On estimates of the first eigen-value in some Sturm-Liouville problems

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A family of Kähler-Einstein manifolds

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Contents

Abstract

In some Sturm-Liouville problems the estimates of the first eigen-values are obtained. In many cases the sharp values are found and the existence of the optimal solution is proved. For the classical Lagrange problem the extremal values of the Lagrange functional are indicated. The functions realizing these extremal values are found. It is proved that these values are extremal globally.

1. On some estimates of the first eigen-value of a Sturm-Liouville problem

Let us consider the dependence of the first eigen-value λ_1 of the Sturm-Liouville problem

$$y''(x) + \lambda q(x) y(x) = 0$$

on the segment $0 \leq x \leq 1$, with the boundary conditions

$$y(0) = 0, \; y(1) = 0$$

on the potential q. Denote R_β the set of real-valued measurable on $(0,1)$ functions q with positive values such that

$$\int_0^1 q(x)^\beta dx = 1,$$

where β is a real number, $\beta \neq 0$. The variational principle implies that the first eigen-value λ_0 can be found as

$$\lambda_1 = \inf_{y \in C_0^\infty(0,1)} \frac{\int_0^1 y'(x)^2 dx}{\int_0^1 q(x) y(x)^2 dx} .$$

We will estimate the values

$$m_\beta = \inf_{q \in R_\beta} \lambda_1, \; M_\beta = \sup_{q \in R_\beta} \lambda_1 .$$

Put

$$L[q,y] = \frac{\int_0^1 y'(x)^2 dx}{\int_0^1 q(x) y(x)^2 dx}, \; G[y] = \frac{\int_0^1 y'(x)^2 dx}{(\int_0^1 y(x)^p dx)^{2/p}} ,$$

where

$$p = \frac{2\beta}{\beta - 1} .$$

The main result of this section is the following

Theorem 1. *If* $\beta > 1$, *then*

$$m_\beta = \frac{(\beta-1)^{1+1/\beta}}{\beta(2\beta-1)^{1/\beta}} B^2(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}),$$

and $M_\beta = \infty$, *where* B *is the Euler's Beta-function:*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \ .$$

There exist functions $u(x)$ *and* $q(x)$ *such that*

$$\inf_y L[q,y] = L[q,u] = m_\beta .$$

If $\beta = 1$, *then* $M_1 = \infty$ *and* $m_1 = 4$.
If $0 < \beta < 1/2$, *then*

$$M_\beta = \frac{(1-\beta)^{1+1/\beta}}{\beta(1-2\beta)^{1/\beta}} B^2(\frac{1}{2}, \frac{1}{2\beta}) \ ,$$

and $m_\beta = 0$. *There exist functions* $u(x)$ *and* $q(x)$ *such that* $\inf_y L[q,y] = L[q,u] = M_\beta$.
If $\beta < 0$, *then*

$$M_\beta = -\frac{(1-\beta)^{1+1/\beta}}{\beta(1-2\beta)^{1/\beta}} B^2(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}) \ ,$$

and $m_\beta = 0$. *There exist functions* $u(x)$ *and* $q(x)$ *such that* $\inf_y L[q,y] = L[q,u] = M_\beta$).
If $1/2 \le \beta < 1$, *then* $M_\beta = \infty$ *and* $m_\beta = 0$.

Proof. 1. If $\beta > 1$, then we have by the Hölder inequality

$$\int_0^1 qy(x)^2dx \le (\int_0^1 q(x)^\beta dx)^{1/\beta}(\int_0^1 |y(x)|^p dx)^{2/p} = (\int_0^1 |y(x)|^p dx)^{2/p} \ ,$$

where $p = 2\beta/(\beta-1)$, for any $y \in H_0^1(0,1)$. Therefore,

$$\lambda_1 \ge m \ ,$$

where $m = inf_y G[y]$ in the class $H_0^1(0,1)$. Remark that the homogeneity allows to assume that

$$\int_0^1 |y(x)|^p dx = 1 \ .$$

Let $\{y_k\}$ be a sequence of functions of this class, such that

$$\int_0^1 y_k'(x)^2 dx \to m \ .$$

This sequence is bounded in $H_0^1(0,1)$, therefore it is weakly compact in this space and compact in $C[0,1]$. We will assume that this sequence is converging uniformly and weakly in $H_0^1(0,1)$ to a function u. Then

