Cohomology of Monoids in Monoidal Categories

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Introduction

It has been known for some time that the cohomology theories of many classical algebraic objects — monoids, groups, associative algebras and Lie algebras for instance — have a common framework in terms of cohomology of internal monoids in a symmetric monoidal category; see for example [24]. But there are also important examples of algebraic structures which occur as monoids in non-symmetric monoidal categories, such as operads, monads, theories, categories, and square rings as described below. In this article we show that these structures are still susceptible to cohomological investigation, by developing the theory in the absence of the symmetry condition. Later we shall assume that the monoidal structure is left distributive over coproducts and the category is an abelian category; this is the case for operads, our original motivating example.

1. Monoids and Modules

We define monoids in monoidal categories and introduce the "module" objects which will be used later as coefficients in the cohomology of such monoids. We also give some of our motivating examples of monoidal categories and the monoids therein.

Let us start by recalling that a monoidal category is a tuple $\mathbb{V} = (\mathbf{V}, \circ, I, a, l, r)$ where $\mathbf{V}$ is a category, $\circ : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ is a functor, I is an object of $\mathbf{V}$, and

$$\begin{aligned} a &= (a_{X,Y,Z} : (X \circ Y) \circ Z \to X \circ (Y \circ Z))_{X,Y,Z \in \mathbf{V}}, \\ l &= (l_X : I \circ X \to X)_{X \in \mathbf{V}}, \\ r &= (r_X : X \circ I \to X)_{X \in \mathbf{V}} \end{aligned}$$

are natural isomorphisms, required to satisfy certain conditions which we omit here (see e.g. [19]). In many examples our monoidal categories will be strictly associative and have strict units, in the sense that all $a_{X,Y,Z}$ and l_X, r_X are identity morphisms. The monoidal category $\mathbb{V}$ is *abelian* if the underlying category $\mathbf{V}$ is an abelian category. Suppose $\mathbf{V}$ has binary coproducts, denoted $X \sqcup Y$; then the monoidal structure is *left distributive* if the canonical natural transformation

$$(X_1 \circ Y) \sqcup (X_2 \circ Y) \to (X_1 \sqcup X_2) \circ Y$$

is an isomorphism. Right distributivity is defined similarly.

Key words and phrases. cohomology, internal monoids, operads.

A strict monoidal functor between monoidal categories is a functor between the underlying categories preserving all the existing structure in the obvious way.

Given such a $\mathbb{V}$, a monoid in $\mathbb{V}$, or a $\mathbb{V}$-monoid, is a triple $\mathcal{G} = (G, \mu, \eta)$ where $G \in \mathbf{V}$, $\mu : G \circ G \to G$, $\eta : I \to G$ must satisfy the identities

$$\begin{array}{rcll} \mu(\mu \circ G) & = & \mu(G \circ \mu)a_{G,G,G} & \text{(associativity)}, \\ \mu(\eta \circ G) & = & l_G & \text{(left unit)}, \\ \mu(G \circ \eta) & = & r_G & \text{(right unit)}. \end{array}$$

Basic examples of monoidal categories are the following:

Example 1.1. Let $\mathbf{C}$ be any category with finite products. Then these products may be used to give it a monoidal structure $\mathbf{C}^{\times} = (\mathbf{C}, \times, 1, a, l, r)$, where $\times$ is the binary product, 1 is the terminal object (which exists as the empty product), and a, l, r are uniquely determined by the universal property of the products. A monoid in this monoidal category is what is usually called an internal monoid in a category with products.

Also in this "cartesian" situation one may define what it means for a monoid $\mathcal{G} = (G, \mu : G \times G \to G, \eta : 1 \to G)$ to be an internal group object: there must exist an endomorphism $\iota : G \to G$ satisfying

$$\mu(G \times \iota)d \quad = \quad \eta p \quad = \quad \mu(\iota \times G)d$$

where $d : X \to X \times X$ and $p : X \to 1$ are the canonical morphisms (which are only available in the cartesian case).

In particular, taking $\mathbf{C}$ to be the category $\mathbf{Ens}$ of sets and functions, one obtains just monoids and groups in the ordinary sense; or, taking the categories of spaces, simplicial sets, etc., one obtains topological or simplicial monoids and groups.

Example 1.2. The category R-**mod** of modules over a commutative ring R may be given a monoidal structure using the tensor product over R. We shall denote this monoidal category by $R\text{-}\mathbf{mod}^{\otimes} = (R\text{-}\mathbf{mod}, \otimes_R, R, a, l, r)$. Here a, l, r are the obvious isomorphisms. Monoids in this example are the associative R-algebras with unit.

These are in fact examples of *symmetric* monoidal categories, i. e. they admit additional structure consisting of natural isomorphisms $c = (c_{X,Y} : X \circ Y \to Y \circ X)_{X,Y}$ satisfying further coherence conditions (see [19] for these). In the symmetric situation one may also talk about commutative monoids: (G, μ, η) is commutative if

$$\mu c_{G,G} \quad = \quad \mu$$

holds. In particular, in the cartesian situation of the example 1.1 one has the notion of an internal commutative, or abelian, group. We write $\mathbf{Ab(C)}$ for the category of abelian group objects in the cartesian monoidal category $\mathbf{C}$.

There is also an important relaxation of the symmetric structure called braiding (the same $c_{X,Y}$, but satisfying less stringent coherence conditions; see e.g. [16] for numerous examples of monoidal categories of this kind).

We are going to define cohomology of $\mathbb{V}$-monoids; hence we must first determine what are the coefficients for such a cohomology theory. For this we recall (see e.g. [27]) that a general notion of coefficients for the cohomology of an object X in a category $\mathbf{C}$ is given by internal abelian group objects in the slice category $\mathbf{C}/X$. Here $\mathbf{C}/X$ is the category whose objects are morphisms $Y \to X$ in $\mathbf{C}$ and whose morphisms are commutative triangles of the obvious kind. In order to speak about internal abelian groups in the slice categories one has to assume that the $\mathbf{C}/X$ have finite products, or equivalently that $\mathbf{C}$ has pullbacks.

Given a monoidal category $\mathbb{V}$, there is an obvious notion of a morphism between $\mathbb{V}$-monoids, so we have the category $\mathbf{Mon}(\mathbb{V})$ of monoids and their morphisms, equipped with a forgetful functor $U : \mathbf{Mon}(\mathbb{V}) \to \mathbf{V}$. And if we assume existence of pullbacks in $\mathbf{V}$, the same will be true for $\mathbf{Mon}(\mathbb{V})$. Indeed, one has

Lemma 1.3. For any monoidal category $\mathbb{V} = (\mathbf{V}, ...)$, the forgetful functor

$$U : \mathbf{Mon}(\mathbb{V}) \to \mathbf{V}$$

reflects any inverse limits that exist in $\mathbf{V}$.

Proof. Consider any diagram $((\mathcal{G}_i)_{i \in \mathrm{I}}, (f_\iota : \mathcal{G}_i \to \mathcal{G}_{i'})_{\iota : i \to i'})$ in $\mathbf{Mon}(\mathbb{V})$, where $\mathcal{G}_i = (G_i, \mu_i, \eta_i)$ are $\mathbb{V}$-monoids. Suppose we are given a limiting cone $(f_i : G \to G_i)_{i \in \mathrm{I}}$ over this diagram, considered as a diagram in $\mathbf{V}$. One easily sees that $(G \circ G \xrightarrow{f_i \circ f_i} G_i \circ G_i \xrightarrow{\mu_i} G_i)_{i \in \mathrm{I}}$ and $(I \xrightarrow{\eta_i} G_i)_{i \in \mathrm{I}}$ are cones in $\mathbf{V}$, hence they determine maps $\mu : G \circ G \to G$ and $\eta : I \to G$, respectively. And one then checks that this gives a structure of a limiting cone in $\mathbf{Mon}(\mathbb{V})$. ∎

Note that for any monoid $\mathcal{G} = (G, \mu, \eta)$ in $\mathbb{V}$, there is a natural monoidal structure on $\mathbf{V}/G$, which we will denote by $\mathbb{V}/\mathcal{G} = (\mathbf{V}/G, \circ_\mu, I_\eta, a, l, r)$. Here the functor $\circ_\mu$ is determined by $(X \xrightarrow{x} G) \circ_\mu (Y \xrightarrow{y} G) = (X \circ Y \xrightarrow{x \circ y} G \circ G \xrightarrow{\mu} G)$; I_η is just $I \xrightarrow{\eta} M$; and a, l and r are those of $\mathbb{V}$ (in fact there is a one-to-one correspondence between monoid structures on an object G and those monoidal structures on $\mathbf{V}/G$ which turn the forgetful functor $U : \mathbf{V}/G \to \mathbf{V}$ into a strict monoidal functor). With respect to this monoidal structure one has the equivalence of categories $\mathbf{Mon}(\mathbb{V}/\mathcal{G}) \simeq \mathbf{Mon}(\mathbb{V})/\mathcal{G}$.

So we shall assume henceforward that our category $\mathbf{V}$ has pullbacks, and, for a $\mathbb{V}$-monoid $\mathcal{G} = (G, \mu, \eta)$ we choose the category $\mathbf{Ab}(\mathbf{Mon}(\mathbb{V})/\mathcal{G})$ of internal abelian groups in $\mathbf{Mon}(\mathbb{V})/\mathcal{G}$ and their homomorphisms to be the category of coefficients for the cohomology of $\mathcal{G}$. Fortunately, this category has a much simpler description, up to equivalence. This description involves the notion of action of a monoid on an object:

Definition 1.4. A left action of a $\mathbb{V}$-monoid $\mathcal{G} = (G, \mu, \eta)$ on an object A of $\mathbf{V}$ is a morphism $u : G \circ A \to A$ satisfying

$$\begin{aligned} u(\mu \circ A) &= u(G \circ u)a_{G,G,A}, \\ u(\eta \circ A) &= l_A. \end{aligned}$$

We will also say that A is a left $\mathcal{G}$-object. Similarly, a right action of a monoid $\mathcal{G}' = (G', \mu', \eta')$ on A is a morphism $u' : A \circ G' \to A$ satisfying analogous identities. And given two such actions we say that they are compatible, or that A is an $\mathcal{G}$-$\mathcal{G}'$-biobject, if

$$u'(u \circ G') = u(G \circ u')a_{G,A,G'}.$$

For example, given any monoid $\mathcal{G} = (G, \mu, \eta)$, there is an evident $\mathcal{G}$-$\mathcal{G}$-biobject structure on G itself.

It is obvious how to define a morphism of left $\mathcal{G}$-, right $\mathcal{G}'$-, or $\mathcal{G}$-$\mathcal{G}'$-biobjects; the corresponding categories will be denoted by ${}^{\mathcal{G}}\mathbf{V}$, $\mathbf{V}^{\mathcal{G}'}$, and ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}'}$, respectively. All these categories come with forgetful functors to $\mathbf{V}$ (which will be denoted by the same letter U); and just as in the lemma above, these forgetful functors reflect all the limits that happen to exist in $\mathbf{V}$. Hence we also can talk about internal abelian groups in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}$. And we have

Proposition 1.5. For any monoid $\mathcal{G}$ in $\mathbf{V}$, there is an equivalence of categories

$$\mathbf{Ab}(\mathbf{Mon}(\mathbb{V})/\mathcal{G}) \simeq \mathbf{Ab}({}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G).$$

Proof. To simplify exposition, we will prove the proposition in the particular case when the monoid in question is the terminal object 1 of $\mathbf{V}$, with its unique monoid structure. That is we will prove that there is an equivalence

$$\mathbf{Ab}(\mathbf{Mon}(\mathbb{V})) \simeq \mathbf{Ab}({}^{1}\mathbf{V}^{1}).$$

By the above remarks on slice categories, this will suffice: for any monoid $\mathcal{G}$, the underlying object G (more precisely, its identity map) is clearly terminal in $\mathbf{V}/G$.

Now an object of the category $\mathbf{Ab}(\mathbf{Mon}(\mathbb{V}))$ looks like $(A, \mu : A \circ A \to A, \eta : I \to A, + : A \times A \to A, 0 : 1 \to A, - : A \to A)$. First of all note that 0 must be a morphism of monoids, in particular $\eta = (I \to 1 \xrightarrow{0} A)$, so that η is in fact determined by 0. As for μ, one has the commutative diagram

$$\begin{tikzcd}
(A \times A) \circ (A \times A) \arrow[r, "\mu_\times"] \arrow[d, "+\circ+"'] & A \times A \arrow[d, "+"] \\
A \circ A \arrow[r, "\mu"] & A
\end{tikzcd}$$

where $\mu_\times$ is the monoid structure on $A \times A$ which, by a particular case of lemma 1.3, equals

$$(A \times A) \circ (A \times A) \xrightarrow{(p_1 \circ p_1, p_2 \circ p_2)} (A \circ A) \times (A \circ A) \xrightarrow{\mu \times \mu} A \times A.$$

Composing all this with $A \circ A \cong (A \times 1) \circ (1 \times A) \xrightarrow{(A \times 0) \circ (0 \times A)} (A \times A) \circ (A \times A)$ reveals that μ is equal to the composite

$$A \circ A \xrightarrow{(A \circ p, p \circ A)} (A \circ 1) \times (1 \circ A) \xrightarrow{v \times u} A \times A \xrightarrow{+} A,$$

where p is the unique morphism from A to 1, and $u : 1 \circ A \xrightarrow{0 \circ A} A \circ A \xrightarrow{\mu} A$, $v : A \circ 1 \xrightarrow{A \circ 0} A \circ A \xrightarrow{\mu} A$ are easily seen to define a 1-1-biobject structure on A, compatible with the abelian group structure. Hence μ is determined by these structures.

Conversely, given an object $(A, u : 1 \circ A \to A, v : A \circ 1 \to A, + : A \times A \to A, 0 : 1 \to A, - : A \to A)$ of $\mathbf{Ab}({}^{1}\mathbf{V}^{1})$, one equips it with a $\mathbb{V}$-monoid structure via $A \circ A \xrightarrow{(A \circ p, p \circ A)} (A \circ 1) \times (1 \circ A) \xrightarrow{v \times u} A \times A \xrightarrow{+} A$ and $I \to 1 \xrightarrow{0} A$ and checks that this is compatible with the abelian group structure. ■

Hence we are left with $\mathbf{Ab}({}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G)$ as our category of coefficients for the cohomology of the $\mathbb{V}$-monoid $\mathcal{G}$. In the next section we will simplify the category of coefficients even more by imposing the conditions that $\mathbf{V}$ be abelian with left distributive monoidal structure.

We finish this section with the examples of monoids in non-symmetric monoidal categories which mainly motivated the results in this paper.

