^*$.

¿From the point of view of the G-action, $\mathfrak{E}$ is the union of conormal bundles. The following is a kind of 'global' characterization:

$$L{\cdot}\mathfrak{l}(1) \cap (\mathfrak{l}(1) \oplus \mathfrak{l}(-1)) = \mathfrak{E}\ . \tag{3.3}$$

Actually, we shall prove a more precise statement whose consequence is the above formula. To this end, recall some results from [11] and [8]. At the very beginning, we have introduced the parabolic subgroup P with *aura* whose standard Levi factor is G. Since $\mathfrak{l}(1)$ is P-stable and $\mathfrak{l}(1)\oplus\mathfrak{l}(-1)$ is G-stable, it suffices to work with a system of representatives of $G\backslash L/P$ while studying $L{\cdot}\mathfrak{l}(1)\cap(\mathfrak{l}(1)\oplus\mathfrak{l}(-1))$. The double coset space $G\backslash L/P$ is finite and a system of representatives is described in [11, sect. 3]:

For each i let $u_{-\beta_i}$ be a nontrivial element in the one-parameter unipotent subgroup $U_{-\beta_i}$ corresponding to $-\beta_i$, and let w_{β_i} be the reflection in W corresponding to β_i, realized as an element of $N_L(T)$. For $0\le i\le j\le r$, let $z_{ij}=\prod\limits_{t=i+1}^{j}u_{-\beta_t}\cdot\prod\limits_{s=1}^{i}w_{\beta_s}$. Then $L=\bigsqcup\limits_{i,j}Gz_{ij}P$. This description is valid with *any* maximal orthogonal sequence of long roots in $\Delta(1)$. For future convenience, we have stated it with the l.c.s. $\beta_1,\dots,\beta_r$. Let $\mathcal{K}_{ij}=Gz_{ij}P\subset L$. (Our indexing and notation differ from those of [11].)

We have the irreducible decomposition $\mathfrak{E}=\cup_{i=1}^{r}\mathfrak{E}_i$. Take any $i,j\in[1,r]$ $(i\le j)$. By [8, 5.8], $\mathfrak{E}_{ij}:=\mathfrak{E}_i\cap\mathfrak{E}_j$ is irreducible and $\mathfrak{E}_{ij}=\mathfrak{E}\cap(\overline{\mathcal{O}_i}\times\overline{\mathcal{O}^*_{r-j}})$. Moreover, if C is an irreducible G-stable subvariety of $\mathfrak{E}$ such that the projection of C to $\mathfrak{l}(1)$ is $\overline{\mathcal{O}_i}$ and to $\mathfrak{l}(-1)$ is $\overline{\mathcal{O}^*_{r-j}}$, then $C=\mathfrak{E}_{ij}$.

3.4 Theorem. *For $0\le i\le j\le r$, one has $\mathcal{K}_{ij}{\cdot}\mathfrak{l}(1)\cap(\mathfrak{l}(1)\oplus\mathfrak{l}(-1))=\mathfrak{E}_{r-j,r-i}$.*

Proof. Since $\mathcal{K}_{ij}{\cdot}\mathfrak{l}(1)\cap(\mathfrak{l}(1)\oplus\mathfrak{l}(-1))=Gz_{ij}{\cdot}\mathfrak{l}(1)\cap(\mathfrak{l}(1)\oplus\mathfrak{l}(-1))=G{\cdot}\{z_{ij}{\cdot}\mathfrak{l}(1)\cap(\mathfrak{l}(1)\oplus\mathfrak{l}(-1))\}$, we need to realize what is $z_{ij}{\cdot}\mathfrak{l}(1)$. To simplify notation, let $u_{ij}=\prod\limits_{t=i+1}^{j}u_{-\beta_t}$. Then $z_{ij}=u_{ij}z_{ii}$. To understand the action of z_{ij}, let us introduce the partition of $\Delta(1)$ corresponding to three values of the function $\alpha\mapsto(\alpha,\beta_1+\ldots+\beta_i)$ (cf. 2.7) :

$$\Delta(1)=\bigsqcup_{m=0}^{2}\Delta(1)_{i,m},\quad\text{where }\Delta(1)_{i,m}:=\{\alpha\in\Delta(1)\mid(\alpha,\beta_1+\ldots+\beta_i)=\tfrac{m}{2}(\beta,\beta)\}.$$

Consider the respective decomposition $\mathfrak{l}(1)=V_{i,0}\oplus V_{i,1}\oplus V_{i,2}$. Then z_{ii} acts trivially on $V_{i,0}$, $z_{ii}{\cdot}V_{i,1}\subset\mathfrak{l}(0)$, and $z_{ii}{\cdot}V_{i,2}\subset\mathfrak{l}(-1)$. For $x=x_0+x_1+x_2\in\mathfrak{l}(1)$, we have $z_{ij}{\cdot}x=u_{ij}(x_0+z_{ii}{\cdot}x_1+z_{ii}{\cdot}x_2)$. The unipotent transformation u_{ij} takes x_0 to x_0+y+y' for some $y\in\mathfrak{l}(0)$ and $y'\in\mathfrak{l}(-1)$, $z_{ii}{\cdot}x_1$ to $z_{ii}{\cdot}x_1+v$ for some $v\in\mathfrak{l}(-1)$, and $z_{ii}{\cdot}x_2$ to itself. It is easily seen that the summands lying in $\mathfrak{l}(0)$, $z_{ii}{\cdot}x_1$ and y, are supported by disjoint sets of roots in $\Delta(0)$, see also Fig. 1 below. Hence the condition $z_{ij}{\cdot}x\in\mathfrak{l}(1)\oplus\mathfrak{l}(-1)$ forces $x_1=0$ and $y=0$. The last equality clearly implies that $y'=0$. Then $z_{ij}{\cdot}x=x_0+z_{ii}{\cdot}x_2$ and the summands commute. This already proves the inclusion "$\subset$" in (3.3). To prove the theorem in full strength, one has to look more carefully at those $x_0\in V_{i,0}$ that are stabilized by u_{ij}. A straightforward calculation shows that $u_{ij}{\cdot}x_0=x_0$ if and only if x_0 is supported by those roots in $\Delta(1)_{i,0}$ that are orthogonal to $\beta_{i+1},\dots,\beta_j$, i.e., $x_0\in V_{j,0}\subset V_{i,0}$.

