Mixed motives over $\mathbb{Z}$ and p-adic L-functions

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MIXED MOTIVES OVER $\mathbb{Z}$ AND p-ADIC L-FUNCTIONS
(comments and conjectures)

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Introduction

Let M be a pure motive over $\mathbb{Q}$ and $h = h(p, M) := [P_{N,p}(d^{\pm}) - P_H(d^{\pm})] + 1$ denote corresponding h-invariant of M. At critical points, Deligne's conjecture [Del] relates the value of an L-function of M to a certain period and in this case we have formulated a conjecture [Dab], [Pa1] according to which for each prime p there exists a $h(M)$-admissible measure μ on the Galois group G_p, the Mellin transform L_μ of which is a $\mathbb{C}_p$-analytic function of the $o(log^h)$ type which interpolates the special values of the L-function of the motive M at the critical points. J. Coates [Co1,2] described, in the p-ordinary critical case, an invariant form of the conjecture. At non-critical points, the conjectures of Beilinson [Bel] describe the leading coefficients of L-function in terms of regulators. A. Scholl [Scho] showed how periods and regulators may be interpreted as periods of mixed motives and gave an interpretation of Beilinson's conjectures in terms of a Deligne conjecture for critical mixed motives.

In this paper we describe a conjectural generalization of the above construction of $h(M)$-admissible measure to arbitrary (possibly non-critical) pure motives. We also propose a second possible construction of p-adic L-function using ideas from [Pa2], [GrS], [MaW]. At the end we give some remarks on the central critical values of L-series.

Summary

1 Mixed motives (over $\mathbb{Z}$)

1.1 Category of mixed realizations

Let $\mathcal{MM}_{\mathbb{Q}}$ denote a category of mixed motives over $\mathbb{Q}$ (see [Jan], where such a category has been constructed using absolute Hodge cycles). It is an abelian category, containing $\mathcal{M}$, a category of pure motives over $\mathbb{Q}$, as a full subcategory.

Let's recall the necessary definitions. Associated to a mixed motive $M \in \mathcal{MM}_{\mathbb{Q}}$ is (in a functorial way) a collection of de Rham M_{DR}, Betti M_B and l-adic M_l (for all primes l) mixed realizations, which are respectively vector spaces (of finite dimension $d = d(M)$) over $\mathbb{Q}$, $\mathbb{Q}$, $\mathbb{Q}_l$ respectively (here we take for $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). These realizations are endowed with the additional structures:

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

(i) M_{DR} is equipped with a decreasing exhaustive filtration $\{F^k\}_{k\in\mathbf{Z}}$ (the Hodge filtration) and an increasing filtration $\{W_m\}_{m\in\mathbf{Z}}$ (the weight filtration),

(ii) M_B is a mixed $\mathbb{Q}$-Hodge structure, i.e. it admits an involution F_∞ and there is an increasing filtration $\{W_m\}_{m\in Z}$ (the weight filtration) on M_B and a decreasing filtration $\{F^k\}_{k\in\mathbf{Z}}$ (the Hodge filtration) on $M_B\otimes_{\mathbb{Q}}\mathbb{C}$, which induces a $\mathbb{Q}$-structure of weight m on $Gr_m^W M_B = W_m M_B/W_{m-1}M_B$, that is

$$Gr_m^W M_B \otimes \mathbb{C} = \oplus_{p+q=m} M^{p,q}$$

with

$$F_\infty(M^{p,q}) = M^{p,q} \;\; and \;\; F^p Gr_m^W M_B \otimes \mathbb{C} = \oplus_{p'\geq p} M^{p',q'}$$

(iii)$\forall_l$ prime M_l is a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ - module with an ($Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant) filtration $\{W_m\}_{m\in\mathbf{Z}}$ (the weight filtration)

(iv) There is a "natural" $Gal(\mathbb{C}/\mathbb{R})$-isomorphism of $\mathbb{C}$-vector spaces $I_\infty : M_B\otimes\mathbb{C}\to M_{DR}\otimes\mathbb{C}$ identifying the filtrations induced by the Hodge filtrations (respectively, the weight filtrations) on both sides

(v) There exists comparison isomorphisms $I_l : M_B\otimes\mathbb{Q}_l \to M_l$ transforming the weight filtration of M_B into the weight filtration of M_l and F_∞ into complex conjugation.

Remarks.

(i) Let $\mathcal{V}_{\mathbb{Q}}^0$ be the category of smooth quasi-projective varieties over $\mathbb{Q}$ and $\mathcal{MR}_{\mathbb{Q}}$ the (abelian) category of mixed realizations. Jannsen [Jan] has constructed functors $H^n : \mathcal{V}_{\mathbb{Q}}^0 \to \mathcal{MR}_{\mathbb{Q}}$, $\forall_{n\in\mathbf{Z}}$, associating to each $X\in\mathcal{V}_{\mathbb{Q}}^0$ its n-th realization

$$H^n(X) = (M_{DR}^n(X), M_B^n(X), M_l^n(X); I_\infty, I_l)$$

Define the functor $H : \mathcal{V}_{\mathbb{Q}}^0 \to \mathcal{MR}_{\mathbb{Q}}$ by $H(X) := \oplus_{n\geq 0} H^n(X)$. Then $\mathcal{MM}_{\mathbb{Q}}$ (for absolute Hodge cycles) is (by definition) the Tannakian subcategory of $\mathcal{MR}_{\mathbb{Q}}$ generated by the image of H.

(ii) We define a mixed realization H to be pure of weight m, if $W_m H = H$, $W_{m-1} = 0$, and the category $\mathcal{R}_{\mathbb{Q}}$ of realizations to be the full subcategory of $\mathcal{MR}_{\mathbb{Q}}$ consisting of direct sums of pure realizations. Then any $H\in\mathcal{MR}_{\mathbb{Q}}$ is a succesive extension of the pure realizations $Gr_m^W H$.

(iii) $W_m M \in \mathcal{MM}_{\mathbb{Q}}$ and $Gr_m^W M$ are pure motives for any $M\in\mathcal{MM}_{\mathbb{Q}}$. In particular, any mixed motive is a succesive extension of pure motives.

We would like to say a few words how one can construct mixed motives explicitely [Har].

The idea is in the use of Shimura varieties; the Hecke algebra of operators acting on the Shimura variety S then cut it into the mixed motives. In general Shimura varieties are not projective. So we have to compactify them; the Baily-Borel- and the toroidal compactifications are known. One can explicitely construct the canonical models of these compactifications (R. Pink); in the case of Baily-Borel compactification it is a highly singular space, which has a stratification by a union of canonical models of Shimura varieties (belonging to smaller reductive groups). To each individual piece in the cohomology of the boundary one can attach the corresponding

Eisenstein series of a complex variable s. To understand the Betti realisation we have to investigate the image of the global cohomology of $S(\mathbb{C})$ (with coefficients in suitable sheaf) and it depends on whether the Eisenstein series is holomorphic at a suitable value s.

The L-function of M is defined as the Euler product

$$L(M,s) := \prod_{p\ prime} L_p(M,p^{-s}) ,$$

with $L_p(M,p^{-s})^{-1} := det(1 - p^{-s} \cdot r_l(Fr_p^{-1})|M_l^{I_p}),\ \ l \neq p$ (conjecturally independent of l) and where Fr_p is the Frobenius element at p, defined modulo the inertia group I_p; r_l denotes the l-adic representation in the definition (iii) above.

We assume the existence of the meromorphic continuation and functional equation.

Notice, that in general, if $M \in \mathcal{MM}_{\mathbf{Q}}$ then $L(M,s)$ and $\prod_j L(Gr_j^W M,s)$ will differ by a finite number of Euler factors. The equality we have, for example, in the following important case of mixed motives over $\mathbb{Z}$.

Definition. *(Scholl) $M \in \mathcal{MM}_{\mathbf{Q}}$ is a mixed motive over $\mathbb{Z}$ if the weight filtration on M_l splits over $\mathbb{Q}_p^{nr}$ for every l, p with $l \neq p$.*

The mixed motives over $\mathbb{Z}$ form a full subcategory $\mathcal{MM}_{\mathbf{Z}}$ of $\mathcal{MM}_{\mathbf{Q}}$: $\mathcal{M} \subset \mathcal{MM}_{\mathbf{Z}} \subset \mathcal{MM}_{\mathbf{Q}}$.

1.2 Period conjecture

For $M \in \mathcal{MM}_{\mathbf{Q}}$ we define the period maps

$$I_\infty^+ : M_B^+ \otimes \mathbb{R} \to \frac{M_{DR}}{F^0} \otimes \mathbb{R}$$

and

$$I_\infty^- : M_B^- \otimes \mathbb{R} \to (M_{DR}/F^1 M_{DR}) \otimes \mathbb{R}$$

in the same way as for pure motives.

