ON THE FOURTH ADJOINT CONTRACTIONS OF DIVISORIAL AND FIBER TYPES

SHU GILBERT NAKAMURA

ABSTRACT. In this article, we give a rough classification on the ray contractions of divisorial and fiber types for nonsingular projective varieties of dimension five over the complex numbers. In order to begin the geometry of this dimension, it is inevitable to show that a general positive-dimensional fiber is irreducible for the divisorial type. This was once claimed in [2], but there was a gap in his proof. In our proof, we introduce completely a different method which reduces the difficulty of proof to Fujita's sectional genus conjecture (see Conjecture 1.8). This conjecture has been proven in [6] only for normal varieties of dimension ≤ 3 by virtue of the 3-dimensional Minimal Model Program of [13]. We employ the concept of adjoint contraction (see Definition 1.3) as the technical frame work which is convenient in clarifying things, in particular, observing the necessity of restriction to the dimension five. (We do not show the existence of the cases.)

1. INTRODUCTION AND MAIN RESULTS

Let X be a 5-dimensional smooth projective variety over the complex number field $\mathbb{C}$. Assume that a canonical divisor K_X of X is not nef. Then, Mori Theory ([10], [12]) provides an *extremal ray* R of X which is defined by $R = H^{\perp} \cap \overline{NE}(X)$, where H is a nef divisor of X (called a *supporting divisor* of R) and $\overline{NE}(X)$ is the cone of curves of X, such that for any $\gamma \in R\backslash\{0\}$, $K_X \cdot \gamma < 0$ and that $\dim_{\mathbb{R}} R \otimes \mathbb{R} = 1$. Moreover, the linear system $\mid mH \mid$, $m >> 0$, gives rise to a morphism $f : X \to Y$ onto a normal projective variety Y with only connected fibers such that $f(C)$ is a point if and only if the numerical class $[C]$ is in R (or equivalently, $H \cdot C = 0$) for any integral curve C in X, and that $H = f^*\mathcal{A}$ for some ample Cartier divisor $\mathcal{A}$ of Y. We call (the isomorphism class of) f the *ray contraction* of R. If the dimension of the exceptional locus of f is equal to $\dim X - 1$ or $\dim X$, f is said to be *of divisorial type*, *of fiber type*, respectively. The main result in this paper follows:

Theorem 1.1. *The tables* **A** *and* **B** *list all the possible cases for the ray contractions $f : X \to Y$ of divisorial and fiber types for smooth five folds X.*
A. *Let $f : X \to Y$ be of divisorial type. Let F denote a general fiber of $E \to f(E)$. If $\dim f(E) = 1$ and $F \cong \mathbb{P}^3$, then F can be any fiber of $E \to f(E)$ over a smooth point of $f(E)$.*

$\dim f(E)$	*structure of a positive-dimensional fiber of f*	*singularities of Y*
0	*E is a Mukai variety and $K_X\|_E \cong \mathcal{N}_{E/X}$*	*1-factorial, terminal*
	E is a Del Pezzo variety and $2K_X\|_E \cong \mathcal{N}_{E/X}$	*2-factorial, terminal*
	E is a Del Pezzo variety and $K_X\|_E \cong 2\mathcal{N}_{E/X}$	*1-factorial, terminal*
	$E \cong \boldsymbol{Q}^4, K_X\|_E \cong \mathcal{O}_{\boldsymbol{Q}}(-1),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\boldsymbol{Q}}(-3)$	*3-factorial, terminal*
	$E \cong \boldsymbol{Q}^4, K_X\|_E \cong \mathcal{N}_{E/X} \cong \mathcal{O}_{\boldsymbol{Q}}(-2)$	*2-factorial, terminal*
	$E \cong \boldsymbol{Q}^4, K_X\|_E \cong \mathcal{O}_{\boldsymbol{Q}}(-3),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\boldsymbol{Q}}(-1)$	*1-factorial, terminal*
	$E \cong \mathbb{P}^4, K_X\|_E \cong \mathcal{O}_{\mathbb{P}}(-1),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}}(-4)$	*4-factorial, terminal*
	$E \cong \mathbb{P}^4, K_X\|_E \cong \mathcal{O}_{\mathbb{P}}(-2),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}}(-3)$	*3-factorial, terminal*
	$E \cong \mathbb{P}^4, K_X\|_E \cong \mathcal{O}_{\mathbb{P}}(-3),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}}(-2)$	*2-factorial, terminal*
	$E \cong \mathbb{P}^4, K_X\|_E \cong \mathcal{O}_{\mathbb{P}}(-4),$ *and* $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}}(-1)$	*smooth*

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1	F^3 *is a Del Pezzo variety and* $K_X\|_F \cong \mathcal{N}_{E/X}\|_F$	*1-factorial, terminal*
	$F \cong \boldsymbol{Q}^3, K_X\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-1),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-2)$	*2-factorial, terminal*
	$F \cong \boldsymbol{Q}^3, K_X\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-2),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-1)$	*1-factorial, terminal*
	$F \cong \mathbb{P}^3, K_X\|_F \cong \mathcal{O}_{\mathbb{P}}(-1),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-3)$	*3-factorial, terminal*
	$F \cong \mathbb{P}^3, K_X\|_F \cong \mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-2)$	*2-factorial, terminal*
	$F \cong \mathbb{P}^3, K_X\|_F \cong \mathcal{O}_{\mathbb{P}}(-3),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-1)$	*1-factorial, terminal*
2	$F \cong \boldsymbol{Q}^2, K_X\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-1),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-1)$	*1-factorial, terminal*
	$F \cong \mathbb{P}^2, K_X\|_F \cong \mathcal{O}_{\mathbb{P}}(-1),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-2)$	*2-factorial, terminal*
	$F \cong \mathbb{P}^2, K_X\|_F \cong \mathcal{O}_{\mathbb{P}}(-2),$ *and* $\mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-1)$	*1-factorial, terminal*
3	$F \cong \mathbb{P}^1, K_X\|_F \cong \mathcal{N}_{E/X}\|_F \cong \mathcal{O}_{\mathbb{P}}(-1)$	*1-factorial, terminal*

B. *Let $f : X \to Y$ be of fiber type. Let F denote a general fiber of f.*

$\dim Y$	*structure of a positive-dimensional fiber of f*	*singularities of Y*
0	*X is a Fano manifold of co-index five*	*smooth*
1	*F^4 is a Fano manifold of co-index four*	*smooth*
2	*F^3 is a Mukai manifold*	*1-factorial, normal*
3	*F^2 is a Del Pezzo manifold*	*1-factorial, normal*
4	$F \cong \boldsymbol{Q}^1$ *(smooth), and* $K_X\|_F \cong \mathcal{O}_{\boldsymbol{Q}}(-1)$	*1-factorial, normal*

where $\mathcal{N}_{E/X}$ denotes the normal sheaf of E to X, and $\boldsymbol{Q}$ denotes a hyperquadric in a projective space. For the definitions of Fano variety, co-index, Del Pezzo variety and Mukai variety, see Definition 2.11. (We do not show the existence of any case in the above table.)

The way we see the problem is somewhat more general than that of the 5-dimensional ray contraction. This idea may be explained as follows: Note that $H - K_X$ is f-ample. Call this L, i.e., $H = K_X + 1 \cdot L = K_X + (n-k)L$, where $n = \dim X = 5$ and $k = 4$. We can take L to be ample, and thus, our object becomes a polarized manifold (X, L) of dimension n with an adjoint divisor $K_X + (n-k)L$, where $k = 4$. Furthermore, we do not want L to be "numerically" some multiple of another ample divisor A, since, for instance, if $L \approx 2A$, then $K_X + (n-k)L \approx K_X + 2(n-k)A \approx K_X + (n-k')A$, where $k' < k$. Hence, this case would essentially be reduced to an easier adjoint system: $K_X + (n-k')A$. This argument leads to a good interaction between both Mori and Adjunction Theories and produces the following proposition whose proof is given in Appendix.

Proposition 1.2. *Let X be an n-dimensional smooth projective variety over the complex number field $\mathbb{C}$ with a canonical divisor K_X not nef ($n \geq 2$). Let R be an arbitrary extremal ray on X, and let $f : X \to Y$ be the ray contraction of R. Then there is an ample Cartier divisor A on X and a unique integer $-1 \leq k \leq n-1$ such that*

1. *$R = (K_X + (n-k)A)^{\perp} \cap \overline{NE}(X)$, and*
2. *A is numerically reduced in the sense that there is no ample Cartier divisor A' satisfying $A \cdot C > A' \cdot C$ for a nonzero effective curve C whose numerical class is in R.*

Definition 1.3. In the setting of Proposition 1.2, we call $f : X \to Y$ the k-th *adjoint contraction* with a supporting divisor $K_X + (n-k)A$. In particular, f is called the *fourth adjoint contraction* if $k = 4$. (We note that this definition has already divided all the ray contractions into $(n+1)$-distinct families as k varies from -1 to $n-1$.)

All the k-th adjoint contractions for $-1 \leq k \leq 3$ are listed in [15]. One wishes to extend the list to the case $k = 4$ for the fourth adjoint contractions. As part of this attempt, we obtain a list of all the fourth adjoint contractions of fiber and divisorial types in Theorem 1.4 below. (Those of the flipping type are not considered.)

Theorem 1.4. *Let $f : X \to Y$ be an n-dimensional fourth adjoint contraction of fiber type or of divisorial type. (From Proposition 1.2, it follows that $n \geq 5$.) Then, all the possible cases of f are listed in the following table: If f is of fiber type, F denotes a general fiber of f. If f is of divisorial type, E denotes a unique exceptional prime divisor of f, and F denotes a general fiber of $f|_E : E \to f(E)$ (or any fiber of $f|_E$ over a smooth point of $f(E)$ if in addition $r = 1$ and $F \cong \mathbb{P}^{n-2}$), where $r = \dim f(E)$.*

$\dim Y$	*structure of a positive-dimensional fiber of f*	*singularities of Y*
0	(X, A) *is a Fano manifold of co-index five*	*smooth*
1	(F^{n-1}, A_F) *is a Fano manifold of co-index four*	*smooth*
2	(F^{n-2}, A_F) *is a Mukai manifold*	*1-factorial, normal*
3	(F^{n-3}, A_F) *is a Del Pezzo manifold*	*1-factorial, normal*
4	$(F, A_F) \cong (\boldsymbol{Q}^{n-4}, \mathcal{O}(1))$, *where* $\boldsymbol{Q}^{n-4}$ *is smooth*	*1-factorial, normal*
5	$(F, A_F) \cong (\mathbb{P}^{n-5}, \mathcal{O}(1))$, $(n \geq 6)$	*1-factorial, normal*
n	$r = 0$ *and* (E, A_E) *is a Mukai variety such that* $\Delta(E, A_E) = {A_E}^{n-1}/2 \geq 1$ *and* $\mathcal{N}_{E/X} \cong -A_E$	*1-factorial, terminal*
	$r = 0$ *and* (E, A_E) *is a Del Pezzo variety such that* $\mathcal{N}_{E/X} \cong -2A_E$	*2-factorial, terminal*
	$r = 0$ *and* $(E, A_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-3))$	*3-factorial, terminal*
	$r = 0$ *and* $(E, A_E, \mathcal{N}_{E/X}) \cong (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-4))$	*4-factorial, terminal*
	$r = 1$ *and* $\mathcal{N}_{E/X}\|_F \cong -A_F$. *If either* $n = 5$ *or* F *is irreducible, then* (F^{n-2}, A_F) *is a Del Pezzo variety.*	*1-factorial, terminal*
	$r = 1$ *and* $(F, A_F, \mathcal{N}_{E/X}\|_F) \cong (\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-2))$	*2-factorial, terminal*
	$r = 1$ *and* $(F, A_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-3))$	*3-factorial, terminal*
	$r = 2$ *and* $(F, A_F, \mathcal{N}_{E/X}\|_F) \cong (\boldsymbol{Q}^{n-3}, \mathcal{O}(1), \mathcal{O}(-1))$	*1-factorial, terminal*
	$r = 2$ *and* $(F, A_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^{n-3}, \mathcal{O}(1), \mathcal{O}(-2))$	*2-factorial, terminal*
	$r = 3$ *and* $(F, A_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^{n-4}, \mathcal{O}(1), \mathcal{O}(-1))$	*1-factorial, terminal*

If we do not impose the numerical reducedness on the polarization, that is, if ray contractions $f : X \to Y$ have a supporting divisor $K_X + (n-4)L$ for a merely ample divisor L, then it is expected that the list of these ray contractions may contain some extra cases. Indeed, Proposition 1.5 provides a table for such extra contractions (X, L) which is recovered from the tables of the "rigid" ones, the k-th adjoint contractions (X, A) of divisorial type for $k = 1, 2$ and 3 in [15], Proposition 1.2 and Theorem 1.3.

