The universal coefficient theorem for quadratic functors

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THE UNIVERSAL COEFFICIENT THEOREM FOR QUADRATIC FUNCTORS

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Let X be a simplicial abelian group and let F be a functor which carries abelian groups to abelian groups so that FX is again a simplicial abelian group. The homotopy groups $\pi_* X$ and $\pi_* FX$ are defined as the homology of the corresponding Moore complexes. Since the work of Kan and Dold-Puppe it is a well known problem to compute $\pi_* FX$ in terms of $\pi_* X$ and invariants of the functor F. If F is an additive functor there is a classical solution which is derived from the universal coefficient theorem for homology groups. We here show that a similar kind of coefficient theorem also holds if F is a quadratic functor. We use quadratic modules [1] to formulate explicitly the graded quadratic tensor and torsion products needed. As an application we compute in § 5 the homology [11] in the variety of groups of nilpotency degree 2. We also compute the quadratic functors [8,9] of Ellis in § 6.

§ 1 <u>UNIVERSAL COEFFICIENT THEOREM FOR ADDITIVE FUNCTORS</u>

The Dold-Kan equivalence shows that the category of simplicial abelian groups is equivalent to the category of non-negative chain complexes. The equivalence carries X to the Moore chain complex $C = NX$ so that $\pi_* X = H_* C$ is the homology of C. Let $\mathbf{Ab}$ be the category of abelian groups and let $F : \mathbf{Ab} \to \mathbf{Ab}$ be an additive functor which preserves direct limits of direct systems. Then one has for free abelian groups A the natural isomorphism

$$F(A) = A \otimes M \quad \text{where} \quad M = F(\mathbb{Z}). \tag{1.1}$$

Moreover, if X consists of free abelian groups then $FX = X \otimes M$ and $NFX = C \otimes M$ so that one gets the well known <u>universal coefficient theorem</u> given by the following short exact sequences.

$$\begin{array}{ccccccccc}
 & & & & \pi_* FX & & & & \\
 & & & & \| & & & & \\
0 & \longrightarrow & (\pi_* X) \otimes M & \xrightarrow{\Delta} & \pi_*(X \otimes M) & \xrightarrow{\mu} & (\pi_* X) * M & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & (H_* C) \otimes M & \longrightarrow & H_*(C \otimes M) & \longrightarrow & (H_* C) * M & \longrightarrow & 0
\end{array}$$

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Here the right hand side is defined by the torsion product of abelian groups $A * B = Tor_1(A, B)$. The rows are binatural short exact and split (unnaturally). The map Δ is of degree 0 and μ is of degree -1. The bottom row is the classical universal coefficient theorem for the homology of chain complexes; see for example [5] and [15].

§ 2 Quadratic functors and quadratic modules

A functor $F : \mathbf{A} \to \mathbf{B}$ between two additive categories is termed quadratic if $F(0) = 0$ and if the cross effect

$$(2.1) \qquad F(A \mid B) = \mathrm{kernel}(F(A \oplus B) \to FA \oplus FB)$$

is biadditive. This yields the binatural isomorphism

$$F(A \oplus B) = F(A) \oplus F(B) \oplus F(A \mid B)$$

Moreover, for any object A one gets the diagram

$$(2.2) \qquad F\{A\} = (F(A) \xrightarrow{H} F(A \mid A) \xrightarrow{P} F(A))$$

where H is induced by the diagonal map $A \to A \oplus A$ and P is induced by the codiagonal $A \oplus A \to A$.

(2.3) Definition. A quadratic module in $\mathbf{B}$ is a diagram

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

satisfying $HPH = 2H$ and $PHP = 2P$. For example $F\{A\}$ is always a quadratic module in $\mathbf{B}$.

Now assume $F : \mathbf{Ab} \to \mathbf{Ab}$ is a quadratic functor which preserves direct limits of direct systems. Then one has for free abelian groups A a natural isomorphism which is the quadratic analogue of (1.1):

$$(2.4) \qquad F(A) = A \otimes M \quad \text{where} \quad M = F\{\mathbb{Z}\}$$

Here $A \otimes M$ denotes the quadratic tensor product in [1] defined as follows:

(2.5) Definition. Let A be an abelian group and let M be a quadratic module in $\mathbf{Ab}$. Then $A \otimes M$ is the abelian group with generator $a \otimes m$, $[a, b] \otimes n$ for $a, b \in A$, $m \in M_e$, $n \in M_{ee}$ and relations

$$(a + b) \otimes m = a \otimes m + b \otimes m + [a, b] \otimes Hm$$
$$[a, a] \otimes n = a \otimes P(n)$$

where $a \otimes m$ is linear in m and $[a,b] \otimes n$ is linear in a, b and n.