Example 1.6 (Bimodules). For any associative ring R, the category R-R-**Mod** of R-R-bimodules has a non-symmetric monoidal structure given by $\otimes_R$. A monoid G in this monoidal category may be identified with an R-ring, that is, a ring equipped with a ring homomorphism from R. The coefficients for the cohomology of an R-ring G turn out to be G-G-bimodules, as we will see later.

Example 1.7 (Monads). For any category $\mathbf{C}$, the category $\mathbf{End}(\mathbf{C})$ of endofunctors on $\mathbf{C}$ carries a monoidal structure induced by composition of endofunctors; we denote the corresponding monoidal category by $\mathbf{End}(\mathbf{C})^{\circ} = (\mathbf{End}(\mathbf{C}), \circ, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}, \mathrm{id}, \mathrm{id})$. This is an example of a *strict* monoidal category — the associativity and unit natural transformations are all identities. Note also that as soon as $\mathbf{C}$ has coproducts, $\mathbf{End}(\mathbf{C})^{\circ}$ is automatically left distributive, but almost never right distributive, nor symmetric. Monoids in $\mathbf{End}(\mathbf{C})^{\circ}$ are monads on $\mathbf{C}$.

There are also variations on this example: one may take various full subcategories of $\mathbf{End}(\mathbf{C})$ which are closed under the monoidal structure, e.g. the category of *finitary* endofunctors (that is, those preserving filtered colimits), or the category of cocontinuous endofunctors (preserving all colimits), or the category of endofunctors having a right adjoint. Monoids in these categories are various kinds of monads on $\mathbf{C}$.

Example 1.8 (Theories). Monoids in the category of finitary endofunctors are finitary monads. In the case of finitary endofunctors on **Ens** the category of finitary monads is equivalent to the category of finitary algebraic theories in the sense of Lawvere [20]. In this particular case, coefficients turn out to be the general coefficients for cohomology of algebraic theories briefly mentioned in [14].

Example 1.9 (Operads). In example 1.7, let $\mathbf{C}$ be the category of vector spaces over a characteristic zero field k. Consider the full subcategory of $\mathbf{End}(\mathbf{C})$ consisting of endofunctors which are *analytic*; recall from [15] that these are functors F admitting a decomposition into a Taylor series

$$F(V) = \bigoplus_{n \geqslant 0} F_n \otimes_{\mathfrak{S}_n} V^{\otimes n}$$

where $(F_n)_{n\geqslant 0}$ is some sequence of linear representations of symmetric groups $\mathfrak{S}_n$. Since the analytic endofunctors are closed under composition, one obtains an abelian (in fact also k-linear) left distributive monoidal category. This category is equivalent to that considered in [17]; in particular, its category of monoids is equivalent to the category of k-linear operads. We will identify coefficients in the next section.

Example 1.10 (Square rings). Let $\mathbf{C}$ be the category of groups **Gr** or of abelian groups **Ab**, and consider the full subcategory

$$\mathbf{Degree}_n(\mathbf{C}) \;\subset\; \mathbf{End}(\mathbf{C})$$

whose objects are the finitary endofunctors which preserve cokernels and which have degree n. In particular functors F of degree one, or *linear* functors, are those which carry coproducts to products, i.e. the canonical natural transformation

$$(r_{1*}, r_{2*}) : F(X \sqcup Y) \to F(X) \times F(Y)$$

is an isomorphism. Functors F of degree two, or *quadratic* functors, are those for which the cross effect $F(X|Y) = \ker(r_{1*}, r_{2*})$ is linear as a bifunctor in X and Y. It is shown in [4] that there are canonical equivalences of monoidal categories

$$\mathbf{Ab} \cong \mathbf{Degree}_1(\mathbf{Ab}) \cong \mathbf{Degree}_1(\mathbf{Gr})$$

Moreover $\mathbf{Degree}_2(\mathbf{Ab})$ and $\mathbf{Degree}_2(\mathbf{Gr})$ are equivalent to categories of certain simple algebraic objects termed quadratic $\mathbb{Z}$-modules [1] and square groups [4] respectively. The category $\mathbf{Degree}_n(\mathbf{Ab})$ is equivalent to the category of modules over a certain commutative ring defined by Pirashvili [25] and calculated by Dreckmann [7].

Now unlike linear endofunctors, the quadratic ones are *not* closed under composition. However in the cases considered, the inclusion of the full subcategory of quadratic endofunctors into $\mathbf{End}(\mathbf{C})$ has a left adjoint $(\ )^{\mathrm{quad}}$. So one may define a monoidal structure on $\mathbf{Degree}_2(\mathbf{C})$ by $F \mathbin{\square} G = (F \circ G)^{\mathrm{quad}}$. Monoids in $\mathbf{Degree}_2(\mathbf{Gr})$ correspond under the equivalence with square groups to the *square rings* of [3]. Similarly one can define "rings of degree n" in the category $\mathbf{Degree}_n(\mathbf{Gr})$. Rings of degree 1 are just the classical rings.

Example 1.11 (Categories). Given an object I in a category with pullbacks $\mathbf{S}$, there is a monoidal structure on the slice category $\mathbf{S}/(I \times I)$: the unit object is the diagonal map $d : I \to I \times I$ and for $f : X \to I \times I$, $g : Y \to I \times I$ the object $f \circ g : Z \to I \times I$ is determined by the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & I \times I \times I \arrow[r, "p_2"] \arrow[d, "I \times d \times I"] & I \times I \\
X \times Y \arrow[r, "f \times g"] & I \times I \times I \times I &
\end{tikzcd}$$

in which the square is pullback. This is sometimes termed the "category of matrices", since for $\mathbf{S} = \mathbf{Ens}$ it is equivalent to the category of families $(X_{ij})_{i,j \in I}$ of sets, with the operation

$$(X_{ij}) \circ (Y_{ij}) = (\coprod_k X_{ik} \times Y_{kj})_{i,j \in I}.$$

Now monoids in this monoidal category may be identified with those internal categories in $\mathbf{S}$ having I as the object of objects; and morphisms of monoids are those internal functors which are identity on objects. For any two such categories $\mathcal{C}$ and $\mathcal{D}$, the $\mathcal{C}$-$\mathcal{D}$-biobjects may be identified with *internal profunctors* from $\mathcal{D}$ to $\mathcal{C}$. When $\mathbf{S} = \mathbf{Ens}$, these are just bifunctors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathbf{S}$. In particular, the canonical $\mathcal{C}$-$\mathcal{C}$-biobject structure on $\mathcal{C}$ itself corresponds to its hom bifunctor. Coefficients for the cohomology of an internal category $\mathcal{C}$ are *natural systems* on $\mathcal{C}$, that is, abelian group objects in the category of internal profunctors. For $\mathbf{S} = \mathbf{Ens}$ these are exactly the natural systems in the sense of [5].

Note that even this example may be fitted into the general setting of the example 1.7: each object $X \xrightarrow{f} I \times I$ of $\mathbf{S}/(I \times I)$ determines an endofunctor of the category $\mathbf{S}/I$ as follows:

$$\mathbf{S}/I \xrightarrow{(p_1 f)^*} \mathbf{S}/X \xrightarrow{(p_2 f)_*} \mathbf{S}/I,$$

where $p_1, p_2 : I \times I \to I$ are the projections, $(p_1 f)^*$ is pullback along $p_1 f$, and $(p_2 f)_*$ is composition with $p_2 f$. For $\mathbf{S} = \mathbf{Ens}$, $\mathbf{S}/I$ may be identified with the category of I-indexed families of sets, and then the endofunctor corresponding to the "matrix" (X_{ij}) is given by

$$(V_i)_{i \in I} \mapsto (\coprod_j X_{ij} \times V_j)_{i \in I}.$$

Endofunctors of this kind are obviously closed under composition, and the monoidal structure so obtained coincides with the "matrix multiplication" above.

Example 1.12 (Spectra). According to recent work of Elmendorf-Kriz-Mandell-May [10] the category of spectra can be given a monoidal structure. Moreover the monoids in this category correspond to A_∞-ring spectra; compare 6.2 in [10].

2. Monoids and modules in the abelian left distributive case

Throughout this section $\mathbb{A} = (\mathbf{A}, \square, I)$ will be an abelian left distributive monoidal category. For this case the coefficient objects for a monoid $\mathcal{G} = (G, \mu, \eta)$ in $\mathbb{A}$, given by abelian groups in ${}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G$ according to proposition 1.5, can be further simplified. In fact if the monoidal structure is also right distributive the coefficients are just bimodules:

Proposition 2.1. Let $\mathbb{A}$ be an abelian monoidal category which is both left and right distributive, and suppose $\mathcal{G}$ is a monoid in $\mathbb{A}$. Then there is an equivalence of categories

$$\mathbf{Ab}({}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G) \simeq {}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}$$

This can be readily seen by the arguments below for the left distributive case.

The results in this section can be applied to the following examples.

Examples 2.2. The following are abelian left distributive monoidal categories. Let R be a commutative ring.

(1) Clearly the monoid operation $\otimes_R$ on R-**mod** of example 1.2 is both left and right distributive, and applying proposition 2.1 shows that the coefficients for cohomology of R-algebras G are the G-bimodules. This is the classical case in for example [21].
(2) Let $\mathfrak{S}$ be the symmetric groupoid and let $\mathbf{A} = \mathbf{Cat}(\mathfrak{S}, R\text{-}\mathbf{mod})$ be the category of functors from $\mathfrak{S}$ to R-modules. Then there is a monoidal structure $\square$ on $\mathbf{A}$ such that $\mathbf{Mon}(\mathbf{A})$ is the category of *operads* in $\mathbf{A}$. See example 1.9.
(3) Let $\mathbf{A}$ be the category of endofunctors of R-**mod** which preserve filtered colimits and cokernels. Then composition yields a monoidal structure and $\mathbf{Mon}(\mathbf{A})$ is the category of *monads* on R-**Mod**.
(4) The category $\mathbf{Degree}_n(\mathbf{Ab})$ of example 1.10.

We may consider $(1.) \subseteq (2.) \subseteq (3.)$ as a sequence of inclusions of monoidal categories.

Definition 2.3. Let (G, μ, η) be a monoid in an abelian monoidal category $(\mathbf{A}, \square, I)$, with $\square$ left distributive over $\oplus$. Then a *coefficient G-module* is an object M and morphisms

$$G \square (G \oplus M) \xrightarrow{\lambda} M \qquad\qquad M \square G \xrightarrow{\rho} M$$

in $\mathbf{A}$ with the following properties

(1) λ is *linear* in M:

$$\begin{tikzcd}
G \square (G \oplus M \oplus M) \arrow[r, "(1 \square p_1, 1 \square p_2)"] \arrow[d, "1 \square (1 \oplus +)"'] & G \square (G \oplus M) \oplus G \square (G \oplus M) \arrow[d, "\lambda + \lambda"] \\
G \square (G \oplus M) \arrow[r, "\lambda"'] & M
\end{tikzcd}$$

(2) λ is a *cross-action*:

$$\begin{tikzcd}
I \square (G \oplus M) \arrow[d, "\eta \square 1"'] \arrow[dr, "p_M"] & \\
G \square (G \oplus M) \arrow[r, "\lambda"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
G \square G \square (G \oplus M) \arrow[r, "\mu \square 1"] \arrow[d, "1 \square \alpha"'] & G \square (G \oplus M) \arrow[d, "\lambda"] \\
G \square (G \oplus G \square (G \oplus M)) \arrow[r, "\lambda^2"'] & M
\end{tikzcd}$$

where $\lambda^2 = \lambda(1 \square (1 \oplus \lambda))$ and $\alpha = (\mu(1 \square p_G), 1)$.

(3) ρ is a *right action*:

$$\begin{tikzcd}
M \square I \arrow[d, "1 \square \eta"'] \arrow[dr, "\cong"] & \\
M \square G \arrow[r, "\rho"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \square G \square G \arrow[r, "1 \square \mu"] \arrow[d, "\rho \square 1"'] & M \square G \arrow[d, "\rho"] \\
M \square G \arrow[r, "\rho"'] & M
\end{tikzcd}$$

(4) λ and ρ are compatible:

$$\begin{tikzcd}
G \square (G \oplus M) \square G \arrow[r, "\lambda \square 1"] \arrow[d, "1 \square (\mu \oplus 1)"'] & M \square G \arrow[dr, "\rho"] & \\
 & & M \\
G \square (G \oplus M \square G) \arrow[r, "1 \square (1 \oplus \rho)"'] & G \square (G \oplus M) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Morphisms between coefficient G-modules are morphisms in $\mathbf{A}$ which respect all the structure. We write $\mathbf{Coef}_G$ for the category of coefficient G-modules M over a fixed monoid G in $\mathbf{A}$.

Proposition 2.4. Let $\mathcal{G} = (G, \mu, \eta)$ be a monoid in an abelian left distributive monoidal category $\mathbf{A}$ as above. Then there is an equivalence of categories

$$\mathbf{Ab}({}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G) \;\simeq\; \mathbf{Coef}_G$$

Proof. Let $(A, u, v, +, 0, -)$ be an object of $\mathbf{Ab}({}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G)$. Then the map $p : A \to G$ is split by $0 : G \to A$ and so we can write $A = G \oplus M$ with $p = p_G$ and $0 = i_G$. The addition $+ : A \times_G A \to A$ becomes now $1 \oplus (1,1) : G \oplus M \oplus M \to G \oplus M$ and the actions u, v are given by

$$G \,\square\, (G \oplus M) \xrightarrow{(1 \,\square\, p_G, 1)} G \,\square\, G \oplus G \,\square\, (G \oplus M) \xrightarrow{\mu \oplus \lambda} G \oplus M$$

$$(G \oplus M) \,\square\, G \xrightarrow{\cong} G \,\square\, G \oplus M \,\square\, G \xrightarrow{\mu \oplus \rho} G \oplus M$$

for some $\lambda : G \,\square\, (G \oplus M) \to M$ and $\rho : M \,\square\, G \to M$, where the biobject axioms on u and v are just the (cross-)action and compatibility laws for λ and ρ. Furthermore the compatibility of $+$ with u is equivalent to the linearity of λ. ■

Let $\mathbf{Coef}_G$ be the category of coefficient G-modules, for (G, η, μ) a monoid in $\mathbf{A}$. The *forgetful functor*

$$U : \mathbf{Coef}_G \longrightarrow \mathbf{A}$$

is the functor which takes a coefficient G-module (M, λ, ρ) to M regarded simply as an object of $\mathbf{A}$. We will show that U has a left adjoint F, giving explicitly the *free coefficient G-module* $(F(V), \lambda, \rho)$ on an object V of $\mathbf{A}$. The adjunction gives an isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbf{A}}(V, M) \;\cong\; \mathrm{Hom}_{\mathbf{Coef}_G}(F(V), M)$$

which is natural in $A \in \mathbf{A}$ and $M \in \mathbf{Coef}_G$.