Definition. *A mixed motive M is critical if I_∞^+ is an isomorphism. If this holds, define $c^+(M) := detI_\infty^+$ and $c^-(M) := detI_\infty^-$ $\ (\in \mathbb{R}^\times/\mathbb{Q}^\times)$.*

Note, that for mixed motives the notion of critical does not depend on the Hodge numbers and the action of involution F_∞: if the pure motives $Gr_j^W M$ are critical, then so is M, but the converse is far from true.

Period conjecture. *(Deligne) If M is critical, then* $\frac{L(M,0)}{c^+(M)} \in \mathbb{Q}$.

The conjecture has been verified in the case of Tate motives (Euler), elliptic cusp forms (Shimura,...) and their symmetric squares (Sturm) and cubes (Garrett,...), tensor products of two (and, in some cases, three) elliptic cusp forms (Shimura, Garrett-Harris,...). Also the (we hope "motivic") case of standard zeta functions of Siegel modular forms was treated.

G. Harder showed [Har] that Anderson's mixed motives are critical in a sense of Scholl (see 2.2) and so conjecturally critical.

Remarks.

1. Given a Siegel cusp eigenform f of (even) degree m and of weight k (with respect to some congruence subgroup, and with a Dirichlet character ψ); assume that $k > 2m+2$ and f belong to the orthogonal complement to the generalized Maass subspace. Then one expect [Pa3] that there exists a motive M_f^{st} (over $\mathbb{Q}$, with coeff. in $\mathbb{Q}(f)$) of rank $2m+1$, weight 0, with Hodge structure of the type $(-k+1, k-1) + ...(-k+m, k-m) + (0,0) + (k-m, -k+m) + ... + (k-1, -k+1)$ such that the complex conjugation acts on the $(0,0)$-subspace via $\psi(-1)$, and for any Dirichlet character χ $L_{st}(s, f, \chi) = L(M_f^{st}(\chi), s)$, where $L_{st}(s, f, \chi)$ denote the standard zeta function of f.
By the work of Adrianov, Kalinin, Böcherer and others the function $L_\infty(M_f^{st}(\chi), s) \cdot L_{st}(s, f, \chi)$ admits a meromorphic continuation for all $s \in \mathbb{C}$ with the possible simple pole at $s = 0$ in case $\chi^2\psi^2 = 1$. Description of the critical values of $L_{st}(s, f, \chi)$ and the shape of the corresponding p-adic L-function perfectly match with the general conjecture on critical values and on p-adic L-functions attached to motives.

2. The Saito-Kurokawa conjecture in degree 2 may suggest that for f in the (generalized) Maass space we may expect that there exists (mixed) motive attached to spinor zeta function of f. For f of degree 2 let's try to imagine relations between (conjectural) standard and spinor motives M_f^{st},M_f^{sp}. The identities for Satake parameters may suggest that the motive M_f^{sp} is a direct sum of two 2-dimensional submotives: $M_f^{sp} = M_1 \oplus M_2$ in such a way that $M_f^{st} = \mathbb{Q}(0) \oplus M_1 \otimes M_2(2k-3)$.

The shape of Γ-factor of M_f^{sp} may suggest that, in some cases, $M_1 = M(g_1)$ for g_1 elliptic cusp form of weight $2k-2$ and $M_2 = M(g_2)(2-k)$ for g_2 elliptic cusp form of weight 2. Acually, Yoshida's work [Yo] partially confirm these observations.

3. I would also like to mention the following observation due to W. Kohnen. In [Koh] he stated a general conjecture that the rational structures on spaces of modular forms coming from the rationality of Fourier coefficients and the rationality of periods are not compatible. From this it follows, in particular, that for f elliptic cusp form of level one $c^+(f)$, $c^-(f)$ should be transcendental. It is expected that $c^+(f)c^-(f) \sim < f, f >$ should be transcendental as well. This observation and formulae for $c^\pm(Sym^m f)$, $c^\pm(f_1 \otimes f_2)$,... lead to the following
Question: In case of any (pure) motive M over $\mathbb{Q}$ we should have $c^+(M)c^-(M) \notin \overline{\mathbb{Q}}$.

2 "Universal extensions" and highly critical motives

2.1 Ext-groups

Write $Ext_{\mathbf{Q}}$ for the Ext-groups in $\mathcal{MM}_{\mathbf{Q}}$, and $Ext_{\mathbf{Z}}$ for the Ext-groups in $\mathcal{MM}_{\mathbf{Z}}$.

If $M_1, M_2 \in \mathcal{MM}_{\mathbf{Z}}$ then $Ext^0_{\mathbf{Q}}(M_1, M_2) = Ext^0_{\mathbf{Z}}(M_1, M_2) = Hom(M_1, M_2)$, and $Ext^1_{\mathbf{Z}}(M_1, M_2)$ is the subgroup of $Ext^1_{\mathbf{Q}}(M_1, M_2)$ comprising the classes of those extensions

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

such that for every p and every $l \neq p$, the extension M_l of Galois modules splits over $\mathbb{Q}_p^{nr}$. We should have $Ext^q_{\mathbf{Q}} = Ext^q_{\mathbf{Z}} = 0$ unless $q = 0, 1$. Conjecturally, the groups $Ext_{\mathbf{Z}}$ will be finite-dimensional over $\mathbb{Q}$.

In the case of the Tate motive we should have $Ext^1_{\mathbf{Z}}(\mathbb{Q}(0), \mathbb{Q}(1)) = 0$.

More generally, let $M = h^i(X)(m)$ for X smooth and proper over $\mathbb{Q}$. Then we should have

$$Ext^0_{\mathbf{Z}}(M, \mathbb{Q}(1)) = Ext^0_{\mathbf{Z}}(\mathbb{Q}(0), M^\wedge(1)) = Hom(\mathbb{Q}(m-1-i), h^i(X))$$
$$= \begin{cases} 0 & \text{if } i \neq 2m-2 \\ CH^{i+1-m}(X)/CH^{i+1-m}(X)^0 \otimes \mathbb{Q} & \text{if } i = 2m-2 \end{cases}$$

and

$$Ext^1_{\mathbf{Z}}(M, \mathbb{Q}(1)) = Ext^1_{\mathbf{Z}}(\mathbb{Q}(0), M^\wedge(1))$$
$$= \begin{cases} H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(i+1-m))_{\mathbf{Z}} & \text{if } i \neq 2m-1 \\ CH^{i+1-m}(X)^0 \otimes \mathbb{Q} & \text{if } i = 2m-1. \end{cases}$$

Here $CH^{i+1-m}(X)$ denotes the Chow group of codimension $i+1-m$ on X modulo rational equivalence, and $CH^{i+1-m}(X)^0$ is the subgroup of classes of cycles homologically equivalent to zero, and $H_{\mathcal{M}}$ denotes the motivic cohomology

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = (K_{2j-i}X \otimes \mathbb{Q})^{(j)}$$

and $H^\star_{\mathcal{M}}(X, \cdot)_{\mathbf{Z}}$ is the image in $H^\star_{\mathcal{M}}(X, \cdot)$ of the K-theory of a regular model for X, proper and flat over Z. In particular, we should have $Ext^1_{\mathbf{Z}}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{Z}^\times \otimes_{\mathbf{Z}} \mathbb{Q} = 0$. On the other hand, similarly the above would imply that $Ext^1_{\mathbf{Q}}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{Q}^\times \otimes_{\mathbf{Z}} \mathbb{Q}$. Deligne [De3] conjectures that $K_{2m-1}(\mathbb{Q}) \otimes \mathbb{Q} \cong Ext^1_{\mathbf{Q}}(\mathbb{Q}(0), \mathbb{Q}(m))$. To give meaning to this conjecture one needs not bother with the entire category $\mathcal{MM}_{\mathbf{Q}}$, but it suffices to restrict oneself to the subcategory $\mathcal{TM}_{\mathbf{Q}}$ generated by the objects whose graded quotients are sums of Tate motives (see [De3] for a conjectural description of $\mathcal{TM}_{\mathbf{Q}}$).

Using Bloch's description of motivic cohomology by means of higher Chow groups one can unconditionally associate to elements of $H_{\mathcal{M}}$-groups explicit extensions of cohomology arising from the cohomology of non- compact or singular schemes [DenS]. In the appendix to [DenS] it is shown that for smooth, projective varieties X over $\mathbb{Q}$ there are natural maps (conjecturally isomorphisms) for $p+1 < 2q$

$$H^{p+1}_{\mathcal{M}}(X, \mathbb{Q}(q)) \to Ext^1_{\mathcal{MM}_{\mathbf{Q}}}(\mathbb{Q}(0), H^p(X)(q))$$

and one hopes that the image of $H^{p+1}_{\mathcal{M}}(X,\mathbb{Q}(q))_{\mathbf{Z}}$ is precisely $Ext^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbb{Q}(0), H^p(X)(q))$. On the other hand, it is shown in [Bel] that for X smooth, projective over $\mathbb{R}$ there is a canonical isomorphism (for $p+1<2q$)

$$H^{p+1}_{\mathrm{D}}(X,\mathbb{R}(q)) \to Ext_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), H^p_B(X)(q)),$$

where $H^*_{\mathrm{D}}(X,\mathbb{R}(q))$ denote Deligne cohomology groups and $\mathcal{MH}_{\mathbb{R}}$ is the abelian category of $\mathbb{R}$-mixed Hodge structures with the action of a real Frobenius.