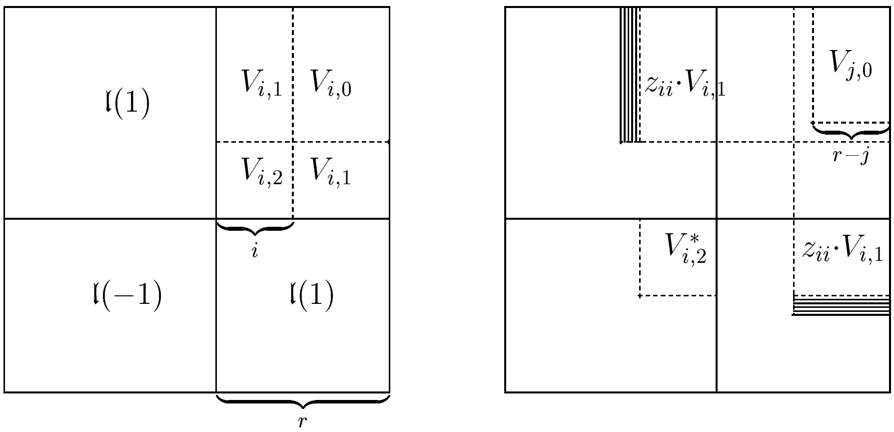
Thus, the previous argument proves that $z_{ij}{\cdot}\mathfrak{l}(1) \cap (\mathfrak{l}(1) \oplus \mathfrak{l}(-1))$ is the direct sum of the spaces $V_{j,0} \subset \mathfrak{l}(1)$ and $V^*_{i,2} := z_{ii}{\cdot}V_{i,2} \subset \mathfrak{l}(-1)$. I claim that both these spaces are B-stable, $G{\cdot}V_{j,0} = \overline{\mathcal{O}}_{r-j}$, and $G{\cdot}V^*_{i,2} = \overline{\mathcal{O}^*_i}$.

(a) For $V^*_{i,2}$: Note that the set of weights of $V^*_{i,2}$ is just $-\Delta(1)_{i,2} = \{\alpha \in \Delta(-1) \mid (\alpha, -\beta_1 - \ldots - \beta_i) = (\beta,\beta)\}$. Therefore for $V^*_{i,2}$ holds the analogue of Proposition 2.2: if one takes the point $y_i = e_{-\beta_1} + \ldots + e_{-\beta_i} \in \mathcal{O}^*_i$, then $V^*_{i,2} = [\mathfrak{b}, y_i]$ and so on.. This is because $-\beta_1, \ldots, -\beta_r$ is the u.c.s. with respect to $\mathfrak{l}(-1)$.

(b) For $V_{j,0}$: We again perform an argument that uses "dual" (or "$\mathfrak{l}(-1)$") versions of some previous results. Take $y_j = e_{-\beta_1} + \ldots + e_{-\beta_j} \in \mathcal{O}^*_j$. Then $[\mathfrak{g}, y_j]^\perp = V_{j,0}$ and this space is B-stable (analogue of 2.8). Therefore the dense orbit in $G{\cdot}V_{j,0}$ is $(\mathcal{O}^*_j)^\vee$, i.e., $\mathcal{O}_{r-j}$ (analogue of 3.1).

The various spaces involved in the proof are depicted in Figure 1. It is the real picture for $\mathfrak{l} = \mathfrak{sl}_N$ and $\mathfrak{g} = \mathfrak{sl}_{N-r} \times \mathfrak{sl}_r \times k$ with $N - r > r$. The shaded strips have width $j - i$. These represent the space that consists of the $\mathfrak{l}(0)$-components of vectors $u_{ij}{\cdot}x_0$ ($x_0 \in V_{i,0}$).

Figure 1:

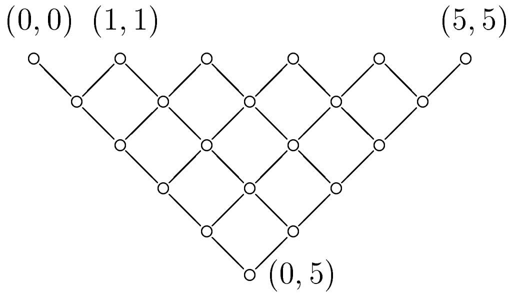


Finally, the LHS in (3.4) is $G{\cdot}(V_{j,0} \oplus V^*_{i,2})$. It follows from the above claims that it is a G-stable closed subset in $\mathfrak{E}$ and its projection to $\mathfrak{l}(1)$ (resp. $\mathfrak{l}(-1)$) is the closure of $\mathcal{O}_{r-j}$ (resp. $\mathcal{O}^*_i$). By a result in [8, 5.8, 5.10], which is stated just before Theorem 3.4, this means the variety in question is $\mathfrak{E}_{r-j,r-i}$. $\square$

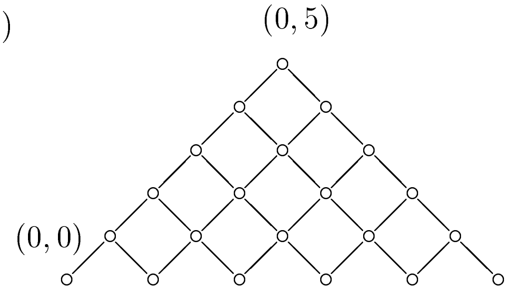
Remarks. 1. As a by-product of the proof, we have the following: if $x \in \mathfrak{l}(1)$ and $g{\cdot}x \in \mathfrak{l}(1) \oplus \mathfrak{l}(-1)$, then there exists a decomposition $x = x' + x''$ ($x', x'' \in \mathfrak{l}(1)$) such that $g{\cdot}x' \in \mathfrak{l}(1)$ and $g{\cdot}x'' \in \mathfrak{l}(-1)$.
2. One may think of $\mathcal{K}_{ij}$'s as the $G \times P$-orbits in L. By [11, 3.7], $\overline{\mathcal{K}_{ij}} = \bigcup_{i \le s \le t \le j} \mathcal{K}_{st}$. On the other side, it was proved in [8, 5.10] that each G-orbit in $\mathfrak{E}$ is dense in some $\mathfrak{E}_{ij}$

and, denoting such orbit $\mathcal{O}_{ij}$, one has $\overline{\mathcal{O}_{ij}} = \bigcup\limits_{s\le i\le j\le t} \mathcal{O}_{st}$. Theorem 3.4 establishes thus a geometric order-reversing bijection between these two posets. The respective Hasse diagrams in case $r=5$ are depicted in Figure 2.

Figure 2:



The poset of G-orbits in $\mathfrak{E}$



The poset of $G\times P$-orbits in L

Taking into account the previous convention on indexing the irreducible components of $\mathfrak{E}$, one sees that $\mathcal{O}_{ir} = \mathcal{O}_i$, the orbit in $\mathfrak{l}(1)$, and $\mathcal{O}_{0,r-i} = \mathcal{O}_i^*$, the orbit in $\mathfrak{l}(-1)$.