We give first an alternative definition of coefficient G-modules using the language of additive functors.

Definition 2.5. (cf example 1.10) Let $F : \mathbf{A} \to \mathbf{A}$ be an endofunctor on an abelian category $\mathbf{A}$. We define for objects A, B of $\mathbf{A}$ the *cross-effect* $F(A|B)$ by the kernel

$$F(A|B) \;=\; \ker(\pi : F(A \oplus B) \to F(A) \oplus F(B))$$

where $\pi = (Fp_A, Fp_B)$ is given by the projections from $A \oplus B$ to A and to B respectively. Clearly $F(A|B)$ is functorial in A and B. We say that F is an *additive* functor if $F(A|B)$ is zero for all A, B. We define natural maps P by

$$P : F(A|A) \hookrightarrow F(A \oplus A) \xrightarrow{F(+)} F(A)$$

where $+$ is the addition map $(1,1): A \oplus A \to A$ for A an object of $\mathbf{A}$. The *additivisation* of F is the additive functor F^{add} defined by the cokernel

$$F^{\mathrm{add}}(A) \;=\; \operatorname{coker}(p : F(A|A) \to F(A))$$

The *quotient map* $q : F \to F^{\mathrm{add}}$ has the universal property that any natural transformation $F \to G$ where G is additive has a unique factorisation $F \to F^{\mathrm{add}} \to G$ through q.

In our situation the left distributivity of the tensor product $\square$ in $\mathbf{A}$ says that each functor $- \square B : A \mapsto A \square B$ is additive. However the functors $A \square - : B \mapsto A \square B$ are not in general additive; for $\mathbf{A} = \mathbf{Cat}(\mathfrak{S}, R\text{-}\mathbf{mod})$ for example the functor $A \square -$ is additive if and only if the object A is concentrated in degree 1.

Consider the functor $L_0 : \mathbf{A} \to \mathbf{A}$ with

$$L_0(X) \;=\; G \square (G \oplus X)$$

and the additive functor $L = L_0^{\mathrm{add}} : \mathbf{A} \longrightarrow \mathbf{A}$ defined by the additivisation of L_0. We note that for a coefficient G-module (M, λ, ρ), the linearity property (2.3)(1) says precisely that $\lambda : G \square (G \oplus M) = L_0(M) \to M$ factors through the quotient map $q : L_0 \to L$. Furthermore the cross-action properties (2.3)(2) may be written as $\lambda(\eta \square 1) = p_M : G \oplus M \to M$ and

$$\text{(2.6)} \qquad \begin{tikzcd} G \square L_0(M) \arrow[r, "\mu \square 1"] \arrow[d, "1 \square \alpha"'] & L_0(M) \arrow[d, "\lambda"] \\ L_0(L_0(M)) \arrow[r, "\lambda^2"'] & M \end{tikzcd}$$

where $\lambda^2 = \lambda L_0(\lambda) = \lambda(1 \square (1 \oplus \lambda))$ and $\alpha = (\mu(1 \square p_G), 1)$.

Lemma 2.7. In the presence of the linearity condition on λ, the commutativity of (2.6) is equivalent to that of

$$\text{(2.8)} \qquad \begin{tikzcd} G \square L_0(M) \arrow[r, "\mu \square 1"] & L_0(M) \arrow[d, "\lambda"] \\ L_0(L_0(M)) \arrow[u, "1 \square \beta"] \arrow[r, "\lambda^2"'] & M \end{tikzcd}$$

$$\begin{array}{llllll} \text{where} & \beta & = & p_2 + (1 \square i_G)\beta' & : G \oplus L_0(M) & \longrightarrow L_0(M) \\ \text{and} & \beta' & = & (\eta \square 1)p_1(1 - \alpha p_2) & : G \oplus L_0(M) & \longrightarrow G \square G \end{array}$$

Proof. Since $p_2\alpha = 1$ the maps $(1 - \alpha p_2)\alpha$ and $\beta'\alpha$ are zero. Thus $\beta\alpha$ is the identity on $L_0(M)$ and the commutativity of (2.8) implies that of (2.6). In the opposite direction, we will show that $(1 \oplus \lambda)\alpha\beta = 1 \oplus \lambda$, so that $\lambda L_0(\lambda)(1 \square \alpha)(1 \square \beta) = \lambda L_0(\lambda)$ and (2.6) will imply (2.8). We have

$$\begin{aligned} p_1\alpha\beta &= p_1\alpha p_2 + p_1\alpha(1 \square i_G)\beta' \\ &= p_1\alpha p_2 + \mu(1 \square p_G)(1 \square i_G)(\eta \square 1)p_1(1 - \alpha p_2) \\ &= p_1\alpha p_2 + p_1(1 - \alpha p_2) \\ &= p_1 \end{aligned}$$

Also $\lambda(1 \square i_G)$ is zero by linearity and so $\lambda p_2\alpha\beta = \lambda\beta = \lambda p_2$. Thus $(1 \oplus \lambda)\alpha\beta = 1 \oplus \lambda$ as required. ∎

We say $\mathbb{A}$ is *right compatible with cokernels* if for each $A \in \mathbf{A}$ the additive functor $A \square - : \mathbf{A} \to \mathbf{A}$ given by $B \mapsto A \square B$ preserves cokernels. If $\mathbb{A}$ has this property one has natural transformations $\eta_{(1)}$, $\mu_{(1)}$ and $\mu_{(2)}$ given by the following commutative diagrams, in which q is the quotient map from $L_0(X) = G \square (G \oplus X)$ to the additivisation $L(X)$, q^2 is $qL_0(q)$ and $\mu' = 1 \square (\mu \oplus 1) : L_0(X) \square G = G \square (G \square G \oplus X \square G) \to L_0(X)$.

$$\begin{tikzcd} G \oplus X \arrow[r, "p_X"] \arrow[d, "\eta \square 1"'] & X \arrow[d, "\eta_{(1)}"] \\ L_0(X) \arrow[r, "q"] & L(X) \end{tikzcd} \qquad \begin{tikzcd} L_0(L_0(X)) \arrow[r, "q^2"] \arrow[d, "\mu \square \beta"'] & L(L(X)) \arrow[d, "\mu_{(1)}"] \\ L_0(X) \arrow[r, "q"] & L(X) \end{tikzcd} \qquad \begin{tikzcd} L_0(X) \square G \arrow[r, "q \square 1"] \arrow[d, "\mu'"'] & L(X) \square G \arrow[d, "\mu_{(2)}"] \\ L_0(X \square G) \arrow[r, "q"] & L(X \square G) \end{tikzcd}$$

Lemma 2.9. The natural transformations $\eta_{(1)}$, $\mu_{(1)}$ and $\mu_{(2)}$ are well-defined.

Proof. Since X is clearly the additivisation of $G \oplus X$ in X, $\eta_{(1)}$ is well defined. Similarly $\mu_{(2)}$ is well defined since $- \square G$ is additive. By the assumption that $\mathbb{A}$ is right compatible with cokernels it follows that $L(L(X))$ is the additivisation of $L_0(L_0(X))$ in X with q^2 the corresponding quotient map. Thus $\mu_{(1)}$ is also well defined. ∎

Using these natural transformations between additive functors we have

Proposition 2.10. A coefficient G-module is equivalently specified by an object M and morphisms

$$L(M) \xrightarrow{\overline{\lambda}} M, \qquad M \square G \xrightarrow{\rho} M$$

such that $\overline{\lambda}\eta_{(1)} = 1_M$, ρ is a right action as in (2.3)(3), and the following diagrams commute:

$$\begin{tikzcd} L(L(M)) \arrow[r, "L(\overline{\lambda})"] \arrow[d, "\mu_{(1)}"'] & L(M) \arrow[d, "\overline{\lambda}"] \\ L(M) \arrow[r, "\overline{\lambda}"'] & M \end{tikzcd} \qquad \begin{tikzcd} L(M) \square G \arrow[r, "\overline{\lambda} \square 1"] \arrow[d, "\mu_{(2)}"'] & M \square G \arrow[dr, "\rho"] & \\ L(M \square G) \arrow[r, "L(\rho)"'] & L(M) \arrow[r, "\overline{\lambda}"'] & M \end{tikzcd}$$

Note that these are just the diagrams in (2.8) and (2.3)(4) made additive.

Proof. Given $\overline{\lambda}$ we obtain λ by the composite

$$\lambda : G \,\square\, (G \oplus M) \xrightarrow{\quad q \quad} L(M) \xrightarrow{\quad \overline{\lambda} \quad} M$$

Then (M, λ, ρ) is a coefficient G-module in the sense of (2.3), as follows from the previous lemmas. Conversely any coefficient G-module M is obtained in this way since the linearity property in (2.3)(1) is equivalent to the existence of $\overline{\lambda}$ with $\lambda = \overline{\lambda} q$. ■

We can now give an explicit construction for free coefficient modules. If the monoidal structure is both right and left distributive, the coefficient modules are just bimodules, and it is well known that the free G-bimodule is given by $F(V) = G \,\square\, V \,\square\, G$ with left and right actions given by the multiplication in G. With the assumption that $\mathbb{A}$ is right compatible with cokernels we have a similar explicit presentation of F in our more general situation.

Proposition 2.11. Let $G = (G, \eta, \mu)$ be a monoid in $\mathbb{A}$. Then the free coefficient G-module on an object V of $\mathbf{A}$ is given by

$$F(V) \quad = \quad L(V \,\square\, G)$$

with the structure maps $\overline{\lambda}$ and ρ given by

$$\overline{\lambda} : L(L(V \,\square\, G)) \xrightarrow{\qquad \mu_{(1)} \qquad} L(V \,\square\, G)$$

$$\rho : L(V \,\square\, G) \,\square\, G \xrightarrow{\quad \mu_{(2)} \quad} L(V \,\square\, G \,\square\, G) \xrightarrow{L(1 \,\square\, \mu)} L(V \,\square\, G)$$

Proof. For an object V of $\mathbf{A}$ and a coefficient G-module $(M, \overline{\lambda}, \rho)$ we have natural maps $V \to UF(V)$ in $\mathbf{A}$ and $F(UM) \to M$ in $\mathbf{Coef}_G$ given by

$$V = V \,\square\, I \xrightarrow{\quad 1 \,\square\, \eta \quad} V \,\square\, G \xrightarrow{\quad \eta_{(1)} \quad} L(V \,\square\, G)$$

$$L(M \,\square\, G) \xrightarrow{\quad L(\rho) \quad} L(M) \xrightarrow{\qquad \overline{\lambda} \qquad} M$$

respectively, and these satisfy the triangle identities required to define an adjunction. ■

We end by interpreting the results of this section for operads, the example promised in (2.2.2). First recall the definition of an operad from e.g. [17].

Let $\mathfrak{S}$ be the symmetric groupoid; that is, $\mathfrak{S}$ is given by the disjoint union of the symmetric groups $\mathfrak{S}_n$, with $\mathfrak{S}_0 = \{*\}$. Let $\mathbf{A} = R\text{-}\mathbf{Mod}$ be the category of R-modules (or R-module chain complexes) for R a commutative ring, with monoidal structure $\otimes = \otimes_R$ and $I = R$. Consider the category $\mathbf{Cat}(\mathfrak{S}, \mathbf{A})$ of $\mathfrak{S}$-objects in $\mathbf{A}$, given by functors A from the symmetric groupoid to $\mathbf{A}$, or equivalently by families $\{A_n\}_{n \geqslant 0}$ together with actions of $\mathfrak{S}_n$. The category $\mathbf{Cat}(\mathfrak{S}, \mathbf{A})$ is clearly abelian, with the sum $A \oplus B$ of $\mathfrak{S}$-objects given by the sum in $\mathbf{A}$

$$(A \oplus B)_n \quad = \quad A_n \oplus B_n$$

The tensor product of $\mathfrak{S}$-objects is defined as follows. Let $\mathcal{P}_n^k$ be the set of partitions of $\{1,\ldots,n\}$ into k disjoint subsets $(J_i)_{i=1}^k$, and write j_i for $|J_i|$. Then for an $\mathfrak{S}$-object B let

$$B_n^k = \bigoplus_{(J_i)\in\mathcal{P}_n^k} B_{j_1}\otimes\ldots\otimes B_{j_k}$$

Clearly $\mathfrak{S}_k$ acts on B_n^k. In fact $\mathfrak{S}_n$ also acts on B_n^k via the $\mathfrak{S}_{j_i}$ actions. Thus the monoidal structure on $\mathbf{Cat}(\mathfrak{S},\mathbf{A})$ can be defined by

$$(A\,\square\,B)_n = \bigoplus_{k=0}^{\infty} A_k\otimes_{\mathfrak{S}_k} B_n^k$$

If $A_0 = B_0 = 0$ this is a finite sum $\bigoplus_{k=1}^n$. The functor $\iota:\mathbf{A}\to\mathbf{Cat}(\mathfrak{S},\mathbf{A})$ with $\iota(C)_1 = C$ and $\iota(C)_n = 0$ for $n\neq 1$ preserves the tensor product, and $I=\iota(R)$ defines a neutral object for $\square$ in $\mathbf{Cat}(\mathfrak{S},\mathbf{A})$. The monoidal structure on $\mathbf{Cat}(\mathfrak{S},\mathbf{A})$ is not symmetric, but it is left distributive. In fact $-\,\square\,B$ preserves all colimits and has a right adjoint $[B,-]$ given by

$$[B,C]_k = \bigoplus_{n=0}^{\infty}\mathbf{A}(B_n^k, C_n)_{\mathfrak{S}_n}$$

where $\mathbf{A}(-,-)_{\mathfrak{S}_n}$ is the object of $\mathfrak{S}_n$-equivariant maps in $\mathbf{A}$.

Definition 2.12. An *operad* in $\mathbf{A}$ is a monoid in $\mathbf{Cat}(\mathfrak{S},\mathbf{A})$, that is, an $\mathfrak{S}$-object A together with morphisms $\eta: I\to A$, $\mu: A\,\square\,A\to A$ satisfying the unit and associativity laws.

Thus an operad is specified by the objects $\{A_n\}_{n\geqslant 0}$ and $\mathfrak{S}_n$-actions, together with operations

$$A_k\otimes A_{j_1}\otimes A_{j_2}\otimes\ldots\otimes A_{j_k}\xrightarrow{\ \mu\ } A_n$$

where $n = j_1+\ldots+j_k$, satisfying the obvious unit and associative laws, together with certain equivariance relations as in May [23].