Proposition 1.5. *Let X be an n-dimensional smooth projective variety. Let $f : X \to Y$ be a ray contraction of divisorial or fiber type. If a supporting divisor of f is $K_X + (n-4)L$ for merely an ample divisor L on X, then all the possible cases of f consist of not only the table of the fourth adjoint contractions in Theorem 1.4 but also the following one, in the latter of which f is of divisorial type, E denotes a unique exceptional prime divisor of f, and F denotes a general fiber of $f|_E : E \to f(E)$ (or any fiber over a smooth point of $f(E)$ if in addition* $\dim f(E) = 1$ *and either $F \cong \mathbb{P}^4$ or $\mathbb{P}^3$).*

$\dim f(E)$	*structure of a positive-dimensional fiber of f*	*singularities of Y*
0	$(E, L_E, \mathcal{N}_{E/X}) \cong (\mathbb{P}^{n-1}, \mathcal{O}(9-n), \mathcal{O}(-1))$ $(n = 5$ *or* $7)$	*smooth*
	$(E, L_E, \mathcal{N}_{E/X}) \cong (\mathbb{P}^{n-1}, \mathcal{O}(8-n), \mathcal{O}(-2))$ $(n = 5$ *or* $6)$	*2-factorial, terminal*
	$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Q}^{n-1}, \mathcal{O}(8-n), \mathcal{O}(-1))$ $(n = 5$ *or* $6)$	*1-factorial, terminal*
	$(E, L_E, \mathcal{N}_{E/X}) \cong (\mathbb{P}^4, \mathcal{O}(2), \mathcal{O}(-3))$ $(n = 5)$	*3-factorial, terminal*
	$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Q}^4, \mathcal{O}(2), \mathcal{O}(-2))$ $(n = 5)$	*2-factorial, terminal*
	$(E, L_E, \mathcal{N}_{E/X}) \cong (E, 2A_E, -A_E)$ $(n = 5)$, *where A is a numerically reduced ample divisor on X, and (E, A_E) is a Del Pezzo variety*	*1-factorial, terminal*
1	$(F, L_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^{n-2}, \mathcal{O}(8-n), \mathcal{O}(-1))$ $(n = 5$ *or* $6)$	*1-factorial, terminal*
	$(F, L_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^3, \mathcal{O}(2), \mathcal{O}(-2))$ $(n = 5)$	*2-factorial, terminal*
	$(F, L_F, \mathcal{N}_{E/X}\|_F) \cong (\boldsymbol{Q}^3, \mathcal{O}(2), \mathcal{O}(-1))$ $(n = 5)$	*1-factorial, terminal*
2	$(F, L_F, \mathcal{N}_{E/X}\|_F) \cong (\mathbb{P}^2, \mathcal{O}(2), \mathcal{O}(-1))$ $(n = 5)$	*1-factorial, terminal*

Proof. The same argument as in the proof of [15], Proposition 2.1 works. *q.e.d.*

We obtain Theorem 1.1 as a corollary of Proposition 1.5.

Proof of Theorem 1.1. The argument is exactly the same as in the proof of [ibid.], Corollary 2.2 by taking $n = 5$ in Proposition 1.5. *q.e.d.*

As for the numerical classification on the dimensions of exceptional divisors and fibers, we rely on the celebrated Theorem (1.1) in [17] due to P. Ionescu and J. A. Wiśniewski. On the other hand, this is the main reason that our ambient space X has to be nonsingular.

The rest of the paper is devoted to proving Theorem 1.4. In Sections 3 through 7, we apply the techniques developed in [15] in order to determine a general positive-dimensional fiber F of the fourth adjoint contraction $f : X \to Y$ of divisorial and fiber types. However, it turns out difficult and is far beyond those techniques of [ibid.] to prove that such a fiber F is irreducible for $f : X \to Y$ of divisorial type with $\dim f(E) = 1$ and $\mathcal{N}_{E/X}|_F \cong -A_F$, where E is a unique exceptional prime divisor of f: If we knew that E is normal, it would be easy to show that a general fiber F of $f|_E : E \to f(E)$ is normal and so irreducible since f has only connected fibers. Unfortunately, we do not know whether E is normal, and in fact, this is an open question.

Question 1.6. Is a unique exceptional prime divisor of a ray contraction of divisorial type always normal?

Aside from the context of ray contraction, we can construct the following example which might "tempt" a possibility of counterexample against the affirmative answer to this Question 1.6.

Example 1.7. (Z. Ran) Let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a natural projection and $\rho : \mathbb{P}^1 \to \mathbb{P}^1$ a double cover defined by $x \to x^2$ in a local coordinate of $\mathbb{P}^1$. Let C be the image of a section of π. Note that C intersects at exactly two points (counting the multiplicity) with each fiber of $h := \rho \circ \pi$. By identifying these two points on C, we construct a variety E and a morphism $g : E \to \mathbb{P}^1$ from $h : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then, E is irreducible and a general fiber of g is connected and reducible.

In [2], Proof of Theorem 2.1, T. Ando asserted that a theorem of Bertini type would imply the irreducibleness of a general fiber F discussed above, but the author did not figure out how to apply it to the possibly non-normal variety E and this does not seem a trivial application of a theorem of this type.

The method we use in Section 8 is totally different from Ando's and does not depend on the Bertini theorem as explained in what follows: Let $f|_E : E \to f(E)$ and F be as above. Let M be the pull back by f of a general hyperplane section V on Y. Then, F is realized as a connected component of the exceptional locus of $f|_M : M \to V$. Assuming that F is not irreducible, we eventually derive a contradiction only when $n := \dim X = 5$. Let G be an arbitrary irreducible component of F, and $\nu := \dim G$ $(= n - 2 = 3)$. We understand F and G as divisors on the nonsingular variety M. Then, from the assumption there exists an *information divisor* D (> 0) of G which is defined as the restriction to G of the other components of F and tells how G intersects with the other components. We obtain a complete table of $(\tilde{G}, A_{\tilde{G}}, \mathcal{N}_{G/M}, D)$ in Lemma 8.10, where $(\tilde{G}, A_{\tilde{G}})$ is the normalization of (G, A_G). Then, the intersection number $F \cdot Z$ with a fixed curve Z can be calculated in two different methods: The first $F \cdot Z$ is based on this table as $F \cdot Z = \sum_G G \cdot Z$. Second $F \cdot Z$ is calculated from the embedding of E into X. In effect, these two methods produce two distinct numbers for the same $F \cdot Z$, hence the expected contradiction. (See the beginning of Section 8 for the precise notations.)

The most difficult part to obtain in this argument is the list of $(\tilde{G}, A_{\tilde{G}})$ in Lemma 8.3. In order to apply T. Fujita's classification theory, we have to show $\Delta(\tilde{G}, A_{\tilde{G}}) = 0$ in Lemma 8.5. (See Theorem 2.3 for Δ-genus.) The key tool to prove this equation is the sectional genera (Definition 2.4), $g(G, A_G)$ and $g(\tilde{G}, A_{\tilde{G}})$ which behave well with severe singularities. In fact, we generalize the sectional genus formula, as evidence of such good behavior, to Cohen-Macaulay pre-polarized varieties in Proposition 2.9. From this, the condition $D > 0$ implies that $g(G, A_G) \leq 0$ in Lemma 8.5. Then, another useful Proposition 2.5, measurement of non-normal locus by the sectional genus, asserts that $g(G, A_G) \geq g(\tilde{G}, A_{\tilde{G}})$. Therefore, we obtain $g(\tilde{G}, A_{\tilde{G}}) \leq 0$. What we need here is exactly Fujita's conjecture regarding sectional genus.

Conjecture 1.8 ([6], Conjecture in Introduction)**.** It should be true that $g \geq 0$ for any quasi-polarized projective variety (Definition 2.2). Moreover, $g = 0$ should imply $\Delta = 0$ if the variety is normal.

Only for varieties of dimension three or less, this conjecture is solved by Fujita in [6], Corollary (4.8), where he essentially reduces the non-negativity of the sectional genus to the nefness of canonical divisor, or the existence of minimal models in the 3-dimensional Minimal Model Program in [13]. Therefore, we are able to obtain the list of $(\tilde{G}, A_{\tilde{G}})$ only when $\nu = \dim \tilde{G} = 3$ or equivalently $n = \dim X = 5$, while the unsolved part of Fujita's Conjecture 1.8 remains in our setting as the following

Conjecture 1.9. A general positive-dimensional fiber F of the fourth adjoint contraction of divisorial type should be irreducible when $\dim f(E) = 1$ and $\mathcal{N}_{E/X}|_F \cong -A_F$ even for $\dim X > 5$.

It is worth noting that even in the case of $\dim X = 4$, the irreducibleness of a general positive-dimensional fiber of a ray contraction of divisorial type is a non-trivial fact and has been proven by H. Takagi in [16], and in a more general context of the third adjoint contraction by myself in [15]. Hence, all the results, for instance in [3], proofs of which depend on this irreducibleness, are secured now.

The existence of each case of the lists is technically rather different business. We just leave the following

Problem 1.10. Determine the existence of each case of the lists in Theorems 1.1, 1.4 and Proposition 1.5.