In this context, the regulator should fit into a commutative diagram

$$\begin{array}{ccc} H^{p+1}_{\mathcal{M}}(X,\mathbb{Q}(q)) & \longrightarrow & H^{p+1}_{\mathrm{D}}(X_{\mathbb{R}},\mathbb{R}(q)) \\ \downarrow & & \downarrow \\ Ext^1_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}(0), H^p(X)(q)) & \longrightarrow & Ext^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), H^p_B(X_{\mathbb{R}})(q)). \end{array}$$

The conjectures of Birch-Swinnerton-Dyer, Tate and Beilinson on the orders of L-series at integer points can be stated in terms of Ext-groups as follows:

Conjecture. *Let $M \in \mathcal{MM}_{\mathbf{Z}}$. Then*

$$ord_{s=0}L(M,s) = dimExt^1_{\mathbf{Z}}(M,\mathbb{Q}(1)) - dimExt^0_{\mathbf{Z}}(M,\mathbb{Q}(1)).$$

Remarks.

1. For a prime number p let $G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and let $B_{cris} = B_{cris\ ,\ p}$ denote the Fontaine ring of p-adic periods. Let l be a prime and V an l-adic representation of $G_{\mathbb{Q}_p}$. We define

$$D(V) = \begin{cases} V^{I_{\mathbb{Q}_p}} & l \neq p \\ (B_{cris}\otimes V)^{G_{\mathbb{Q}_p}} & l = p \end{cases}$$

Let $H^1_f(\mathbb{Q}_l,V)$ be the sub-$\mathbb{Q}_l$-linear space of $H^1(\mathbb{Q}_l,V)$ classifying those extensions V_x of $\mathbb{Q}_l$ by V which are such that the map $D(V_x)\to D(\mathbb{Q}_l)$ is surjective.

Now let V be an l-adic representation of $G_{\mathbf{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. For $x \in H^1(\mathbb{Q},V)$ denote by x_p its image in $H^1(\mathbb{Q}_p,V)$. Set

$$H^1_f(\mathbb{Q},V) = \{x \in H^1(\mathbb{Q},V) | \forall p\ x_p \in H^1_f(\mathbb{Q}_p,V)\}$$

Fontaine and Perrin-Riou [FoP] have formulated the following version of the above conjecture:

$$ord_{s=0}L(M,s) = dim_{\mathbf{Q}}H^1_f(\mathbb{Q},M^\wedge(1)) - dim_{\mathbf{Q}}H^0(\mathbb{Q},M^\wedge(1))$$

where $H^0(\mathbb{Q},M) = Hom(\mathbb{Q}(0),M)$ and $H^1_f(\mathbb{Q},M)$ is the $\mathbb{Q}$-linear space of the classes of extensions of $\mathbb{Q}(0)$ by M in $\mathcal{MM}_{\mathbf{Q}}$ such that $x \in H^1_f(\mathbb{Q},M)$ iff $\forall$ prime number l $x_l \in H^1_f(\mathbb{Q},M_l)$.

2. Also, let's recall the definition of a period given by Fontaine and Perrin-Riou.

Fix $\mathbb{Z}$-structure T on M (it means: we fix lattice T in M_B s.t. image of $T \otimes \mathbb{Z}_l$ in M_l under comparison isomorphism is $G_{\mathbb{Q}}$-stable). Choose a basis of a canonical $\mathbb{Z}$-structure $\mathbb{Z}(1)$ of $\mathbb{Q}(1)$. Also fix bases ω_T^+ resp. $\omega_{T(1)}^+$ of $det_{\mathbb{Z}}T^+$ resp. of $det_{\mathbb{Z}}T(1)^+$.

Take $j \in \mathbb{Z}$ so that $F^{-1}M_{DR}(j) = 0$. Then one can define the period isomorphism

$$\mathbb{R} \otimes H^1_f(\mathbb{Q}, M(j)) \to \mathbb{R} \otimes M_{DR}(j)/M_B(j)^+,$$

and therefore the isomorphism

$$det_{\mathbb{Q}}H^1_f(\mathbb{Q}, M(j)) \otimes (det_{\mathbb{Q}}M_{DR}(j))^* \otimes det_{\mathbb{Q}}M_B(j)^+ \to \mathbb{R}.$$

For $\omega \in (det_{\mathbb{Q}}M_{DR}(j))^* \otimes det_{\mathbb{Q}}H^1_f(\mathbb{Q}, M(j))$ we define $c_{\infty,T(j)}(\omega)$ to be the image of $\omega \otimes \omega^+_{T(j)}$ under that isomorphism.

2.2 Universal extensions

Scholl [Scho] proposes the following algebraic criterion for a mixed motive over $\mathbb{Z}$ to be critical:

Definition. *$M \in \mathcal{MM}_{\mathbb{Z}}$ is called highly critical, if*

$$\forall_{q=0,1} Ext^q_{\mathbb{Z}}(M, \mathbb{Q}(1)) = Ext^q_{\mathbb{Z}}(\mathbb{Q}(0), M) = 0.$$

Conjecture. *(Scholl) If M is highly critical, then it is critical.*

Now we consider "universal extensions" of an arbitrary pure motive M by sums of $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$ in order to obtain a new motive $M^\sim \in \mathcal{MM}_{\mathbb{Z}}$, which is highly critical, and for which $L(M^\sim, s) = \zeta(s)^a \cdot \zeta(s+1)^b \cdot L(M, s)$. Then, up to a nonzero rational, $L(M^\sim, 0)$ (and $c^+(M^\sim)$) will equal the leading coefficient of $L(M, s)$ at $s = 0$, and The Period Conjecture will then be applicable to $M^\sim$.

We consider separately the cases of pure motives of weights $w \leq -2$, $w \geq 0$, $w = -1$:

(i) Let M be a pure motive of weight $w \leq -2$. Then we have

$$Ext^1_{\mathbb{Q}}(M, \mathbb{Q}(1)) = Ext^0_{\mathbb{Q}}(\mathbb{Q}(0), M) = 0.$$

Assume that $Hom(M, \mathbb{Q}(1)) = 0$. Then for the universal extension

$$0 \longrightarrow M \longrightarrow M^\sim \longrightarrow Ext^1_{\mathbb{Z}}(\mathbb{Q}(0), M) \otimes \mathbb{Q}(0) \longrightarrow 0$$

we have

$$Ext^q_{\mathbb{Z}}(M^\sim, \mathbb{Q}(1)) = Ext^q_{\mathbb{Z}}(\mathbb{Q}(0), M^\sim) = 0$$

for $q = 0, 1$.

In this case $L(M^\sim, s) = \zeta(s)^a \cdot L(M, s)$, where $a = a(M) = dim Ext^1_{\mathbb{Z}}(\mathbb{Q}(0), M)$.

(ii) Let M be a pure motive of weight $w \geq 0$; assume that $Hom(\mathbb{Q}(0), M) = 0$. Then, dually to the above, there is a universal extension

$$0 \longrightarrow \mathbb{Q}(1)^b \longrightarrow M^{\sim} \longrightarrow M \longrightarrow 0$$

where $b = b(M) = dimExt^1_{\mathbb{Z}}(M, \mathbb{Q}(1))$. $M^{\sim}$ is highly critical and in this case $L(M^{\sim}, s) = \zeta(s+1)^b \cdot L(M, s)$.

(iii) Assume that M is a pure motive of weight -1. Then there is a unique mixed motive $M^{\sim}$ over $\mathbb{Z}$, which is highly critical and for which $L(M^{\sim}, s) = \zeta(s)^a \cdot \zeta(s+1)^b \cdot L(M, s)$

Remarks.

(i) "$\zeta(3)$ and universal extensions". We have $Hom(\mathbb{Q}(3), \mathbb{Q}(1)) = 0$. Let consider the universal extension

$$0 \longrightarrow \mathbb{Q}(3) \longrightarrow M \longrightarrow \mathbb{Q}(0)^\rho \longrightarrow 0$$

of $\mathbb{Q}(3)$, where $\rho = dimExt^1_{\mathbb{Z}}(\mathbb{Q}(0), \mathbb{Q}(3))$ (= 1, if Deligne's conjecture takes place [De3, p.163]). The period conjecture then implies

$$\frac{\zeta(3)}{c^+(M)} \in \mathbb{Q}.$$

Note, that if $M(n)$ is critical for some even positive integer n, then

$$\zeta(n+3) \sim \pi^{n(d^-(M)-\rho)} \zeta(3).$$

On the other hand, if $M(n)$ is critical for some odd positive integer n, then we can "evaluate" $\zeta(n+3)$:

$$\zeta(n+3) \sim \pi^{n(d^+(M)-\rho)} c^-(M).$$

(ii) "twist with Dirichlet character". Take $M \in \mathcal{M}_{\mathbb{Q}}$; assume $w(M) \leq -2$. Consider the corresponding universal extension

$$0 \to M \to M^{univ} \to \mathbb{Q}(0)^\rho \to 0.$$

Let χ be Dirichlet character. Then the extension

$$0 \to M(\chi) \to M^{univ}(\chi) \to \mathbb{Q}(\chi)^\rho \to 0$$

need not to be universal. However, we have

$$L(M^{univ}(\chi), 0) = L(M(\chi), 0)\ L(\chi, 0)^\rho.$$

It is essential for the formulation of the Conjecture in section 3.4.