(2.6) *Remark.* The category consisting of quadratic functors $F : \mathbf{Ab} \to \mathbf{Ab}$ which preserve direct limits of direct systems and cokernels is equivalent to the category of quadratic modules in $\mathbf{Ab}$. The equivalence carries F to $F\{\mathbb{Z}\}$ with the inverse carrying M to the functor $A \mapsto A \otimes M$.

Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories where $\mathbf{A}$ has enough projective objects. For any functor $F : \mathbf{A} \to \mathbf{B}$ one obtains the derived functors

$$\begin{aligned} &L_n F : \mathbf{A} \to \mathbf{B} \quad \text{with} \\ &(L_n F)(A) = \pi_n(FK(A,0)). \end{aligned} \tag{2.7}$$

Here $K(A,m)$ is a simplicial object in $\mathbf{A}$ with projective components such that $\pi_m K(A,m) = A$, $\pi_i K(A,m) = 0$ for $i \neq m$. If F is quadratic and if the projective dimension of A is ≤ 1 then $L_n F = 0$ for $n \geq 3$. Moreover in this case given a projective resolution

$$0 \longrightarrow A_1 \xrightarrow{d} A_0 \longrightarrow A \longrightarrow 0 \tag{1}$$

of A we obtain the chain complex

$$F(A_1 \mid A_1) \xrightarrow{\delta_2} F(A_1) \oplus F(A_1 \mid A_0) \xrightarrow{\delta_1} F(A_0) \tag{2}$$

with $\delta_1 = (F(d), PF(d \mid 1))$ and $\delta_2 = (P, -F(1 \mid d))$ which satisfies

$$(L_n F)A = \begin{cases} \operatorname{cokernel} \delta_1 & n = 0 \\ \operatorname{kernel} \delta_1 / \operatorname{image} \delta_2 & n = 1 \\ \operatorname{kernel} \delta_2 & n = 2 \end{cases} \tag{3}$$

If $F : \mathbf{Ab} \to \mathbf{Ab}$ is given by $F(A) = A \otimes M$ we have for $A, B \in \mathbf{Ab}$ the cross effect

$$F(A \mid B) = (A \mid B) \otimes M = A \otimes B \otimes M_{ee}$$

and in this case (2) has the form

$$A_1 \otimes A_1 \otimes M_{ee} \xrightarrow{\delta_2} A_1 \otimes M \oplus A_1 \otimes A_0 \otimes M_{ee} \xrightarrow{\delta_1} A_0 \otimes M. \tag{4}$$

For $a, a' \in A_1$, $b \in A_0$, $m \in M_e$, $n \in M_{ee}$ we obtain δ_1 and δ_2 by

$$\begin{aligned} \delta_1(a \otimes m) &= (da) \otimes m \\ \delta_1([a,a'] \otimes n) &= [da, da'] \otimes n \\ \delta_1(a \otimes b \otimes n) &= [da, b] \otimes n \\ \delta_2(a \otimes a' \otimes n) &= -a \otimes da' \otimes n + [a, da'] \otimes n \end{aligned}$$

Now we have $(L_0F)A = A \otimes M$ and we obtain the quadratic torsion functors [1]

$$(5) \qquad (L_1F)A = A *' M \quad \text{and} \quad (L_2F)A = A *'' M.$$

The proof of (2.7) (3) is based on 4.23 in [6] since the normalization of $FK(A,0)$ in low degrees coincides with (2.7) (2); compare also [1] where various examples of quadratic torsion products are computed.

§ 3 Universal coefficient theorem for quadratic functors

We introduce the graded quadratic tensor and torsion products needed for the quadratic analogue of the universal coefficient theorem in § 1. Each quadratic functor F yields the associated chain complex

$$(3.1) \qquad F_*(A) = \{F(A) \xleftarrow{P} F(A \mid A) \xleftarrow{1-T} F(A \mid A) \xleftarrow{1+T} F(A \mid A) \xleftarrow{1-T} \ldots\}$$

where $F(A)$ is in degree 0 and the differential d_i is given by $d_1 = P$, $d_{2n} = 1 - T$, $d_{2n+1} = 1 + T$ for $n \geq 1$ with $T = HP - 1$. One readily checks that $TT = 1$ and $dd = 0$. Now assume the functor F is given by $F(A) = A \otimes M$. Then the quadratic functor F_n in (3.1) with $F_n(A) = F(A \mid A)$ for $n \geq 1$ satisfies

$$F_n(A \mid A) = A \otimes A \otimes M_{ee}$$

so that $F_n\{\mathbb{Z}\}$ for $n \geq 1$ is given by the quadratic module

$$(M_{ee} \xrightarrow{(1,1)} M_{ee} \oplus M_{ee} \xrightarrow{(1,1)} M_{ee})$$

Hence the chain complex (3.1) corresponds by the equivalence in (2.6) to the following chain complex associated to M in the category of quadratic modules.