$$\int_0^1 |u(x)|^p dx = 1, \ \int_0^1 u'(x)^2 dx \leq \lim_{k\to\infty} \int_0^1 y_k'(x)^2 dx,$$

and therefore $\int_0^1 u'(x)^2 dx = m$. Since $G[y]$ has the minimal value at $y = u$, we have

$$\frac{d}{dt} G[u + tz] = 0 \text{ at } t = 0$$

for an arbitrary function z of the class $H_0^1(0,1)$. It means that

$$\int_0^1 u'(x)z'(x)dx - m\int_0^1 |u(x)|^{p-2}u(x)z(x)dx = 0$$

for all $z \in H_0^1(0,1)$. This equality yields that the function u' has a generalized derivative, equal to $-m|u|^{p-2}u$, i.e.

$$u'' + m|u|^{p-2}u = 0$$

almost everywhere in $(0,1)$. Since u is a continuous function, we have $u'' \in C[0,1]$, so the equation is true in the classical sense.

Since $G[|y|] = G[y]$ for all y , we can assume that $y_k(x) \geq 0$ and thus $u(x) \geq 0$. Then by the unicity theorem for the Cauchy problem $u(x) > 0$ in $(0,1)$. Multiplying the both sides of the equation

$$u''(x) + mu(x)^{p-1} = 0$$

by $2u'$ and integrating over $(0,x)$, we obtain that

$$u'(x)^2 + \frac{2m}{p} u(x)^p = C \ .$$

Integrating over $(0,1)$ the both sides of this equality and taking into account that

$$\int_0^1 u'(x)^2 dx = m, \ \int_0^1 u(x)^p dx = 1 \ ,$$

we obtain that $m(1+2/p)=C$.

Let b be a point, at which the function u has the maximal value M. Since $u''=-mu^{p-1}<0$, such a point exists and is unique. If $b\neq 1/2$, then we can assume that $b<1/2$, since $u(x)$ can be replaced by $u(1-x)$. The function $u_1(x)=u(2b-x)$ satisfies the same equation on $(b,2b)$ as u and $u(b)=u_1(b)=M$, $u'(b)=u_1'(b)=0$. Therefore, these functions coincide and $u(2b)=u(0)=0$, i.e. $b=1/2$. Since

$$u'(x)=\sqrt{C-\frac{2m}{p}u(x)^p}$$

for $0\leq x\leq \frac{1}{2}$, we have

$$\int_0^M \frac{dy}{\sqrt{C-2my^p/p}}=\frac{1}{2}.$$

Since $u'(1/2)=0$, we have

$$M=u(1/2)=(\frac{pC}{2m})^{1/p}=(1+p/2)^{1/p}\ .$$

Changing the variable of integration $y=Mt$, we obtain the equality

$$M^{1-p/2}\sqrt{\frac{p}{2m}}\int_0^1\frac{dt}{\sqrt{1-t^p}}=\frac{1}{2}\ .$$

Remark that

$$\int_0^1\frac{dt}{\sqrt{1-t^p}}=\frac{1}{p}B(\frac{1}{p},\frac{1}{2})\ ,$$

so that

$$M^{p/2-1}=\sqrt{\frac{2}{pm}}B(\frac{1}{p},\frac{1}{2})\ .$$

The obtained relations allows to find

$$m=C(\beta)=\frac{(\beta-1)^{1+1/\beta}}{\beta(2\beta-1)^{1/\beta}}B^2(1/2,1/2-1/2\beta)\ .$$

2. Let now $\beta=1$. Since

$$\int_0^1 q(x)y(x)^2dx\leq \max y(x)^2,$$

we have

$$\lambda_1 \geq m = \inf_{y \in H_0^1(0,1)} \frac{\int_0^1 y'(x)^2 dx}{\max y(x)^2} .$$

The value of m can be found according to the following Lemma.

Lemma A10 of S.7 implies that $\lambda_1 \geq 4$ if $\beta = 1$.