(iii) It turns out that under some reasonable assumptions it is sufficient to consider only the cases (i) - (iii). More precisely, assume the following

Hypotheses:

(a) $Ext^1_{\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(1))$ is generated by the classes of 1-motives $[\phi : \mathbb{Z} \longrightarrow \mathbb{G}_m]$, $\phi(1) = p$

(b) If M is pure of weight -1, then

$$ord_{s=0} L(M, s) \geq dimExt^1_{\mathbb{Z}}(M, \mathbb{Q}(1)) = dimExt^1_{\mathbb{Q}}(M, \mathbb{Q}(1))$$

(c) If M is pure of weight ≤ -2, then the realisation map

$$Ext^1_{\mathbb{Z}}(\mathbb{Q}(0), M) \otimes \mathbb{R} \longrightarrow Ext^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), M_{\mathbb{R}})$$

is injective.

Then we have the following

Theorem. *[Scho] Let $M \in \mathcal{MM}_{\mathbb{Q}}$ be critical, with $L(M,0) \neq 0$. Then there is an increasing filtration $K.$ of M with the following properties:*
(a) $M_i = Gr_i^K M$ are critical motives with $L(M,0) \in \mathbb{R}^\times$,
(b)$\prod L(M_i,0) \cdot L(M,0)^{-1} \in \mathbb{Q}^\times$,
(c) each M_i is one of the following:
(i) an extension of a pure motive M of weight ≥ 0 by a sum of copies of $\mathbb{Q}(1)$, with $Hom(\mathbb{Q}(0), M) = 0$,
(ii) an extension of a sum of copies of $\mathbb{Q}(0)$ by a pure motive M of weight ≤ -2 with $Hom(\mathbb{Q}(1), M) = 0$,
(iii) a motive whose nonzero graded pieces in the weight filtration are a sum of copies of $\mathbb{Q}(1)$, a pure motive of weight -1, and a sum of copies of $\mathbb{Q}(0)$,
(iv) the 1-motive $[\phi : \mathbb{Z} \to \mathbb{G}_m]$, $\phi(1) = p$
(d) in cases (i) - (iii) M_i is a motive over $\mathbb{Z}$.

(iv) For other (conjecturally equivalent) definitions of critical motive we refer to [FoP],[Gr],[Schn].

3 A conjecture about p - adic L - functions attached to (possibly non-critical) motives

3.1 Admissible measures and their Mellin transforms

For the rest of this paper we fix a rational prime p and embeddings of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$ and in $\mathbb{C}$; let $\mathbb{Z}_p$ be the ring of p-adic integers. We have $\mathbb{Z}_p^\times \cong Gal_p$, where $Gal_p = Gal(\mathbb{Q}(\{p\})/\mathbb{Q})$ denotes the Galois group of the maximal abelian extension $\mathbb{Q}(\{p\})$ of $\mathbb{Q}$ ramified only at p and infinity. We denote by x_p the inclusion $\mathbb{Z}_p \to \mathbb{C}_p$, where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ is the Tate field.

Let $C^h(Gal_p)$ be the space of $\mathbb{C}_p$-valued functions on Gal_p which can be locally represented by polynomials in the variable x_p of degree less then h.

Definition. *By a h-admissible measure on Gal_p we mean a $\mathbb{C}_p$-linear form $\mu : C^h(Gal_p) \to \mathbb{C}_p$ which satisfies the following growth condition for all $0 \leq i \leq h-1$:*

$$\sup_{a \in Gal_p} \left| \int_{a+(m)} (x_p - a_p)^i d\mu \right|_p = o(|\, m\, |_p^{i-h}).$$

By the integral in the above equality we mean the value of μ on the product of the polynomial in x_p and the characteristic function of the set $a + (m)$.

Let $\mathcal{X}_p := Hom_{cont}(Gal_p, \mathbb{C}_p^\times)$ be the $\mathbb{C}_p$-analytic Lie group consisting of all continuous p-adic characters of the group Gal_p. We regard the elements of finite order $\chi \in \mathcal{X}_p$ as Hecke characters of finite order whose conductors may contain only powers of p. To each h-admissible measure μ one can associate its non-archimedean Mellin transform $L_\mu : \mathcal{X}_p \to \mathbb{C}_p$,

$$L_\mu(\chi) := \int_{Gal_p} \chi d\mu.$$

The function L_μ is of the $o(log^h)$-type. It is known that the special values of the form $L_\mu(\chi x_p^r)$ for $r = 0, 1, ..., h-1$, $\chi \in \mathcal{X}_p^{tors}$ determine L_μ and μ uniquely [Vi].

3.2 A new version of Deligne's period conjecture

Let $M \in \mathcal{M}_{\mathbf{Q}}$ and $M^\sim$ be the corresponding mixed motive over $\mathbb{Z}$ (see section 2). Put $\Omega(\epsilon_0, M^\sim) := (2\pi i)^{r(M^\sim)} c^{\epsilon_0}(M^\sim)$, where $\epsilon_0 := \pm$, $r(M^\sim) := \sum_{j<k} jh(j,k)$, $h(j,k) := dim_C M^{\sim j,k}$. Define $\Lambda_{(\infty)}(M^\sim, s) := E_\infty(M^\sim, s) L(M^\sim, s)$, where the modified Euler factor at infinity is defined as follows: $E_\infty(M^\sim, s) := \prod_u E_\infty(U, s)$, where U runs over direct summands of the Hodge decomposition, and $E_\infty(U, s)$ is given by

(a) if $U = M^{\sim j,k} \oplus M^{\sim k,j}$, $j < k$ then $E_\infty(U, s) := \Gamma_{\mathbf{C}}(s-j)^{h(j,k)}$,

(b) if $U = M^{\sim k,k}$, $k \geq 0$, then $E_\infty := 1$

(c) if $U = M^{\sim k,k}$, $k < 0$ then $E_\infty(U, s) := \frac{\Gamma_{\mathbf{R}}(s-k+\delta)}{i^\delta \Gamma_{\mathbf{R}}(1-s+k-\delta)}$, where $\delta = 0, 1$ is choosen so that F_∞ acts on $M^{k,k}$ as $(-1)^{k+\delta}$, $\Gamma_{\mathbf{C}}(s) := 2(2\pi i)^{-s}\Gamma(s)$, $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$.

For the Dirichlet character χ of conductor $C(\chi)$ we define the Gauss sum $G(\chi) := \sum \chi(x) exp(2\pi i x / C(\chi))$, where x runs over a complete set of relatively prime residue classes modulo $C(\chi)$.

Now we formulate a new version of Deligne's conjecture, which gives a natural variation of the period as m and χ vary [Co1,2]. Note, that we do not assume M to be critical.

Period conjecture (a modified form). *Let $M \in \mathcal{M}_{\mathbf{Q}}$. Then for every pair $(m, \chi) \in \mathbb{Z} \times \mathcal{X}_p^{tors}$ such that $M^\sim(\chi)(m)$ is critical (highly critical), and $\epsilon_0 = \epsilon(\chi) \cdot v$, $v = sgn((-1)^m)$, we have*

$$\Lambda_{(\infty)}(M^\sim(m)(\chi), 0)(G(\chi)^{d^{\epsilon_0}(M^\sim)} \Omega(\epsilon_0, M^\sim))^{-1} \in \mathbb{Q}(\chi).$$

3.3 h-admissible and p-ordinary motives

We consider the local p-polynomial

$$L_p(M, X)^{-1} = 1 + A_1(p)X + ... + A_d(p)X^d \in \mathbb{C}_p[X]$$

of the motive M.

Definition. *(a) The Newton p-polygon $P_{N,p}(u) = P_{N,p}(u, M)$ of the motive M is the convex hull of the points $(i, ord_p(A_i(p)))$, $i = 0, 1, ..., d$.*
(b) The Hodge polygon $P_H(u) = P_H(u, M)$ $(0 \leq u \leq d)$ by definition passes through the points $(0,0), ..., (\sum_{i' \leq i} h(i', j), \sum_{i' \leq i} i' h(i', j))$, ... i.e. the lenght of the horizontal segment with slope i is equal to $\sum_{w_k} h(i, w_k - i)$, where $w_k = w(Gr_k^W M)$
(c) We call M a p-admissible motive if $P_{N,p}(d^\pm) = P_H(d^\pm)$
(d) $h = h(p, M) := [P_{N,p}(d^\pm) - P_H(d^\pm)] + 1$

Definition. *We call $M \in \mathcal{MM}_{\mathbb{Q}}$ a p-ordinary, if*
(i) $I_p \subset G_p$ acts trivially on each $Gr^W_m M_l$, $l \neq p$
(ii) $\forall_m \exists$ a decreasing filtration $F^i_{p,m}$ on $V_m := Gr^W_m M_p$, of G_p-stable $\mathbb{Q}_p$-subspaces such that

$$\forall_{i \in \mathbf{Z}}\ G_p \text{ acts on } F^i_{p,m} V_m / F^{i+1}_{p,m} V_m$$

via some power of the cyclotomic character, say $\psi_p^{-e_i(m)}$. Then $e_1(m) \geq \ldots \geq e_t(m)$, and the following properties take place:
(a) $dim_{\mathbb{Q}_p} F^i_{p,m} V_m / F^{i+1}_{p,m} V_m = h(e_i(m), w_m - e_i(m))$,
(b) $P_{N,p}(u, M) = P_H(u, M)$.