The set $\{\mathcal{O}_{ij}\}$ has an interesting connection with some nilpotent L-orbits in $\mathfrak{l}$. To describe it, let us look at the short grading from another point of view. The decomposition $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$, where $\mathfrak{l}_0 = \mathfrak{l}(0)$ and $\mathfrak{l}_1 = \mathfrak{l}(1) \oplus \mathfrak{l}(-1)$, is a $\mathbb{Z}_2$-grading. The principal results on "orbits and invariants" associated with $\mathbb{Z}_2$-gradings are due to Kostant and Rallis [4]. Some complementary results, which are also valid for more general gradings, are due to E.B. Vinberg [14]. Later, we need the following properties:

- if $z \in \mathfrak{l}_1$, then $\dim L{\cdot}z = 2\dim G{\cdot}z$, see [4, Prop. 5].
- if $\tilde{\mathcal{O}} \subset \mathfrak{l}$ is any L-orbit, then each irreducible component of $\tilde{\mathcal{O}} \cap \mathfrak{l}_1$ is a G-orbit. The same is true for $\tilde{\mathcal{O}} \cap \mathfrak{l}(1)$. (See [14, Lemma in §2]).

Let us start with a G-orbit $\mathcal{O}_i \subset \mathfrak{l}(1)$. It generates the L-orbit $\tilde{\mathcal{O}}_i = L{\cdot}\mathcal{O}_i$. The question is what is $\tilde{\mathcal{O}}_i \cap \mathfrak{l}_1$? An immediate consequence of the aforementioned properties is that $\mathcal{O}_i$ is an irreducible component of this intersection and that $\dim \tilde{\mathcal{O}}_i = 2\dim \mathcal{O}_i$. It is also clear that $\mathcal{O}_i^*$ is another component of the intersection. The complete answer is given by the following

3.5 Theorem. $\tilde{\mathcal{O}}_i \cap \mathfrak{l}_1 = \bigsqcup\limits_{s=0}^{i} \mathcal{O}_{s,r-i+s}$.

Proof. By (3.3), the LHS lies in $\mathfrak{E}$. Being G-stable, it is the union of some $\mathcal{O}_{st}$. To realize which $\mathcal{O}_{st}$ do occur, it suffices to find out what is the dense L-orbit in $L{\cdot}\mathfrak{E}_{st}$. Keep the notation of the proof of (3.4). It was shown therein that $\mathfrak{E}_{r-j,r-i} = G{\cdot}(V_{j,0} \oplus V_{i,2}^*)$ for $r-j \le r-i$. Since $z_{ii} \in L$ takes $V_{i,2}^*$ to $V_{i,2}$ and $V_{j,0}$ to itself, we have $L{\cdot}\mathfrak{E}_{r-j,r-i} =$

$L{\cdot}(V_{j,0}\oplus V_{i,2})$. Invoking the definition of the sets $\Delta(1)_{i,m}$ and Lemma 1.2(ii), one sees that the set of weights of $V_{j,0}\oplus V_{i,2}$, say $\Gamma_{i,j}$, is contained in the set of weights in $\Delta(1)$ that are orthogonal to $\beta_{i+1},\dots,\beta_j$. On the other side, $\Gamma_{i,j}$ contains an orthogonal sequence of length $r-j+i$. It then follows from (2.1) and (3.2) that $\mathcal{O}_{r-j+i}\cap(V_{j,0}\oplus V_{i,2})$ is dense in $V_{j,0}\oplus V_{i,2}$ and therefore $\tilde{\mathcal{O}}_{r-j+i}$ is dense in $L{\cdot}\mathfrak{E}_{r-j,r-i}$. Thus, $L{\cdot}\mathcal{O}_{r-j,r-i}=\tilde{\mathcal{O}}_{r-(r-i)+(r-j)}$, which is exactly what is needed. $\square$

3.6 Corollary. *Dimension of $\mathfrak{E}_{ij}$ depends only on $j-i$.*

Proof. Indeed, $\dim\mathfrak{E}_{ij}=\dim\mathcal{O}_{ij}=\frac{1}{2}\dim\tilde{\mathcal{O}}_{r-j+i}$. $\square$

Recall that the $\mathfrak{E}_{ij}$'s were defined as the intersection of two irreducible components of $\mathfrak{E}$: $\mathfrak{E}_{ij}=\mathfrak{E}_i\cap\mathfrak{E}_j$. We have thus obtained a surprising result that dimension of the intersection depends only on "distance" between irreducible components. Looking at the Hasse diagram of the poset $\mathfrak{E}$, we see that the components of the intersection $\tilde{\mathcal{O}}_i\cap\mathfrak{l}(1)$ occupy the ith row, and the Corollary says that dimension is constant along the rows.

Our last goal in this section is to construct a G-equivariant resolution of $\mathfrak{E}_{ij}$. Since $\mathfrak{E}_{ij}\subset\mathfrak{l}(1)\oplus\mathfrak{l}(-1)$ projects onto $\overline{\mathcal{O}_i}$ and $\overline{\mathcal{O}^*_{r-j}}$ respectively, it is natural to suggest that a resolution we are searching for has something to do with known resolutions of both orbit closures. And this is really so. A resolution of $\overline{\mathcal{O}_i}$ is constructed in sect. 2 and a resolution of $\overline{\mathcal{O}^*_{r-j}}$ can be described in the 'dual' fashion. We reproduce the essential steps just in order to fix the related notation. Let $y_{r-j}=e_{-\beta_1}+\ldots+e_{-\beta_{r-j}}\in\mathcal{O}^*_{r-j}$. The string $-\beta_1,\dots,-\beta_r$ plays the same rôle for $\mathfrak{l}(-1)$ as the u.c.s. for $\mathfrak{l}(1)$. Therefore the following analogues of Proposition 2.2 and Theorem 2.6 hold: $V^*_{r-j}:=[\mathfrak{b},y_{r-j}]$ is B-stable and the mapping $\tau^*_j: G*_{R^*_{r-j}}V^*_{r-j}\to\overline{\mathcal{O}^*_{r-j}}$, where R^*_{r-j} is the normalizer in G of V^*_{r-j}, is a G-equivariant resolution of singularities. Set $R_{ij}:=R_i\cap R^*_{r-j}$. It is a standard parabolic subgroup of G and $V_i\oplus V^*_{r-j}$ is a R_{ij}-module. Before stating the next theorem, it is worth to observe a relationship between the spaces V_i, V^*_{r-j} that are needed for resolutions of singularities and the spaces $V_{i,m}$ depicted in Fig. 1. The proofs are easy and left to the reader.