3. Let $\beta < 0$. Put

$$q(x) = \begin{cases} (1-\varepsilon)^{1/\beta}\varepsilon^{-1/\beta}, & \text{if } 0 < x < \varepsilon, \\ (1-\varepsilon)^{-1/\beta}\varepsilon^{1/\beta}, & \text{if } \varepsilon < x < 1, \end{cases}$$

where $\varepsilon > 0$ is a small number. Let $y_0(x) = 1/2 - |x - 1/2|$. Then

$$\lambda_1 \leq \frac{1}{\int_0^1 q(x) y_0(x)^2 dx} \leq C\varepsilon^{-1/\beta}.$$

Therefore λ_1 can be arbitrary small.

4. Let $0 < \beta < 1$. Put

$$q(x) = \begin{cases} (2\varepsilon)^{-1/\beta}, & \text{if } |x - 1/2| < \varepsilon, \\ 0, & \text{if } |x - 1/2| < \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a small number. Let y_0 be a smooth function, vanishing in the points $x = 0$ and $x = 1$, which is equal to 1 in $(1/3, 2/3)$. Then

$$\lambda_1 \leq \frac{C}{\int_{1/2-\varepsilon}^{1/2+\varepsilon} (2\varepsilon)^{-1/\beta} dx} = C_1 \varepsilon^{1/\beta - 1} \to 0$$

as $\varepsilon \to 0$. Therefore λ_1 can be arbitrary small.

5. Let $\beta > 1/2$. Put

$$q(x) = \begin{cases} \varepsilon^{-1/\beta}, & \text{if } 0 < x < \varepsilon, \\ 0, & \text{if } \varepsilon < x < 1, \end{cases}$$

where $\varepsilon > 0$ is a small number. Then

$$\int_0^1 q(x) y(x)^2 dx = \varepsilon^{-1/\beta} \int_0^\varepsilon y(x)^2 dx \leq \varepsilon^{2-1/\beta} \int_0^\varepsilon y'(x)^2 dx$$

and thence

$$\lambda_1 \geq \varepsilon^{1/\beta - 2}.$$

Therefore in this case $M_\beta = \infty$.

6. If $\beta = 1/2$, we can put

$$q(x) = Cx^{\varepsilon-2},$$

where $C = \varepsilon^2/4$, so that $\int_0^1 q(x)^{1/2}dx = 1$. Then

$$\int_0^1 q(x)y(x)^2dx = \frac{\varepsilon^2}{4}\int_0^1 x^{\varepsilon-2}y(x)^2dx \le C_1\varepsilon^2 \int_0^1 y'(x)^2dx.$$

Therefore $\lambda_1 \ge C_1^{-1}\varepsilon^{-2}$ and $M_{1/2} = \infty$.

7. Let $0 < \beta < 1/2$. Then by the Hölder inequality

$$\int_0^1 q(x)^\beta dx \le (\int_0^1 q(x)y(x)^2dx)^\beta(\int_0^1 y(x)^p dx)^{1-\beta},$$

where $p = 2\beta/(\beta-1)$ so that $0 > p > -2$. Therefore,

$$L[q,y] \le G[y].$$

Put $y_0(x) = x^\gamma$ for $0 < x < 1/2$ and $y_0(x) = (1-x)^\gamma$ for $1/2 < x < 1$, where $-1/p > \gamma > 1/2$ so that $y_0 \in H_0^1(0,1)$. Then the integral $\int_0^1 y_0(x)^p dx$ is converging and thus

$$\lambda_1 \le C_1.$$

Let $m = \inf_{y\in H_0^1(0,1)} G[y]$. Consider a minimizing sequence $\{y_k\}$ such that

$$\int_0^1 y_k(x)^p dx = 1,\ \int_0^1 y_k'(x)^2dx \to m.$$

There exists a subsequence $\{y_{n_k}\}$ uniformly converging to a function $u \in H_0^1(0,1)$. By the Fubini theorem, we have $\int_0^1 u(x)^p dx \le 1$ and $\int_0^1 u'(x)^2dx \le m$. Therefore, $G[u] \le m$ and since m is the minimal possible value of G, we have $G[u] = m$. Since $G[|y|] = G[y]$ for all y , we can assume that $y_k(x) \ge 0$ and thus $u(x) \ge 0$. The function u satisfies the same equation as in the S.1, i.e. the equation

$$u'' + m|u|^{p-2}u = 0$$

almost everywhere in $(0,1)$. Since u is a continuous function, we see that the equation is true in the classical sense in each interval where $u \ne 0$. If $u(x_0) =$