(3.2)
$$M_* = \left\{ \begin{tikzcd} M_{ee} \arrow[d, shift right] & M_{ee} \oplus M_{ee} \arrow[l, "{(1,T)}"'] \arrow[d, shift right] & M_{ee} \oplus M_{ee} \arrow[l, "{\binom{1,-T}{-T,1}}"'] \arrow[d, shift right] & M_{ee} \oplus M_{ee} \arrow[l, "{\binom{1,T}{T,1}}"'] \arrow[d, shift right] & \arrow[l] \\ M_e \arrow[u, shift right] & M_{ee} \arrow[l, "P"] \arrow[u, shift right] & M_{ee} \arrow[l, "1-T"] \arrow[u, shift right] & M_{ee} \arrow[l, "1+T"] \arrow[u, shift right] & \arrow[l] \end{tikzcd} \quad \cdots \right\}$$

Clearly $A \otimes M_* = F_*(A)$ for $A \otimes M = F(A)$. We can also write

$$M_* = (M_*^e \xrightarrow{H} M_*^{ee} \xrightarrow{P} M_*^e)$$

as a quadratic module in the category of chain complexes. Here M_*^e is the bottom row and M_*^{ee} is the top row in (3.2). For a chain complex C in an abelian category let Z_*C be the graded object of cycles with $Z_nC = \text{kernel}(d_n : C_n \to C_{n-1})$. In particular we shall use in definition (3.4) the graded object of cycles Z_*M_* of the chain complex M_* above. More explicitly one gets for $k \geq 1$

$$
(3.3) \qquad Z_n M_* = \begin{cases} M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e & n = 0 \\ \operatorname{Ker} p \rightarrowtail M_{ee} \xrightarrow{2-HP} \operatorname{Ker} P & n = 1 \\ \operatorname{Ker}(2-HP) \rightarrowtail M_{ee} \xrightarrow{HP} \operatorname{Ker}(2-HP) & n = 2k \\ \operatorname{Ker}(HP) \rightarrowtail M_{ee} \xrightarrow{2-HP} \operatorname{Ker}(HP) & n = 2k+1 \end{cases}
$$

where $\rightarrowtail$ denotes the inclusion. We also shall use the homology groups $H_n\, M_*^e$. The homology $H_n\, M_*^{ee} = 0$ is trivial.

(3.4) Definition of graded tensor products. Let $\mathbf{Ab}_*$ be the category of graded abelian groups A with $A_i = 0$ for $i < 0$. As usual we define for $A, B \in \mathbf{Ab}_*$ the graded tensor product $A \otimes B$ with

$$
(1) \qquad (A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j .
$$

Let $A \overset{>}{\otimes} B$ be the ordered tensor product with

$$
(2) \qquad (A \overset{>}{\otimes} B)_n = \bigoplus_{\substack{i+j=n \\ i>j}} A_i \otimes B_j .
$$

We define the graded quadratic tensor product $A \otimes M$ of $A \in \mathbf{Ab}_*$ and a quadratic module M in $\mathbf{Ab}$ by

$$
(3) \quad (A \otimes M)_n = (A \overset{>}{\otimes} A)_n \otimes M_{ee} \oplus (A \overset{>}{\otimes} H_* M_*^e)_n \oplus \begin{cases} A_m \otimes Z_m M_* & \text{if} \quad n = 2m, \\ 0 & \text{if} \quad n = 2m+1. \end{cases}
$$

Hence we obtain for each M the quadratic functor $\mathbf{Ab}_* \to \mathbf{Ab}_*$ which carries A to $A \otimes M$. One readily checks that for $A, B \in \mathbf{Ab}_*$ the cross effect satisfies

$$
(4) \qquad (A \mid B) \otimes M = A \otimes B \otimes M_{ee}
$$

yielding a relationship of the graded quadratic tensor product and the graded tensor product above. Since $\mathbf{Ab}_*$ has global dimension 1 we can apply (2.7) (2) for the definition of the graded quadratic torsion products as derived functors of (3):

$$
(5) \qquad \begin{cases} A *' M = L_1(- \otimes M)(A) \\ A *'' M = L_2(- \otimes M)(A) \end{cases}
$$

Moreover $A \otimes M = L_0(- \otimes M)(A)$ and $L_n(- \otimes M)(A) = 0$ for $n \geq 3$. Given a bifunctor F on abelian groups like the tensor product $\otimes$ or the torsion product $*$

one obtains extensions of F to graded abelian groups A, B in the same way as in (1) and (2) so that $F(A,B) \in \mathbf{Ab}_*$ and $\overset{>}{F}(A,B) \in \mathbf{Ab}_*$ are defined. Let Trp be the triple torsion product of Mac Lane; see the Notes on page 393 in [12]. Then the cross effects of (5) are given by

$$
(6) \qquad \begin{cases} (A \mid B) *' M = Trp(A, B, M_{ee}), \\ (A \mid B) *'' M = A * B * M_{ee}. \end{cases}
$$