- $V_{r-j}\subset V_{j,0}$ and these are equal if and only if the equivalent conditions of (1.5) hold.
- $V^*_{i,2}=V^*_i$.

3.7 Theorem. *Let $1\le i\le j\le r$. Then*
(i) *The natural mapping $\tau_{ij}: G*_{R_{ij}}(V_i\oplus V^*_{r-j})\to\mathfrak{E}_{ij}$ is a G-equivariant resolution of singularities;*
(ii) *R_{ij}-module $V_i\oplus V^*_{r-j}$ is completely reducible.*

Proof. (i) $z_{r-j,r-j}=\prod_{s=1}^{r-j}w_{\beta_s}$ takes V^*_{r-j} into $\mathfrak{l}(1)$ and keeps V_i intact. (The inequality $i\le j$ is needed at this point.) Therefore $V_i\oplus V^*_{r-j}\subset\mathfrak{E}$. Hence the image of τ_{ij} is a G-stable irreducible subvariety of $\mathfrak{E}$. By construction, its projection to $\mathfrak{l}(1)$ (resp. $\mathfrak{l}(-1)$) is $\overline{\mathcal{O}_i}$ (resp. $\overline{\mathcal{O}^*_{r-j}}$). Thus, the image has to be equal to $\mathfrak{E}_{ij}$. Since $G_{x_i}\subset R_i$

and $G_{y_{r-j}} \subset R^*_{r-j}$, we have $G_{x_i+y_{r-j}} = G_{x_i} \cap G_{y_{r-j}} \subset R_{ij}$. This inclusion guarantee us birationality of τ_{ij}, for the G-orbit of $x_i + y_{r-j}$ is dense in $\mathfrak{E}_{ij}$.
(ii) The unipotent radical of R_{ij} is the product of the unipotent radicals of R_i and R^*_{r-j}. By symmetry, it suffices to show that N_i, the unipotent radical of R_i, acts trivially on $V_i \oplus V^*_{r-j}$. By (2.5), N_i acts trivially on V_i. A similar argument applies to V^*_{r-j}: all the roots μ corresponding to V^*_{r-j} satisfy the condition $(\mu, \gamma_1 + \ldots + \gamma_i) = 0$, while $(\nu, \gamma_1+\ldots+\gamma_i) > 0$ for each $\nu \in \Delta(N_i)$. Since $(\alpha, \gamma_1+\ldots+\gamma_i) \le 0$ for each $\alpha \in \Delta(-1)$, we have $\nu + \mu$ is never a root. $\square$

Again, as a consequence of Kempf's theorem and complete reducibility, we get the assertion that the varieties $\mathfrak{E}_{ij}$ are normal and have rational singularities. Another proof is found in [8].

4 The algebra of covariants on $\mathfrak{l}(1)$ and beyond

Let U be the unipotent radical of B. It is well known that the algebra of covariants $k[\mathfrak{l}(1)]^U$ is polynomial. For instance, it already follows from the fact that $\mathfrak{l}(1)$ is a spherical G-module (for, the B-orbit of $e_{\beta_1} + \ldots + e_{\beta_r}$ is dense in $\mathfrak{l}(1)$). There are a plenty of papers, where this algebra is explicitly described, see e.g. [12],[2],[5]. Nevertheless, I believe Theorem 4.1 yet deserves to be stated and proved. In this section, the description of $k[\mathfrak{l}(1)]^U$ will be obtained as a consequence of a restriction theorem (see [6, §2] for representations and a general version in [7, §1]).

Denote by F the subspace of $\mathfrak{l}(1)$ with basis $e_{\beta_1}, \ldots, e_{\beta_r}$. Let $f_1, \ldots, f_r$ be the coordinates in this basis.

4.1 Theorem. *The restriction homomorphism $k[\mathfrak{l}(1)] \to k[F]$ maps $k[\mathfrak{l}(1)]^U$ isomorphically and T-equivariantly onto the subalgebra generated by $f_1, f_1f_2, \ldots, f_1\cdot\ldots\cdot f_r$.*

We postpone a bit the proof and show that the theorem immediately implies the basic properties of $k[\mathfrak{l}(1)]^U$.

4.2 Corollary. *1. $k[\mathfrak{l}(1)]^U$ is polynomial and the weights of T-homogeneous free generators are $\lambda_1, \ldots, \lambda_r$, where $\lambda_i = -(\beta_1 + \ldots + \beta_i)$.*
2. The G-module $k[\mathfrak{l}(1)]$ is multiplicity free and if $\lambda = \sum_{i=1}^r a_i\lambda_i$ ($a_i \in \mathbb{Z}_{\ge 0}$), then the simple G-module with highest weight λ occurs in the space of polynomials of degree $\sum_{i=1}^r ia_i$. $\square$

<u>Proof of (4.1)</u>. Roughly speaking, the restriction theorem for U-invariants applies as follows:

1. One has to find generic stabilizer for the G-module $\mathfrak{l}(1) \oplus \mathfrak{l}(-1)$. Of course, such a stabilizer is determined up to a conjugation in G. We need the so-called "canonical" generic stabilizer that can be determined via an explicit procedure. Let S be the canonical stabilizer. It is a reductive subgroup of G, which is normalized by T.

2. Form the connected reductive group $Z_G(S)^0$. Then $\tilde{U} := U \cap Z_G(S)^0$ is a maximal unipotent subgroup in it.

3. The restriction theorem says that the restriction homomorphism $k[\mathfrak{l}(1)] \to k[\mathfrak{l}(1)^S]$ maps $k[\mathfrak{l}(1)]^U$ isomorphically and T-equivariantly onto the algebra $(k[\mathfrak{l}(1)^S]^{\tilde{U}})_{\mathcal{T}}$, where $\mathcal{T}$ is the monoid consisting of those dominant weights of G that vanish on $T \cap S$ and the subscript "$\mathcal{T}$" means that one takes only weight spaces whose weights lie in $\mathcal{T}$.

An inductive procedure for finding S is described in [6, §1]. In our case, it amounts essentially to constructing the l.c.s. Indeed, one starts with the (unique) lowest weight vector $e_{\beta_1} \in \mathfrak{l}(1)$. Let $\hat{G}_1$ be the standard Levi factor of the parabolic subgroup stabilizing the line $\langle e_{\beta_1} \rangle$ and let $\mathfrak{l}(1)_1$ be the $\hat{G}_1$-stable complement to $[\mathfrak{g}, e_{\beta_1}]$. Then $\Delta(\hat{G}_1)$ consists of all roots in $\Delta(0)$ that are orthogonal to β_1. A specific feature of the 'Abelian' situation is that the set of weights of $\mathfrak{l}(1)_1$ coincide with $\Delta_1(1)$ (see notation in (1.1)). Therefore on the second step one has the *unique* lowest weight vector e_{β_2} and so on... The procedure terminates when one arrives at the Levi subgroup $\hat{G}_r \subset G$ and no lowest weight vectors are available. The root system of $\hat{G}_r$ is $\Delta_r(0)$ in the notation of (1.1). Then

(4.3)
$$S = \{g \in \hat{G}_r \mid \beta_i(g) = 1, \quad i = 1, \dots, r\}$$

Here the β_i's are being considered as characters of $\hat{G}_r$.