It should be mentioned that the following case be included in [15], Proposition 1.2.

k	$\dim Y$	structure of a positive-dimensional fiber of f	singularities of Y
1	2	f is a scroll ($n \geq 3$)	smooth

2. Basic concepts

Definition 2.1. A *variety* means a separated integral scheme of finite type over the complex number field $\mathbb{C}$ unless otherwise stated. A *subvariety* means an integral subscheme of a scheme. A *point* means a closed point. Locally free sheaves and vector bundles are used interchangeably. ω_X^0 denotes the dualizing sheaf of a complete scheme X. K_X denotes a canonical divisor or the canonical sheaf of a normal variety X. A normal variety is said to have *q-factorial singularities* if there is an integer $q \geq 1$ such that for every Weil divisor Z, qZ is a Cartier divisor. For convenience, 0-*factorial* means non-singular. (In our case, 1-factoriality is equivalent to local factoriality, as shown in [1], Proposition (3.10) on p. 139.) We employ the intersection theory due to Kleiman in [11] as well as Definition 2.7. The linear equivalence between divisors D_1 and D_2 is denoted by $D_1 \sim D_2$.

We make two contributions, Propositions 2.5 and 2.9, to the Theory of Polarized Varieties after T. Fujita.

Definition 2.2 (cf. [6], Introduction)**.** Let X be a complete variety over an algebraically closed field, and L a line bundle on it. A pair (X, L) is called a *polarized (quasi-polarized, pre-polarized) variety* if L is an ample (a nef and big, a merely) line bundle, respectively.

Theorem 2.3 ([ibid.], Theorem (1.1))**.** *Let $\Delta(X, L)$ be the* delta genus *of an n-dimensional quasi-polarized variety (X, L), i.e., $\Delta(X, L) = n + L^n - h^0(X, L)$. If X is a projective variety, then $\Delta(X, L)$ is a non-negative integer.*

Definition 2.4 (cf. [ibid.], Introduction)**.** Let (X, L) be an n-dimensional pre-polarized variety $(n \geq 1)$. We define integers χ_i by $\chi(X, L^{\otimes t}) = \sum_{i=0}^{n} (\chi_i / i!) t^{[i]}$, where $\chi(X, L^{\otimes t})$ is the Euler characteristic, $t^{[i]} = t(t+1)\cdots(t+i-1)$ for $i > 0$ and $t^{[0]} = 1$. Then, the *sectional genus* $g(X, L)$ of (X, L) is defined by $g(X, L) = 1 - \chi_{n-1}$ $(\in \mathbb{Z})$.

The sectional genus detects whether the size of the non-normal locus of a polarized variety is large. We need the following proposition to prove Lemma 8.5 in Section 8.

Proposition 2.5. *Let $\mu : \tilde{X} \to X$ be the normalization of a complete variety X of dimension ≥ 1, over an algebraically closed field, and L an ample invertible sheaf on X. Then, we have*

1. *$g(\tilde{X}, \mu^* L) = g(X, L)$ if the non-normal locus of X has codimension > 1, and*
2. *$g(\tilde{X}, \mu^* L) < g(X, L)$ if the non-normal locus of X has codimension $= 1$.*

Proof. We go through a Leray spectral sequence argument. Let $n = \dim X$. Set $E_2^{p,q} = H^p(X, R^q\mu_*\mu^*(L^{\otimes t}))$. Since $R^q\mu_*\mu^*(L^{\otimes t}) = 0$ for $q > 0$, it follows that

$$E_2^{p,q} = \begin{cases} H^p(X, \mu_*\mathcal{O}_{\tilde{X}} \otimes L^{\otimes t}) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Hence, $E_2^{p,q} \cong E_\infty^{p,q}$ and thus $H^i(\tilde{X}, \mu^*(L^{\otimes t})) \cong \oplus_{p+q=i} E_\infty^{p,q} = E_\infty^{i,0} = H^i(X, \mu_*\mathcal{O}_{\tilde{X}} \otimes L^{\otimes t})$. Therefore, we obtain the relation between two Euler characteristics on X and $\tilde{X}$ as follows:

$$\chi(\tilde{X}, \mu^*L^{\otimes t}) = \chi(X, \mu_*\mathcal{O}_{\tilde{X}} \otimes L^{\otimes t}). \tag{2.1}$$

There is a natural short exact sequence, $0 \to \mathcal{O}_X \to \mu_*\mathcal{O}_{\tilde{X}} \to \mathcal{F} \to 0$, where $\mathcal{F}$ is a coherent torsion sheaf. Hence, form (2.1),

$$\chi(\tilde{X}, \mu^*L^{\otimes t}) = \chi(X, L^{\otimes t}) + \chi(X, \mathcal{F} \otimes L^{\otimes t}). \tag{2.2}$$

Since $\dim \operatorname{supp} \mathcal{F} \leq n-1$,

$$\chi(X, \mathcal{F} \otimes L^{\otimes t}) = [a_{n-1}/(n-1)!]t^{n-1} + \cdots + a_0 \quad (a_i \in \mathbb{Q}). \tag{2.3}$$

From Definition 2.4, (2.2) and (2.3), we deduce $g(\tilde{X}, \mu^*L) = g(X, L) - a_{n-1}$. Therefore, to finish the proof, it suffices to show the following Claim 2.6 since the non-normal locus coincides with $\operatorname{supp} \mathcal{F}$.

Claim 2.6. $a_{n-1} = 0$ or > 0 according that $s := \dim \operatorname{supp} \mathcal{F} < n-1$ or $= n-1$, respectively.

From [11], Proposition 1 on p. 302 it follows that $\chi(X, \mathcal{F} \otimes L^{\otimes t}) = [\deg \mathcal{F}/s!]t^s +$ (terms of lower degree), and $\deg \mathcal{F} > 0$ since L is ample. Simple comparison of this equation with (2.3) proves Claim 2.6. *q.e.d.*

The following definition provides the intersection number of a coherent sheaf of rank one with an invertible sheaf. Note that the projectiveness imposed on a variety in [15], Definition 3.6 is loosened to the completeness, and the amplitude on an invertible sheaf in [ibid.] is dropped in the new definition.

Definition and Lemma 2.7. Let X be a complete variety over an algebraically closed field, of dimension $n \geq 1$. Let $\mathcal{F}$ be any coherent sheaf of rank one, and let A and B be any invertible sheaves on X.

1. Then, we obtain $\chi(X, \mathcal{F} \otimes B^{\otimes t}) = \dfrac{B^n}{n!}t^n + O(t^{n-1})$, where $O(t^d)$ denotes a polynomial in $\mathbb{Q}[t]$ of degree $\leq d$. Therefore, we can define the *intersection number* $\mathcal{F} \cdot B^{n-1}$ $(\in \mathbb{Z})$ of a coherent sheaf $\mathcal{F}$ of rank one with an invertible sheaf B by the following equation:
$$\chi(X, \mathcal{F} \otimes B^{\otimes t}) - \chi(X, B^{\otimes t}) = \frac{\mathcal{F} \cdot B^{n-1}}{(n-1)!}t^{n-1} + O(t^{n-2}).$$
2. Moreover, applying this definition of intersection number, we recover partially [11], Proposition 2 on p. 296 in the following form: $(\mathcal{F} \otimes A) \cdot B^{n-1} = \mathcal{F} \cdot B^{n-1} + A \cdot B^{n-1}$.

Remark 2.8. When $\mathcal{F}$ is an invertible sheaf $\mathcal{L}$, our definition $\mathcal{F} \cdot B^{n-1}$ coincides with Kleiman's one $(\mathcal{L} \cdot B^{n-1} \bullet \mathcal{O}_X)$ (see [ibid.], Proposition 2 on p. 302).

Proof. Note that $\dim \operatorname{supp} \mathcal{F} = n$ for a coherent sheaf $\mathcal{F}$ of rank one. From [11], Theorem on p. 295 and Lemma 1 on p. 296, there are integers $a_{i,j}$ and b_i such that for any integers s and t,

$$\begin{aligned} \chi(X, \mathcal{F} \otimes A^{\otimes s} \otimes B^{\otimes t}) &= \textstyle\sum_{i,j \geq 0,\, i+j \leq n} a_{i,j} \binom{s+i}{i} \binom{t+j}{j} \\ &= \frac{a_{0,n}}{n!}t^n + \frac{1}{(n-1)!}\left[\frac{(n+1)a_{0,n}}{2} + a_{0,n-1} + a_{1,n-1}s\right]t^{n-1} + O(t^{n-2}), \qquad (2.a) \\ \chi(X, B^{\otimes t}) &= \textstyle\sum_{i=0}^n b_i \binom{t+i}{i} \\ &= (b_n/n!)t^n + [1/(n-1)!][b_{n-1} + (n+1)b_n/2]t^{n-1} + O(t^{n-2}). \qquad (2.b) \end{aligned}$$

Take $s = 0$ in $(2.a)$, then it holds that

$$\chi(X, \mathcal{F} \otimes B^{\otimes t}) = (a_{0,n}/n!)t^n + [1/(n-1)!][a_{0,n-1} + (n+1)a_{0,n}/2]t^{n-1} + O(t^{n-2}). \tag{2.c}$$

Substituting $t = t_1 + \cdots + t_{n-1}$ for $(2.a)$ and $t = t_1 + \cdots + t_n$ for both $(2.b)$ and $(2.c)$, Kleiman's definition of intersection number in [ibid.], p. 296 implies that

$$a_{1,n-1} = (A \cdot B^{n-1} \bullet \mathcal{F}) = A \cdot B^{n-1}, b_n = (B^n \bullet \mathcal{O}_X) = B^n \text{ and } a_{0,n} = (B^n \bullet \mathcal{F}) = B^n. \tag{2.d}$$

The last equality, for instance, is shown as follows: Since X is reduced and irreducible, and $\mathcal{F}$ is a coherent sheaf of rank one, it follows that $\mathrm{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi = 1$, where ξ is the generic point of X. Hence, from [ibid.], Corollaries 1 and 2 on p. 298 and $(2.d)$,

$$\begin{aligned} a_{0,n} &= (B^n \bullet \mathcal{F}) \\ &= (\mathrm{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi)(B^n \bullet \mathcal{O}_X) \\ &= B^n. \end{aligned} \tag{2.e}$$

From $(2.c)$ and $(2.d)$, the first statement of the lemma follows.
As for the second statement, take $s = 1$ in $(2.a)$, then it holds that

$$\chi(X, \mathcal{F} \otimes A \otimes B^{\otimes t}) = (a_{0,n}/n!)t^n + [1/(n-1)!][a_{0,n-1} + a_{1,n-1} + (n+1)a_{0,n}/2]t^{n-1} + O(t^{n-2}). \tag{2.f}$$

From $(2.d)$ and $(2.e)$, $b_n = a_{0,n}$. Thus, $(2.b)$, $(2.c)$ and $(2.f)$ imply that

$$\begin{aligned} \chi(X, \mathcal{F} \otimes A \otimes B^{\otimes t}) - \chi(X, B^{\otimes t}) &= [1/(n-1)!](a_{0,n-1} + a_{1,n-1} - b_{n-1})t^{n-1} + O(t^{n-2}), \text{ and} \\ \chi(X, \mathcal{F} \otimes B^{\otimes t}) - \chi(X, B^{\otimes t}) &= [1/(n-1)!](a_{0,n-1} - b_{n-1})t^{n-1} + O(t^{n-2}). \end{aligned}$$

Therefore, applying the above Definition 2.7, we obtain that

$$\left.\begin{aligned} (\mathcal{F} \otimes A) \cdot B^{n-1} &= (a_{0,n-1} - b_{n-1}) + a_{1,n-1}, \text{ and} \\ \mathcal{F} \cdot B^{n-1} &= a_{0,n-1} - b_{n-1}. \end{aligned}\right\} \tag{2.g}$$

Hence, $(2.d)$ and $(2.g)$ provide the second statement. Also, $(2.g)$ shows that $\mathcal{F} \cdot B^{n-1}$ is an integer. *q.e.d.*

We generalize the sectional genus formula to a Cohen-Macaulay pre-polarized variety and use this for the proofs of Lemmas 8.5 and 8.6.