Proposition. *If $M \in \mathcal{M}_{\mathbb{Q}}$ is p-ordinary, then $M^{\sim}$ is p-ordinary.*

The proof follows from the definitions.

Remarks.

(i) $h(p, M)$ does not change if we replace M by its Tate twist, by its dual $\widehat{M}$, and by its twists with Hecke characters of finite order whose conductor is prime to p.

(ii) the quantity $h^*(p, M) := P_{N,P}(d^{\pm}) - P_H(d^{\pm})$ can be considered as generalization to motives of the classical Hasse invariant of elliptic curve: it distinguishes p-ordinary and p- supersingular cases.

(iii) In the case of the motive of elliptic curve E over $\mathbb{Q}$ we have the theorem of Elkies [El]: there exists an infinite number of supersingular prime numbers for E. In fact we have even more: in the CM-case approximately half of primes are supersingular; in the case of elliptic curves without complex multiplication we expect about $C\frac{x^{\frac{1}{2}}}{log\ x}$ supersingular primes up to x, where $C > 0$ some constant (Lang-Trotter [LaT]).

Now let's consider the case of elliptic cusp forms with respect to the congruence subgroups. Here the situation is more complicated.

· In the well examined case of unique normalized elliptic cusp form $\Delta = \sum \tau(n) q^n$ of weight 12 with respect to $SL(2, \mathbb{Z})$ we have the Lehmer conjecture $\tau(p) \neq 0$ for every prime p (checked for $p < 10^{15}$), so the most supersingular case (hypothetically) does not appear. On the other hand, it is known that $p \not| \tau(p)$ for every prime $2 \leq p \leq 65063$, $p \neq 2, 3, 5, 7, 2411$.

· Let

$$\Theta_1 := \sum_{m,n \in \mathbf{Z}} q^{m^2 + mn + 4n^2}, \qquad g(z) := \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

Then $f := g\Theta_1 \in S_3\left(15, \left(\frac{\cdot}{15}\right)\right)$ is normalized Hecke eigenform. Peters, Top and van der Vlugt [PTV] proved the following result.

Proposition. . *Let $f = \sum_{n \geq 1} a_n q^n$ be as above. Then*
(a) $a_p = 0$ iff $\left(\frac{p}{15}\right) = -1$,
(b) a_p, for $p > 5$, $p \equiv 1, 2, 4, 8 (mod 15)$ can be computed by means of the following algorithm

(i) $p \equiv 1$ or $4(mod 15)$: find an integral solution of the equation $x^2+xy+4y^2 = p$; then $a_p = 2x^2 - 7y^2 + 2xy$.

(ii) $p \equiv 2$ or $8(mod 15)$: find an integral solution of $2x^2 + xy + 2y^2 + p$; then $a_p = x^2 + 8xy + y^2$.

It is easy to see that, in this case, the most supersingular case doesn't appear. Indeed, let (x, y) be an integral solution of (i). Then $a_p = 2p - 15y^2$, and $p|a_p \Rightarrow p|y \Rightarrow p|x \Rightarrow p^2|p$. Analogously for (ii). Therefore only two extremal cases appear. Also note that in our example f is with CM and this may suggest that, as in the CM-elliptic curves case, only extremal cases appear.

In the general case of motives over $\mathbb{Q}$ we may consider the attached F-cristals and the results and conjectures probably should be formulated in these terms.

(iv) it is plain that $h(p, M) = h(p, M^\sim)$ where $M^\sim$ denotes the universal extension described in section 2.

(v) any p-ordinary motive is p-admissible.

(vi) note that B. Perrin-Riou [Pe1; 2.3.1] has given another characterization of the p-admissibility condition for a motive M: there exists p-adic subrepresentation M_p' of M_p such that $F^0D(M_p') = 0,\ \ D(M_p/M_p') = F^0D(M_p/M_p')$.

Now we give examples of p-admissible pure motives which are not p-ordinary.

(a) It is easy to see that the motive $M(f) \otimes M(g)$ (where f, g are elliptic cusp forms such that $w(f) > w(g)$ and f is p-ordinary) is of this type. Note, that $M(f) \otimes M(g)$ is p-ordinary iff f and g are p-ordinary.

(b) More generally, we have the following facts:

Proposition. *Let f be p-ordinary CM- elliptic cusp form of weight k and $M \in \mathcal{M}_{\mathbb{Q}}$ be pure motive of weight w and of rank d. Assume $k \gg 0$. Then the motive $Sym^{2m+1}M(f) \otimes M$ is p-admissible, where $m = 0, 1, \ldots$.*

Proof: As $h(p, M) = h(p, M(r))$, we can suppose that $w \geq 0$ and $M_B \otimes \mathbb{C} = M^{0,w} \oplus \ldots \oplus M^{w,0}$. Let α, β denote p-roots of f, $ord_p\alpha = 0$, and $\alpha_1, \ldots, \alpha_d$ are inverse p-roots of M indexed in such a way that $ord_p\alpha_1 \leq \ldots \leq ord_p\alpha_d$. Then, under the above assumption, we have

$ord_p(\alpha^{2m+1}\alpha_1) \leq \ldots \leq ord_p(\alpha^{2m+1}\alpha_d) \leq ord_p(\alpha^{2m}\beta\alpha_1) \leq \ldots \leq ord_p(\alpha^{2m}\beta\alpha_d) \leq \ldots \leq ord_p(\alpha^{m+1}\beta^m\alpha_1) \leq \ldots \leq ord_p(\alpha^{m+1}\beta^m\alpha_d)$.

Now

$$d^+(Sym^{2m+1}M(f) \otimes M) = (m+1)d$$

and

$P_{N,p}((m+1)d, Sym^{2m+1}M(f) \otimes M) = \sum_{i=0}^{m}\sum_{j=1}^{d} ord_p(\alpha^{2m+1-i}\beta^i\alpha_j)$
$= (m+1)\sum_{i=1}^{d} ord_p\alpha_i + d(ord_p\beta + ord_p\beta^2 + \ldots + ord_p\beta^m)$
$= \frac{1}{2}(m+1)dw + \frac{1}{2}m(m+1)(k-1)d$

(here we use the property $\sum_{i=1}^{d} ord_p\alpha_i = \frac{1}{2}dw$).

On the other hand

$(Sym^{2m+1}M(f) \otimes M)_B \otimes \mathbb{C} = \oplus_{r=0}^{m} \oplus_i M^{i+r(k-1),(2m+1-r)(k-1)+w-i} \oplus$ the conjugate part; here i are the same as in the Hodge decomposition $M_B \otimes \mathbb{C} = \oplus_i M^{i,w-i}$ of M. Put $d_i = dim_{\mathbb{C}} M^{i,w-i}$. Then

$P_H((m+1)d, Sym^{2m+1}M(f)\otimes M) = \sum_i id_i + \sum_i(i+k-1)d_i + \sum_i(i+2k-2)d_i + \ldots + \sum_i(i+mk-m)d_i = (m+1)\sum_i id_i + (1+\ldots+m)(k-1)\sum_i d_i = \frac{1}{2}(m+1)dw + \frac{1}{2}m(m+1)(k-1)d.$

Similarly one can prove the following

Proposition. *Let f be p-ordinary elliptic cusp form of weight k and $M \in \mathcal{M}_{\mathbb{Q}}$ be pure motive of weight w and of rank d. Assume $k \gg 0$. Then the motive $M(f)\otimes M$ is p-admissible.*

Note, that in non-CM case $Sym^r M(f)$ is irreducible.

We also note the following result, due to N.M. Katz.