Definition 2.13. A *linear module* over an operad G is a coefficient G-module in $\mathbf{Cat}(\mathfrak{S},\mathbf{A})$, that is, an $\mathfrak{S}$-object M together with a right action $\rho: M\,\square\,G\to M$ and a left cross-action $\lambda: G\,\square\,(G\oplus M)\to M$ with the properties (1)–(4) of definition 2.3.

The functor $L_0(M) = G\,\square\,(G\oplus M)$ may be expanded by the distributivity of the tensor product $\otimes$ in $\mathbf{A}$, and we see that the additivisation $L(M)$ consists of those summands which contain precisely one factor from M. Thus a linear G-module is a family of objects $\{M_n\}_{n\geqslant 0}$ with $\mathfrak{S}_n$-actions, and operations

$$M_k\otimes G_{j_1}\otimes G_{j_2}\otimes\ldots\otimes G_{j_k}\xrightarrow{\ \rho\ } M_n$$

$$G_k\otimes G_{j_1}\otimes\ldots\otimes G_{j_{i-1}}\otimes M_{j_i}\otimes G_{j_{i+1}}\otimes\ldots\otimes G_{j_k}\xrightarrow{\ \lambda_i\ } M_n$$

for $1\leqslant i\leqslant k$ and $n = j_1+\ldots+j_k$, satisfying the obvious action and compatibility laws together with equivariance relations as those for the operad structure. Compare also [22].

3. Cohomology

Let $\mathcal{G} = (G, \mu, \eta)$ be a monoid in a monoidal category $\mathbb{V} = (\mathbf{V}, \square, I)$. We will avoid mentioning the associativity isomorphisms where possible.

We write $G^{\square n}$ for the n-fold iterated tensor product $G \square G \square \cdots \square G$, and let $\mu^n : G^{\square n} \to G$ be given by the iterated multiplication map, with $\mu^0 = \eta$ and μ^1 the identity. We also write μ_i and η_i for the maps given by applying the multiplication and the unit between the ith and $(i+1)$st tensor factors:

$$\mu_i : G^{\square n} \cong G^{\square(i-1)} \square G \square G \square G^{\square(n-i-1)} \xrightarrow{1 \square \mu \square 1} G^{\square(n-1)} \quad (0 < i < n)$$

$$\eta_i : G^{\square n} \cong G^{\square i} \square I \square G^{\square(n-i)} \xrightarrow{1 \square \eta \square 1} G^{\square(n+1)} \quad (0 \leqslant i \leqslant n)$$

Definition 3.1. We denote by $B_\bullet(\mathcal{G})$ the two-sided bar construction [23] in the monoidal category $\mathbb{V}/\mathcal{G}$. This is the simplicial object in $\mathbb{V}/\mathcal{G}$ with

$$B_n(\mathcal{G}) \quad = \quad (G^{\square(n+2)} \xrightarrow{\mu^{n+2}} G)$$

and face and degeneracy maps given by

$$d_i : G^{\square(n+2)} \xrightarrow{\mu_{i+1}} G^{\square(n+1)}$$

$$s_i : G^{\square(n+2)} \xrightarrow{\eta_{i+1}} G^{\square(n+3)}$$

for $0 \leqslant i \leqslant n$. As usual, this in fact defines a simplicial object in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$. There are extra degeneracy operators $s_{-1} = \eta_0 = \eta \square G^{\square(n+2)}$ and $s_{n+1} = \eta_{n+2} = G^{\square(n+2)} \square \eta$ which provide contractions of $B_\bullet(\mathcal{G})$ in $\mathbf{V}^{\mathcal{G}}/G$ and ${}^{\mathcal{G}}\mathbf{V}/G$ respectively, but *not* in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$.

Given an internal abelian group A in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$, we define

Definition 3.2. The cohomology of a monoid $\mathcal{G} \in \mathbf{Mon}(\mathbb{V})$ with coefficients in an internal abelian group $A \in \mathbf{Ab}({}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G) \simeq \mathbf{Ab}(\mathbf{Mon}(\mathbb{V})/\mathcal{G})$, denoted $H^\bullet(\mathcal{G}; A)$, is the cohomology of the cochain complex associated to the cosimplicial abelian group $\mathrm{Hom}_{{}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G}(B_\bullet(\mathcal{G}), A)$.

Now the forgetful functor $U : {}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G \to \mathbf{V}/G$ has a left adjoint F, where in particular

$$F(G^{\square(n)} \xrightarrow{\mu^n} G) \quad = \quad (G^{\square(n+2)} \xrightarrow{\mu^{n+2}} G)$$

Hence there are natural bijections

$$\mathrm{Hom}_{{}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G}(G^{\square(n+2)} \to G, A) \cong \mathrm{Hom}_{\mathbf{V}/G}(G^{\square n} \to G, A)$$

and translating the cosimplicial structure of $B_\bullet(\mathcal{G})$ along these one gets

Proposition 3.3. $H^*(\mathcal{G};A)$ is isomorphic to the cohomology of the complex $C^*(\mathcal{G};A)$ with $C^n(\mathcal{G};A) = \mathrm{Hom}_{\mathbf{V}/G}(G^{\square(n)} \xrightarrow{\mu^n} G, A)$ and differentials

$$d = \sum_{i=0}^{n}(-1)^i d^i : C^{n-1}(\mathcal{G};A) \to C^n(\mathcal{G};A)$$

where

$$d^0(G^{\square(n-1)} \xrightarrow{f} A) = (G^{\square(n)} \xrightarrow{1\square f} G \square A \xrightarrow{u} A),$$
$$d^i(G^{\square(n-1)} \xrightarrow{f} A) = (G^{\square(n)} \xrightarrow{\mu_i} G^{\square(n-1)} \xrightarrow{f} A) \text{ for } 0 < i < n,$$
$$d^n(G^{\square(n-1)} \xrightarrow{f} A) = (G^{\square(n)} \xrightarrow{f\square 1} A \square G \xrightarrow{v} A).$$

Since the forgetful functor $U : {}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G \to \mathbf{V}/G$ is monadic, there is also a standard way to define cohomology in this setting, the so called cotriple cohomology (see [6]). We will show that this leads to the same result:

Proposition 3.4. The cohomology groups $H^*(\mathcal{G};A)$ defined above are isomorphic to the cotriple cohomology groups w. r. t. the cotriple on ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$ induced by the monadic adjunction $(F \dashv U) : {}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G \to \mathbf{V}/G$.

Proof. The standard simplicial object for the cotriple cohomology has $(FU)^n(1_G)$ in dimension n; as $F(X \xrightarrow{f} G) = (G \square X \square G \xrightarrow{1\square f\square 1} G \square G \square G \xrightarrow{\mu^3} G)$, this simplicial object will have $G^{\square(2n+3)}$ in dimension n. In fact direct calculation shows that this simplicial object is exactly the *edgewise subdivision* $\mathrm{Sub}(B_\bullet(\mathcal{G}))$ of $B_\bullet(\mathcal{G})$, in the sense of [28]. Now it is not clear whether a simplicial object in a general category is homotopy equivalent to its edgewise subdivision. But to prove our proposition, it is enough to deal with cosimplicial abelian groups obtained by applying to simplicial objects the contravariant functor $\mathrm{Hom}(-,A)$, for A an internal abelian group. There is an obvious dual notion of subdivision for cosimplicial objects. And analyzing the proof of the particular case in [28], one can modify it to obtain a proof for cosimplicial internal abelian groups. Therefore the proposition will follow from the following lemma.

Lemma 3.5 (Subdivision Lemma). For a cosimplicial abelian group $A^\bullet$ in any category, the cochain complexes corresponding to $A^\bullet$ and $\mathrm{Sub}(A^\bullet)$ are homotopy equivalent.

This lemma is proved in appendix A. ∎

We now identify the simplification of the cochain complex in proposition 3.3 in the special case of monoids in an abelian and left distributive monoidal category $\mathbf{A}$. In this case we know by proposition 2.4 that the coefficients $A \in \mathbf{Ab}({}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G)$ can be replaced by coefficient G-modules $(M,\lambda,\rho) \in \mathbf{Coef}_G$.

Proposition 3.6. Let M be a coefficient G-module. Then there is a cosimplicial abelian group

$$C^n(G,M) \quad = \quad \mathrm{Hom}_{\mathbf{A}}(\underbrace{G \square \ldots \square G}_{n \text{ factors}}, M)$$

The coface and codegeneracy maps are defined on $c \in C^n(G, M)$ by

$$d^0(c) : G^{\square(n+1)} \cong G \square G^{\square n} \xrightarrow{1 \square (\mu^n, c)} G \square (G \oplus M) \xrightarrow{\lambda} M$$

$$d^{n+1}(c) : G^{\square(n+1)} \cong G^{\square n} \square G \xrightarrow{c \square 1} M \square G \xrightarrow{\rho} M$$

$$d^i(c) : G^{\square(n+1)} \cong G^{\square(i-1)} \square G^{\square 2} \square G^{\square(n-i)} \xrightarrow{1 \square \mu \square 1} G^{\square n} \xrightarrow{c} M$$

$$s^i(c) : G^{\square(n-1)} \cong G^{\square i} \square I \square G^{\square(n-i-1)} \xrightarrow{1 \square \eta \square 1} G^{\square n} \xrightarrow{c} M$$

where $G^{\square n}$ is the nth tensor power, $\mu^0 = \eta$, $\mu^1 = 1$ and $\mu^n : G^{\square n} \to G$ for $n \geqslant 2$ is given by the multiplication on G.

Proof. We must check those cosimplicial identities which involve d^0; the others are exactly as in the classical definition of Hochschild cohomology. We have

$$\begin{array}{lrcllcl}
a) & d^1 d^0 & = & d^0 d^0 & \Longleftrightarrow & \lambda(1 \square (\mu^n, c))(\mu \square 1) & = & \lambda[1 \square [\mu^{n+1}, \lambda(1 \square (\mu^n, c))]] \\
b) & d^{n+2} d^0 & = & d^0 d^{n+1} & \Longleftrightarrow & \rho(\lambda(1 \square (\mu^n, c)) \square 1) & = & \lambda(1 \square (\mu^{n+1}, \rho(c \square 1))) \\
c) & d^{i+1} d^0 & = & d^0 d^i & \Longleftrightarrow & \lambda(1 \square (\mu^n, c))\mu_{i+1} & = & \lambda(1 \square (\mu^{n+1}, c\mu_i)) \\
d) & s^0 d^0 & = & 1 & \Longleftrightarrow & \lambda(1 \square (\mu^n, c))(\eta \square 1) & = & c \\
e) & s^{i+1} d^0 & = & d^0 s^i & \Longleftrightarrow & \lambda(1 \square (\mu^n, c))\eta_{i+1} & = & \lambda(1 \square (\mu^{n-1}, c\eta_i))
\end{array}$$

for all $c : G^{\square n} \to M$, where we write $\mu_i : G^{\square(k+1)} \to G^{\square k}$ and $\eta_i : G^{\square(k-1)} \to G^{\square k}$ for the multiplication and unit of G applied at the ith factor. By the cross-action property we know

$$\lambda(\mu \square (\mu^n, c)) = \lambda[1 \square [(\mu(1 \square p_G), \lambda)(1 \square (\mu^n, c))]] = \lambda[1 \square [\mu(1 \square \mu^n), \lambda(1 \square (\mu^n, c))]]$$

and hence (a) follows. Also the left distributivity and the compatibility of λ and ρ give

$$\rho(\lambda \square 1)(1 \square (\mu^n, c) \square 1) = \lambda(1 \square (\mu \oplus \rho))(1 \square (\mu^n \square 1, c \square 1))$$

and hence (b). By the unit law for λ we have $\lambda(\eta \square (\mu^n, c)) = p_M(\mu^n, c) = c$ which gives (d), and (c) and (e) are clear from naturality and the monoid laws.

Finally we note that $\mathrm{Hom}(G^{\square n}, M)$ has an abelian group structure by addition in M, and that d^0 is a group homomorphism by the linearity of λ. ∎

Definition 3.7. Let M be a coefficient G-module as above. Then the *cohomology of G with coefficients in M*, $H^n(G, M)$, is given by the cohomology of the cochain complex (C^*, δ) with $C^n = C^n(G, M)$ the abelian group of homomorphisms $c : G^{\square n} \to M$ under pointwise addition, and the boundary maps given by

$$\delta^n(c) \;=\; \sum_{i=0}^{n+1}(-1)^i d^i(c) \;=\; \lambda(1 \square (\mu^n, c)) + \left(\sum_{i=1}^{n}(-1)^i c\mu_i\right) + (-1)^{n+1}\rho(c \square 1)$$

Below we show that this cohomology is a special case of the cohomology in (3.2). From the usual relations between the cosimplicial maps d^i in the proposition we know that $\delta^{n+1}\delta^n$ is zero. As usual the same cohomology is obtained from the *normalised* cochain complex

$C_N^*(G, M)$ defined by quotienting by the subcomplex of C^* of elements arising as codegeneracies.

Definition 3.8. Assume that $\mathbb{A}$ is right compatible with cokernels as so that we have a free coefficient G-module functor F as in (2.11). We define a simplicial coefficient G-module $B(G)$ termed the *bar resolution* of the monoid G. In the case of R-**mod**, example 1.2, this will be the un-normalised bar resolution described in MacLane [21, X.2]. The objects $B_n(G)$ are given by the free coefficient G-modules on $G^{\square n}$

$$B_n(G) \;=\; F(G^{\square n})$$

The degeneracy maps $s_i : B_n(G) \to B_{n+1}(G)$ and face maps $d_i : B_{n+1}(G) \to B_n(G)$ are given by

$$\begin{aligned} s_i &= F(\eta_i) \quad \text{for } 0 \leqslant i \leqslant n \\ d_i &= F(\mu_i) \quad \text{for } 1 \leqslant i \leqslant n \end{aligned}$$

where η_i and μ_i are defined on $G^{\square k}$ by applying $\eta : I \to G$ and $\mu : G \square G \to G$ at the ith factor. The face maps d_0, $d_{n+1} : F(G^{\square(n+1)}) \to F(G^{\square n})$ are the morphisms of coefficient G-modules corresponding under the adjunction to the following maps d'_0, $d'_{n+1} : G^{\square(n+1)} \to F(G^{\square n})$ in $\mathbf{A}$:

$$\begin{tikzcd} G \square G^{\square n} \arrow[d, "1 \square (\mu^n, 1')"'] \arrow[dr, "d'_0"] & \\ G \square (G \oplus F(G^{\square n})) \arrow[r, "\lambda"'] & F(G^{\square n}) \end{tikzcd} \qquad \begin{tikzcd} G^{\square n} \square G \arrow[d, "1' \square 1"'] \arrow[dr, "d'_{n+1}"] & \\ F(G^{\square n}) \square G \arrow[r, "\rho"'] & F(G^{\square n}) \end{tikzcd}$$

where $1' : G^{\square n} \to F(G^{\square n})$ in $\mathbf{A}$ corresponds to the identity on $F(G^{\square n})$ in $\mathbf{Coef}_G$.