Proposition 2.9. *Let L be an invertible sheaf on an n-dimensional complete variety X over an algebraically closed field ($n \geq 1$). If X is locally Cohen-Macaulay, then the sectional genus formula holds: $2g(X, L) - 2 = [\omega_X^0 \otimes L^{\otimes(n-1)}] \cdot L^{n-1}$, where we use the intersection number defined in Definition 2.7. One can express the right hand side of the formula additively as long as one keeps the arithmetic rules in Lemma 2.7 and Remark 2.8:*

$$2g(X, L) - 2 = [\omega_X^0 + (n-1)L] \cdot L^{n-1}.$$

Proof. Since X is proper over an algebraically closed field, the dualizing sheaf ω_X^0 exists (see [7] and [9]). From Definition 2.4, it follows that

$$\begin{aligned} \chi(X, L^{\otimes t}) &= \textstyle\sum_{i=0}^n (1/i!)\chi_i\, t^{[i]} \\ &= (1/n!)\chi_n\, t^n + [1/2(n-1)!][(n-1)\chi_n + 2\chi_{n-1}]t^{n-1} + O(t^{n-2}). \end{aligned} \tag{2.4}$$

From the Serre duality theorem for complete Cohen-Macaulay varieties (see [ibid.]),

$$\begin{aligned} \chi(X, \omega_X^0 \otimes L^{\otimes t}) &= (-1)^n \chi(X, L^{\otimes(-t)}) \\ &= (1/n!)\chi_n t^n - [1/2(n-1)!][(n-1)\chi_n + 2\chi_{n-1}]t^{n-1} + O(t^{n-2}). \end{aligned}$$

Hence,

$$\chi(X, \omega_X^0 \otimes L^{\otimes t}) - \chi(X, L^{\otimes t}) = -\frac{(n-1)\chi_n + 2\chi_{n-1}}{(n-1)!} t^{n-1} + O(t^{n-2}). \tag{2.5}$$

Therefore, from Definition 2.4 and Definition 2.7, we derive that

$$\omega_X^0 \cdot L^{n-1} = -(n-1)\chi_n - 2\chi_{n-1} = -(n-1)L^n - 2(1 - g(X, L)).$$

Hence, $2g(X, L) - 2 = \omega_X^0 \cdot L^{n-1} + (n-1)L^n$, and the second statement in Lemma 2.7 thus completes the proof by substituting $\mathcal{F} = \omega_X^0$ and $A = B = L$. *q.e.d.*

The following is the key theorem to prove the irreducibleness of a general positive-dimensional fiber of the fourth adjoint contraction of divisorial type when its dimension is five.

Theorem 2.10 ([6], Corollary (4.8)). *Let (V, L) be a quasi-polarized projective variety over an algebraically closed field of characteristic zero. If V is normal and $\dim V \leq 3$, then the sectional genus satisfies $g(V, L) \geq 0$. Moreover, if $g(V, L) = 0$, it follows that $\Delta(V, L) = 0$.*

Finally, we define special varieties arising in this paper.

Definition 2.11. A projective variety X with Gorenstein singularities is said to be a *Fano variety* if $-\omega_X^0$ is ample. If $-\omega_X^0 \sim mL$ (linear equivalence) for an ample Cartier divisor L on X and an integer $m > 0$, then we call $c = n + 1 - m$ the *co-index* of a Fano variety X. In particular, a Fano variety of co-index two or three is called a *Del Pezzo variety* or a *Mukai variety*, respectively.

Remark 2.12. In the above Definition 2.11, we have relaxed, for simplicity, the condition on co-index which is defined in a stricter sense previously in [15], Definition 3.8. In fact, among all the lists in this paper, only a Fano manifold (X, A) of co-index five in Theorem 1.4 is known to be of co-index five in the strict sense of [ibid.].

Lemma 2.13. *Let X be a Fano variety which has log-terminal singularities (over algebraically closed field of characteristic zero). If $-K_X \sim mL$ for an ample Cartier divisor L on X and an integer $m > 0$, then, we have*

$$H^i(X, tL) = 0 \text{ for any } t \in \mathbb{Z} \text{ and any } 0 < i < \dim X. \tag{2.6}$$

Proof. From the Serre duality theorem, it follows that $H^i(X, tL) = H^{n-i}(X, K_X - tL)$, where $n = \dim X$. Apply the Kawamata-Viehweg vanishing theorem, [10], Theorem 1-2-5 and Remark 1-2-6 to each side of this equality. *q.e.d.*

Remark 2.14. Let $f : X \to Y$ be an n-dimensional fourth adjoint contraction of divisorial type, and F a general positive-dimensional fiber of f. From [15], § 4, we deduce that $H^i(F, tA_F) = 0$ for any $0 < i < \dim X$ and any $t \in \mathbb{Z}$ satisfying either $t \leq -q$ or $-(n-4) \leq t$, where q is defined by $E|_F \cong -qA_F$ for the exceptional divisor E of f. We see later that q takes only 1 or 2 when Del Pezzo and Mukai varieties arise for F. (They appear only in Sections 4 and 5.) Therefore, together with the above Lemma 2.13, we conclude that all the Del Pezzo and Mukai varieties in every list in this paper satisfy (2.6), or the traditional definitions of these varieties as in [15], Definition 3.8 and [2], § 2.

3. Numerical classification of the fourth adjoint contractions of divisorial type and the proof for the fiber type

Proposition 3.1. *If $f : X \to Y$ is the fourth adjoint contraction of divisorial type, then we have $r = \dim f(E) = 0, 1, 2$ or 3.*

Proof. By virtue of [17], Theorem (1.1), the same argument as in the proof of [15], Proposition 6.1 implies the desired result. *q.e.d.*

Proof for the fiber type. Let F be a general fiber of $f : X \to Y$. Then, applying the Bertini theorem, [8], Corollary 10.9 and Remark 10.9.1 in Chap. III, it follows that F is smooth and irreducible since f has only connected fibers. As in [15], § 6, we derive that $\dim F = n, n-1, n-2, n-3, n-4, n-5$ from [17], Theorem (1.1), and that $K_F + (n-4)A_F \cong \mathcal{O}_F$. Hence, Definition 2.11 determines F. The proof of [10], Lemma 5-1-5 works for our setting and produces the statement of singularities of Y. *q.e.d.*

4. Proof for the fourth adjoint contractions of divisorial type with $\dim f(E) = 0$

In this case, (F, A_F) is a polarized variety (E, A_E). We use the same argument as in [15], § 7. Let $\nu = \dim E$ $(= n - 1 \geq 4)$, $d = {A_E}^\nu$ and $P(t) = \chi(E, tA_E)$. From [ibid.], Lemma 4.5, it follows that

$$P(t) = (d/\nu!)(t+1)\cdots(t+\nu-3)(t+\alpha)(t+\beta)(t+\gamma) \tag{4.1}$$

for some complex numbers α, β and γ,

$$\nu(\nu-1)(\nu-2) = d\alpha\beta\gamma, \tag{4.2}$$

$$h^0(E, A_E) = [d/\nu(\nu-1)][1 + (\alpha+\beta+\gamma) + (\beta\gamma+\gamma\alpha+\alpha\beta) + \alpha\beta\gamma], \tag{4.3}$$

and

$$P(t) = (-1)^\nu P(-t-(\nu-3)-q). \tag{4.4}$$

From (4.1) and (4.4), we have two expressions for roots of $P(t)=0$:

$$\{-1 \geq \cdots \geq -(\nu-3), -\alpha, -\beta, -\gamma\} =$$
$$\{-1+(1-q) \geq \cdots \geq -(\nu-3)+(1-q), \alpha-(\nu-3)-q, \beta-(\nu-3)-q, \gamma-(\nu-3)-q\}. \tag{4.5}$$

By adding both sides of (4.5), it follows that

$$\alpha+\beta+\gamma = \nu-3+q\nu/2. \tag{4.6}$$

By multiplying both sides of (4.5),

$$(-1)^\nu(\nu-3)!\alpha\beta\gamma = (-1)^{\nu-3}[(\nu+q-4)!/(q-1)!]$$
$$\cdot[\alpha\beta\gamma-(\nu-3+q)(\beta\gamma+\gamma\alpha+\alpha\beta)+(\nu-3+q)^2(\alpha+\beta+\gamma)-(\nu-3+q)^3].$$

Hence,

$$\beta\gamma+\gamma\alpha+\alpha\beta = [1/(\nu-3+q)][1+(q-1)!(\nu-3)!/(\nu+q-4)!]\alpha\beta\gamma$$
$$+(\nu-3+q)(\alpha+\beta+\gamma)-(\nu-3+q)^2. \tag{4.7}$$

Substitute (4.2), (4.6) and (4.7) into (4.3), then

$$h^0(E,A_E) = [d/\nu(\nu-1)]\{1-(\nu-3+q)^2+(\nu-2+q)(\nu-3+q\nu/2)$$
$$+[1/(\nu-3+q)][\nu-2+q+(q-1)!(\nu-3)!/(\nu+q-4)!][\nu(\nu-1)(\nu-2)/d]\}$$
$$= [(\nu-2+q)(q\nu-2q+2)/(2\nu(\nu-1))]d$$
$$+[(\nu-2)/(\nu-3+q)][\nu-2+q+(q-1)!(\nu-3)!/(\nu+q-4)!]. \tag{4.8}$$

[ibid.], Lemma 4.3 implies

$$\omega_E^0 \cong -(\nu-3+q)A_E. \tag{4.9}$$

Hence, from [ibid.], Lemma 5.1, we derive that $\nu-3+q \leq \nu+1$, or $q \leq 4$.
We proceed to examine each of the four cases: $q=1,2,3,4$ below.

Case 1) $q=1$: From [15], Lemma 4.6, Y has 1-factorial, terminal singularities, and [ibid.], Lemma 4.2 implies that $\mathcal{N}_{E/X} \cong -A_E$. From (4.8), we deduce $h^0(E,A_E) = d/2+\nu$ and thus $\Delta(E,A_E) = d/2$. From (4.9), $\omega_E^0 \cong -(\nu-2)A_E$. Since E has Gorenstein singularities, (E,A_E) is a Mukai variety such that $\Delta(E,A_E) = d/2 \geq 1$. (See Definition 2.11 and Remark 2.12 for the relation with the traditional definition of Mukai variety.)

Case 2) $q=2$: As above Y has 2-factorial, terminal singularities, and $\mathcal{N}_{E/X} \cong -2A_E$. From (4.9), $\omega_E^0 \cong -(\nu-1)A_E$. Since E has Gorenstein singularities, (E,A_E) is a Del Pezzo variety. (See Definition 2.11 and Remark 2.12 for the relation with the traditional definition of Del Pezzo variety.)