Proposition. *([Ka1]) Let $M_1, M_2 \in \mathcal{M}_{\mathbb{Q}}$. Then $M_1 \otimes M_2$ is of Hodge-Witt type iff M_1 is p-ordinary and M_2 is of Hodge-Witt type.*

3.4 A formulation of the Conjecture

We use the following notation:

$N^{\sim} := M^{\sim}(\chi)(m)$,
$m_* := max\{j | \exists_{j,k}\ j < k \text{ such that } h(j,k) \neq 0\} + 1,$
$m^* := min\{j | \exists_{j,k}\ j > k \text{ such that } h(j,k) \neq 0\},$

$$A_p(N^{\sim}) := \begin{cases} \prod_{i=d^+(N^{\sim})+1}^{d} \left(1 - \chi(p)\alpha^{(i)}(p)\cdot p^{-m}\right) & \cdot \\ \cdot\prod_{i=1}^{d^+(N^{\sim})} \left(1 - \chi(p)^{-1}\alpha^{(i)}(p)^{-1}\cdot p^{m-1}\right) & \text{if } p \not| C(\chi) \\ \prod_{i=1}^{d^+(N^{\sim})} \left(\frac{p^m}{\alpha^{(i)}(p)}\right)^{ord_p C(\chi)} & \text{if } p | C(\chi) \end{cases}$$

where inverse roots of the local p-polynomial $L_p(M^{\sim}, X)^{-1}$ are indexed in such a way that $ord_p\alpha^{(1)}(p) \leq ord_p\alpha^{(2)}(p) \leq \ldots \leq ord_p\alpha^{(d)}(p)$.

Conjecture. *For every $\epsilon_0 \in \{\pm\}$ there exists a $\mathbb{C}_p$-analytic function $L_{(p)}^{(\epsilon_0)} : \mathcal{X}_p \to \mathbb{C}_p$ such that*
(i) For all but a finite number of pairs $(m,\chi) \in \mathbb{Z}\times\mathcal{X}_p^{tors}$ such that $N^{\sim} = M^{\sim}(\chi)(m)$ is critical at $s=0$ and $\epsilon_0 = \epsilon(\chi)\cdot v$, $v = sgn((-1)^m)$, we have

$$L_{(p)}^{(\epsilon_0)}\left(\chi x_p^m\right) = \frac{A_p(N^{\sim})}{G(\chi)^{d^{\epsilon_0}(M^{\sim})}} \cdot \frac{\Lambda_{(\infty)}(M^{\sim}(\chi)(m),0)}{\Omega(\epsilon_0, M^{\sim})}$$

(ii) $L_{(p)}^{(\epsilon_0)}(\chi)$ is holomorphic on $\mathcal{X}_p$ if $M^{k,k}=0$; otherwise there exists a finite subset $\Xi \subset \mathcal{X}_p$ and natural numbers $n(\xi)$, $\xi \in \Xi$ such that $\forall_{g_0} \in G_p$ the function

$$\prod_{\xi\in\Xi}(\chi(g_0) - \xi(g_0))^{n(\xi)} \cdot L_{(p)}^{(\epsilon_0)}(\chi)$$

is holomorphic on $\mathcal{X}_p$.

(iii) The function in (ii) is holomorphic of type $o(\log^h)$.

(iv) If $h \le m^* - m_* + 1$ *then the above conditions (i) and (ii) uniquely determine the function* $L_{(p)}^{(\epsilon_0)}$.

(v) If the motive M is p-admissible then there exists an unique bounded $\mathbb{C}_p$*-analytic function* $L_{(p)}^{(\epsilon_0)}$ *satisfying the conditions (i) and (ii).*

Remarks.

(a) In the p-ordinary critical case the conjecture was formulated by J. Coates and B. Perrin-Riou [Co1,2].

(b) In the critical case the conjecture was formulated in [Da], [Pa1].

(c) The conjecture has been verified in the case of Tate motives (Kubota-Leopoldt), elliptic cusp forms (Manin, Vishik), symmetric squares (Schmidt, Panchishkin) and tensor products of two elliptic cusp forms (Hida), symmetric powers of elliptic cusp forms of CM-type, "standard motives" attached to Siegel cusp eigenforms (Panchishkin).

(d) 3.3. gives us the examples of p-admissible (not necessary p -ordinary) motives, for which it should exist a bounded $\mathbb{C}_p$-analytic function as described in (v) above.

(e). For motivic Iwasawa theory and its relation to p-adic L-functions, p-adic Bloch-Kato, etc. we refer to [Fo],[FoP],[Gr1,2],[Ka],[Pe1,2],[Schn].

3.5 An invariant form of the Conjecture

Our aim here is to formulate an invariant analogue of the conjecture 3.4 using certain complex representations of the Weil-Deligne group. In the case of p-ordinary critical pure motives such a description was done by J. Coates [Co2].

Suppose $M \in \mathcal{M}_{\mathbb{Q}}$. For any prime number p let $W_p \subset G_p$ be the Weil group. We fix an element $\Phi \in G_p$, whose image in G_p/I_p is the geometric Frobenius. For each $s \in \mathbb{C}$ let

$$\omega_s : W_p \to \mathbb{C}^\times$$

be the homomorphism which is trivial on I_p, and such that $\omega_s(\Phi) = p^{-s}$.

We also fix a prime number $l \neq p$.

Let W_p' be the Weil-Deligne group of $\mathbb{Q}_p$. Recall (see [De2]) that a finite-dimensional representation of W_p' over $\mathbb{Q}_p$ is, by definition, a pair $\Theta = (\gamma, N)$, where V is a finite-dimensional linear space over $\mathbb{Q}_l$ and

(i) $\gamma : W_p \to GL(V)$ is a homomorphism, whose kernel contains an open subgroup of I_p

(ii) N is a nilpotent endomorphism of V such that

$$\forall \sigma \in W_p \;\; \gamma(\sigma) N \gamma(\sigma)^{-1} = \omega_l(\sigma) N$$

Consider l-adic representation of W_p given by its natural action on M_l. Then, by Grothendieck theorem, this representation gives rise to a unique representation

$\Theta = (\gamma, N)$ of the Weil-Deligne group W'_p (see [De2], sec.8). Put

$$R_p(M,s) := \frac{L_p(M,s)}{\varepsilon_p(M,s)L_p(M^\wedge, 1-s)}$$

and similarly

$$R_p(\Theta,s) := \frac{L_p(\Theta,s)}{\varepsilon_p(\Theta,s)L_p(\Theta^\wedge, 1-s)}$$

where $L_p(M,s)$ denotes the Euler factor at p, and $L_p(\Theta,s) := Z_p(\Theta, p^{-s})$, for

$$Z_p(\Theta,X) := det(1-\gamma(\Phi)X|ker(N)^{\gamma(I)})^{-1}$$

and $\varepsilon_p(M,s), \varepsilon_p(\Theta,s)$ are defined in [Co2], [De1,2].

Lemma. *[Co2] There exists a representation $\Theta' = (\gamma', N')$ of the Weil-Deligne group in M_l, which satisfies*

(i) $N' = 0$

(ii) If we extend scalars from $\mathbb{Q}_l$ to $\mathbb{C}$ via a fixed embedding $\mathbb{Q}_l \to \mathbb{C}$, then γ' is a semisimple complex representation of W_p

(iii) we have

$$R_p(M,s) = R_p(\Theta', s).$$

Let $\gamma' : W_p \to GL(Y)$, where $Y := M_l \otimes_{\mathbb{Q}_l} \mathbb{C}$, be the semisimple complex representation of the Weil group given by the above Lemma. Let $Y = \oplus U$ be the decomposition of Y into irreducible complex representations of W_p. By [De2] it is known that each such U is of the form $\xi_U \otimes \omega_{s(U)}$, where $s(U) \in \mathbb{C}$ and ξ_U is a complex representation of W_p such that $\xi_U(W_p)$ is a finite group. Consequently, the inverse roots of the polynomial $P(X,U) = det(1-\Phi\cdot X|_U)$ are all of the form a root of unity times one fixed root. Assuming that these roots are algebraic numbers, we define $ord_p(U) := ord_p(\alpha)$ for any inverse root α of the above polynomial, where ord_p is normalized so that $ord_p(p) = 1$. Note, that $ord_p(U)$ does not depend of the choice of Φ. Define $R_p(U,s)$ by the same formula as for $R_p(M,s)$.

Basic for what follows will be the following

Hope. *It should exist a "natural" filtration $\{F_l^k\}$ of W'_p-subspaces of Y which agrees with the Hodge filtration $\{F^k\}$ on $M_B \otimes \mathbb{C}$ in the sense that $dim_{\mathbb{C}}F^k = dim_{\mathbb{C}}F_l^k$.*

In CM-case it is given by comparison isomorphisms. In general case it seems to be no natural candidate.

For the rest of this section we shall assume the Hope takes place.