Proposition 3.9. Let M be a coefficient G-module. Then there is a natural isomorphism

$$\psi : C(G, M) \;\cong\; \mathrm{Hom}_{\mathbf{Coef}_G}(B(G), M)$$

and hence the cohomology of G is determined by maps from the bar resolution

$$H^*(G, M) \;\cong\; H^*\mathrm{Hom}_{\mathbf{Coef}_G}(B(G), M)$$

Proof. The free/forget adjunction gives natural isomorphisms

$$\psi_n : C_n(G, M) = \mathrm{Hom}_{\mathbf{A}}(G^{\square n}, M) \cong \mathrm{Hom}_{\mathbf{Coef}_G}(F(G^{\square n}), M) = \mathrm{Hom}_{\mathbf{Coef}_G}(B_n(G), M)$$

and we must check these respect the (co)simplicial structures. We have

$$\psi_n(s^i) = \psi_n \mathrm{Hom}_{\mathbf{A}}(\eta_i, M) = \mathrm{Hom}_{\mathbf{Coef}_G}(F(\eta_i), M) = \mathrm{Hom}_{\mathbf{Coef}_G}(s_i, M)$$

and similarly $\psi_n(d^i) = \mathrm{Hom}_{\mathbf{Coef}_G}(d_i, M)$ for $i \neq 0, n+1$. Let d'_0 and $1'$ be as in the definition of d_0 above, and let $c : F(G^{\square n}) \to M$ be a morphism of coefficient G-modules. Then naturality of the adjunction implies $\psi^{-1}(d_0^* c) = c d'_0$ and $\psi^{-1}(c) = c1'$, and

$$d^0(c1') = \lambda(1 \square (\mu^n, c1')) = \lambda(1 \square (1 \oplus c))(1 \square (\mu^n, 1')) = c\lambda(1 \square (\mu^n, 1')) = cd'_0$$

Thus $d^0\psi^{-1}(c) = \psi^{-1}(d_0^* c)$. One shows $d^{n+1}\psi^{-1}(c) = \psi^{-1}(d_{n+1}^* c)$ in the same way. ■

Finally we show that the definition of cohomology in (3.7) is a special case of that in (3.2).

Proposition 3.10. For $\mathcal{G} = (G, \mu, \eta)$ a monoid in $\mathbf{A}$ the cochain complexes of propositions 3.3 and 3.6 are isomorphic

$$\theta : C^*(\mathcal{G}; A_M) \cong C(G, M)$$

where M is any coefficient G-module and $A_M \in \mathbf{Ab}({}^{\mathcal{G}}\mathbf{A}^{\mathcal{G}}/G)$ is given by M under the equivalence of proposition 2.4.

Proof. Recall first that $A_M = (p_G : G \oplus M \to G)$, and that the structure maps satisfy

$$p_M u = \lambda : G \square (G \oplus M) \to M, \qquad p_M v = \rho(p_M \square 1) : (G \oplus M) \square G \to M.$$

Now a morphism $c : G^{\square n} \to M$ in $\mathbf{A}$ determines a morphism $(\mu^n, c) : G^{\square n} \to G \oplus M$ in $\mathbf{A}/G$, and conversely a morphism $f : G^{\square n} \to A$ in the slice category gives a morphism $p_M f : G^{\square n} \to M$ in $\mathbf{A}$.

$$\begin{tikzcd} & G^{\square n} \arrow[dl, "\mu^n"'] \arrow[d, "f"] \arrow[dr, "c"] & \\ G & G \oplus M \arrow[l, "p_G"] \arrow[r, "p_M"'] & M \end{tikzcd}$$

Clearly this gives isomorphisms of abelian groups

$$\theta_n : C^n(\mathcal{G}; A_M) = \mathrm{Hom}_{\mathbf{A}/G}(\mu^n, p_G) \cong \mathrm{Hom}_{\mathbf{A}}(G^{\square n}, M) = C^n(G, M)$$

and we must check the cosimplicial structures coincide. For cochains f, c with $\theta_n f = c$ we have

$$\begin{aligned} \theta_{n+1}(d^0 f) &= p_M u(1 \square f) &= \lambda(1 \square (\mu^n, c)) &= d^0 c \\ \theta_{n+1}(d^{n+1} f) &= p_M v(f \square 1) &= \rho(p_M f \square 1) &= d^{n+1} c \end{aligned}$$

and the results for the other cofaces and the codegenacies are straightforward. ■

Remark 3.11. Particular examples of the cohomology defined by (3.2) or (3.7) above coincide with various cohomologies in the literature.

(1) For $\mathbf{V} = R$-**mod** in example 1.2 the cohomology $H^*(G, A)$ is the same as the classical cohomology of an R-algebra G; see [21, X.3]. We saw in proposition 2.1 that the coefficients A are G-bimodules.

(2) Consider the monoidal category $\mathbf{V} = R$-R-**mod** of bimodules over an arbitrary ring R, as in example 1.6. The cohomology $H^*(G, A)$ we obtain is the R-relative Hochschild cohomology from [12]. Indeed, direct comparison shows that in this case our complex coincides with the one used by Gerstenhaber and Schack in [12] to define the R-relative Hochschild cohomology groups.

(3) For $\mathbf{V} = \mathbf{Cat}(\mathfrak{S}, R\text{-}\mathbf{Mod})$ in example 2.2.2 the cohomology $H^*(G, A)$ is the cohomology of an operad with coefficients as described in proposition 2.13. These have also appeared in [11, 22].

(4) For $\mathbf{V} = \mathbf{Ens}/I \times I$ in example 1.11 the cohomology $H^*(G, A)$ coincides with the cohomology of a category G with coefficients in a natural system A, see [5].

(5) For $\mathbf{V}$ the category of finitary endofunctors of **Ens** in example 1.8 the cohomology $H^\bullet(G, A)$ is the cohomology of a finitary theory G considered briefly in [14].

4. Derivations, extensions and torsors

We now turn to the interpretation of elements in cohomology groups. We first consider abelian and left distributive monoidal categories $\mathbb{A}$ and the low degree cohomology of monoids in $\mathbb{A}$, which we interpret in terms of derivations and extensions. In the second part of this section we deal with the case of a general monoidal category $\mathbb{V}$ and the cohomology of monoids in $\mathbb{V}$ which in low degrees can be interpreted using torsors.

Recall that for the cohomology of a monoid $\mathcal{G} = (G, \mu, \eta)$ in $\mathbb{A}$ we use the coefficient G-modules (M, λ, ρ) of definition 2.3.

Definition 4.1. A *derivation* (or *crossed homomorphism*) from a monoid G to a coefficient G-module M is a morphism $\Delta : G \to M$ in $\mathbf{A}$ which satisfies $\Delta\mu = \lambda(1 \square (1, \Delta)) + \rho(\Delta \square 1)$.

$$\begin{tikzcd} G \square G \arrow[r, "(1 \square (1,\Delta),\, \Delta \square 1)"] \arrow[d, "\mu"'] & G \square (G \oplus M) \oplus M \square G \arrow[d, "\lambda + \rho"] \\ G \arrow[r, "\Delta"] & M \end{tikzcd}$$

The abelian group of derivations from G to M is written $\mathrm{Der}(G, M)$.

In particular a morphism $\phi : I \to M$ in $\mathbf{A}$ defines an *inner derivation* $\mathrm{Inn}(\phi) : G \to M$ by $\mathrm{Inn}(\phi) = \lambda\phi_0 - \rho\phi_1$ where

$$\phi_0 = 1 \square (\eta, \phi) : G \square I \to G \square (G \oplus M) \quad \text{and} \quad \phi_1 = \phi \square 1 : I \square G \to M \square G$$

We thus have a homomorphism

$$\mathrm{Hom}(I, M) \longrightarrow \mathrm{Der}(G, M)$$

whose image is the subgroup $\mathrm{Inn}(G, M)$ of inner derivations. The kernel consists of those ϕ with

$$\lambda(1 \square (\eta, \phi)) \;=\; \rho(\phi \square 1)$$

This may be thought of as the subgroup M^G of *G-invariant* morphisms $I \to M$.

Proposition 4.2. There are isomorphisms

$$H^0(G, M) \;\cong\; M^G \quad \text{and} \quad H^1(G, M) \;\cong\; \mathrm{Der}(G, M)/\mathrm{Inn}(G, M)$$

and an exact sequence of abelian groups

$$0 \longrightarrow H^0(G, M) \longrightarrow \mathrm{Hom}(I, M) \longrightarrow \mathrm{Der}(G, M) \longrightarrow H^1(G, M) \longrightarrow 0$$

Proof. The derivation property is $\delta^1\Delta = 0$, so derivations are just 1-cocycles. Also the inner derivation map $\phi \mapsto \mathrm{Inn}(\phi)$ is just the coboundary map δ^0. ∎

We now describe the theory of extensions of monoids (G, η, μ) in $\mathbb{A}$. Our exposition will be parallel to and will extend the classical description for the case $\mathbf{A} = R$-**Mod** of example 1.2, where the tensor $\otimes_R$ preserves colimits on both sides; see for example MacLane [21].

Definition 4.3. An *extension* of a monoid $\mathcal{G}$ in $\mathbb{A}$ is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\ i\ } A \xrightarrow{\ p\ } G \longrightarrow 0$$

in the abelian category $\mathbf{A}$ together with a monoid structure on A such that p is a morphism of monoids. The extension is $\mathbf{A}$-*split* if there is an $s : G \to A$ in $\mathbf{A}$ which is right inverse to p, $ps = 1_G$. The extension is termed *singular* if the following conditions hold.

(1) The map $\mu_A(i \square 1) : M \square A \to A$ is zero on the kernel of $1 \square p : M \square A \to M \square G$
(2) The maps $\mu_A(1 \square +)$, $\mu_A(1 \square p_1) + \mu_A(1 \square p_2)$: $A \square (A \oplus_G A) \longrightarrow A$ are equal.

$$\begin{tikzcd}
A \square (A \oplus_G A) \arrow[r, "1 \square +"] \arrow[d, "{(1 \square p_1, 1 \square p_2)}"'] & A \square A \arrow[d, "\mu_A"] \\
A \square A \oplus_G A \square A \arrow[r, "\mu_A + \mu_A"] & A
\end{tikzcd}$$

Extensions A, A' are *equivalent* if there is a morphism $\varepsilon : A \to A'$ of monoids with $\varepsilon i = i'$ and $p'\varepsilon = p$.

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, "p"] \arrow[d, "\varepsilon"', "\cong"] & G \arrow[d, equal] \\
M \arrow[r, "i'"] & A' \arrow[r, "p'"] & G
\end{tikzcd}$$

Fixing a monoid G and a coefficient G-module M, we write $\operatorname{Ext}(G, M)$ for the set of equivalence classes of $\mathbf{A}$-split singular extensions.

Suppose $M \xrightarrow{i} A \xrightarrow{p} G$ is an $\mathbf{A}$-split singular extension with section s as above. Let $d = s\eta_G - \eta_A : I \to A$, then by replacing s by $s - \mu_A(d \square 1)$ if necessary we can assume that s respects the units of G and A. Also the map $s + i : G \oplus M \to A$ is a map of short exact sequences and hence an isomorphism in $\mathbf{A}$ by the 5-lemma.

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \arrow[d, equal] \\
M \arrow[r, "i_M"] & G \oplus M \arrow[u, "s+i", "\cong"'] \arrow[r, shift left, "p_G"] & G \arrow[l, shift left, "i_G"]
\end{tikzcd}$$

Using the isomorphism $s+i$ we obtain a coefficient G-module structure (λ, ρ) on M as follows. The maps $\mu_A(s\square(s+i) - s\square(s+0)) : G\square(G\oplus M) \to A\square A \to A$ and $\mu_A(i\square s) : M\square G \to A\square A \to A$

factor through $\ker(p)$ and define

$$G \,\Box\, (G \oplus M) \xrightarrow{\lambda} M, \qquad M \,\Box\, G \xrightarrow{\rho} M$$

respectively. The singularity conditions show that λ and ρ are independent of the choice of splitting s and that λ is linear in the sense of (2.3.1). The action and compatibility laws follow by associativity of μ_A.

Conversely, suppose M is a coefficient G-module and $M \xrightarrow{i} A \xrightarrow{p} G$ is an extension of G. Then it is a singular extension if and only if the monoid structure on A extends the coefficient G-module structure on M:

$$\begin{tikzcd} M \,\Box\, A \arrow[r, "1 \,\Box\, p"] \arrow[d, "i \,\Box\, 1"'] & M \,\Box\, G \arrow[r, "\rho"] & M \arrow[d, "i"] \\ A \,\Box\, A \arrow[rr, "\mu_A"'] & & A \end{tikzcd} \qquad \begin{tikzcd} A \,\Box\, (A \oplus M) \arrow[r, "p\Box(p\oplus 1)"] \arrow[d, "1\Box(1\oplus i)"'] & G \,\Box\, (G \oplus M) \arrow[r, "\lambda"] & M \arrow[d, "i"] \\ A \,\Box\, (A \oplus A) \arrow[r, "\kappa"'] & A \,\Box\, A \arrow[r, "\mu_A"'] & A \end{tikzcd}$$

where $\kappa = 1 \,\Box\, (1+1) - 1 \,\Box\, (1+0) : A \,\Box\, (A \oplus A) \to A \,\Box\, A$.

The simplest example of an $\mathbf{A}$-split singular extension is the *trivial extension* or *semi-direct sum* given by $A = G \oplus M$ with unit $i_G\eta_G$ and multiplication

$$(\mu(p_G \,\Box\, p_G)\ , \lambda(p_G \,\Box\, 1) + \rho(p_M \,\Box\, p_G)) : (G \oplus M) \,\Box\, (G \oplus M) \to G \oplus M$$

Any $\mathbf{A}$-split singular extension for which p is split by a morphism of monoids is equivalent to the semi-direct sum. More generally each splitting s of a singular extension defines a *factor set* $c_s : G \,\Box\, G \to M$, or 2-cochain of $C^*(G, M)$, by

$$\mu_A(s \,\Box\, s) \;=\; s\mu_G + ic_s$$

which is *normalised* if $s\eta_G = \eta_A$ and is zero if s is a monoid homomorphism. The factor set c_t given by a different choice of splitting t differs from c_s by a coboundary: one can define $\Delta : G \to M$ by $t = s + i\Delta$, and then

$$\begin{aligned} ic_t - ic_s &= \mu_A((s+i\Delta) \,\Box\, (s+i\Delta)) - \mu_A(s \,\Box\, s) - (s+i\Delta)\mu_G + s\mu_G \\ &= i\lambda(1 \,\Box\, (1,\Delta)) + i\rho(\Delta \,\Box\, 1) - i\Delta\mu_G \;=\; i\delta\Delta \end{aligned}$$

This process also respects equivalent extensions since given an equivalence $\varepsilon : A \to A'$ and a splitting s for A, then εs is a splitting for A' and the factor sets c_s and $c_{\varepsilon s}$ are equal.