Case 3) $q=3$: Y has 3-factorial, terminal singularities, and $\mathcal{N}_{E/X} \cong -3A_E$. From (4.8),

$$h^0(E,A_E) = [(3\nu^2-\nu-4)d+(2\nu^3-4\nu^2-2\nu+8)]/2\nu(\nu-1)$$

and thus

$$\Delta(E,A_E) = (2-d)(\nu^2-\nu-4)/2\nu(\nu-1). \tag{4.10}$$

Since $\nu \geq 4$, the non-negativity of delta genus in Theorem 2.3 implies that $d=1$ or 2. If $d=1$, then from (4.10), $\nu[(\nu+1)-2(\nu-1)\Delta(E,A_E)] = 4$. From this, it would hold that $\nu=4$ and $3\Delta(E,A_E)=2$, which contradicts that a delta genus is an integer. Hence, $d=2$ and $\Delta(E,A_E)=0$. Therefore, from [4], Corollary (4.3), it follows that $(E,A_E,\mathcal{N}_{E/X}) \cong (\boldsymbol{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-3))$.

Case 4) $q=4$: Y has 4-factorial, terminal singularities, and $\mathcal{N}_{E/X} \cong -4A_E$. From (4.8),

$$h^0(E,A_E) = [(2\nu^2+\nu-6)d+(\nu^3-2\nu^2-2\nu+6)]/\nu(\nu-1)$$

and thus $\Delta(E,A_E) = (1-d)(\nu^2+2\nu-6)/\nu(\nu-1)$. As above $d=1$ and $\Delta(E,A_E)=0$. Therefore, from [4], Corollary (4.3), it follows that $(E,A_E,\mathcal{N}_{E/X}) \cong (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-4))$.

5. Proof for the fourth adjoint contractions of divisorial type with $\dim f(E) = 1$

We begin with applying the same argument as in [15], § 8. Let F be a general fiber of $E \to f(E)$, $\nu = \dim F$ $(= n-2 \geq 3)$, $d = {A_F}^{\nu}$ and $P(t) = \chi(F, tA_F)$. From [15], Lemma 4.5 it follows that

$$P(t) = (d/\nu!)(t+1)\cdots(t+\nu-2)(t+\alpha)(t+\beta)$$

for some complex numbers α and β, $\nu(\nu-1) = d\alpha\beta$, $h^0(F, A_F) = d/\nu + (d/\nu)(\alpha+\beta) + (\nu-1)$, and

$$\{-1 \geq \cdots \geq -(\nu-2), -\alpha, -\beta\} =$$
$$\{-1+(1-q) \geq \cdots \geq -(\nu-2)+(1-q), \alpha-(\nu-2)-q, \beta-(\nu-2)-q\},$$

since $P(t) = (-1)^{\nu}P(-t-(\nu-2)-q)$. By adding both sides of the last equality, $\alpha+\beta = (\nu/2)(q+1)-1$. Hence, $h^0(F, A_F) = (d/2)(q+1)+(\nu-1)$. Formally, we have

$$\Delta(F, A_F) = (d/2)(1-q)+1. \tag{5.1}$$

From [15], Lemma 4.3, it holds that

$$\omega_F^0 \cong -(\nu-2+q)A_F. \tag{5.2}$$

From [ibid.], Lemma 5.1, there are three cases to examine: $q = 1, 2, 3$.

Case 1) $q = 1$**:** From [ibid.], Lemmas 4.6 and 4.2, Y has 1-factorial, terminal singularities, and $\mathcal{N}_{E/X}|_F \cong -A_F$. If we assume the irreducibleness of F, then from the same argument as in Case 2) in Section 4, it follows that (F, A_F) is a Del Pezzo variety. On the one hand, F2 in Theorem 8.1 shows the irreducibleness of F only for $n = \dim X = 5$. Therefore, we obtain the statement in Theorem 1.4.

Note that F1 in Theorem 8.1 guarantees that F is irreducible for Cases 2) and 3) of any dimension ≥ 5 below.

Case 2) $q = 2$**:** As above Y has 2-factorial, terminal singularities, and $\mathcal{N}_{E/X}|_F \cong -2A_F$. The fact that F is irreducible enables us to use the non-negativity of the delta genus. Hence, from (5.1), we deduce that $d = 2$ and thus $\Delta(F, A_F) = 0$. Hence, from [4], Corollary (4.3), we obtain that $(F, A_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-2))$.

Case 3) $q = 3$**:** As above Y has 3-factorial, terminal singularities, and $\mathcal{N}_{E/X}|_F \cong -3A_F$. From (5.1), $d = 1$ and thus $\Delta(F, A_F) = 0$. Hence, $(F, A_F, \mathcal{N}_{E/X}|_F) \cong (\mathbb{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-3))$. As in the proof of [5], Theorem 4, we can improve the description on a fiber F.

6. Proof for the fourth adjoint contractions of divisorial type with $\dim f(E) = 2$

Let F be a general fiber of $E \to f(E)$, $\nu = \dim F$ $(= n-3 \geq 2)$, $d = {A_F}^{\nu}$ and $P(t) = \chi(F, tA_F)$. From [15], Lemma 4.5, as before, it holds that $P(t) = (d/\nu!)(t+1)\cdots(t+\nu-1)(t+\alpha)$ for some complex number α, $\nu = d\alpha$, $h^0(F, A_F) = d+\nu$, and

$$\{-1 \geq \cdots \geq -(\nu-1), -\alpha\} = \{-1+(1-q) \geq \cdots \geq -(\nu-1)+(1-q), \alpha-(\nu-1)-q\},$$

since $P(t) = (-1)^{\nu}P(-t-(\nu-1)-q)$. By adding both sides, $\alpha = q\nu/2$ and thus $\nu = qd\nu/2$, or $2 = qd$. Hence, $(q, d) = (2, 1)$ or $(1, 2)$. Note that from F1 in Theorem 8.1, F is irreducible for Cases 1) and 2) below.

Case 1) $(q, d) = (2, 1)$: Y has 2-factorial, terminal singularities, and $\mathcal{N}_{E/X}|_F \cong -2A_F$. Since $\Delta(F, A_F) = 0$, we derive that $(F, A_F, \mathcal{N}_{E/X}|_F) \cong (\mathbb{P}^{n-3}, \mathcal{O}(1), \mathcal{O}(-2))$.

Case 2) $(q, d) = (1, 2)$: Y has 1-factorial, terminal singularities, and $\mathcal{N}_{E/X}|_F \cong -A_F$. As above, $\Delta(F, A_F) = 0$ implies that $(F, A_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{Q}^{n-3}, \mathcal{O}(1), \mathcal{O}(-1))$.

7. Proof for the fourth adjoint contractions of divisorial type with $\dim f(E) = 3$

Let F be a general fiber of $E \to f(E)$, $\nu = \dim F$ $(= n-4 \geq 1)$, $d = {A_F}^{\nu}$ and $P(t) = \chi(F, tA_F)$. As above, it holds that $P(t) = (d/\nu!)(t+1)\cdots(t+\nu)$, $d = 1$, $h^0(F, A_F) = \nu+1$, and

$$\{-1 \geq \cdots \geq -\nu\} = \{-1+(1-q) \geq \cdots \geq -\nu+(1-q)\},$$

since $P(t) = (-1)^{\nu}P(-t-\nu-q)$. From these, we deduce that F is irreducible, $\Delta(F, A_F) = 0$ and $q = 1$. Hence, Y has 1-factorial, terminal singularities, and $(F, A_F, \mathcal{N}_{E/X}|_F) \cong (\mathbb{P}^{n-4}, \mathcal{O}(1), \mathcal{O}(-1))$.

8. Proof of the irreducibleness of a general fiber F of $E \to f(E)$

First of all, we make clear the situation to be dealt with. Let $f : X \to Y$ be the fourth adjoint contraction of divisorial type with $r := \dim f(E)$, and let E be a unique exceptional divisor for f. Recall [15], Lemma 4.3: For general hyperplane sections $V_1, \ldots, V_r$ on Y, we define $M_i = f^*V_i$, a total transform of V_i, $M = M_1 \cap \cdots \cap M_r$ and $V = V_1 \cap \cdots \cap V_r$. Let F be a general fiber of $f|_E : E \to f(E)$. Note that F can be considered as a connected component of $M \cap E$. Let $\nu := \dim F = n - 1 - r$ ($n = \dim X$). A positive integer q is defined by $E|_F \cong -qA_F$. Note also that F is a connected projective reduced scheme of equidimension ν and a local complete intersection in X. In order to complete the arguments in the previous sections, we have to show

Theorem 8.1. *Let the notations be as defined above.*

F1: *F is irreducible for all $(r, q) = (1, 2), (1, 3), (2, 1)$.*
F2: *F is irreducible for $(r, q) = (1, 1)$ if $n = 5$ (or equivalently, $\nu = 3$).*

We begin the proof by posing

Assumption 8.2. F is not irreducible.

In what follows, this assumption is kept until we obtain a contradiction at the end of this section. Note that the scheme structure of F requires F to be an integral divisor on the smooth variety M and to have the decomposition into the components G_i's which are distinct prime divisors on M: $F = \sum_{i \geq 1} G_i$. Let G be an arbitrary irreducible component of F, say $G = G_1$, and $G' = \sum_{i \geq 2} G_i$. Restrict $F = G + G'$ to G, then $F|_G = \mathcal{N}_{G/M} + D$, where $D := G'|_G$. Note also that the Assumption 8.2 forces G' to be a nonzero effective Cartier divisor on M, and thus the connectedness of F implies that D is a nonzero effective Cartier divisor on G, or $D > 0$. On the one hand, [15], Lemma 4.2 (2) implies

$$F|_G \cong (F|_F)|_G \cong \mathcal{N}_{F/M}|_G \cong (\mathcal{N}_{E/X}|_F)|_G \cong -qA_G. \tag{8.1}$$

Therefore, altogether we obtain

$$-qA_G - \mathcal{N}_{G/M} = D > 0 \text{ in } \mathrm{Div}(G), \tag{8.2}$$

where $\mathrm{Div}(G)$ denotes the set of all Cartier divisors on G. From the adjunction formula,

$$\omega_G^0 \cong (K_M + G)|_G \cong [(K_X + M)|_M + G]|_G = K_X|_G + G|_G.$$

Apply [ibid.], Lemma 4.2 (1) to this, and from (8.2),

$$\begin{aligned} \omega_G^0 &= -(\nu + r - 3)A_G + \mathcal{N}_{G/M} \\ &= -(\nu + r + q - 3)A_G - D \end{aligned} \tag{8.3}$$

Let $g : S \to G$ be a desingularization of G, and $A_S := g^*A_G$. Then, from [ibid.], (4) in Proof of Lemma 5.1,

$$(g_*K_S) \cdot {A_G}^{\nu-1} \geq K_S \cdot {A_S}^{\nu-1} \tag{8.4}$$

The same argument as in [ibid.], (1), (2) and (3) in Proof of Lemma 5.1 can apply to ω_G^0 and results in

$$\omega_G^0 \cdot {A_G}^{\nu-1} \geq g_*K_S \cdot {A_G}^{\nu-1} \tag{8.5}$$

Let $\mu : \tilde{G} \to G$ be the normalization of an arbitrary irreducible component G of F, and $A_{\tilde{G}} = \mu^*A_G$.