We can describe the graded quadratic torsion products explicitely by the quadratic torsion products (2.7) (5) as follows:

(7)

$$
(A *' M)_n = \overset{>}{Trp}(A, A, M_{ee})_n \oplus (A \overset{>}{*} H_* M_*)_n \oplus \begin{cases} A_m *' Z_m M_* & \text{if} \quad n = 2m, \\ 0 & \text{if} \quad n = 2m+1 \end{cases}
$$

$$
(8) \qquad (A *'' M)_n = (A \overset{>}{*} A)_n * M_{ee} \oplus \begin{cases} A_m *'' Z_m M_* & \text{if} \quad n = 2m, \\ 0 & \text{if} \quad n = 2m+1 \end{cases}
$$

We now are ready to formulate the universal coefficient theorem for quadratic functors.

(3.5) <u>Theorem</u>. *Let X be a simplicial abelian group which is free abelian in each degree and let $F : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ be a quadratic functor given by $F(A) = A \otimes M$ where M is a quadratic module in $\underline{\underline{Ab}}$. Moreover assume that M_{ee} is torsion free. Then there is a short exact sequence, $n \in \mathbb{Z}$,*

$$
\begin{array}{ccccccccc}
 & & & & \pi_n(FX) & & & & \\
 & & & & \| & & & & \\
0 & \longrightarrow & ((\pi_* X) \otimes M)_n & \xrightarrow{\Delta} & \pi_n(X \otimes M) & \xrightarrow{\mu} & ((\pi_* X) *' M)_{n-1} & \longrightarrow & 0
\end{array}
$$

which is natural in X and M. Here the left hand side is the graded quadratic tensor product and the right hand side is the graded quadratic torsion product in (3.4). The explicit description of the inclusion map Δ is given in the remark (3.8) below.

If M_{ee} is not free abelian one gets the following spectral sequence.

(3.6) <u>Addendum</u>. Let X, F and M be given as in (3.5). If M_{ee} is not torsion free there is a homological first quadrant spectral sequence $E^2_{pq} \Rightarrow \pi_{p+q}(X \otimes M)$ with $E^3 = E^\infty$ and with differentials $d_2 : E^2_{pq} \to E^2_{p-2,q+1}$. Moreover

$$
\begin{aligned}
E^2_{0*} &= (\pi_* X) \otimes M \\
E^2_{1*} &= (\pi_* X) *' M \\
E^2_{2*} &= (\pi_* X) *'' M
\end{aligned}
$$

and $E^2_{pq} = 0$ for $p \succ 2$. The only possible non-trivial differential is $d_2 : E^2_{2,q} \to E^2_{0,q+1}$. This Addendum and the theorem are consequences of the spectral sequence in § 7 below.

(3.7) <u>*Suspension homomorphism*</u>. By Dold-Puppe [DP] one has for any functor $T : \mathbf{Ab} \to \mathbf{Ab}$ with $T(0) = 0$ the natural suspension homomorphism of degree $+1$

$$\sigma : \pi_*(TX) \to \pi_* T(\Sigma X)$$

The suspensions ΣX of the simplicial abelian group X is given by the quotient

$$\Sigma X = (\mathbb{Z}S^1) \otimes X / * \otimes X$$

where S^1 is the pointed simplicial circle and $\mathbb{Z}S^1$ is obtained by applying the free abelian group functor. The suspension homomorphism is compatible with the universal coefficient theorem (3.5); namely there exist natural transformations σ of degree $+1$ together with a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & (\pi_* X) \otimes M \arrow[r, "\Delta"] \arrow[d, "\sigma"] & \pi_*(X \otimes M) \arrow[r, "\mu"] \arrow[d, "\sigma"] & (\pi_* X) *' M \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & (s\pi_* X) \otimes M \arrow[r, "\Delta"] & \pi_*((\Sigma X) \otimes M) \arrow[r, "\mu"] & (s\pi_* X) *' M \arrow[r] & 0
\end{tikzcd}$$

For a graded abelian group A let sA be given by $(sA)_{n+1} = A_n$ so that the identity is a map $s : A \cong sA$ of degree $+1$. Then it is well known that

$$\pi_*(\Sigma X) = s(\pi_* X)$$

so that the bottom row of the diagram is given by the universal coefficient theorem for ΣX. We now describe explicitly the natural transformations of degree $+1$

$$\sigma : A \otimes M \to (sA) \otimes M,$$
$$\sigma : A *' M \to (sA) *' M$$

used in the diagram. We only define σ on $A \otimes M$; the map σ on $A *' M$ is obtained accordingly. The map σ is trivial on the direct summand $A \overset{>}{\otimes} A \otimes M_{ee}$ of $A \otimes M$. Moreover on $A_i \otimes H_j M^e_*$, $i + j = n$, $i \succ j$, let σ be given by the map

$$s \otimes id : A_i \otimes H_j M^e_* = (sA)_{i+1} \otimes H_j M^e_*.$$

Finally let σ on $A_m \otimes Z_m M_*$, $n = 2m$, be given by

$$s \otimes q : A_m \otimes Z_m M_* \to (sA)_{m+1} \otimes H_m M_*$$

where $q : Z_m M_* \to H_m M_*$ is the projection from cycles to homology classes.