Theorem 4.4. Let G be a monoid and M a coefficient G-module. Then assigning factor sets to $\mathbf{A}$-split singular extensions induces a bijection between the equivalence classes of such extensions and the cohomology classes of cocycles $G \,\Box\, G \to M$

$$\Phi \;:\; \mathrm{Ext}(G, M) \;\cong\; H^2(G, M)$$

under which the class of the trivial extension corresponds to zero.

Proof. We construct an inverse Ψ to Φ. Given a 2-cocycle $c : G\square G \to M$, there is an extension given by $A = G \oplus M$ with unit $i_G\eta_G$ and multiplication μ_c as follows:

$$\mu_c = (\mu_G, \lambda(p_G \square 1) + \rho(p_M \square p_G) + c_G) : (G \oplus M) \square (G \oplus M) \longrightarrow G \oplus M$$

where $\mu_G = \mu(p_G \square p_G) = p_G\mu_c$ and $c_G = c(p_G \square p_G)$. Clearly p_G is a monoid homomorphism, and the monoid structure on $G \oplus M$ extends the coefficient G-module structure on M. If cocycles c and d differ by a coboundary $\delta\Delta$ for $\Delta : G \to M$, then the map $\varepsilon : G\oplus M \to G\oplus M$ given by $(p_G, p_M + \Delta p_G)$ shows that the extensions $\Psi(c)$ and $\Psi(d)$ are equivalent.

For associativity of μ_c we note first

$$p_G\mu_c(\mu_c \square 1) = \mu(\mu_G \square p_G) = \mu(p_G \square \mu_G) = p_G\mu_c(1 \square \mu_c)$$

by associativity of μ. Now $p_M\mu_c(\mu_c \square 1) = \lambda(\mu_G \square 1) + \rho(p_M\mu_c \square p_G) + c(\mu_G, p_G)$ which is

$$\lambda(\mu_G \square 1) + \rho(\lambda(p_G \square 1) \square p_G) + \rho(\rho(p_M \square p_G) \square p_G) + \rho(c_G \square p_G) + c(\mu_G \square p_G)$$

and $p_M\mu_c(1 \square \mu_c) = \lambda(p_G \square \mu_c) + \rho(p_M \square \mu_G) + c(p_G \square \mu_G)$ which by linearity of λ in M is

$$\lambda(p_G \square (\mu_G, \lambda(p_G \square 1))) + \lambda(p_G \square (\mu_G, \rho(p_M \square p_G))) + \lambda(p_G \square (\mu_G, c_G)) + \rho(p_M \square \mu_G) + c(p_G \square \mu_G)$$

Now evaluate these on the inclusions $i_G \square i_G \square 1$, $i_G \square 1 \square i_G$, $i_M \square i_G \square i_G$ and $i_G \square i_G \square i_G$. Then since λ is linear in G we see that $p_M\mu_c(\mu_c \square 1) = p_M\mu_c(1 \square \mu_c)$ if and only if the following relations hold:

$$\begin{array}{lrcl}
a) & \lambda(\mu \square 1) & = & \lambda(1 \square (\mu(1 \square p_G), \lambda)) \\
b) & \rho(\lambda \square 1) & = & \lambda(1 \square (\mu \oplus \rho)) \\
c) & \rho(\rho \square 1) & = & \rho(1 \square \mu) \\
d) & \rho(c \square 1) + c(\mu \square 1) & = & \lambda(1 \square (\mu, c)) + c(1 \square \mu)
\end{array}$$

But $(a), (b), (c)$ are respectively just the cross-action, compatibility and right action laws for λ and ρ, and (d) is the cocycle condition $\delta c = 0$. ∎

We now give similar interpretations of low degree cohomology of monoids in the case of a general monoidal category $\mathbb{V}$. Note that there is already a general interpretation of cotriple cohomology by Duskin [8, 9] as in the following remark, which applies to our cohomology by proposition 3.4. Let $\mathcal{G}$ be a monoid in $\mathbb{V}$ and A an internal abelian group in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$. Let $A_{\mathbf{Mon}}$ be the corresponding abelian group in $\mathbf{Mon}(\mathbf{V})/\mathcal{G}$ according to proposition 1.5.

Remark 4.5. Let $K(A, n)$ be the Eilenberg-MacLane object of A in degree n. Then a $K(A, n)$-torsor relative to the forgetful functor $U : {}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G \to \mathbf{V}$ is a simplicial object $X_\bullet$ in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$, together with a simplicial map $\chi : X_\bullet \to K(A, n)$, such that

(1) $X_\bullet$ is isomorphic to the coskeleton of the nth truncation of $X_\bullet$,
(2) χ satisfies the Kan fibration condition *exactly* in dimension $\geqslant n$,
(3) $U(X_\bullet)$ has a contracting homotopy in $\mathbf{V}/G$.

Duskin proves in [9, section 5.2] that there is a natural bijection between the set of equivalence classes of $K(A, n)$-torsors and the nth cotriple cohomology of G with coefficients in A.

Simplification is possible since it turns out that in degrees $n = 1, 2$ elements of $H^n(\mathcal{G}; A)$ can also be interpreted using $K(A_{\mathbf{Mon}}, n-1)$-torsors. For higher degrees we make the following observations. Suppose we have a left adjoint to the forgetful functor $U : \mathbf{Mon}(\mathbf{V}/G) \to \mathbf{V}/G$, giving a free monoid functor. We construct explicitly the free monoid functor in appendix B, if the monoidal category satisfies some reasonable conditions. Thus we can assume the cotriple cohomology groups $H^*(\mathcal{G}; A_{\mathbf{Mon}})$ are defined. Suppose further that for $\mathcal{G}$ a free monoid our cohomology groups $H^n(\mathcal{G}; A)$ are trivial for $n > 1$. Then an analysis of the proof of Theorem C of [14] shows that one has isomorphisms

$$H^n(\mathcal{G}; A) \cong H^{n-1}(\mathcal{G}; A_{\mathbf{Mon}}),\ n > 1,$$

and under the assumptions above interpretation of $H^n(\mathcal{G}; A)$ by $K(A_{\mathbf{Mon}}, n-1)$-torsors is valid in all degrees.

Let us begin with degree 0; we give an explicit interpretation generalising that for the abelian case above. For any $A \xrightarrow{p} G$ in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$, let $A^{\mathcal{G}}$ denote the set of *$\mathcal{G}$-invariant elements of A*, that is, $A^{\mathcal{G}}$ is the subset of those morphisms $a \in \mathrm{Hom}_{\mathbf{V}}(I, A)$ satisfying $pa = \eta : I \to G$ and

$$(G \xrightarrow{l_G^{-1}} I \circ G \xrightarrow{a \circ G} A \circ G \xrightarrow{v} A) = (G \xrightarrow{r_G^{-1}} G \circ I \xrightarrow{G \circ a} G \circ A \xrightarrow{u} A).$$

Then inspection of the complex in proposition 3.3 gives

Proposition 4.6. There is a natural bijection $H^0(\mathcal{G}; A) \cong A^{\mathcal{G}}$.

Clearly the $\mathcal{G}$-invariant elements correspond to morphisms from 1_G to $A \xrightarrow{p} G$ in ${}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G$; these are just the $K(A, 0)$-torsors of Duskin.

Turning to degree 1 we make the following definition.

Definition 4.7. For $A \xrightarrow{p} G$ in $\mathbf{Ab}({}^{\mathcal{G}}\mathbf{V}^{\mathcal{G}}/G)$, a *derivation* is a morphism $\Delta : G \to A$ in $\mathbf{V}$ satisfying $p\Delta = 1_G$ and

$$\begin{tikzcd}
G \circ G \arrow[rr, "\mu"] \arrow[d, "{(G \circ \Delta, \Delta \circ G)}"'] & & G \arrow[d, "\Delta"] \\
G \circ A \times A \circ G \arrow[r, "u \times v"] & A \times A \arrow[r, "+"] & A
\end{tikzcd}$$

Write $\mathrm{Der}(\mathcal{G}; A)$ for the set of derivations, and define a map $\mathrm{Inn}() : \mathrm{Hom}_{\mathbf{V}/G}(I \xrightarrow{\eta} G, A \xrightarrow{p} G) \to \mathrm{Der}(G; A)$ by

$$\mathrm{Inn}(I \xrightarrow{a} A) = (G \xrightarrow{(r_G^{-1}, l_G^{-1})} G \circ I \times I \circ G \xrightarrow{G \circ a \times a \circ G} G \circ A \times A \circ G \xrightarrow{u \times v} A \times A \xrightarrow{-} A).$$

Proposition 4.8. There is an exact sequence of abelian groups

$$0 \to H^0(\mathcal{G}; A) \to \mathrm{Hom}_{\mathbf{V}/G}(I \xrightarrow{\eta} G, A \xrightarrow{p} G) \xrightarrow{\mathrm{Inn}()} \mathrm{Der}(\mathcal{G}; A) \to H^1(\mathcal{G}; A) \to 0.$$

Proof. Straightforward, on noting that $\mathrm{Hom}_{\mathbf{V}/G}(I \xrightarrow{\eta} G, A \xrightarrow{p} G) \xrightarrow{\mathrm{Inn}()} \mathrm{Der}(\mathcal{G}; A)$ may be identified with $C^0(\mathcal{G}; A) \xrightarrow{d} \ker(C^1(\mathcal{G}; A) \xrightarrow{d} C^2(\mathcal{G}; A))$. ■

Clearly (4.7) and (4.8) reduce to (4.1) and (4.2) in the abelian situation above, where $A = G \oplus M$.

One readily sees that

$$\mathrm{Der}(\mathcal{G}; A) = \mathrm{Hom}_{\mathbf{Mon}(\mathbb{V})/\mathcal{G}}(1_{\mathcal{G}}, A_{\mathbf{Mon}})$$

whose elements are the $K(A_{\mathbf{Mon}}, 0)$-torsors relative to $U : \mathbf{Mon}(\mathbb{V})/\mathcal{G} \to \mathbf{V}/G$.

For degree two we make the following definition.

Definition 4.9. Let $U : \mathbf{C} \to \mathbf{D}$ be a product-preserving functor between categories with finite products, and let A be an internal group object in $\mathbf{C}$. An *A-torsor* relative to U is an object T of $\mathbf{C}$ together with

- morphisms
$$T \times A \xrightarrow{\ +\ } T, \qquad T \times T \xrightarrow{\ -\ } A$$
in $\mathbf{C}$, such that $+$ is a right action and the morphisms $(p_1, +) : T \times A \to T \times T$, $(p_1, -) : T \times T \to T \times A$ are mutually inverse isomorphisms, and
- a morphism $s : 1 \to U(T)$ where 1 is the terminal object in $\mathbf{D}$.

As in Duskin [9, section 3] the A-torsors relative to U can be identified with the $K(A, 1)$-torsors relative to U.

For A-torsors with $A = A_{\mathbf{Mon}}$ as above we now show

Proposition 4.10. There is a one-to-one correspondence between $H^2(\mathcal{G}; A)$ and the set of isomorphism classes of $A_{\mathbf{Mon}}$-torsors relative to the forgetful functor $U : \mathbf{Mon}(\mathbb{V})/\mathcal{G} \to \mathbf{V}/G$.

More explicitly, an $A_{\mathbf{Mon}}$-torsor relative to the forgetful functor U in 4.10 is a $\mathbb{V}$-monoid T, equipped with monoid homomorphisms

$$p : T \to G, \qquad + : T \times_G A \to T, \qquad - : T \times_G T \to A$$

with properties as above, and a section $s : G \to T$, $ps = 1_G$, in $\mathbf{V}$. A morphism of torsors is a monoid homomorphism respecting p, $+$ and $-$.

Proof. Given an $A_{\mathbf{Mon}}$-torsor T with s as above, assign to it the map

$$f_T = (G \circ G \xrightarrow{(f_1, f_2)} T \times_G T \xrightarrow{-} A),$$

where $f_1 = (G \circ G \xrightarrow{\mu_G} G \xrightarrow{s} T)$ and $f_2 = (G \circ G \xrightarrow{s \circ s} T \circ T \xrightarrow{\mu_T} T)$. One checks easily that f_T is a cocycle, that a different choice of s would give a cohomologous cocycle, and any morphism $T_1 \to T_2$ of torsors produces a 1-cochain whose coboundary is equal to $f_{T_1} - f_{T_2}$.

Conversely, for a 2-cocycle $f : G \circ G \to A$, define a new $\circ$-monoid multiplication on A by

$$\mu_f = (A \circ A \xrightarrow{(1_{A \circ A}, p \circ p)} A \circ A \times_G G \circ G \xrightarrow{\mu \times f} A \times_G A \xrightarrow{+} A).$$

One then checks that this together with $+ : A \times_G A \to A$, $- : A \times_G A \to A$ defines a $A_{\mathbf{Mon}}$-torsor T_f, and cohomologous cocycles yield isomorphic torsors.

Finally, it is straightforward to check that any torsor T is isomorphic to T_{f_T} and any cocycle f is cohomologous to f_{T_f}. ■

Examples 4.11. In the example of categories, 1.11, one easily sees that the $A_{\mathbf{Mon}}$-torsors correspond exactly to linear extensions of categories from [5] so that (4.10) corresponds to the result of [5] that the elements of the second cohomology of a category $\mathbf{C}$ classify linear extensions of $\mathbf{C}$. In the example 1.8 one recovers extensions of theories from [14].

Note that in these examples there are also interpretations of the third cohomology, see [2, 13, 26], for example in terms of linear track extensions of categories. These suggest that at least in the presence of a free monoid functor there is an interpretation of $H^3(\mathcal{G};A)$ by $K(A_{\mathbf{Mon}},2)$-torsors. In fact we might expect there to be an explicit correspondence between $K(A,n)$-torsors and $K(A_{\mathbf{Mon}},n-1)$-torsors, without appealing to cocycles.