Lemma 8.3. *If $(r, q) = (1, 2), (1, 3)$ or $(2, 1)$, then $(\tilde{G}, A_{\tilde{G}}) \cong (\mathbb{P}^\nu, \mathcal{O}(1))$.*
If $(r, q) = (1, 1)$ and if $n = \dim X = 5$ or $\nu = 3$, then $(\tilde{G}, A_{\tilde{G}}) \cong (\mathbb{P}^\nu, \mathcal{O}(1))$ or $(\boldsymbol{Q}^\nu, \mathcal{O}(1))$.

Proof. The proof is rather lengthy. We assume that $(\tilde{G}, A_{\tilde{G}}) \not\cong (\mathbb{P}^\nu, \mathcal{O}(1))$. Then, [6], Theorem (2.2) implies one of the following two conditions:

$$\text{There exists an integer } t \text{ with } 1 \leq t < \nu \text{ such that } H^0(S, K_S + tA_S) \neq 0. \tag{8.a}$$

$$H^0(S, K_S + \nu A_S) \neq 0 \text{ and } H^0(S, K_S + tA_S) = 0 \text{ for any } 1 \leq t < \nu. \tag{8.b}$$

Claim 8.4. Case (8.a) does not occur.

Proof. If Case (8.a) did occur, $(K_S+(\nu-1)A_S)\cdot {A_S}^{\nu-1}\geq 0$. Hence, from (8.4), (8.5) and (8.3), it follows that

$$\begin{array}{rcl}(\nu-1){A_S}^{\nu} & \geq & -K_S\cdot {A_S}^{\nu-1} \\ & \geq & -(g_*K_S)\cdot {A_G}^{\nu-1} \\ & \geq & -\omega_G^0\cdot {A_G}^{\nu-1} \\ & = & (\nu+r+q-3){A_G}^{\nu}+D\cdot {A_G}^{\nu-1} \\ & > & (\nu+r+q-3){A_G}^{\nu},\end{array}$$

where we used $D>0$ for the last inequality. Thus, $r+q<2$, a contradiction. *q.e.d.*

Therefore, we are in Case (8.b). Since $H^0(S,K_S+\nu A_S)\neq 0$, it holds that $(K_S+\nu A_S)\cdot {A_S}^{\nu-1}\geq 0$. Hence, exactly the same argument as in the proof of Claim 8.4 implies that $\nu {A_S}^{\nu}>(\nu+r+q-3){A_G}^{\nu}$ and thus

$$r+q<3. \tag{8.6}$$

If $(r,q)=(1,2),(1,3)$ or $(2,1)$, then (8.6) implies a contradiction. Therefore, $(\tilde{G},A_{\tilde{G}})\cong(\mathbb{P}^{\nu},\mathcal{O}(1))$ for these three cases, and the first statement of Lemma 8.3 is proven.

From now on, we prove the second statement of Lemma 8.3. To do so, it suffices to show that $(\tilde{G},A_{\tilde{G}})\cong(\boldsymbol{Q}^{\nu},\mathcal{O}(1))$ under the conditions $(r,q)=(1,1)$, $\nu=3$ and (8.b) as well as $(\tilde{G},A_{\tilde{G}})\not\cong(\mathbb{P}^{\nu},\mathcal{O}(1))$.

Lemma 8.5. *If $h^0(S,K_S+\nu A_S)\neq 1$, then $g(G,A_G)=g(\tilde{G},A_{\tilde{G}})=\Delta(\tilde{G},A_{\tilde{G}})=0$, and ${A_{\tilde{G}}}^{\nu}>2$.*

Proof. From [6], Corollary (2.8), $h^0(S,K_S+\nu A_S)\neq 1$ implies $(K_S+\nu A_S)\cdot {A_S}^{\nu-1}>0$. As in the proof of Claim 8.4,

$$\nu {A_S}^{\nu}>-K_S\cdot {A_S}^{\nu-1}\geq -\omega_G^0\cdot {A_G}^{\nu-1}>(\nu-1){A_G}^{\nu}.$$

Thus, $-{A_G}^{\nu}<(\omega_G^0+(\nu-1)A_G)\cdot {A_G}^{\nu-1}<0$. Note that the scheme structure of F enforces (G,A_G) to be a locally Gorenstein pre-polarized variety (Definition 2.2). Hence, we are able to apply Proposition 2.9 in order to have $-{A_G}^{\nu}<2g(G,A_G)-2<0$, or

$$1-\frac{{A_G}^{\nu}}{2}<g(G,A_G)<1. \tag{8.7}$$

In particular, ${A_{\tilde{G}}}^{\nu}={A_G}^{\nu}>2$. On the one hand, from Proposition 2.5, it follows that $g(G,A_G)\geq g(\tilde{G},A_{\tilde{G}})$. On the other hand, the restriction that $\nu=3$ enables us to apply Theorem 2.10 ([6], Corollary (4.8)) to a normal polarized variety $(\tilde{G},A_{\tilde{G}})$ of dimension three which asserts that $g(\tilde{G},A_{\tilde{G}})\geq 0$, and that if $g(\tilde{G},A_{\tilde{G}})=0$, then $\Delta(\tilde{G},A_{\tilde{G}})=0$. Therefore, (8.7) yields the desired result in Lemma 8.5. *q.e.d.*

Let $\tilde{D}=\mu^*D$, the pull-back of the positive Cartier divisor D. Then, $\tilde{D}$ is also a positive Cartier divisor on $\tilde{G}$ since locally, $\mu_*\mathcal{O}_{\tilde{G}}$ is the integral closure of the integral domain $\mathcal{O}_G$.

Lemma 8.6. *If $g(G,A_G)=g(\tilde{G},A_{\tilde{G}})=0$, then $K_{\tilde{G}}\cong\mu^*\omega_G^0\cong-(\nu-1)A_{\tilde{G}}-\tilde{D}$, and $\tilde{D}\cdot {A_{\tilde{G}}}^{\nu-1}=2$.*

Proof. From Proposition 2.5, $g(G,A_G)=g(\tilde{G},A_{\tilde{G}})$ implies that the non-normal locus G_{non} of G has dimension less than $\nu-1$, or $\dim G_{\mathrm{non}}<\nu-1$. On the one hand,

$$\mu|_{\tilde{G}\setminus\mu^{-1}G_{\mathrm{non}}}:\tilde{G}\setminus\mu^{-1}G_{\mathrm{non}}\to G\setminus G_{\mathrm{non}}$$

is an isomorphism. Hence, $K_{\tilde{G}}\cong\mu^*\omega_G^0$ on $\tilde{G}\setminus\mu^{-1}G_{\mathrm{non}}$. However, this isomorphism extends to the whole normal variety $\tilde{G}$ since $\dim\mu^{-1}G_{\mathrm{non}}<\nu-1$, and $\mu^*\omega_G^0$ is a reflexive sheaf of rank one on $\tilde{G}$. The rest of the first assertion follows from (8.3) and $(r,q)=(1,1)$.

As for the second assertion, combine the assumption with [6], Lemma (1.8), then $g(S,A_S)=g(\tilde{G},A_{\tilde{G}})=0$. Hence, we obtain $(K_S+(\nu-1)A_S)\cdot {A_S}^{\nu-1}=2g(S,A_S)-2=-2$. This equality, (8.4), (8.5) and (8.3) imply

$$\begin{array}{rcl}(\nu-1){A_S}^{\nu} & = & -K_S\cdot {A_S}^{\nu-1}-2 \\ & \geq & -(g_*K_S)\cdot {A_G}^{\nu-1}-2 \\ & \geq & -\omega_G^0\cdot {A_G}^{\nu-1}-2 \\ & = & (\nu-1){A_G}^{\nu}+D\cdot {A_G}^{\nu-1}-2 \\ & \geq & (\nu-1){A_G}^{\nu}-1.\end{array}$$

Therefore, $-\omega_G^0 \cdot {A_G}^{\nu-1} - 2 = (\nu-1){A_G}^{\nu} - 1$ or $(\nu-1){A_G}^{\nu}$. Note that the first possibility is easily ruled out. Otherwise, from Proposition 2.9, $2g(G, A_G) - 2 = (\omega_G^0 + (\nu-1)A_G) \cdot {A_G}^{\nu-1} = -1$. This is impossible since $g(G, A_G) \in \mathbb{Z}$. From the second case, it follows that $D \cdot {A_G}^{\nu-1} = 2$, and thus $\tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = 2$. *q.e.d.*

Lemma 8.7. *Assume that the condition* (8.b) *holds,* $(r, q) = (1, 1)$ *and* $\nu = 3$*. Then,* $h^0(S, K_S + \nu A_S) = 1$*.*

Proof. If $h^0(S, K_S + \nu A_S) \neq 1$, then from the previous Lemmas 8.5 and 8.6, it would follow that $g(\tilde{G}, A_{\tilde{G}}) = \Delta(\tilde{G}, A_{\tilde{G}}) = 0$, ${A_{\tilde{G}}}^{\nu} > \tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = 2$, and $K_{\tilde{G}} \cong -(\nu-1)A_{\tilde{G}} - \tilde{D}$ $\quad(\tilde{D} > 0,\ \nu = 3)$. From $\Delta(\tilde{G}, A_{\tilde{G}}) = 0$, [4], Theorem (4.11) provides the structure of the normal polarized variety $(\tilde{G}, A_{\tilde{G}})$ as follows. Since $A_{\tilde{G}}$ is very ample, $|\ A_{\tilde{G}}\ |$ induces the embedding $\tilde{G} \hookrightarrow \mathbb{P}^N$, where $N = {A_{\tilde{G}}}^{\nu} + \nu - 1$, and we identify $\tilde{G}$ with its image in $\mathbb{P}^N$. Note that $A_{\tilde{G}} \cong \mathcal{O}_{\mathbb{P}^N}(1)|_{\tilde{G}}$, and $\dim \tilde{G}_{\text{sing}} \leq \nu - 2$, where $\tilde{G}_{\text{sing}}$ is the singular locus of $\tilde{G}$. From [ibid.], there is a linear subspace T of $\mathbb{P}^N$ such that

1. $\operatorname{codim}(T, \mathbb{P}^N) = \dim \tilde{G}_{\text{sing}} + 1$,
2. $M := \tilde{G} \cap T$ is a smooth subvariety of $\tilde{G}$ with $\Delta(M, A_M) = 0$, where $A_M := A_{\tilde{G}}|_M$, and
3. $\tilde{G} = M * \tilde{G}_{\text{sing}}$, the union of lines each of which goes through two different points, one in M and the other in $\tilde{G}_{\text{sing}}$.

We note that if $m = \dim M < \dim \tilde{G} = \nu$, then $\tilde{G}$ must be singular. Therefore, from [4], Corollary (4.13), it follows that $\operatorname{Pic}(\tilde{G}) \cong \mathbb{Z} \cdot A_{\tilde{G}}$. Hence, there is a positive integer l such that $\tilde{D} \sim lA_{\tilde{G}}$, and thus

$$\tilde{D}_M := \tilde{D}|_M \sim lA_M \quad (0 < l \in \mathbb{Z}). \tag{8.8}$$

From [ibid.], Theorems (4.9) and (4.11), the condition that ${A_{\tilde{G}}}^{\nu} \geq 3$ possibly allows (M, A_M) to be only either a Veronese surface or a scroll over a 1-dimensional projective space. We show that neither of these two cases occurs, which shows, in turn, $h^0(S, K_S + \nu A_S) = 1$ and finishes the proof of Lemma 8.7.