4.4 Lemma.
(i) $\mathfrak{l}(1)^S = F$;
(ii) $Z_G(S)^0$ *is the central torus in* $\hat{G}_r$.

Proof. (i) Since the roots of $\hat{G}_r$ are orthogonal to $\beta_1, \dots, \beta_r$, we have $\mathfrak{l}(1)^S \supset F$. On the other side, $\mathfrak{l}(1)^{S \cap T} = \oplus_{\alpha \in M} \mathfrak{l}(1)_\alpha$, where $M = \Delta(1) \cap (\mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_r) = \{\beta_1, \dots, \beta_r\}$.
(ii) It suffices to prove that $N_G(S)^0 = \hat{G}_r$. Clearly, $N_G(S)^0 \supset \hat{G}_r$. Conversely, it is easy to see that the root system of $N_G(S)^0$ is $\Delta_r(0) = \Delta(\hat{G}_r)$. □

It follows that $k[\mathfrak{l}(1)]^U$ is isomorphic to $k[F]_{\mathcal{T}}$ and it remains to realize what is $\mathcal{T}$. Denote by $\mathcal{X}_+$ the monoid of dominant weights of G (relative to U). It follows from Eq. (4.3) and part 3 in the exposition of the restriction theorem that $\mathcal{T} = \mathcal{X}_+ \cap (\mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_r)$.

4.5 Lemma. $\mathcal{X}_+ \cap (\mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_r) =$
$$= \begin{cases} \mathbb{Z}_{\ge 0}(-\beta_1) + \ldots + \mathbb{Z}_{\ge 0}(-\beta_1 - \ldots - \beta_r), & \textit{if } \beta_r \ne \gamma, \\ \mathbb{Z}_{\ge 0}(-\beta_1) + \ldots + \mathbb{Z}_{\ge 0}(-\beta_1 - \ldots - \beta_{r-1}) + \mathbb{Z}(-\beta_1 - \ldots - \beta_r), & \textit{if } \beta_r = \gamma. \end{cases}$$

Proof. An equivalent formulation is:
Let $l_i \in \mathbb{Z}$. Then $\sum_{i=1}^r l_i\beta_i \in \mathcal{X}_+$ if and only if $l_1 \le l_2 \le \ldots \le l_r \le 0$ in the first case or $l_1 \le l_2 \le \ldots \le l_r$ in the second case.

(a) It follows from Corollary 1.3 that $\lambda_i = -\beta_1 - \ldots - \beta_i$ lies in $\mathcal{T}$. If $\beta_r = \gamma$, then $\lambda_r = -2\varphi$ is orthogonal to $\Delta(0)$ (see Proposition 1.5) and in this case the coefficient of λ_r can be arbitrary. This proves the inclusion "$\supset$" in both cases.

(b) Suppose $\lambda = \sum_i l_i\beta_i \in \mathcal{X}_+$. By the first part, $\lambda + a\lambda_r \in \mathcal{X}_+$ for any $a \geq 0$. Taking $a \gg 0$, one may assume that all $l_i - a < 0$. Let σ be a permutation such that $l_{\sigma(1)} \leq l_{\sigma(2)} \ldots \leq l_{\sigma(r)}$. By [11, 2.8], any permutation of the sequence $\beta_1, \ldots, \beta_r$ can be achieved by some element in $W(0)$. Take $w \in W(0)$ that realizes the permutation σ^{-1}, i.e., $w.\beta_i = \beta_{\sigma^{-1}(i)}$. Then $w.(\lambda + a\lambda_r) = w.\lambda + a\lambda_r = \sum_{i=1}^r (l_{\sigma(i)} - a)\beta_i$. It then follows from the first part of proof that $w.(\lambda + a\lambda_r) \in \mathcal{X}_+$. Therefore $w.(\lambda + a\lambda_r) = \lambda + a\lambda_r$ and hence $w.\lambda = \lambda$. That is, $l_1 \leq \ldots \leq l_r$. This is enough for the second case. In the first case, i.e., if $\beta_r \neq \gamma$, we see that there exists $\alpha \in \Delta(0)^+$ such that $(\alpha, \beta_1) = \ldots = (\alpha, \beta_{r-1})$ and $(\alpha, \beta_r) < 0$ (by the very definition of the l.c.s.). Then the condition $(\alpha, \lambda) \geq 0$ implies $l_r \leq 0$. □

Anyway, the dichotomy in the Lemma 4.5 does not affect the description of $k[F]_{\mathcal{T}}$, for the condition $l_r \leq 0$ is automatically satisfied for the weights of all monomials in $k[F]$. Since the function f_i has weight $-\beta_i$, it follows that $k[F]_{\mathcal{T}} = k[f_1, f_1 f_2, \ldots, f_1 \cdot \ldots \cdot f_r]$. Thus, Theorem 4.1 is proved. □

4.6 Theorem. *$k[\mathfrak{l}(1)]$ is a free $k[\mathfrak{l}(1)]^U$-module.*

Proof. Obviously, the weights λ_i $(i = 1, \ldots, r)$ are linearly independent. Another property satisfied by them is that if $(\alpha, \lambda_j) > 0$ for some $\alpha \in \Pi(0)$ and $j \in [1, r]$, then $(\alpha, \lambda_m) = 0$ for all $m \neq j$. Indeed, this follows from Lemma 1.2 and Theorem 1.4. For, $(\alpha, \lambda_j) > 0$ implies $(\alpha, \beta_j) < 0$. If $j < r$, then $(\alpha, \beta_{j+1}) > 0$ and these two inner products are the only nonzero ones. If $j = r$, then one has the unique nonzero product and again everything is clear.
An equivalent formulation of the second property is that each simple reflection $w_\alpha \in W(0)$ affects at most one weight λ_j: $\#\{j \mid w_\alpha.\lambda_j \neq \lambda_j\} \leq 1$ for each $\alpha \in \Pi(0)$.
It then follows from Theorem 5.5 and these two properties of the λ_i's that $k[\mathfrak{l}(1)]$ is a flat $k[\mathfrak{l}(1)]^U$-module. As is well known, the word "flat" can be replaced by "free" in the graded situation. □

5 On equidimensional quotient mappings

This section is completely independent of the previous parts. We will work with G without an overgroup L and therefore the notation will partially be changed (simplified). Here G is a connected reductive algebraic group and $U \subset G$ is a maximal unipotent subgroup; $B = N_G(U) = TU$. For an affine G-variety X, our aim is to prove a sufficient condition for $k[X]$ to be a flat $k[X]^U$-module.