(3.8) <u>Remark</u>. Let X be a simplical abelian group and let M be a quadratic module in $\underline{\underline{Ab}}$. Then one has natural homomorphisms

$$\pi_i(X) \otimes \pi_j(X) \otimes M_{ee} \xrightarrow{\nabla} \pi_{i+j}(X \otimes M),$$
$$\pi_i(X) \otimes Z_i M_* \xrightarrow{\gamma} \pi_{2i}(X \otimes M),$$
$$\pi_i(X) \otimes H_j M_*^e \xrightarrow{\delta} \pi_{i+j}(X \otimes M),\ i \succ j,$$

which are defined on generators by the formulas below. In fact, these homomorphisms yield in the obvious way the inclusion Δ in the universal coefficient theorem (3.5). The homomorphism ∇ is induced by the classical shuffle, that is,

$$\nabla(\{x\} \otimes \{y\} \otimes n) = \sum_{(a:b)} \pm [s_b x, s_a y] \otimes n.$$

Here $\{x\} \in \pi_2 X$ is represented by x and the sum ranges over all (i,j)-schuffles $(a:b)$; compare for example 5.6 in [4]. Next we obtain γ for $i \succ 0$ by the formulas

$$\gamma\left([\{x\},\{y\}] \otimes n\right) = \nabla\left(\{x\} \otimes \{y\} \otimes n\right),$$
$$\gamma\left(\{x\} \otimes m\right) = \sum_{\substack{(a:b)\\ a_1=0}} \pm [s_b x, s_a x] \otimes m.$$

Here the sum is taken over all (i,i) -shuffles $(a:b) = (a_1 < \ldots < a_2 : b_1 < \ldots < b_2)$ which are permutations of $\{0, \ldots, 2i-1\}$ with $a_1 = 0$. If $i = 0$ there is an obvious map $\gamma : \pi_0(X) \otimes M \to \pi_0(X \otimes M)$. Finally we obtain δ for $i \succ j \succ 0$ by

$$\delta\left(\{x\} \otimes \{m\}\right) = \sum_{\substack{(a:b)\\ a_1=0}} \pm [s_b x, s_a x] \otimes m$$

where the sum is taken over all (j,j)-shuffles $(a:b)$ with $a_1 = 0$. For $i \succ j = 0$ there is a canonical map $\delta : \pi_i(X) \otimes H_0 M_*^e = \pi_i(X) \otimes \mathrm{cok}(P) \to \pi_i(X \otimes M)$ given by $\delta(\{x\} \otimes \{m\}) = \{x \otimes m\}$. For $M = \mathbb{Z}^s \otimes \mathbb{Z}/2$ the function δ corresponds to the operation $\bar{\delta}_i$ considered by Goerss in 3.4 [10].

§ 4 Examples

The classical functors $\bigotimes^2, P^2, \Lambda^2 S^2, \Gamma^2$ (tensor square, quadratic construction, exterior square, symmetric square, Whitehead's Γ-functor [16]) have the following associated quadratic $\mathbb{Z}$-modules; compare [1].

F	$F\{\mathbb{Z}\} = \left(F(\mathbb{Z}) \xrightarrow{H} F(\mathbb{Z}/\mathbb{Z}) \xrightarrow{P} F(\mathbb{Z})\right)$
$\otimes^2$	$\mathbb{Z}^{\otimes} = \left(\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}\right)$
P^2	$\mathbb{Z}^{P} = \left(\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z}\right)$
Λ^2	$\mathbb{Z}^{\Lambda} = (0 \to \mathbb{Z} \to 0)$
S^2	$\mathbb{Z}^{S} = \left(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}\right)$
Γ	$\mathbb{Z}^{\Gamma} = \left(\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}\right)$

Therefore we obtain easily from (3.2) and (3.3) the following list of associated groups $H_* M_*$ and associated quadratic modules $Z_n M_*$ which define for $M = F\{\mathbb{Z}\}$ the graded quadratic tensor product $A \otimes M$ in (3.4).