Claim 8.8. (M, A_M) cannot be a Veronese surface $(\mathbb{P}^2, \mathcal{O}(2))$.

Proof. If this were the case, it would follow that $(M, A_M) \cong (\mathbb{P}^2, \mathcal{O}(2))$, and thus $m = 2 < 3 = \nu$. Hence, from (8.8), $\tilde{D}_M \cong \mathcal{O}_{\mathbb{P}^2}(2l)$. Therefore, $\tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = \tilde{D}_M \cdot {A_M}^{\nu-2} = 2l \cdot 2 = 4l$. This contradicts that $\tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = 2$. q.e.d. of Claim 8.8.

Claim 8.9. (M, A_M) cannot be a scroll.

Proof. If so, it would follow that $(M, A_M) \cong (\mathbb{P}(\mathcal{E}), \xi)$ which has a $\mathbb{P}^{m-1}$-bundle structure $\pi : M \longrightarrow \mathbb{P}^1$, where $m = \dim M$, $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)$, and $a_i > 0$ for any i. Since $\operatorname{codim}(T, \mathbb{P}^N) \leq \nu - 1$, we have $m = \dim M \geq \nu - (\nu - 1) = 1$, and thus $m = 1, 2$ or $3 = \nu$.

Case $m = 1$**:** $h^1(M, \mathcal{O}_M) = g(M, A_M) = 0$ since $g(\tilde{G}, A_{\tilde{G}}) = 0$. Hence, $M \cong \mathbb{P}^1$. On the one hand, $\deg A_M = {A_{\tilde{G}}}^{\nu} \geq 3$, and thus $A_M \cong \mathcal{O}_{\mathbb{P}^1}(d)$, $(d \geq 3)$. Since $m = 1 < 3 = \nu$, from (8.8), $\tilde{D}_M \cong \mathcal{O}_{\mathbb{P}^1}(ld)$. Hence, $\tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = \deg \tilde{D}_M = ld \geq 3$, a contradiction to $\tilde{D} \cdot {A_{\tilde{G}}}^{\nu-1} = 2$.

Case $m \geq 2$**:** We do not use (8.8), since this case includes the one that $\tilde{G} = M$ or $\tilde{G}$ is nonsingular, and [4], Corollary (4.13) is not applicable. From [8], Exercise 7.9 in Chap. II, $\operatorname{Pic}(M) \cong \operatorname{Pic}(\mathbb{P}^1) \times \mathbb{Z} \cong \mathbb{Z}\mathbf{f} + \mathbb{Z}A_M$, where $\mathbf{f}$ is a fiber of π. From the adjunction formula, $K_M \cong [K_{\tilde{G}} + (\nu - m)A_{\tilde{G}}]|_M$. Hence, $K_M \cong [-(\nu-1)A_{\tilde{G}} - \tilde{D} + (\nu - m)A_{\tilde{G}}]|_M = -(m-1)A_M - \tilde{D}_M$. Since $\tilde{D} > 0$, $\tilde{D}_M \cong a\mathbf{f} + bA_M$ for some integers $a, b \geq 0$. Hence,

$$K_M \cong -a\mathbf{f} - (m + b - 1)A_M. \tag{8.9}$$

From the canonical bundle formula,

$$\begin{aligned} K_M &\cong \pi^*(K_{\mathbb{P}^1} + \det \mathcal{E}) - m\xi \\ &\cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(-2) + \mathcal{O}_{\mathbb{P}^1}(\textstyle\sum_{i=1}^m a_i)) - mA_M \\ &\cong (\textstyle\sum_{i=1}^m a_i - 2)\mathbf{f} - mA_M. \end{aligned} \tag{8.10}$$

Compare (8.10) with (8.9), then $\sum_{i=1}^m a_i + a = 2$. This is absurd since $\sum_{i=1}^m a_i = \deg(\det \mathcal{E}) = {A_M}^m \geq 3$. Hence, Case $m \geq 2$ does not occur. q.e.d. of Claim 8.9.

From Claims 8.8 and 8.9, we have proven that $h^0(S, K_S + \nu A_S) = 1$. q.e.d. of Lemma 8.7.

Recall that we are in Case (8.b) in the proof of the second statement of Lemma 8.3. From this and Lemma 8.7, we have obtained the following conditions:

$$H^0(S, K_S + tA_S) = 0 \text{ for any } 1 \leq t < \nu, \text{ and } h^0(S, K_S + \nu A_S) = 1.$$

Thus, [6], Theorem (2.3) implies either

1. ${A_{\tilde{G}}}^\nu = g(\tilde{G}, A_{\tilde{G}}) = 1$, or
2. $(\tilde{G}, A_{\tilde{G}}) \cong (\boldsymbol{Q}^\nu, \mathcal{O}(1))$.

If the first case occurred, it would follow from [ibid.], Lemma (1.8) that $(K_S+(\nu-1)A_S)\cdot {A_S}^{\nu-1} = 2g(S, A_S)-2 = 2g(\tilde{G}, A_{\tilde{G}}) - 2 = 0$. Hence, $(\nu-1){A_S}^\nu = -K_S \cdot {A_S}^{\nu-1}$, which leads to a contradiction exactly as in the proof of Claim 8.4. Therefore, only the second case above possibly occurs, and we have proven Lemma 8.3. *q.e.d.*

We can express $\mathcal{N}_{G/M}$ and D by A_G so as to get a more detailed list of $(\tilde{G}, A_{\tilde{G}})$.

Lemma 8.10. *The normalization $\mu : \tilde{G} \to G$ of an arbitrary irreducible component G of F satisfies the following list:*

$$(\tilde{G}, A_{\tilde{G}}, \mathcal{N}_{G/M}, D) \cong \begin{cases} (\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(1), -3A_G, 0A_G) & \textit{if } (r,q) = (1,3) \\ (\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(1), -(4-r)A_G, A_G) & \textit{if } (r,q) = (1,2) \textit{ or } (2,1) \\ (\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(1), -3A_G, 2A_G), & \\ \textit{or} & \textit{if } (r,q) = (1,1) \textit{ and if } \nu = 3 \\ (\boldsymbol{Q}^\nu, \mathcal{O}_{\boldsymbol{Q}^\nu}(1), -2A_G, A_G) & \end{cases}$$

Proof. Either case in Lemma 8.3 implies that $g(\tilde{G}, A_{\tilde{G}}) = 0$. On the other hand, from Proposition 2.9 and (8.3), $2g(G, A_G) - 2 = (\omega_G^0 + (\nu-1)A_G) \cdot {A_G}^{\nu-1} = -(r+q-2){A_G}^\nu - D \cdot {A_G}^{\nu-1} < 0$, and thus $g(G, A_G) \leq 0$. From Proposition 2.5, $g(G, A_G) \geq g(\tilde{G}, A_{\tilde{G}})$. Hence, $g(G, A_G) = g(\tilde{G}, A_{\tilde{G}}) = 0$, and from Proposition 2.5, this implies

$$\dim G_{\text{non}} < \nu - 1. \tag{8.c}$$

Therefore, the same argument as in Lemma 8.6 shows that $K_{\tilde{G}} \cong \mu^*\omega_G^0 \cong -(\nu+r+q-3)A_{\tilde{G}} - \tilde{D}$. Hence, from Lemma 8.3, it follows that

$$\tilde{D} \cong \begin{cases} 0A_{\tilde{G}} & \text{if } (r,q) = (1,3) \\ A_{\tilde{G}} & \text{if } (r,q) = (1,2) \text{ or } (2,1) \\ 2A_{\tilde{G}}, & \\ \text{or} & \text{if } (r,q) = (1,1) \text{ and if } \nu = 3 \\ A_{\tilde{G}} & \end{cases}$$

To obtain the result on D in the statement of Lemma 8.10, it suffices to show the following

Claim 8.11. Let B be any Cartier divisor on G. If $\mu^*B \cong \mathcal{O}_{\tilde{G}}$, then $B \cong \mathcal{O}_G$.

Proof. We apply [14], Proposition on p. 65:

Fact 8.12. For two Cartier divisors D_1 and D_2 on a Noetherian scheme X, $D_1 = D_2$ as divisors if and only if the images of D_1 and D_2 in the stalk $(\mathcal{K}_X^*/\mathcal{O}_X^*)_p$ are equal for all $p \in X$ wherever $\operatorname{depth}(\mathcal{O}_{X,p}) = 1$, where $\mathcal{K}_X$ denotes the sheaf of total quotient rings of X.

Since $\mu^*B \cong \mathcal{O}_{\tilde{G}}$, there is a rational function $\tilde{f} \in H^0(\tilde{G}, \mathcal{K}_{\tilde{G}}^*)$ such that $\mu^*B = \operatorname{div}(\tilde{f})$, where $\operatorname{div} : H^0(\tilde{G}, \mathcal{K}_{\tilde{G}}^*) \to H^0(\tilde{G}, \mathcal{K}_{\tilde{G}}^*/\mathcal{O}_{\tilde{G}}^*)$ is the natural homomorphism. Since $\mu : \tilde{G} \to G$ is birational, $\mu^* : H^0(G, \mathcal{K}_G^*) \to H^0(\tilde{G}, \mathcal{K}_{\tilde{G}}^*)$ is an isomorphism. Hence, there is a rational function $f \in H^0(G, \mathcal{K}_G^*)$ such that $\mu^*(f) = \tilde{f}$, and thus $\mu^*B = \mu^*\operatorname{div}(f)$. Note that through the isomorphism $\mu|_{\tilde{G}\setminus\mu^{-1}G_{\text{non}}} : \tilde{G} \setminus \mu^{-1}G_{\text{non}} \to G \setminus G_{\text{non}}$, it follows that $B = \operatorname{div}(f)$ on $G \setminus G_{\text{non}}$. In other words, the condition in the above Fact 8.12 is satisfied except on the non-normal locus G_{non} of G. However, we can show that $\operatorname{depth}\mathcal{O}_{G,p} \geq 2$ for any $p \in G_{\text{non}}$. Because G is locally Cohen-Macaulay, $\mathcal{O}_{G,p}$ is a Cohen-Macaulay local ring, and hence $\operatorname{depth}\mathcal{O}_{G,p} = \dim \mathcal{O}_{G,p} \geq 2$, where we applied (8.c) to the last inequality. Therefore, from Fact 8.12, we derive that $B = \operatorname{div}(f)$ on the whole G, or $B \cong \mathcal{O}_G$. q.e.d. of Claim 8.11.

We simply apply (8.2) to calculate $\mathcal{N}_{G/M}$ based on the above results on D. q.e.d. of Lemma 8.10.

Let us finish up the proof of Theorem 8.1. The results of Lemma 8.10 apply to the investigation of the three cases: $q = 3, 2, 1$. Recall that the irreducible components G's of F are prime Cartier divisors on the smooth variety M, and each of the components intersects at least one of the others.

Case $q = 3$**:** From (8.2), $D > 0$, a contradiction.