Appendix A. Proof of subdivision lemma 3.5

Proof. First of all, recall $(d^i_{\mathrm{Sub}(A)}:\mathrm{Sub}(A)^{n-1}\to\mathrm{Sub}(A)^n)=(d_A^{2n+1-i}d_A^i:A^{2n-1}\to A^{2n+1})$. There is a cosimplicial morphism $f:A^\bullet\to\mathrm{Sub}(A^\bullet)$ defined by $f_n=d^{2n+1}d^{2n}\cdots d^{n+1}:A^n\to A^{2n+1}$, which induces the map of the corresponding cochain complexes. We will construct its homotopy inverse g by induction. Put

$g_0=s^0:A^1\to A^0$,

$g_n=(1\sqcup g_{n-1})s^{2n}+(-1)^ns^0(1\sqcup g_{n-1}\sqcup 1)$,

where $1\sqcup(-)$, resp. $(-)\sqcup 1$, is induced by the functor $\Delta\to\Delta$ adding to a finite linear order an extra smallest, resp. greatest, element. So, $1\sqcup(d^i:A^{k-1}\to A^k)=(d^{i+1}:A^k\to A^{k+1})$, $1\sqcup(s^i:A^k\to A^{k-1})=(s^{i+1}:A^{k+1}\to A^k)$. As for $(-)\sqcup 1$, it does not affect anything on the formal level; we will take advantage of this by not mentioning this functor at all.

First let us prove that g is compatible with differentials. Now for the differential $d_n: A^{n-1}\to A^n$ one has

$$d_n=\sum_{i=0}^n(-1)^id^i=d^0-1\sqcup d_{n-1}$$

and similarly $d'_n:\mathrm{Sub}(A)^{n-1}\to\mathrm{Sub}(A)^n$ is given by

$$d'_n=d^{2n+1}d^0-(1\sqcup d'_{n-1}):A^{2n-1}\to A^{2n+1}.$$

We have to prove $g_nd'_n=d_ng_{n-1}$. Starting with $n=1$, $g_1d'_1=d_1g_0$, one checks directly $(s^1s^2-s^0s^1)(d^3d^0-d^2d^1)=(d^0-d^1)s^0$. Now given $g_{n-1}d'_{n-1}=d_{n-1}g_{n-2}$, one has

$$\begin{aligned}g_nd'_n&=(1\sqcup g_{n-1})s^{2n}+(-1)^ns^0(1\sqcup g_{n-1})(d^{2n+1}d^0-1\sqcup d'_{n-1})\\&=(1\sqcup g_{n-1})d^0+(-1)^ns^0(1\sqcup g_{n-1})d^{2n+1}d^0-(1\sqcup g_{n-1})s^{2n}(1\sqcup d'_{n-1})\\&\quad-(-1)^ns^0(1\sqcup g_{n-1})(1\sqcup d'_{n-1}).\end{aligned}$$

Now one easily sees that $(1\sqcup x)d^0=d^0x$ for any x whatsoever, in particular $(1\sqcup g_{n-1})d^0=d^0g_{n-1}$. In fact all the summands in g_{n-1} are composites of n entries of type s^i, with $i\leqslant 2n-2$, hence one also has $(1\sqcup g_{n-1})d^{2n+1}=d^{n+1}(1\sqcup g_{n-1})$. Using this, $s^0(1\sqcup g_{n-1})d^{2n+1}d^0=s^0d^{n+1}(1\sqcup g_{n-1})d^0=s^0d^{n+1}d^0g_{n-1}=s^0d^0d^ng_{n-1}=d^ng_{n-1}$. Also $(1\sqcup g_{n-1})(1\sqcup d'_{n-1})=1\sqcup(g_{n-1}d'_{n-1})=1\sqcup(d_{n-1}g_{n-2})$, by the induction hypothesis. Taking all this into account gives

$$g_nd'_n=d^0g_{n-1}+(-1)^nd^ng_{n-1}-(1\sqcup g_{n-1})s^{2n}(1\sqcup d'_{n-1})-(-1)^ns^0(1\sqcup d_{n-1})(1\sqcup g_{n-2}).$$

Now turning to $d_n g_{n-1}$, one has

$$\begin{aligned}
d_n g_{n-1} &= (d^0 - 1 \sqcup d_{n-1}) g_{n-1} \\
&= d^0 g_{n-1} - (1 \sqcup d_{n-1})((1 \sqcup g_{n-2})s^{2n-2} + (-1)^{n-1} s^0 (1 \sqcup g_{n-2})) \\
&= d^0 g_{n-1} - (1 \sqcup d_{n-1})(1 \sqcup g_{n-2})s^{2n-2} - (-1)^{n-1}(1 \sqcup d_{n-1}) s^0 (1 \sqcup g_{n-2})) \\
&= d^0 g_{n-1} - (1 \sqcup g_{n-1})(1 \sqcup d'_{n-1})s^{2n-2} + (-1)^{n}(1 \sqcup d_{n-1}) s^0 (1 \sqcup g_{n-2})).
\end{aligned}$$

Comparing these two expressions one gets

$$\begin{aligned}
g_n d'_n - d_n g_{n-1} = (-1)^n d^n g_{n-1} - (1 \sqcup g_{n-1})(s^{2n}(1 \sqcup d'_{n-1}) - (1 \sqcup d'_{n-1})s^{2n-2}) \\
- (-1)^n (s^0(1 \sqcup d_{n-1}) + (1 \sqcup d_{n-1})s^0)(1 \sqcup g_{n-2}).
\end{aligned}$$

Now recalling the formulæ for d_n and d'_n one easily gets
$s^{2n}(1 \sqcup d'_{n-1}) - (1 \sqcup d'_{n-1})s^{2n-2} = d^1(1 - d^{2n-1}s^{2n-2})$,
$s^0(1 \sqcup d_{n-1}) + (1 \sqcup d_{n-1})s^0 = 1 - (-1)^n d^n s^0$.
Hence

$$g_n d'_n - d_n g_{n-1} = (-1)^n d^n g_{n-1} - (1 \sqcup g_{n-1}) d^1 (1 - d^{2n-1}s^{2n-2}) - (-1)^n (1 - (-1)^n d^n s^0)(1 \sqcup g_{n-2}).$$

Now substituting
$g_{n-1} = (1 \sqcup g_{n-2})s^{2n-2} + (-1)^{n-1}s^0(1 \sqcup g_{n-2})$,
$1 \sqcup g_{n-1} = (1 \sqcup 1 \sqcup g_{n-2})s^{2n-1} + (-1)^{n-1}s^1(1 \sqcup 1 \sqcup g_{n-2})$
one gets

$$\begin{aligned}
g_n d'_n - d_n g_{n-1} = (-1)^n d^n (1 \sqcup g_{n-2})s^{2n-2} + (-1)^n(-1)^{n-1} d^n s^0 (1 \sqcup g_{n-2}) \\
- (1 \sqcup 1 \sqcup g_{n-2})s^{2n-1} d^1 (1 - d^{2n-1}s^{2n-2}) - (-1)^{n-1}s^1(1 \sqcup 1 \sqcup g_{n-2}) d^1 (1 - d^{2n-1}s^{2n-2}) \\
- (-1)^n (1 \sqcup g_{n-2}) + (-1)^n(-1)^n d^n s^0 (1 \sqcup g_{n-2}).
\end{aligned}$$

Now as before, $d^n(1 \sqcup g_{n-2}) = (1 \sqcup g_{n-2})d^{2n-1}$, $(1 \sqcup 1 \sqcup g_{n-2})d^1 = d^1(1 \sqcup g_{n-2})$, so we arrive at

$$\begin{aligned}
(-1)^n (1 \sqcup g_{n-2}) d^{2n-1}s^{2n-2} - d^1(1 \sqcup g_{n-2})s^{2n-2}(1 - d^{2n-1}s^{2n-2}) \\
- (-1)^{n-1}(1 \sqcup g_{n-2})(1 - d^{2n-1}s^{2n-2}) - (-1)^n (1 \sqcup g_{n-2})
\end{aligned}$$

and this easily leads to zero.

We now turn to construction of homotopies from fg and gf to the identity morphisms. First note that similarly to g, also f has an inductive definition, $f_0 = d^1$, $f_n = d^{2n+1}(1 \sqcup f_{n-1})$. Using this fact we also determine inductively $e = gf : A^\bullet \to A^\bullet$. It has
$e_0 = s^0 d^1 = 1$,
$e_n = g_n f_n = ((1 \sqcup g_{n-1})s^{2n} + (-1)^n s^0 (1 \sqcup g_{n-1}))d^{2n+1}(1 \sqcup f_{n-1})$
$= 1 \sqcup e_{n-1} + (-1)^n s^0 d^{n+1}(1 \sqcup e_{n-1}) = (1 + (-1)^n d^n s^0)(1 \sqcup e_{n-1})$.

This implies

$$e_n = (1 + (-1)^n d^n s^0)(1 - (-1)^n d^n s^1) \cdots (1 + d^n s^{n-2})(1 - d^n s^{n-1}),$$

so that one may write $e_n = 1 + (-1)^n d^n h_n$, with $h_0 = 0 : A^0 \to$(trivial group), and $h_n : A^n \to A^{n-1}$, $n \geqslant 1$. We will show that h is a homotopy from e to the identity, i. e. that $h_{n+1}d_{n+1} + d_n h_n = e_n - 1$. First let us produce an inductive expression for h_n:

$$(-1)^n d^n h_n = e_n - 1 = (1 + (-1)^n d^n s^0)(1 \sqcup (1 + (-1)^{n-1} d^{n-1} h_{n-1})) - 1$$
$$= (1+(-1)^n d^n s^0)(1-(-1)^n 1 \sqcup (d^{n-1}h_{n-1})) - 1 = (1+(-1)^n d^n s^0)(1-(-1)^n d^n (1 \sqcup h_{n-1})) - 1$$
$$= -(-1)^n d^n (1 \sqcup h_{n-1}) + (-1)^n d^n s^0 - d^n s^0 d^n (1 \sqcup h_{n-1})$$
$$= (-1)^n d^n (-1 \sqcup h_{n-1} + s^0 - (-1)^n d^{n-1} s^0 (1 \sqcup h_{n-1})),$$

i. e.

$$h_n = -1 \sqcup h_{n-1} + s^0 - (-1)^n d^{n-1} s^0 (1 \sqcup h_{n-1}).$$

We now proceed by induction. For $n = 0$, $h_1 d_1 = s^0(d^0 - d^1) = 0 = e_0 - 1$. Now given $h_n d_n + d_{n-1}h_{n-1} = e_{n-1} - 1 = (-1)^{n-1}d^{n-1}h_{n-1}$, we must deduce $h_{n+1}d_{n+1} + d_n h_n = e_n - 1 = (-1)^n d^n h_n$. Moving summands around, this means, given $h_n d_n = ((-1)^{n-1}d^{n-1} - d_{n-1})h_{n-1}$, one must deduce $h_{n+1}d_{n+1} = ((-1)^n d^n - d_n)h_n$. One has

$$h_{n+1}d_{n+1} = h_{n+1}d^0 - h_{n+1}(1 \sqcup d_n) = (-1 \sqcup h_n + s^0 - (-1)^{n+1} d^n s^0 (1 \sqcup h_n))d^0 - h_{n+1}(1 \sqcup d_n)$$
$$= -(1 \sqcup h_n)d^0 + 1 - (-1)^{n+1} d^n s^0 (1 \sqcup h_n) d^0 - h_{n+1}(1 \sqcup d_n)$$
$$= -d^0 h_n + 1 - (-1)^{n+1} d^n h_n - h_{n+1}(1 \sqcup d_n)$$

and

$$((-1)^n d^n - d_n)h_n = (-1)^n d^n h_n - (d^0 - 1 \sqcup d_{n-1})h_n = (-1)^n d^n h_n - d^0 h_n + (1 \sqcup d_{n-1})h_n.$$

Comparing these two expressions we see that we have to prove

$$1 - h_{n+1}(1 \sqcup d_n) = (1 \sqcup d_{n-1})h_n.$$

The left hand side expands to

$$1-(-1 \sqcup h_n + s^0 - (-1)^{n+1} d^n s^0 (1 \sqcup h_n))(1 \sqcup d_n) = 1 + 1 \sqcup h_n d_n - s^0(1 \sqcup d_n) + (-1)^{n+1} d^n s^0 (1 \sqcup h_n d_n);$$

we now use the induction hypothesis and the obvious identity $-s^0(1 \sqcup d_n) = -1 + (1 \sqcup d_{n-1})s^0$ to obtain

$$-1 \sqcup d_{n-1}h_{n-1} + (-1)^{n-1}(1 \sqcup d^{n-1}h_{n-1}) + (1 \sqcup d_{n-1})s^0 + (-1)^{n+1} d^n s^0 (1 \sqcup h_n d_n).$$

Whereas on the right we have

$$(1 \sqcup d_{n-1})h_n = -1 \sqcup d_{n-1}h_{n-1} + (1 \sqcup d_{n-1})s^0 - (-1)^n (1 \sqcup d_{n-1}) d^{n-1} s^0 (1 \sqcup h_{n-1}).$$

Comparing again, we are left with

$$(-1)^{n-1}(1 \sqcup d^{n-1}h_{n-1}) + (-1)^{n+1} d^n s^0 (1 \sqcup h_n d_n) = -(-1)^n (1 \sqcup d_{n-1}) d^{n-1} s^0 (1 \sqcup h_{n-1})$$

to prove, i. e.

$$d^n(1 \sqcup h_{n-1}) + d^n s^0 (1 \sqcup h_n d_n) = (1 \sqcup d_{n-1}) d^{n-1} s^0 (1 \sqcup h_{n-1}).$$

Once again using the induction hypothesis, the left hand side is

$$d^n(1 \sqcup h_{n-1}) + (-1)^{n-1} d^n s^0 (1 \sqcup d^{n-1} h_{n-1}) - d^n s^0 (1 \sqcup d_{n-1} h_{n-1}),$$

or

$$d^n(1 \sqcup h_{n-1}) + (-1)^{n-1} d^n s^0 d^n (1 \sqcup h_{n-1}) - d^n s^0 (1 \sqcup d_{n-1})(1 \sqcup h_{n-1}),$$

so it suffices to prove

$$d^n + (-1)^{n-1} d^n s^0 d^n - d^n s^0 (1 \sqcup d_{n-1}) = (1 \sqcup d_{n-1}) d^{n-1} s^0,$$

and this is straightforward.