First, consider a special class of G-varieties. For any dominant weight λ, let V_λ denote the corresponding irreducible G-module and $v_\lambda \in V_\lambda$ a highest weight vector. Let $\lambda_1, \ldots, \lambda_r$ be linearly independent dominant weights. Set $V =: \langle v_{\lambda_1} \rangle \oplus \ldots \oplus \langle v_{\lambda_r} \rangle \subset V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}$. Then V is B-stable and its normalizer in G is a standard parabolic subgroup, say P.

Let $C := \overline{G{\cdot}V}$. These and even more general varieties—when $\lambda_1,\dots,\lambda_r$ are arbitrary—has been studied in [15]. We shall use the result that in our case the G-orbits in C corresponds to the subsets of $[1,r]$. Namely, $\{v_J := \sum_{j\in J} v_{\lambda_j} \mid J\subset [1,r]\}$ is a set of representatives of the G-orbits. Since $\lambda_1,\dots,\lambda_r$ are linearly independent, the algebra $k[C]^U$ is polynomial, of Krull dimension r. It will be more convenient for us to work with $k[C]^{U_-}$, where U_- is opposite to U. Denoting by x_i the coordinate in v_{λ_i} for a (any) weight basis of V_{λ_i}, we have $k[C]^{U_-}\simeq k[x_1,\dots,x_r]$. In other words, the restriction homomorphism $k[C]\to k[V]$ induces an isomorphism $k[C]^{U_-}\simeq k[V]$.
Let W denote the Weyl group of G, $w_1,\dots,w_n$ the simple reflections in W, and $l: W\to \mathbb{Z}_{\ge 0}$ the usual length function.

5.1 Theorem. *The following conditions are equivalent:*

(i) $\pi_C: C\to C/\!\!/U_-$ *is equidimensional;*

(ii) $\#\{j \mid w_i{\cdot}\lambda_j\ne\lambda_j\}\le 1$ *for each* $i\in[1,n]$.

We need a description of U_--orbits in G/P. It is similar with that of U-orbits, but the presentation requires some alterations. Let $I\subset[1,n]$ be the set corresponding to the simple roots of the standard Levi factor of P. Set $W_I=\langle w_i\mid i\in I\rangle$ and $W_I''=\{w\in W \mid l(ww_i)<l(w)\quad \forall i\in I\}$. It is easy to see that W_I'' is the set of representatives of *maximal* length for W/W_I. For $w\in W_I''$, let $\mathcal{O}(w)=U_-wP\subset G/P$. Let Δ be the root system of G and Δ_I the root system of the Levi factor of P. The following lemma is an easy consequence of the well-known properties of the set of representatives of *minimal* length.

5.2 Lemma.
1. If w_I^o is the longest element in W_I and $w\in W_I''$, then ww_I^o is the representative of minimal length and $l(ww_I^o)=l(w)-l(w_I^o)=l(w)-\#\Delta_I^+$.
2. $w\in W_I''$ if and only if $w(\Delta_I^+)\subset\Delta^-$;
3. the natural map $W_I''\times W_I\to W$, $(w'',w)\mapsto w''w$ is bijective;
4. $G/P=\sqcup_{w\in W_I''}\mathcal{O}(w)$;
5. $\dim\mathcal{O}(w)=\dim U-l(w)$. □

Proof of (5.1). Since C is a cone, equidimensionality of $\pi=\pi_C$ is equivalent to that $\dim\pi^{-1}\pi(0)=\dim C-r=\dim G/P$. Let $\mathfrak{N}=\pi^{-1}\pi(0)$. Our main tool for estimating $\dim\mathfrak{N}$ is the following diagram:
$$\begin{array}{ccc} G*_P V & \xrightarrow{\tau} & C \\ \big\downarrow\phi & & \big\downarrow\pi \\ G/P & & C/\!\!/U_- \end{array},$$
where $\phi(g*v):=gP$ and $\tau(g*v):=g{\cdot}v$. It is easily seen that τ is a resolution of singularities of C. First, deduce a useful formula related to the diagram. It is assumed that each $w\in W$ is realized as an element of $N_G(T)$. Let $\bar g\in\mathcal{O}(w)\subset G/P$, where $w\in W_I''$. That is, $\bar g=\tilde u wP$ for some $\tilde u\in U_-$. Then

$$\begin{aligned}\phi^{-1}(\bar g)\cap\tau^{-1}(\mathfrak{N}) &= \{(\tilde u w * v)\in G*_P V \mid x_j(\tilde u w{\cdot}v)=0 \quad \forall j\}\\ &= \{(\tilde u w * v)\in G*_P V \mid (w^{-1}{\cdot}x_j)(v)=0 \quad \forall j\}.\end{aligned}$$

It follows that $\phi^{-1}(\bar g)\cap\tau^{-1}(\mathfrak{N})$ is isomorphic to a subspace in V of dimension $\#\{j \mid w.\lambda_j\ne\lambda_j\}$. Therefore

(5.3) $$\dim\{\phi^{-1}(\mathcal{O}(w))\cap\tau^{-1}(\mathfrak{N})\} = \dim\mathcal{O}(w) + \#\{j \mid w.\lambda_j\ne\lambda_j\}\ .$$

(ii)$\Rightarrow$(i). Since both the number of G-orbits in C and of U_--orbits in G/P is finite, it suffices to prove that

$$\dim\{\tau(\phi^{-1}(\mathcal{O}(w)))\cap\mathfrak{N}\cap G{\cdot}v_J\}\le\dim G/P$$

for any $w\in W_I''$ and $J\subset[1,r]$. Actually, it will be proven that

(5.4) $$\dim\{\phi^{-1}(\mathcal{O}(w))\cap\tau^{-1}(\mathfrak{N}\cap G{\cdot}v_J)\}\le\dim G/P\ .$$