Case $q = 2$**:** $\tilde{D} \cong A_{\tilde{G}} \cong \mathcal{O}_{\mathbb{P}^\nu}(1)$. This implies that D is a prime divisor. Hence, each component intersects with only one of the other components. Since F is connected, there are only two irreducible components total: $F = G+G'$. Thus, taking an integral curve Z in $G \cap G'$, we obtain that $F \cdot Z = G \cdot Z + G' \cdot Z = \mathcal{N}_{G/M} \cdot Z + \mathcal{N}_{G'/M} \cdot Z = -3A_G \cdot Z - 3A_{G'} \cdot Z$. On the other hand, from (8.1), $F \cdot Z = -qA_G \cdot Z = -2A_G \cdot Z$. This is absurd.

Case $q = 1$**:** There are possibly three kinds of irreducible components of F:

$$(\tilde{G}, A_{\tilde{G}}, \mathcal{N}_{G/M}, D) \cong \begin{cases} a) & (\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(1), -2A_G, A_G) \\ b) & (\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(1), -3A_G, 2A_G) \\ c) & (\boldsymbol{Q}^\nu, \mathcal{O}_{\boldsymbol{Q}^\nu}(1), -2A_G, A_G) \end{cases} \tag{8.11}$$

For this situation, we prepare the following

Claim 8.13. F has at most three irreducible components.

Proof. Note first that from (8.11), the information divisor D allows each irreducible component of F to intersect at most two of the other irreducible components. Assume that G, G' and G'' are distinct irreducible components of F and that G intersects with both G' and G''. Since $\tilde{G}$ is either $\mathbb{P}^\nu$ or $\boldsymbol{Q}^\nu$, the pull-back of the positive Cartier divisors $G'|_G$ and $G''|_G$ of G are ample. Hence, $G'|_G$ and $G''|_G$ are ample since μ is a finite morphism. This tells that if there are three distinct irreducible components G, G' and G'' whose union $G \cup G' \cup G''$ is connected, then, we have

$$\dim G \cap G' \cap G'' \geq 1 \tag{8.12}$$

If there were the fourth irreducible component G''' which meets $G \cup G' \cup G''$, say $G''' \cap G \neq \emptyset$, then, from (8.12), G''' would have to meet $G \cap G' \cap G''$ since $G'''|_G$ is an ample Cartier divisor on G by the same argument as above. This contradicts the restriction posed by D at the beginning of the proof. q.e.d. of Claim 8.13.

From the Claim 8.13, $F = G + G'$ or $G + G' + G''$. Note that (8.12) enables us to take an integral curve Z inside $G \cap G'$ or $G \cap G' \cap G''$ in either case, and thus

$$F \cdot Z = G \cdot Z + G' \cdot Z \text{ or } G \cdot Z + G' \cdot Z + G'' \cdot Z. \tag{8.13}$$

Letting $G^\#$ stand for G, G' or G'', it follows from (8.11) that $G^\# \cdot Z = -iA_{G^\#} \cdot Z$ for $i = 2$ or 3. On the other hand, (8.1) implies that $F \cdot Z = -A_{G^\#} \cdot Z$. This value $-A_{G^\#} \cdot Z$ is always strictly larger than either value of the right hand side in (8.13) for any $i = 1, 2$. This is a contradiction, and we have thus excluded all the possible cases arising from the Assumption 8.2 that F is not irreducible.
Therefore, F must be irreducible, and the proof of Theorem 8.1 is completed.

9. Appendix: Proof of Proposition 1.2

In this appendix, we prove Proposition 1.2 which supports Definition 1.3 of the k-th adjoint contraction. ([15] does not include the proof.) Recall, first of all, the definition of numerical reducedness.

Definition 9.1 (cf. [15], Definition 1.1)**.** Let $f : X \to Y$ be the ray contraction for an extremal ray R. An f-ample Cartier divisor A is called to be *numerically reduced* if there is no ample Cartier divisor A' such that

$$A \cdot C > A' \cdot C$$

for a nonzero effective curve C whose numerical class is in R, where a curve means an element in $Z_1(X)$, the free Abelian group generated by one-dimensional closed integral subvarieties of X.

Given a nonzero effective curve C in R, we define

$$M(C) \overset{\text{def}}{=} \{A' \cdot C | A' \text{ is an } f\text{-ample Cartier divisor on } X\}.$$

$A' \cdot C$ are all positive integers and thus there is an f-ample Cartier divisor A on X such that

$$A \cdot C = \min M(C) \tag{9.1}$$

Lemma 9.2 (cf. [15], Lemma 4.1)**.** *Let A be a Cartier divisor defined by (9.1).*

1. *A is numerically reduced, and thus a numerically reduced divisor always exists for a given ray R.*
2. *For any Cartier divisor D on X, there is a unique integer δ such that*
$$D \equiv \delta A \mod f^*\mathrm{Div}(Y).$$
In particular, a numerically reduced divisor A is unique modulo $f^*\mathrm{Div}(Y)$, and does not depend on the choice of a curve C whose numerical class is in R.
3. *We can choose a numerically reduced divisor A to be ample by adding $f^*(m\mathcal{A})$, where $\mathcal{A}$ is an ample Cartier divisor on Y and m is a large enough integer $m >> 1$.*

Proof. 1. If A were not numerically reduced, then there would be an f-ample Cartier divisor A', a nonzero effective curve C' in R and an integer $p > 1$ such that

$$A \cdot C' = pA' \cdot C'.$$

Note that there is a positive real number a such that $C' \approx aC$ in R, where C is the fixed curve used in $M(C)$, and $\approx$ denotes the numerical equivalence for curves on X. Hence, $A \cdot C = pA' \cdot C$, a contradiction to $A \cdot C = \min M(C)$. Therefore, A must be numerically reduced.

2. If $D \cdot C < 0$, we would consider $-D$ instead of D. So, we can assume that $D \cdot C \geq 0$. Divide $D \cdot C$ by $A \cdot C$, and we obtain unique integers δ and r such that

$$D \cdot C = \delta A \cdot C + r, \quad 0 \leq r < A \cdot C \quad \text{ and } \delta \geq 0.$$

Then, $r = 0$. Otherwise, $(D - \delta A) \cdot C = r > 0$, and thus $D - \delta A$ is f-ample. However, $(D - \delta A) \cdot C < A \cdot C$, which contradicts the choice of A in (9.1). Hence, $D \cdot C = \delta A \cdot C$. From [10], Lemma 3-2-5, $D \equiv \delta A \mod f^*\mathrm{Div}(Y)$.

3. This is nothing but one of the characterizations for A to be f-ample. *q.e.d.*

Lemma 9.3. *Let $f : X \to Y$ be the ray contraction for an extremal ray R. There is a numerically reduced, ample Cartier divisor A on X which is unique modulo $f^*\mathrm{Div}(Y)$, and there is a unique integer k with $-1 \leq k \leq n - 1$ such that*

$$R = (K_X + (n - k)A)^\perp \cap \overline{NE}(X). \tag{9.2}$$

Proof. Note that from the first statement in Lemma 9.2, we have a numerically reduced divisor A for $f : X \to Y$. From [10], Theorem 3-2-1, there is a nef Cartier divisor H such that $R = H^\perp \cap \overline{NE}(X)$, and $K_X \cdot C < 0$ for any nonzero curve C in R. Hence,

$$(H - K_X) \cdot C = -K_X \cdot C > 0,$$

and thus $H - K_X$ is f-ample. Therefore, from the second statement in Lemma 9.2, there exists a unique integer $l \geq 1$ such that $H - K_X \equiv lA \mod f^*\mathrm{Div}(Y)$, or

$$H \equiv K_X + lA \qquad \mod f^*\mathrm{Div}(Y). \tag{9.3}$$

To finish the proof, it suffices to show that

$$R = (K_X + lA)^\perp \cap \overline{NE}(X), \tag{9.4}$$

since from [10], Theorem 4-2-1, we have $l \leq n + 1$, and then the number k be defined to be $n - l$.
To show $\subset$ in (9.4), take $\gamma \in R$. Since γ is the zero map on $f^*\mathrm{Div}(Y)$, (9.3) implies that

$$0 = H \cdot \gamma = (K_X + lA) \cdot \gamma.$$

Thus, $\gamma \in (K_X + lA)^\perp \cap \overline{NE}(X)$.
On the one hand, from (9.3), it follows that

$$H + f^*\mathcal{A} \sim K_X + lA \text{ for some } \mathcal{A} \in \mathrm{Div}(Y),$$

where $\sim$ denotes the linear equivalence. Take an ample Cartier divisor $\mathcal{L}$ on Y, and add $f^*(lm\mathcal{L})$, then

$$H + f^*(\mathcal{A} + lm\mathcal{L}) \sim K_X + l(A + f^*(m\mathcal{L})).$$

We can choose an integer m large enough that $\mathcal{A} + lm\mathcal{L}$ is ample. Hence, writing $\mathcal{A}$ and A for $\mathcal{A} + lm\mathcal{L}$ and $A + f^*(m\mathcal{L})$, respectively, we obtain

$$H + f^*\mathcal{A} \sim K_X + lA, \tag{9.5}$$

where A is a numerically reduced ample divisor, and $f^*\mathcal{A}$ and $K_X + lA$ are nef. We are ready to show $\supset$ in (9.4). Take any non-zero $\gamma \in (K_X + lA)^\perp \cap \overline{NE}(X)$. Then, from the cone theorem ([10], Theorem 4-2-1), $\gamma \in \overline{NE}(X)$ implies

$$\gamma \approx \gamma^+ + \sum_{j=1}^{r} a_j C_j,$$

where γ^+ is an element in $\overline{NE}_{K_X}(X)$, C_j are integral curves which generate different extremal rays, a_j are non-negative real numbers. Since $\gamma \in (K_X + lA)^\perp$,

$$0 = (K_X + lA) \cdot \gamma = (K_X + lA) \cdot \gamma^+ + \sum_{j=1}^{r} a_j (K_X + lA) \cdot C_j.$$

Now that $K_X + lA$ is nef, it follows that $0 = (K_X + lA) \cdot \gamma^+ \geq A \cdot \gamma^+$, and thus $\gamma^+ \approx 0$. Hence, we obtain

$$\gamma \approx \sum_{j=1}^{r} a_j C_j.$$

From (9.5),

$$0 = H \cdot \gamma + (f^*\mathcal{A}) \cdot \gamma.$$

Since H and $f^*\mathcal{A}$ are both nef, we have

$$0 = H \cdot \gamma = \sum_{j=1}^{r} a_j H \cdot C_j,$$

and $a_j H \cdot C_j = 0$ for all $j = 1, \ldots, r$. If there were more than one j such that $a_j > 0$, then $H \cdot C_j = 0$ and $C_j \in R$ for more than one C_j, a contradiction to the choice that $C_1, \ldots, C_r$ generate distinct extremal rays. Hence, there is only one non-zero a_{j_0} and all the others are zero, i.e., $\gamma \approx a_{j_0} C_{j_0} \in R$. *q.e.d.*

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Current address: Max Planck Institute for Mathematics, P.O. Box 7280, D-53072 Bonn, Germany
E-mail address: `ngilbert@mpim-bonn.mpg.de`