We define modified Euler factor at p:

$$E_p(M,s) = \prod_{U\ irred} E_p(U,s),$$

where

$$E_p(U,s) = \begin{cases} 1 & \text{if } U \not\subset F_l^0 \\ R_p(U,s) & \text{if } U \subset F_l^0 \end{cases}.$$

Let $\chi \in \mathcal{X}_p^{tors}$. For every irreducible summand $U \subset Y$ we put

$A_p(U(\chi), \{F_l^k\}, s) :=$

$$\begin{cases} \prod_{P(\alpha,U)=0}(1-\alpha(p)p^{-s}) & \text{if } U \not\subset F_l^0 \textit{ and } \chi \textit{ trivial} \\ \prod_{P(\alpha,U)=0}(1-\alpha(p)^{-1}p^{s-1}) & \text{if } U \subset F_l^k \textit{ and } \chi \textit{ trivial} \\ 1 & \text{if } U \not\subset F_l^k \textit{ and } \chi \textit{ nontrivial} \\ G(\chi)^{-dim\ U} \prod_{P(\alpha,U)=0} \left(\frac{p^s}{\alpha(p)}\right)^{ord_p C(\chi)} & \text{if } U \subset F_l^k \textit{ and } \chi \textit{ nontrivial} \end{cases}$$

Then we have the following (conditional) result (compare [Co2], Lemma 7):

Proposition. *Assume M has good reduction at p. Then*

$$\frac{E_p(M(\chi),s)}{L_p(M(\chi),s)} = \prod_{U_{irr} \subset Y} A_p(U(\chi), \{F_l^k\}, s).$$

Proof: Using the same argument as in [Co2] we obtain

$$A_p(U(\chi), \{F_l^k\}, s) = \frac{E_p(U(\chi),s)}{L_p(U(\chi),s)}.$$

The assertion follows.

We propose the following

Definition.

$$h^{\sim}(p,M) := [\sum_{U \subset F_l^0} (dim\ U) \cdot ord_p U - P_H(d^+)] + 1.$$

Note that $h(p,M) \leq h^{\sim}(p,M)$.

Proceeding as in [Co2] we define the modified L-function

$$\Lambda_{(\infty,p)}(M, \{F_l^k\}, s) := A_\infty(M,s) \cdot E_p(M, \{F_l^k\}, s) \cdot \prod_{q \neq p,\infty} L_q(M,s)$$

Now we are ready to formulate an invariant form of the conjecture 3.4 (compare [Co1,2]).

Conjecture. *For every $\epsilon_0 \in \{\pm\}$ there exists a $\mathbb{C}_p$-analytic function $L_{(p)}^{(\epsilon_0)} : \mathcal{X}_p \to \mathbb{C}_p$ such that*
(i) For all but a finite number of pairs $(m, \chi) \in \mathbb{Z} \times \mathcal{X}_p^{tors}$ such that $N^\sim = M^\sim(\chi)(m)$ is critical at $s = 0$ and $\epsilon_0 = \epsilon(\chi) \cdot v$, $v = sgn((-1)^m)$, we have

$$L_{(p)}^{(\epsilon_0)}(\chi x_p^m) = \frac{\Lambda_{(p,\infty)}(M^\sim(\chi)(m), \{F_l^k\}, 0)}{\Omega(\epsilon_0, M^\sim)}$$

(ii) $L_{(p)}^{(\epsilon_0)}(\chi)$ is holomorphic on $\mathcal{X}_p$ if $M^{k,k} = 0$; otherwise there exists a finite subset $\Xi \subset \mathcal{X}_p$ and natural numbers $n(\xi)$, $\xi \in \Xi$ such that $\forall_{g_0} \in G_p$ the function

$$\prod_{\xi \in \Xi} (\chi(g_0) - \xi(g_0))^{n(\xi)} \cdot L_{(p)}^{(\epsilon_0)}(\chi)$$

is holomorphic on $\mathcal{X}_p$.
(iii) The function in (ii) is holomorphic of type $o(\log^{h^\sim})$.
(iv) If $h^\sim \leq m^* - m_* + 1$ then the above conditions (i) and (ii) uniquely determine the function $L_{(p)}^{(\epsilon_0)}$.
(v) If the motive M is p-admissible then there exists an unique bounded $\mathbb{C}_p$-analytic function $L_{(p)}^{(\epsilon_0)}$ satisfying the conditions (i) and (ii).

4 p-adic families attached to motives

Hida [Hi1,2] obtained some p-adic analytic families of p-ordinary cusp forms, and corresponding L-functions. A conjectural generalization of the Hida's construction to arbitrary critical pure motives has been formulated in a recent paper of A.A. Panchishkin [Pa2].

4.1 p-adic families of Galois representations attached to motives

Let $\mathcal{Y}_p := Hom_{cont}(\mathbb{Z}_p^\times \times Gal_p, \mathbb{C}_p^\times)$ be the $\mathbb{C}_p$-analytic Lie group consisting of all continuous p-adic characters of the group $\mathbb{Z}_p^\times \times Gal_p$, which contains the $\mathbb{C}_p$-analytic Lie subgroup $\mathcal{X}_p$ (the cyclotomic line) via the projection $\mathbb{Z}_p^\times \times Gal_p \to Gal_p$. $\mathcal{Y}_p$ contains the discrete subgroup A of arithmetical characters of the type $\chi \cdot \eta \cdot x_p^m = (\chi, \eta, m)$, where $\chi \in \mathcal{Y}_p^{tors}$, η is algebraic character of $\mathbb{Z}_p^\times$, $m \in \mathbb{Z}$. Let $\mathcal{O}_p$ denotes the ring of integers of the Tate field $\mathbb{C}_p$. For any $P = (\chi, \eta, m) \in A$ we have a homomorphism

$$\nu : \mathbb{Z}_p^\times \times Gal_p \to \mathcal{O}_p$$

defined by the corresponding group homomorphism

$$P : \mathbb{Z}_p^\times \times Gal_p \to \mathcal{O}_p^\times \hookrightarrow \mathbb{C}_p^\times.$$

For a $\mathcal{O}_p[[\mathbb{Z}_p^\times \times Gal_p]]$ - module N and $P \in A$ we define "the reduction of N modulo P":

$$N_P := N \bigotimes_{\mathcal{O}_p[[\mathbb{Z}_p^\times \times Gal_p]]\ ,\ \nu_P} \mathcal{O}_p.$$

Then for each Galois representation

$$r_N : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(N)$$

we have defined its reduction mod P as the natural composition

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(N) \to GL(N_P).$$

Conjecture. *(compare [Pa2]). For every $M \in \mathcal{MM}_{\mathbb{Z}}$ there exists a free $\mathcal{O}_p[[\mathbb{Z}_p^\times \times Gal_p]]$-module $M_\star$ of the rank $d = rank\ M$, a Galois representation*

$$r_\star : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(M_\star)$$

an infinite subset $A' \subset A$ of "positive" characters, and a distinguished point $P_0 \in A$ such that
(a) the reduced Galois representation

$$r_{\star,P_0} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(M_{\star,P_0})$$

is equivalent over $\mathbb{C}_p$ to the p-adic representation $r_{M,p}$ of M,
(b) for every $P \in A'$ there exists a motive $M_P \in \mathcal{MM}_{\mathbb{Z}}$ such that its Galois representation is equivalent over $\mathbb{C}_p$ to the reduction

$$r_{\star,P} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(M_{\star,P}).$$

4.2 Families of p-adic L-functions

Now we are ready to describe some families of p-adic L-functions associated with p-adic families of Galois representations coming from (possibly non-critical) motives.

For $M \in \mathcal{M}_{\mathbb{Q}}$ denote by $M^\sim$ the associated "universal extension"; put

$$\Lambda_{(p,\infty)}(M(m)(\chi), s) = G(\chi)^{-d^{\epsilon_0}(M(m)(\chi))} A_p(M(m)(\chi), s) \cdot \Lambda_{(\infty)}(M(m)(\chi), s).$$

Conjecture. *(compare [Pa2]). For a canonical choice of periods $\Omega(P) \in \mathbb{C}^\times$, $P \in A'$, there exists a $\mathbb{C}_p$-analytic function*

$$L_{(p)} : \mathcal{Y}_p \to \mathbb{C}_p$$

such that
(i) for almost all $P \in A'$ we have

$$L_{(p)}(P) = \frac{\Lambda_{(p,\infty)}(M^\sim(\chi)(m), 0)}{\Omega(P)}$$

(ii) For arithmetic points of type

$$P = (\chi, \eta_0, m) \in A'$$

with η_0 fixed there exists a finite subset $\Xi \subset \mathcal{Y}_p$ and natural numbers $n(\xi)$, $\xi \in \Xi$ such that $\forall_{g_0} \in G_p$ the function

$$\prod_{\xi \in \Xi} \left(x(g_0) - \xi(g_0)\right)^{n(\xi)} L_{(p)}(x \cdot P)$$

is holomorphic on $\mathcal{Y}_p$.
(iii) The function in (ii) is holomorphic of type $o\left(log^{h_0}\right)$ where $h_0 = h(M_P)$ for P as in (ii).
(iv) Consider arithmetic points of type

$$P = (\chi, \eta_0, m) \in A'$$

with η_0 fixed. If the motive M_P is p-admissible then the function in (ii) is bounded.

Examples of p-adic families satisfying conjectures 4.1 and 4.2 are given by the work of Katz [Ka2], Hida [Hi1,2]; see also [GrS],[MaV].