$0, u(x_1) = 0, u(x) > 0$ for $x_0 < x < x_1$ and $0 \le x_0 < x_1 \le 1, x_1 - x_0 = \kappa < 1$, then we can consider the function $v(x) = u(x_0 + \kappa x)$ and since

$$v''(x) + m\kappa^2 v = 0, v(0) = 0, v(1) = 0,$$

we see that $G[v] = m\kappa^2 < m$, what is impossible. So $u(x) > 0$ in $(0,1)$. Moreover then the equation

$$u'(x)^2 + \frac{2m}{p} u(x)^p = C$$

holds for $0 < x < 1$ with $C = m(1 + 2/p) < 0$.

Let b be a point, at which the function u has the maximal value M. Since $u'' = -mu^{p-1} < 0$, such a point exists and is unique. The function $u_1(x) = u(2b - x)$ satisfies the same equation on $(b, 2b)$ as u and $u(b) = u_1(b) = M$, $u'(b) = u_1'(b) = 0$. Therefore, by the unicity theorem for the Cauchy problem these functions coincide and $u(2b) = u(0) = 0$, i.e. $b = 1/2$. We have

$$\int_0^M \frac{dy}{\sqrt{C - 2my^p/p}} = \frac{1}{2}.$$

Since $u'(1/2) = 0$, we have

$$M = u(1/2) = (\frac{pC}{2m})^{1/p} = (1 + p/2)^{1/p}.$$

Changing the variable of integration $y = Mt$, we obtain the equality

$$M^{1-p/2}\sqrt{\frac{-p}{2m}} \int_0^1 \frac{dt}{\sqrt{t^p - 1}} = \frac{1}{2}.$$

Remark that

$$\int_0^1 \frac{dt}{\sqrt{t^p - 1}} = -\frac{1}{p} B(\frac{1}{2}, \frac{1}{2} - \frac{1}{p}),$$

so that

$$m = C(\beta) = \frac{(1-\beta)^{1+1/\beta}}{\beta(1-2\beta)^{1/\beta}} B^2(1/2, 1/2\beta).$$

Since

$$\inf_y L[q, y] \le \inf_y G[y]$$

and $L[u^{p-2}, u] = C(\beta)$, we see that $M_\beta = C(\beta)$.

8. If $\beta < 0$, then by the Hölder inequality

$$\int_0^1 y(x)^{2\beta/(\beta-1)}dx \leq (\int_0^1 q(x)y(x)^2dx)^{\beta/(\beta-1)}(\int_0^1 q(x)^\beta dx)^{1/(1-\beta)}.$$

Therefore,

$$L[q,y] \leq G[y].$$

Put $y_0(x) = |x-1/2|-1/2$. Then

$$\lambda_1 \leq \frac{1}{(\int_0^1(|x-1/2|-1/2)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}}$$

and so $M_\beta < \infty$. Consider a minimizing sequence $\{y_k\}$ such that

$$\int_0^1 y_k(x)^p dx = 1,\ \int_0^1 y_k'(x)^2 dx \to m.$$

There exists a subsequence $\{y_{n_k}\}$ uniformly converging to a function $z \in H_0^1(0,1)$. Since $p > 0$, the sequence $\{y_{n_k}^p\}$ converges uniformly to u^p. Therefore, $\int_0^1 u(x)^p dx \leq m^{-p/2}$, and $\int_0^1 u'(x)^2 dx \leq 1$. Therefore, $G[u] \leq m$ and since m is the minimal possible value of G, we have $G[u] = m$. The function u satisfies the same equation as in the S.1, and $C(a)$ is defined by the same formula as for $\beta > 1$. Since

$$\inf_y L[q,y] \leq \inf_y G[y] = C(\beta),$$

we see that $M_\beta = C(\beta)$. □

2. On other estimates of the first eigen-value

Let us consider the dependence of the first eigen-value λ_1 of the Sturm-Liouville problem