These two inequalities are equivalent for the dense G-orbit, i.e., for $J=[1,r]$, because τ is bijective over it, while for all other G-orbits the second inequality is stronger. Suppose $g{\cdot}v_J\in\mathfrak{N}$ for $g=\tilde u wp$ ($\tilde u\in U_-$, $w\in W_I''$, and $p\in P$), that is, $x_j(wp{\cdot}v_J)=0$ for all j. This implies $w.\lambda_j\ne\lambda_j$ for $j\in J$. However, it may happen that also $w.\lambda_i\ne\lambda_i$ for some $i\notin J$. I claim that $l(w)\ge\#\{j\mid w.\lambda_j\ne\lambda_j\}+\#\Delta_I^+$. Indeed, $l(w)=l(ww_I^o)+\#\Delta_I^+$ (see Lemma 5.2(1)) and one still has $ww_I^o.\lambda_j\ne\lambda_j$ for the same set of indices j, since $w_I^o.\lambda_i=\lambda_i$ for all $i\in[1,r]$. Now, condition (ii) implies $l(ww_I^o)\ge\#\{j\mid w.\lambda_j\ne\lambda_j\}$. In other words, if $g{\cdot}v_J\in\mathfrak{N}$, then $\bar g\in G/P$ belongs to a U_--orbit $\mathcal{O}(w)$ which is of codimension $\ge\#\{j\mid w.\lambda_j\ne\lambda_j\}$ in G/P. Thus, Eq. (5.4) follows from Eq. (5.3).

(i)$\Rightarrow$(ii). Suppose (ii) is not satisfied. Then there exists a subset $K\subset[1,n]\setminus I$ such that $\#K<r$ and $w.\lambda_j\ne\lambda_j$ for all j, where $w=\prod_{i\in K}w_i$. clearly $w.\alpha\in\Delta^+$ for any $\alpha\in\Delta_I^+$. Hence $w'':=ww_I^o\in W_I''$ and $l(w'')<r+\#\Delta_I^+$. It then follows from Eq. (5.3) and the previous part of the proof that

$$\dim\{\phi^{-1}(\mathcal{O}(w''))\cap\tau^{-1}(\mathfrak{N}\cap G{\cdot}v_{[1,r]})\}=\dim U-l(w'')+r>\dim G/P\ .$$

Since τ is bijective over $G{\cdot}v_{[1,r]}$, we also have

$$\dim\{\tau(\phi^{-1}(\mathcal{O}(w'')))\cap\mathfrak{N}\cap G{\cdot}v_{[1,r]}\}>\dim G/P\ .$$

Thus, $\dim\mathfrak{N}>\dim G/P$. □

Remarks. 1. It is not hard to realize that the above proof yields the following conclusions as well: if π is equidimensional, then each irreducible component of $\mathfrak{N}$ has a nonempty intersection with the dense G-orbit in C and the number of irreducible components is equal to the number of $w\in W$ such that $l(w)=r$ and $w.\lambda_i\ne\lambda_i$ for all $i\in[1,r]$.
2. Since $C/\!\!/U$ is an affine space and C has rational singularities by [3, thm. 0], the equidimensionality condition is equivalent to that π_C is flat.

Next theorem applies to a wider class of varieties. However, it provides only a sufficient condition of flatness.

5.5 Theorem. *Let X be an affine G-variety such that $k[X]^U$ is polynomial and $\lambda_1, \dots, \lambda_r$ the weights of T-homogeneous free generators of $k[X]^U$. Suppose $\lambda_1, \dots, \lambda_r$ are linearly independent and $\#\{j \mid w_i.\lambda_j \neq \lambda_j\} \le 1$ for each $i \in [1, n]$. Then $k[X]$ is a flat $k[X]^U$-module.*

Proof. By the assumptions of the theorem, $k[X]$ is a multiplicity free G-module, i.e., X is a spherical G-variety. In particular, $k[X]^G = k$ and X has a dense G-orbit. The following argument uses deformation results from [9]. We refer to that paper for precise definitions and generalities concerning $\operatorname{gr} X$, etc.
By [9, §5], there exists a G-variety Y and a $q \in k[Y]^G$ such that $k[Y]/(q-a) \simeq k[X]$ for any $a \in k^*$, $k[Y][q^{-1}] \simeq k[X][q, q^{-1}]$, and $k[Y]/(q) \simeq k[\operatorname{gr} X]$. One can take the last equality as a definition of the G-variety $\operatorname{gr} X$. But the meaning of this construction is that $\operatorname{gr} X$ can be defined directly via a filtration of $k[X]$ and that it enjoys a number of nice properties. In our case, $\operatorname{gr} X = C$ is the variety considered above. (It stems from the following facts: $\operatorname{gr} X$ is again spherical, $k[\operatorname{gr} X]^U \simeq k[X]^U$, and the stabilizer of a point in the dense G-orbit in $\operatorname{gr} X$ contains a maximal unipotent subgroup.) We need some details concerning Y. One considers an ascending filtration of $k[X]$:

$$\{0\} \subset k[X]_{(0)} \subset k[X]_{(1)} \dots \subset k[X]_{(n)} \dots$$

The whole description is unimportant for us now, we shall only use the fact that $k[X]_{(0)} \simeq k[X]^G$, which is just k in our situation. Then $k[Y]$ is defined as the following subalgebra of $k[X][q]$:

$$k[Y] := \bigoplus_{n=0}^{\infty} k[X]_{(n)} q^n \ .$$

Let $f_1, \dots, f_r$ be free generators of $k[X]^U$. Define m_i to be the least integer such that $f_i \in k[X]_{(m_i)}$. It is then easy to see that $k[Y]^U \simeq k[q, q^{m_1} f_1, \dots, q^{m_r} f_r]$ is a polynomial algebra, of Krull dimension $r+1$. One has the following commutative diagram:

$$\begin{tikzcd}
C \simeq & \operatorname{gr} X \arrow[r, hook] \arrow[d, "\pi_C"] & Y \arrow[r, hook] \arrow[d, "\pi_Y"] & X \times \mathbb{A}^1 \\
\mathbb{A}^r \simeq & (\operatorname{gr} X)/\!\!/U \arrow[r, hook] \arrow[d] & Y/\!\!/U \arrow[d, "q"] & \simeq \mathbb{A}^{r+1} \\
& \{0\} \arrow[r, hook] & \mathbb{A}^1 &
\end{tikzcd}$$

Looking at it, one sees that $(\mathfrak{N} =)\, \pi_C^{-1}(\pi_C(0)) = \pi_Y^{-1}(\pi_Y(0))$, where $0 \in C \in Y$ is the unique $G \times k^*$-fixed point in Y (k^* acts on $\mathbb{A}^1$ and hence on Y by homotheties).