M	Z_1M_*	Z_2M_*	Z_3M_*	$H_0(M_*)_e$	$H_1(M_*)_e$	$H_2(M_*)_e$	$H_3(M_*)_e$
$\mathbb{Z}^{\otimes}$	$\mathbb{Z}^{\otimes}$	$\mathbb{Z}^{\otimes}$	$\mathbb{Z}^{\otimes}$	0	0	0	0
$\mathbb{Z}^{P}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}^{\Gamma}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0
$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}^{\Gamma}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}^{\Gamma}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$\mathbb{Z}^{S}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}^{\Gamma}$	$\mathbb{Z}^{\Lambda}$	0	0	$\mathbb{Z}/2$	0
$\mathbb{Z}^{\Gamma}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}^{\Gamma}$	$\mathbb{Z}^{\Lambda}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

For all quadratic modules M and Z_iM_* in this list we have $A *'' M = 0$ since M_{ee} is free abelian. Moreover the quadratic torsion $A *' M$ is given by the classical functors

$$A *' \mathbb{Z}^{\otimes} = A * A$$
$$A *' \mathbb{Z}^{\Gamma} = R(A) = H_5K(A,2)$$
$$A *' \mathbb{Z}^{\Lambda} = \Omega(A) = H_7K(A,3)/(\mathbb{Z}/3\mathbb{Z} \otimes A)$$

where R and Ω are functors of Eilenberg-Mac Lane [7] with $R(A \mid B) = \Omega(A \mid B) = A * B$ and $R(\mathbb{Z}) = \Omega(\mathbb{Z}) = 0$ and $R(\mathbb{Z}/n) = \mathbb{Z}/(2,n)$, $\Omega(\mathbb{Z}/n) = \mathbb{Z}/n$. Moreover

$$A *' \mathbb{Z}^{S} = A *' \mathbb{Z}^{P} = A * A/\{\tau_n(a,a),\ a \in A\}$$

has the cross effect $A * B$ and satisfies $(\mathbb{Z}/n) *' \mathbb{Z}^{S} = 0$ for $n \geq 0$.

Using the list above one obtains readily by (3.5) the universal coefficient theorem for $\pi_n(FX)$ where $F = \otimes^2,\ P^2, \Lambda^2, S, \Gamma$. For example for $F = \Lambda^2$ we have the short exact sequence

$$(4.1) \qquad 0 \longrightarrow ((\pi_* X) \otimes \mathbb{Z}^\Lambda)_n \xrightarrow{\Delta} \pi_*(\Lambda^2 X) \xrightarrow{\mu} ((\pi_* X) *' \mathbb{Z}^\Lambda)_{n-1} \longrightarrow 0$$

with

$$((\pi_* X) \otimes \mathbb{Z}^\Lambda)_n = (\pi_* X \overset{>}{\otimes} \pi_* X)_n \oplus (\pi_* X \overset{>}{\otimes} (\mathbb{Z}/2)_{odd})_n$$

$$\oplus \begin{cases} \Lambda(\pi_m X),\ n = 2m,\ m & \text{even} \\ \Gamma(\pi_m X),\ n = 2m,\ m & \text{odd} \end{cases}$$

$$((\pi_* X) *' \mathbb{Z}^\Lambda)_{n-1} = ((\pi_* X) \overset{>}{*} (\pi_* X))_{n-1} \oplus ((\pi_* X) \overset{>}{*} (\mathbb{Z}/2)_{odd})_{n-1}$$

$$\oplus \begin{cases} \Omega(\pi_m X),\ n-1 = 2m,\ m & \text{even} \\ R(\pi_m X),\ n-1 = 2m,\ m & \text{odd} \end{cases}$$

Here $(\mathbb{Z}/2)_{odd}$ is the graded abelian group which is $\mathbb{Z}/2$ in odd degrees ≥ 1 and trivial otherwise.

(4.2) Remark. We point out that the universal coefficient sequence (4.1) for Λ^2 is split (unnaturally), also the coefficient sequences for $\otimes^2$, P^2, S^2, Γ are split (unnaturally). To see this it is enough to consider only $\pi_n F\, K(A,m)$ for such functors F since X is a sum of such $K(A,m)$. Then the canonical map $\gamma : K(A,m) \to K'(A,m)$ yields the retraction. Here $K'(A,m)$ is the simplicial group for which the normalization is concentrated in degree m and $(NK'(A,m))_m = A$.

§ 5 Homology in the variety of groups of nilpotency degree two

The homology in varieties of groups is studied by Leedham-Green [11]. We here compute this homology if the variety is the category **Nil** of groups of nilpotency degree 2. Let $G \in$ **Nil** and let $K(G,0)^{Nil}$ be a simplicial object in **Nil** such that each group $K(G,0)^{Nil}$, $n \geq 0$, is a free object in the category **Nil** and such that

$$\pi_0\, K(G,0)^{Nil} = G \quad \text{and} \quad \pi_i\, K(G,0)^{Nil} = 0 \quad \text{for} \quad i \geq 1.$$

Let H_1 : **Nil** $\to$ **Ab** be the abelianization functor. Then the homology of G in the variety **Nil** is defined by

$$(5.1) \qquad H_n^{Nil}(G) = \pi_{n-1}(H_1\, K(G,0)^{Nil}).$$

For any free object F in **Nil** one has the natural central extension

$$0 \to \Lambda^2(H_1 F) \to F \to H_1 F \to 0$$

which yields a short exact sequence of simplicial groups

$$(5.2) \qquad 0 \to \Lambda^2 X \to K(G,0)^{Nil} \to X \to 0$$

with $X = H_1 K(G,0)^{Nil}$. Hence the long exact sequence of homotopy groups associated to (5.2) determines for $n \geq 2$ the isomorphisms

$$(5.3) \qquad H^{Nil}_{n+1}(G) = \pi_n(X) \overset{\partial}{\cong} \pi_{n-1}(\Lambda X).$$

where the right hand side is embedded in the universal coefficient sequence (4.1). We obtain the following result which can be used to compute the groups $H^{Nil}_n(G)$ completely.