$$(p(x)y')' + \lambda y(x) = 0$$

on the segment $0 \leq x \leq 1$, with the boundary conditions

$$y(0) = 0,\ y(1) = 0$$

on the function p. Let us denote R_α the set of real-valued positive measurable functions p on $[0,1]$ such that

$$\int_0^1 p(x)^\alpha dx = 1,$$

where α is a real number, $\alpha \neq 0$. Put

$$L[p,y] = \frac{\int_0^1 p(x)y'(x)^2dx}{\int_0^1 y(x)^2dx}, \; G[y] = \frac{(\int_0^1 y'(x)^r dx)^{2/r}}{\int_0^1 y(x)^2dx}, \; r = \frac{2\alpha}{\alpha - 1}.$$

Let $K_p(a,b)$ for real $p \neq 0$ be the set of non-decreasing real functions y defined on $[a,b]$, absolutely continuous on $[a, b-\varepsilon]$ for any $\varepsilon > 0$ and such that $y(0) \geq 0$,

$$\int_a^b y'(x)^p dx < \infty, \; \int_a^b y(x)^2 dx < \infty.$$

Let $K_p(a,b,c)$ be the set of real functions y defined on $[a,c]$ and such that $y \in K_p(a,b)$, $y(-x) \in K_p(-c,-b)$, $\int_a^b |y'(x)|^p dx < \infty$ and $\int_b^c |y'(x)|^p dx < \infty$ The main result of this section is the following

Theorem 2. *Let*

$$M_\alpha = \sup_{p \in R_\alpha} \lambda_1, \; m_\alpha = \inf_{p \in R_\alpha} \lambda_1,$$

$$C(r) = \frac{3r-2}{r}\left(\frac{2r-2}{3r-2}\right)^{2/r} B^2\left(\frac{1}{2}, 1 - \frac{1}{r}\right).$$

If $\alpha > -1/2$, $\alpha \neq 0$, then $M_\alpha = C(r)$ and $m_\alpha = 0$. There exist functions $p \in R_\alpha$, $z \in H_0^1(0,1)$ such that $z'(x)^2 = p(x)^{\alpha-1}$ and

$$inf_y L[p,y] = L[p,z] = C(r).$$

If $\alpha \leq -1$, then $m_\alpha = C(r)$ and $M_\alpha = \infty$. There exist functions $p \in R_\alpha$, $z \in H_0^1(0,1)$ such that $z'(x)^2 = p(x)^{\alpha-1}$ and

$$inf_y L[p,y] = L[p,z] = C(r).$$

If $-1 < \alpha \leq -1/2$, then $M_\alpha = \infty$ and $m_\alpha = 0$.

The proof of Theorem 2 is based on the variational principle, according to which $\lambda_1 = \inf_{y \in H_0^1(0,1)} L[p,y]$.

Proof. 1. If $\alpha > 0$, then we can take a function y vanishing in $[0,1/2]$ and such that $\int_0^1 y^2 dx = 1$. Since the function p can have arbitrarily small values in $[1/2, 1]$, the value of λ_1 cannot be bounded from below by a positive constant.

2. Let $0 > \alpha > -1$. Let us show that in this case also λ_1 cannot be bounded from below by a positive constant.

Put for that

$$y(x) = \begin{cases} x/\varepsilon, & \text{if } 0 < x < \varepsilon, \\ 1, & \text{if } \varepsilon < x < 1-\varepsilon, \\ (1-x)/\varepsilon, & \text{if } 1-\varepsilon < x < 1, \end{cases}$$

$$p(x) = \begin{cases} \delta, & \text{if } 0 < x < \varepsilon \text{ or } 1-\varepsilon < x < 1, \\ \varepsilon^{-1}, & \text{if } \varepsilon < x < 1-\varepsilon, \end{cases}$$

where δ is a number such that

$$\int_0^1 p(x)^\alpha dx = 2\varepsilon\delta^\alpha + (1-2\varepsilon)\varepsilon^{-\alpha} = 1,$$

i.e. $\delta \le C_1\varepsilon^{-1/\alpha}$. It is evident that

$$\int_0^1 y(x)^2 dx = 1 - 2\varepsilon + 2\varepsilon/3 = 1 - 4\varepsilon/3.$$

On the other hand,

$$\int_0^1 p(x)y'(x)^2 dx = 2\delta/\varepsilon \le C_2\varepsilon^{-1-1/\alpha}.$$

Therefore,

$$\lambda_1 \le \frac{\int_0^1 p(x)y'(x)^2 dx}{\int_0^1 y(x)^2 dx} \le C\varepsilon^{-1-1/\alpha},$$

and since $-1-1/\alpha > 0$, the value of λ_1 can be arbitrarily small.