¿From Theorem 5.1, it then follows that $\dim \pi_Y^{-1}(\pi_Y(0)) = \dim Y - \dim Y/\!\!/U$. The algebra $k[Y]$ is $\mathbb{Z}_{\ge 0}$-graded and $k[Y]_0 = k[X]_{(0)} = k$, therefore π_Y is equidimensional. By [9, Thm. 6], Y has rational singularities and in particular is Cohen-Macaulay. Hence

$k[Y]$ is a flat $k[Y]^U$-module. Taking localisation, one obtains $k[Y][q^{-1}] \simeq k[X] \otimes k[q, q^{-1}]$ and $k[Y]^U[q^{-1}] \simeq k[X]^U \otimes k[q, q^{-1}]$. Whence, $k[X]$ is a flat $k[X]^U$-module. $\square$

Appendix

Now G is a simple algebraic group and $U \subset G$ is as above. We give the classification of irreducible representations V such that $k[V]$ is a free $k[V]^U$-module. As is well known, the last condition implies that $k[V]^U$ is polynomial. Representations with the last property were classified by M. Brion. Therefore our task is to look through the table in p. 13 in [1] and to realize for which representations in it π_V is equidimensional, i.e., free generators of $k[V]^U$ form a regular sequence in $k[V]$. Brion's table contains 19 items and we numerate them according to their ordering in the table.

1. For items 1–3, 5–8, 10, 13, 14, 16, and 18, flatness follows by Theorem 5.5. For, V is a spherical $G \times k^*$-module in these cases (k^* acts by homotheties) and the weights of generators are as required. Note that items 1–3, 5, 6, 10, 13, 14, and 18 arise from parabolic subgroups with *aura*.

2. For all other items, free generators of $k[V]^U$ do not form a regular sequence. Namely, one can always find a triple of generators $f_1, f_2, f_3 \in k[V]^U$ such that $f_3 = d_1 f_2 - d_2 f_1$ for some $d_1, d_2 \in k[V]$. The argument, which applies to all cases, relies on the fact that there exist always 2 generators of the same fundamental weight φ_i and a third generator of weight $2\varphi_i - \alpha_i$, where α_i is the simple root corresponding to φ_i. For instance, consider $G = Sp_6$ and $V = V_{\varphi_3}$. We exploit the following generators of $k[V]^U$: f_1 of degree 1 and weight φ_3, f_2 of degree 3 and weight φ_3, and f_3 of degree 4 and weight $2\varphi_2$. The generators are determined, up to a scalar factor, by degree and weight in this (and any other) case. The functions f_1, f_2 generate isomorphic G-submodules in $k[V]$. Introduce 'one level down' vectors $d_i := e_{-\alpha_3} f_i$ $(i = 1, 2)$ in these modules, e_α being a nonzero root vector in $\mathfrak{g}$. Then $d_1 f_2 - d_2 f_1$ has degree 4 and weight $2\varphi_3 - \alpha_3 = 2\varphi_2$. Obviously, it is a nonzero U-invariant and therefore, up to a scalar factor, it is equal to f_3.

A posteriori, the classification can be stated in the following nice form:

A.1 Theorem. *For an irreducible representation V of a simple algebraic group G, the following conditions are equivalent:*

(1) $k[V]$ is a free $k[V]^U$-module;

(2) A generic G-orbits in V is spherical. $\square$

(It is not hard to prove that the second condition implies $k[V]^U$ is polynomial.) However, implication (2)$\Rightarrow$(1) is no longer true for semisimple groups. For instance, let $G = SL_4 \times Sp_4$ and let V be the tensor product of tautological 4-dimensional representations. The same idea as above allows us to isolate a suitable triple of generators and then to prove that π_V is not equidimensional. It is likely that (1) always implies (2), but I was unable to find a proof.

References

[1] Brion, M.: Invariants d'un sous-groupe unipotent maximal d'un groupe semi-simple. *Ann. Inst. Fourier* **33**(1983), 1–27.

[2] Johnson, K.: On a ring of invariant polynomials on a Hermitian symmetric space, *J. Algebra* **67**(1980), 72–81.

[3] Kempf, G.: On the collapsing of homogeneous bundles, *Invent. Math.* **37**(1976), 229–239.

[4] Kostant, B.; Rallis, S.: Orbits and representations associated with symmetric spaces, *Amer. J. Math.* **93**(1971), 753–809.

[5] Muller, I.; Rubenthaller, H.; Schiffmann, G.: Structure des espaces préhomogènes associés a certaines algèbres de Lie graduées, *Math. Annalen* **274**(1986), 95–123.

[6] Panyushev, D.: Orbits of maximal dimension of solvable subgroups of reductive algebraic groups and reduction for U-invariants, *Matem. Sbornik* **132**(1987), 371–382. (Russian). English translation: *Math. USSR-Sb.* **60**(1988), 365–375.

[7] Panyushev, D.: A restriction theorem and the Poincaré series for U-invariants, *Math. Annalen* **301**(1995), 655–675.

[8] Panyushev, D.: On the conormal bundle of an orbit, (*to appear*).

[9] Popov, V.L.: Contractions of actions of reductive algebraic groups, *Matem. Sbornik* **130**(1986), 310–334 (Russian). English translation: *Math. USSR Sb.* **58**(1987), 311–335.

[10] Pyasetskii, V.S.: Linear Lie groups acting with finitely many orbits, *Funkt. analiz i priloz.* **9**:4 (1975), 85–86 (Russian). English translation: *Functional Anal. Appl.* **9**(1975), 351-353.

[11] Richardson, R.; Röhrle, G.; Steinberg, R.: Parabolic subgroups with Abelian unipotent radical, *Invent. Math.* **110**(1992), 649–671.

[12] Schmid, W.: Die randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, *Invent. Math.* **9**(1969), 61–80.

[13] Springer, T.A.; Steinberg, R.: Conjugacy classes, In: *"Seminar on algebraic groups and related finite groups"*. Lecture notes in Math. **131**, pp.167–266, Springer Verlag, Berlin Heidelberg New York 1970.

[14] Vinberg, E.B.: The Weyl group of a graded Lie algebra, *Izv. Akad. Nauk SSSR, Ser. Mat.* **40**(1976), 488–526 (Russian). English translation: *Math. USSR-Izv.* **10**(1976), 463–495.

[15] Vinberg, E.B.; Popov, V.L.: On a class of quasihomogeneous affine varieties, *Izv. AN SSSR. Ser. Matem.*, **36**(1972), 749–764 (Russian). English translation: *Math. USSR-Izv.* **6**(1972), 743–758.

ul. Akad. Anokhina, d.30, kor.1, kv.7
Moscow 117602 Russia

Current address: *Max-Planck-Institut für Mathematik*
Gottfried-Claren-Str. 26
D-53225 Bonn, Deutschland

e-mail: panyush@dpa.msk.ru panyush@mpim-bonn.mpg.de