Finally, we construct a homotopy between $e'_n = f_n g_n$ and the identity. Therefore, introduce some auxiliary notation: $c_n = 1 \sqcup f_{n-1} = d^{2n} d^{2n-1} \cdots d^{n+1}$. So $c_1 = d^2$, $c_n = d^{2n}(1 \sqcup c_{n-1})$, and $d^{2n+1} c_n = f_n$. We now define

$h'_1 = s^0 s^1 : A^3 \to A^1$,

$h'_n = -1 \sqcup h'_{n-1} + c_{n-1} s^0 (1 \sqcup g_{n-1}) : A^{2n+1} \to A^{2n-1}$

and prove that h'_n, considered as maps $\mathrm{Sub}(A)^{n-1} \to \mathrm{Sub}(A)^n$, constitute a homotopy between e'_n and the identity, i. e. $h'_{n+1} d'_{n+1} + d'_n h'_n = e'_n - 1$. For $n = 0$ this means $s^0 s^1 (d^3 d^0 - d^2 d^1) = d^1 s^0 - 1$. Further by induction: given $h'_n d'_n + d'_{n-1} h'_{n-1} = e'_{n-1} - 1$, one has

$$\begin{aligned} h'_{n+1} d'_{n+1} &= (-1 \sqcup h'_n + c_n s^0 (1 \sqcup g_n))(d^{2n+3} d^0 - 1 \sqcup d'_n) \\ &= -(1 \sqcup h'_n) d^{2n+3} d^0 + c_n s^0 (1 \sqcup g_n) d^{2n+3} d^0 - c_n s^0 (1 \sqcup g_n d'_n) + 1 \sqcup h'_n d'_n; \end{aligned}$$

As we noted before, $(1 \sqcup x) d^0 = d^0 x$; since g is a morphism of complexes, $g_n d'_n = d_n g_{n-1}$; and $h'_n d'_n = -d'_{n-1} h'_{n-1} - 1 + d^{2n-1} c_{n-1} g_{n-1}$ by the induction hypothesis. Hence one obtains

$$h'_{n+1} d'_{n+1} = -d^0 h'_n d^{2n+2} + c_n g_n d^{2n+2} - c_n s^0 (1 \sqcup d_n g_{n-1}) - 1 \sqcup d'_{n-1} h'_{n-1} - 1 + 1 \sqcup d^{2n-1} c_{n-1} g_{n-1}.$$

Similarly

$$\begin{aligned} d'_n h'_n &= (d^{2n+1} d^0 - 1 \sqcup d'_{n-1}) h'_n \\ &= d^{2n+1} d^0 h'_n - (1 \sqcup d'_{n-1})(-1 \sqcup h'_{n-1} + c_{n-1} s^0 (1 \sqcup g_{n-1})) \\ &= d^0 d^{2n} h'_n + 1 \sqcup d'_{n-1} h'_{n-1} - (1 \sqcup d'_{n-1}) c_{n-1} s^0 (1 \sqcup g_{n-1}). \end{aligned}$$

Collecting these together, one sees that the thing to prove is

$$\begin{aligned} d^0 d^{2n} h'_n - d^0 h'_n d^{2n+2} + c_n g_n d^{2n+2} - c_n s^0 (1 \sqcup d_n g_{n-1}) + c_n (1 \sqcup g_{n-1}) - (1 \sqcup d'_{n-1}) c_{n-1} s^0 (1 \sqcup g_{n-1}) \\ = d^{2n+1} c_n g_n. \end{aligned}$$

Now an easy inductive argument shows that $h'_n d^{2n+2} = d^{2n} h'_n$; we saw before that $g_n d^{2n+2} = d^{n+1} g_n$; and trivially $c_n d^{n+1} = d^{2n+1} c_n$. All this leaves us with

$$-c_n s^0 (1 \sqcup d_n g_{n-1}) + c_n (1 \sqcup g_{n-1}) - (1 \sqcup d'_{n-1}) c_{n-1} s^0 (1 \sqcup g_{n-1}) = 0$$

to prove. For that, it is sufficient to omit $(1 \sqcup g_{n-1})$ on the right, obtaining

$$-c_n s^0 (1 \sqcup d_n) + c_n - (1 \sqcup d'_{n-1}) c_{n-1} s^0 = 0;$$

and since, as we noted earlier, $s^0(1 \sqcup d_n) = 1 - (1 \sqcup d_{n-1})s^0$, this amounts to

$$c_n(1 \sqcup d_{n-1}) = (1 \sqcup d'_{n-1})c_{n-1}.$$

And recalling that $c_n = 1 \sqcup f_{n-1}$, this just expresses the fact that f is a morphism of complexes. ■

Appendix B. Free monoids

Let $(\mathbf{C}, \square, I)$ be a monoidal category in which the monoid operation $\square$ is left distributive over coproducts $\sqcup$ and preserves filtered colimits. In this case we are going to define an explicit free monoid functor which is the left adjoint of the forgetful functor

$$\mathbf{Mon(C)} \xrightarrow{U} \mathbf{C}$$

If $\mathbf{C} = R$-**Mod** then the free monoid on $V \in \mathbf{C}$ is the classical tensor algebra $T(V)$. The assumptions on $\mathbf{C}$ also hold for the monoidal category $\mathbf{C} = \mathbf{Cat}(\mathfrak{S}, R\text{-}\mathbf{Mod})$ in which monoids are operads. In this case the free monoid is the free operad on an $\mathfrak{S}$-object in R-**Mod** which is used for the definition of the bar construction of operads in [18].

Let V be an object of $\mathbf{C}$ and define a sequence of objects V_n by $V_0 = I$ and inductively $V_{n+1} = I \sqcup V \square V_n$. The first few terms are:

$$V_0 = I, \qquad V_1 = I \sqcup V, \qquad V_2 = I \sqcup V \square (I \sqcup V), \qquad V_3 = I \sqcup V \square (I \sqcup V \square (I \sqcup V)), \quad \ldots$$

There are maps $i_n : V_{n-1} \to V_n$ given inductively by $i_{n+1} = 1 \sqcup 1 \square i_n$, with $i_1 : I \to I \sqcup V$ the natural inclusion of the summand. We define V_∞ by the colimit

$$V_\infty \;=\; \operatorname{colim}(V_0 \to V_1 \to V_2 \to V_3 \to \cdots)$$

We will write i for any of the maps $V_n \to V_m$ for $n < m \leqslant \infty$.

There are also maps $\mu_{n,m} : V_n \square V_m \longrightarrow V_{n+m}$ as follows. Let $\mu_{0,m} = 1_{V_m}$. If $n \geqslant 1$ then $V_n \square V_m = (I \sqcup V \square V_{n-1}) \square V_m = V_m \sqcup V \square V_{n-1} \square V_m$ and we define $\mu_{n,m}$ inductively by

$$V_n \square V_m = V_m \sqcup V \square V_{n-1} \square V_m \xrightarrow{\mu_{n,m} = (i,\, j_{n+m}(1 \square \mu_{n-1,m}))} V_{n+m}$$

Here $j_k : V \square V_{k-1} \to V_k = I \sqcup V \square V_{k-1}$ is the inclusion of the direct summand.

Proposition B.1. Suppose the tensor product $\square$ in $\mathbf{C}$ is left distributive over coproducts and preserves filtered colimits, and let V be an object of $\mathbf{C}$. Then the free monoid on V is $T(V) = (V_\infty, \eta, \mu)$, with unit η given by the map $i : I = V_0 \to T(V)$ and multiplication $\mu : T(V) \square T(V) \to T(V)$ induced by the maps $i\mu_{n,m} : V_n \square V_m \to T(V)$.

We also write $T_{\leqslant n}(V)$ for V_n. Note that for $\mathbf{C} = R$-**Mod** the category of R-modules the tensor product is distributive on both sides and we have $T_{\leqslant n}(V) = \bigoplus_{k \leqslant n} V^{\otimes k}$. In this situation the maps i_n are the natural inclusions of summands, and the multiplication structure is given by the isomorphisms $V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes(n+m)}$.

Proof. To show that the multiplication is well defined on the colimit we need the relations $\mu_{n+1,m-1}(i_{n+1} \square 1) = i_k\mu_{n,m-1} = \mu_{n,m}(1 \square i_m)$ where $k = n + m$. For $n = 0$ this becomes $(i_m, j_m)(i_1 \square 1) = i_m = I \square i_m$. For $n \geqslant 1$ we have

$$\begin{aligned}
\mu_{n+1,m-1}(i_{n+1} \square 1) &= (i, j_k(1 \square \mu_{n,m-1}))(1 \sqcup 1 \square i_n \square 1) &= (i, j_k(1 \square \mu_{n,m-1}(i_n \square 1))) \\
\mu_{n,m}(1 \square i_m) &= (i, j_k(1 \square \mu_{n-1,m}))(i_m \sqcup 1 \square i_m) &= (i, j_k(1 \square \mu_{n-1,m}(1 \square i_m)))
\end{aligned}$$

which are both equal to $(i, j_k(1 \square i_{k-1})(1 \square \mu_{n-1,m-1}))$ by the inductive hypothesis. Since $j_k(1 \square i_{k-1}) = i_k j_{k-1} : V \square V_{k-2} \to V_k$ this is just $(i, i_k j_{k-1}(1 \square \mu_{n-1,m-1}))$ which equals $i_k\mu_{n,m-1}$ as required. For the identity laws $\mu(\eta \square 1) = 1 = \mu(1 \square \eta)$ we note that $\mu_{0,m} = 1 = \mu_{n,0}$, where $\mu_{n,0} = 1$ follows inductively from the fact that (i, j_n) is the identity on $V_n \square V_0 = I \sqcup V \square V_{n-1}$. For the associative law we note that $j_{n+m}(1 \square \mu_{n-1,m}) = \mu_{n,m}(j_n \square 1) : V \square V_{n-1} \square V_m \to V_{n+m}$, and $i\mu_{q,r} = \mu_{p+q,r}(i \square 1)$ as above, so that we have inductively

$$\begin{aligned}
\mu_{p,q+r}(1 \square \mu_{q,r}) &= (i, j_{p+q+r}(1 \square \mu_{p-1,q+r}))(\mu_{q,r} \sqcup 1 \square \mu_{q,r}) \\
&= (i\mu_{q,r},\ j_{p+q+r}(1 \square \mu_{p-1,q+r}(1 \square \mu_{q,r}))) \\
&= (i\mu_{q,r},\ j_{p+q+r}(1 \square \mu_{p+q-1,r})(1 \square \mu_{p-1,q} \square 1)) \\
&= \mu_{p+q,r}((i \square 1), (j_{p+q} \square 1)(1 \square \mu_{p-1,q} \square 1)) \\
&= \mu_{p+q,r}(\mu_{p,q} \square 1)
\end{aligned}$$

This construction is functorial. If $f : V \to W$ is a morphism in $\mathbf{C}$ then $T(f)$ is defined by maps $f_n : V_n \to W_n$ where $f_0 = 1_I$ and $f_n = 1 \sqcup f \square f_{n-1}$. The map $T(f)$ is well defined since $i_n f_{n-1} = f_n i_n$ is clear inductively. Using this and $j_k(f \square f_{k-1}) = f_k j_k$ we have

$$f_{n+m}\mu_{n,m} = f_{n+m}(i, j_{n+m}(1 \square \mu_{n-1,m})) = (if_m, j_{n+m}(f \square f_{n+m-1}\mu_{n-1,m}))$$

which if $f_{n+m-1}\mu_{n-1,m} = \mu_{n-1,m}(f_{n-1} \square f_m)$ becomes $(i, j_{n+m}(1 \square \mu_{n-1,m}))(f_m \sqcup f \square f_{n-1} \square f_m)$ which is just $\mu_{n,m}(f_n \square f_m)$. By induction $T(f)$ is thus a monoid homomorphism.

There is a natural monoid homomorphism $\phi_A : T(A) \longrightarrow A$ for (A, η_A, μ_A) a monoid in $\mathbf{C}$ defined as follows. Let $\phi_0 = \eta_A$ and $\phi_n = (\eta_A, \mu_A(1 \square \phi_{n-1}))$ for $n \geqslant 1$. Then $\phi_1 i_1 = \eta_A = \phi_0$, and $\phi_{n+1}i_{n+1} = (\eta_A, \mu_A(1 \square \phi_n i_n)) = \phi_n$ if $\phi_n i_n = \phi_{n-1}$, so the ϕ_n give a well-defined ϕ_A on $T(A) = A_\infty$. Clearly $\phi_A\eta = \eta_A$. By the unit and associativity laws for A and by the relations $\phi_{n+m}i = \phi_m$ and $\phi_k j_k = \mu_A(1 \square \phi_{k-1})$ we have

$$\begin{aligned}
\mu_A(\phi_n \square \phi_m) &= \mu_A((\eta_A, \mu_A(1 \square \phi_{n-1})) \square \phi_m) &= (\phi_m, \mu_A(1 \square \mu_A(\phi_{n-1} \square \phi_m))) \\
\phi_{n+m}\mu_{n,m} &= (\phi_{n+m}i, \phi_{n+m}j_{n+m}(1 \square \mu_{n-1,m})) &= (\phi_m, \mu_A(1 \square \phi_{n+m-1})(1 \square \mu_{n-1,m}))
\end{aligned}$$

Thus $\mu_A(\phi_A \square \phi_A) = \phi_A\mu$ follows inductively and ϕ_A is a monoid homomorphism. For any object V of $\mathbf{C}$ we also have a natural map $\psi_V = ij_1 : V \to T(V)$. The freeness of $T(V)$ will now follow from showing that the composites

$$A \xrightarrow{\psi_A} T(A) \xrightarrow{\phi_A} A \qquad\qquad T(V) \xrightarrow{T(\psi_V)} T(T(V)) \xrightarrow{\phi_{T(V)}} T(V)$$

are the identity. The first of these is clear: $\phi_A\psi_A = \phi_A ij_1 = \phi_1 j_1 = \mu_A(1 \square \eta_A) = 1$. Consider the maps $(\psi_V)_n : T_{\leqslant n}(V) \to T_{\leqslant n}(T(V))$ and $\phi_n : T_{\leqslant n}(T(V)) \to T(V)$ which define

$T(\psi_V)$ and $\phi_{T(V)}$. Then $\phi_0(\psi_V)_0 = \eta = i : I \to T(V)$, and assuming inductively that $\phi_{n-1}(\psi_V)_{n-1} = i : V_{n-1} \to T(V)$ we have

$$\phi_n(\psi_V)_n = (\eta, \mu(1 \square \phi_{n-1}))(1 \sqcup (ij_1) \square (\psi_V)_{n-1}) = (\eta, i\mu_{1,n-1}(j_1 \square 1)) = i(i, j_n) = i$$

since $\mu_{1,n-1}(j_1 \square 1) = j_n : V \square V_{n-1} \to V_n$. Thus $\phi_{T(V)}T(\psi_V) = 1$. ∎

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