4.3 "False" p-adic L-functions attached to motives

Let $\mathcal{T}$ denotes the Hecke algebra, and let $\lambda : \mathcal{T} \to \overline{\mathbb{Q}}_p$ be a continuous homomorphism of weight k, $k \geq 2$ an integer. Let $P = ker\lambda$, $\mathcal{O} = \mathcal{T}/P \hookleftarrow \mathbb{Z}_p$, $F =$ the field of fractions of $\mathcal{O}$. Assume that the residual representation attached to P is irreducible. Hida [Hi1] has shown that the corresponding two-dimensional representation $\rho_P : G_{\mathbf{Q}} \longrightarrow GL_2(F)$ is a Deligne representation attached to a cuspidal newform of weight k. Let $L_{(p)}(k,s)$, $k,s \in \mathbb{Z}_p$, denotes a two variable p-adic L-function attached to ordinary Λ-adic cusp form [GrS]. Then $L_{(p)}(k,s)$ is analytic on $\mathbb{Z}_p^2$ and interpolates the one variable p-adic L-functions $L_{(p)}(M(f_k),s)$ associated to the newforms f_k, $k \geq 2$, in the following sense: $\forall k \geq 2$ there is a "period" $\Omega_k \in \mathbb{Q}_p$ such that $L_{(p)}(k,s) = \Omega_k \cdot L_{(p)}(f_k,s)$.

Therefore, in the naive sense, the limit case $k = 1$ should correspond to a "twisted" modular form of weight one. More precisely, put formally

$$L^{\sim}_{(p)}(s) := L_{(p)}(1,s)$$

Question. *Does exist an invertible function $\Omega_1(s)$ such that $L^{\sim}_{(p)}(s) = \Omega_1(s) \cdot L_{(p)}(s)$, where $L_{(p)}$ is the product of twisted p-adic L-functions of Kubota-Leopoldt-Iwasawa ?*

Note, that Mazur and Wiles [MaW] gave examples when the specialization to weight one cannot be the p-adic representation attached to a classical modular form of weight one. They show that in these examples the corresponding p-adic Hodge structure is not even semi-simple. On the other hand from their work we also know that for $k = 1$ the λ -adic Hodge twists of ρ_P are $(0,0)$.

In general, the weight one specjalization of the two variable p-adic L-function is a new one-variable p-adic L-function. As R. Greenberg informed the author, in some cases a specialization of the 2-variable p-adic L-function should be related to a product of two Kubota-Leopoldt p-adic L-functions. But the appropriate specialization is then some p-adic value of k, where the Λ-adic specialization becomes reducible as a $\mathbb{Q}_p$-representation.

Our idea is to try to produce some motivic p-adic L-functions by applying the Conjecture from section 4.2 to appropriate family of Galois representations.

First let's consider the following fact which can be considered as a generalization of a result of Blasius ([Bl]) where the case $M = M(f_1) \otimes ... \otimes M(f_t)$ was treated.

Proposition. *Let M be any pure motive over $\mathbb{Q}$ (with coefficients in $\mathbb{Q}$) and f_k be fixed elliptic cusp form of weight k. Then we have, for $k \gg_M 0$:*

$$c^+(M \otimes M(f_k)) = c^+(f_k)^{d^+(M)} c^-(f_k)^{d^-(M)} \delta(M),$$

$$c^-(M \otimes M(f_k)) = c^+(f_k)^{d^-(M)} c^-(f_k)^{d^+(M)} \delta(M).$$

Proof: By the assumption we have

$$F^{\pm}(M \otimes M(f_k)) = H_{DR}(M) \otimes F^{\pm}(M(f_k))$$

Let $\{u^{\pm}\}$ (resp. $\{v_1, ..., v_m\}$) be a basis of $H_B^{\pm}(M(f_k))$ (resp. of $H_B^{\pm}(M)$). Let $\{e^-\}$ be a basis of $F^-(M(f_k))$; take e^+ so that $\{e^+, e^-\}$ form a basis of $H_B(M(f_k))$. Let $\{d_1, ..., d_m\}$ be a basis of $H_{DR}(M)$. Let

$$I_{f_k} : H_B(M(f_k)) \otimes \mathbb{C} \to H_{DR}(M(f_k)) \otimes \mathbb{C}\ ,\ I_M : H_B(M) \otimes \mathbb{C} \to H_{DR}(M) \otimes \mathbb{C}$$

be canonical isomorphisms. Then $I_{f_k}(u^+) = x_{11}e^+ + x_{12}e^-$, $I_{f_k}(u^-) = x_{21}e^+ + x_{22}e^-$, $I_M(v_i) = \sum_{j=1}^m y_{ij}d_j$ and $c^+(f_k) = x_{11}$, $c^-(f_k) = x_{21}$, $\delta(M) = det(y_{ij})$. Let $\{v_1, ..., v_t\}$ (resp. $\{v_{t+1}, ..., v_m\}$) be a basis of $H_B^+(M)$ (resp. $H_B^-(M)$), where $t = d^+(M)$. Then $d_l \otimes e^+$ mod $F^-(M \otimes M(f_k))$, $1 \leq l \leq m$, form a basis of $H_{DR}^{\pm}(M \otimes M(f_k))$. We have

$$(I_M \otimes I_{f_k})(v_i \otimes u^+) = \sum_{l=1}^m y_{il}x_{11}(d_l \otimes e^+)\ modulo\ F^-(M \otimes M(f_k)) \otimes \mathbb{C}$$

$$(I_M \otimes I_{f_k})(v_i \otimes u^-) = \sum_{l=1}^m y_{il}x_{21}(d_l \otimes e^+)\ modulo\ F^-(M \otimes M(f_k)) \otimes \mathbb{C}$$

Therefore

$$c^+(M \otimes M(f_k)) = det \begin{pmatrix} x_{11} & Y_1 \\ x_{21} & Y_2 \end{pmatrix},\ c^-(M \otimes M(f_k)) = det \begin{pmatrix} x_{11} & Y_2 \\ x_{21} & Y_1 \end{pmatrix}$$

where $(y_{ij}) = Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $Y_1 \in M_{t \times m}(\mathbb{C})$, $Y_2 \in M_{(m-t) \times m}(\mathbb{C})$

Corollary. *Under the above notation if $M^{r,r} = 0$ for $k \gg 0$,*

$$c^{\pm}(M \otimes M(f_k)) = < f_k, f_k >^{\frac{1}{2}d(M)} \delta(M).$$

Now the idea is to include a motive M into p-adic (critical) family and to apply 4.2. Let f_k denote fixed elliptic cusp form of weight k. We assume that $\exists_{A=A(M)}$ $\forall_{k \geq A}$ $M \otimes M(f_k)$ have critical points.

Let's specialize the conjecture from 4.2 to our situation

Conjecture. *There exists a two variable p-adic L-function $L_{(p)}(M; k, s)$ on $\mathbb{Z}_p^2$, which interpolates the one variable p-adic L-functions $L_{(p)}(M \otimes M(f_k), s)$, $k \geq A(M)$.*

Similarly, one can ask if a specialization to some p-adic value of k of $L_{(p)}(M, k, s)$ should be related to a product of twisted p-adic L-functions attached to M.

4.4 "False" p-adic L-functions attached to motives II

Let λ be a fixed algebraic Hecke character of an imaginary quadratic field K. It turns out that the algebraic part of central critical values of Hecke L-series $L(\lambda^{2k+1}, k+1)$ are (up to the standard factors) squares in a fixed finite extension of $\mathbb{Q}$ (a theorem of Rodriques-Villegas and Zagier [R-VZ] provides a formula for the square roots). Recently A. Sofer [So] proved that these squares can be p-adically interpolated (such a posibility was conjectured by Koblitz [Kob]).

Now take any $M \in \mathcal{M}_{\mathbb{Q}}$. Assume that there exists $A = A(M)$ such that $[\lambda^r] \otimes M$ have critical points for $r \geq A$. It is interesting to ask whether exists any analogue of a result of Sofer.

In a case of three cusp forms we have the following result of Harris - Kudla

Theorem. *[HaK]. Let f, g, h are elliptic cusp forms satisfying $w(f) \geq w(g) + w(h)$; put $w = w(f) + w(g) + w(h) - 3$. Then*

$$\frac{L(f \otimes g \otimes h, \frac{1}{2}(w+1))}{< f\ ,\ f >^2} \times a\ product\ of\ bad\ local\ factors$$

is a square in $\mathbb{Q}(f, g, h)$.

Harris and Tilouine recently proved that these squares can be p-adically interpolated in the following sense: the corresponding p-adic L-function is a function on Hida's family containing f.

We suggest the following question:

Conjecture. *(i) Let $M \in \mathcal{M}_{\mathbb{Q}}$, $w(M)$ even. Then there exists p-adic continuous function which interpolates the square roots of algebraic part of special values*

$$L\left([\lambda^{2k+1}] \otimes M, k+1+\frac{w}{2}\right),\ \ k \gg 0,$$

(ii) Let $M \in \mathcal{M}_{\mathbb{Q}}$, $w(M)$ odd. Then there exists p-adic continuous function which interpolates the square roots of algebraic part of special values

$$L\left([\lambda^{2k}] \otimes M, k + \frac{w+1}{2}\right), \quad k \gg 0.$$

Here one can consider any Hida's family instead of $\{\lambda^m\}$.

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