(5.4) Theorem. *Let $G \in$ **Nil**. Then one has*

$$\begin{aligned} H^{Nil}_1(G) &= H_1(G) = \text{ abelianization of } G \\ H^{Nil}_2(G) &= Ker\,(\Lambda^2 H_1(G) \xrightarrow{w} [G,G]) \end{aligned}$$

where w is the commutator map. Moreover for $n \geq 2$ one has the split short exact sequence

$$0 \longrightarrow (H^{Nil}_{*+1}(G) \otimes \mathbb{Z}^\Lambda)_n \longrightarrow H^{Nil}_{n+2}(G) \longrightarrow (H^{Nil}_{*+1}(G) *' \mathbb{Z}^\Lambda)_{n-1} \longrightarrow 0$$

Here H_{*+1} is the graded object with $(H_{*+1})_n = H_{n+1}$. Inductively the sequence determines all groups $H_n = H^{Nil}_n(G)$. For example one gets the following split exact sequences.

$$\begin{aligned} &0 \longrightarrow H_2 \otimes H_1 \longrightarrow H_3 \longrightarrow \Omega(H_1) \longrightarrow 0 \\ &0 \longrightarrow H_3 \otimes H_1 \oplus \Gamma(H_1) \longrightarrow H_4 \longrightarrow H_2 * H_1 \longrightarrow 0 \\ &0 \longrightarrow H_4 \otimes H_1 \oplus H_3 \otimes (H_2 \oplus \mathbb{Z}/2) \longrightarrow H_5 \longrightarrow H_3 * H_1 \oplus R(H_2) \longrightarrow 0 \end{aligned}$$

§ 6 Universal quadratic functors of Ellis

Let G be a group. In [8,9] Ellis introduces the groups $\Gamma_n G$ and $J_n G$, $n \geq 2$, which are related with the homology $H_n G$ by an exact sequence

$$(6.1) \qquad 0 \longleftarrow H_2 G \longleftarrow J_2 G \longleftarrow \Gamma_2 G \longleftarrow H_3 G \longleftarrow \dots$$

Moreover one has $\Gamma_2 G = \Gamma(H_1 G)$. We now describe $\Gamma_n G$ for $n \geq 2$ in terms of the homology of G. Let $K(G,0)$ be a free simplicial group with $\pi_0 K(G,0) = G$ and $\pi_n K(G,0) = 0$ for $n \succ 0$. Then the homology of G is given by

$$(6.2) \qquad H_{n+1}G = \pi_n(X) \quad \text{where} \quad X = H_1K(G,0).$$

Moreover for $n \geq 2$ the group $\Gamma_n G$ is given by

$$(6.3) \qquad \Gamma_n G = \pi_{n-2}(\Gamma X)$$

so that we can apply the universal coefficient theorem for the functor Γ. For this we use the last row in the second table of § 4 and theorem (3.4). For example one gets for $H_n = H_n G$ the following short exact sequences which are split.

$$\begin{aligned}
&0 \longrightarrow H_2 \otimes (H_1 \oplus \mathbb{Z}/2) \longrightarrow \Gamma_3 G \longrightarrow R(H_1) \longrightarrow 0 \\
&0 \longrightarrow H_3 \otimes (H_1 \oplus \mathbb{Z}/2) \oplus \Lambda^2 H_2 \longrightarrow \Gamma_4 G \longrightarrow H_2 * (H_1 \oplus \mathbb{Z}/2) \longrightarrow 0 \\
&0 \longrightarrow H_4 \otimes (H_1 \oplus \mathbb{Z}/2) \oplus H_3 \otimes H_2 \longrightarrow \Gamma_5 G \longrightarrow H_3 * (H_1 \oplus \mathbb{Z}/2) \oplus \Omega(H_2) \longrightarrow 0
\end{aligned}$$

From the first sequence one gets the group

$$\Gamma_3(\mathbb{Z}/m\mathbb{Z}) = R(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/(2,m)$$

which was also considered in [9] (where this group was calculated incorrectly).

§ 7 A SPECTRAL SEQUENCE FOR QUADRATIC FUNCTORS

Let $\mathbf{A}$ be an abelian category with enough projectives. For any biadditive functor $G : \mathbf{A} \times \mathbf{A} \to \mathbf{Ab}$ we denote by L_*G the total derived bifunctor [3]. Recall that $L_*G(A,B) = H_n\, Tot\,(G(P_*, R_*))$ where $P_* \to A$ and $R_* \to B$ are projective resolutions. Let $G_\Delta : \mathbf{A} \to \mathbf{Ab}$ be the diagonal of G, that is $G_\Delta(A) = G(A,A)$. Then it is well known that

$$(7.1) \qquad (L_n G_\Delta)(A) = (L_n G)(A,A).$$

This is a consequence of Eilenberg-Zilber-Cartier theorem [6] (2.9). Given a quadratic functor $F : \mathbf{A} \to \mathbf{Ab}$ we obtain the biadditive functor G as a cross effect of F and one gets $L_nF(A \mid B) = (L_nG)(A,B)$. Moreover we have by F the chain complex $F_*(A)$ as in (3.1) which yields the functors $A \mapsto H_iF_*(A)$ and $A \mapsto Z_iF_*(A)$ by the homology and cycles respectively.

(7.2) Theorem. *Let $F : \mathbf{A} \to \mathbf{Ab}$ be a quadratic functor and let X be a component-wise projective simplicial object in $\mathbf{A}$. Then there is a spectral sequence*

$$E^r_{pq} \Longrightarrow \pi_{p+q}F(X)$$

with differentials $d_r : E^r_{pq} \to E^r_{p-r,q+r-1}$. Moreover the E^2 -term is naturally given by the formula

$$E^2_{pq} = \bigoplus_{\substack{i+j=q \\ i>j}} [L_p(H_jF_*)(\pi_i X) \oplus L_pF(\pi_i X \mid \pi_j X)]$$

$$\bigoplus \begin{cases} L_p(Z_m F_*)(\pi_m X), & \text{if} \quad q = 2m \\ 0 & , \quad \text{if} \quad q \quad \text{odd} \end{cases}$$

Proof: *Case 1.* First we consider the case when $X = K(Q,n) = K'(Q,n)$ is given by a projective object Q. Here $K'(Q,n)$ is just the simplical object in $\mathbf{A}$ with normalization concentrated in degree n and $N_n K(Q,n) = Q$. By definition we get for $n = 0$

$$\pi_i F\, K(Q,0) = (L_i F)(Q) \tag{1}$$

with $(L_0F)(Q) = F(Q)$ and $(L_iF)(Q) = 0$ for $i \succ 0$. In fact, for $n = 0$ the simplicial object $K(Q,0)$ and $FK(Q,0)$ are both constant simplicial objects. If $n \succ 0$ one has by 4.23 in [6]

$$\pi_i F\, K(Q,n) = 0 \quad \text{for} \quad i \succ 2n \quad \text{or} \quad i < n. \tag{2}$$

Moreover for $n \succ 0$ one has the isomorphisms

$$\pi_{n+i} FK(Q,n) \cong (H_i F_*)(Q) \quad \text{for} \quad 0 \leq i < n, \tag{3}$$

$$\pi_{2n} FK(Q,n) \cong (Z_n F_*)(Q). \tag{4}$$

For a proof of formula (3) see § 4 in [14]; compare also [2]. For a proof of (4) we first observe that

$$\pi_{2n} F(K(Q,n) \mid K(Q,n)) = F(Q \mid Q)$$

and the operators H and P of the cross effect $F(\quad|\quad)$ yield homomorphisms

$$\pi_{2n} FK(Q,n) \xrightarrow{\beta_{2n}} F(Q \mid Q) \xrightarrow{\alpha_{2n}} \pi_{2n} FK(Q,n).$$

By 8.8 in [6] β_{2n+2} is a monomorphism with

$$\text{image}\,(\beta_{2n+2}) = \text{kernel}\,(\alpha_{2n}). \tag{5}$$

This shows that formula (4) holds. Using (2), (3) and (4) we see that for $X = K(Q,n)$ we have $\pi_* F(X) = E^2_{0*}$ and hence the theorem holds for such X since L_p vanishes on projective objects for $p \succ 0$.

Case 2. Now consider the case when X has homotopy groups $\pi_n X$ which are projective objects in $\mathbf{A}$. Then X is of the form $X \simeq \oplus K(Q_n, n)$ where Q_n is projective so that $Q_n = \pi_n X$. Then using the decomposition of FX by cross effects we obtain the theorem also for such X.

Case 3. For general X we use proposition 17.1.2 in [3] which shows that there is a bisimplicial projective resolution $Q_{**} \to X$ such that the induced map

$$\pi_n^{vertical}(Q_{**}) \to \pi_n X \tag{6}$$

is again a simplicial projective resolution. Hence the theorem now follows from the spectral sequence for FQ_{**} by the computations for case 2 above.

q.e.d.

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