3. Let $\alpha \le -1$. Then $1 \le r \le 2$ and by the Hölder inequality

$$\int_0^1 y'(x)^r dx = \int_0^1 p(x)^{r/2} y'(x)^r \cdot p(x)^{-r/2} dx$$

$$\le (\int_0^1 p(x)y'(x)^2 dx)^{r/2} (\int_0^1 p(x)^\alpha dx)^{1/(1-\alpha)},$$

where $r = 2\alpha/(\alpha-1)$. Therefore, for any admissible p we have $L[p,y] \ge G[y]$. Let $m = \inf_y G[y]$. Since $y(x) = \int_0^x y'(t)dt$, we have

$$\int_0^1 y(x)^2 dx \le \int_0^1 (\int_0^x |y'(t)|dt)^2 dx \le (\int_0^1 |y'(t)|^r dt)^{2/r}$$

and thus $m \geq 1$.

Consider a minimizing sequence $\{y_k\}$ such that

$$\int_0^1 |y_k'(x)|^r dx = 1, \int_0^1 |y_k(x)|^2 dx \to 1/m.$$

There exists a subsequence $\{y_{n_k}\}$ converging uniformly to a function $z \in H_0^1(0,1)$ such that $\int_0^1 |z'(x)|^r dx = 1$, $\int_0^1 |z(x)|^2 dx = 1/m$. The function z satisfies the Euler-Lagrange equation

$$(|z'(x)|^{r-2} z'(x))' + mz(x) = 0.$$

Multiplying it by z' and integrating we obtain

$$\frac{r-1}{r}|z'(x)|^r + \frac{m}{2}z^2 = C.$$

Integrating the last equality over $(0,1)$ we see that $C = 3/2 - 1/r > 0$. The function z is even with respect to $x = 1/2$, increasing in $(0, 1/2)$ from 0 to M and decreasing in $(1/2, 1)$. Since $z'(1/2) = 0$, we have

$$mM^2 = 2C = 3 - 2/r.$$

We have

$$\int_0^M \frac{dz}{(C - mz^2/2)^{1/r}} = \frac{1}{2}(\frac{r}{r-1})^{1/r}.$$

Substituting $z = My$, we see that

$$\int_0^1 \frac{dy}{(1-y^2)^{1/r}} = \frac{1}{2}(\frac{r}{r-1})^{1/r}\frac{C^{1/r}}{M}.$$

Remark that

$$\int_0^1 \frac{dy}{(1-y^2)^{1/r}} = \frac{1}{2}B(\frac{1}{2}, 1 - \frac{1}{r}).$$

Therefore,

$$M^{-1} = (\frac{r-1}{r})^{1/r} C^{-1/r} B(\frac{1}{2}, 1 - \frac{1}{r})$$

and $m = 2CM^{-2} = C(r)$. Since $L[p,y] \geq C(r)$ and $L[z'^{2/(\alpha-1)}, z] = C(r)$, it follows that $m_\alpha = C(r)$.

4. If $\alpha > 1$, then by the Hölder inequality

$$\int_0^1 p(x)y'(x)^2dx \leq (\int_0^1 p(x)^\alpha dx)^{1/\alpha}(\int_0^1 y'(x)^r dx)^{2/r},$$

where $r = 2\alpha/(\alpha-1) > 2$. Put $y_0(x) = 1/2 - |x - 1/2|$. Then

$$\lambda_1 \leq \frac{(\int_0^1 y_0'(x)^r)^{2/r}}{\int_0^1 y_0(x)^2dx} = C.$$

We can repeat the same arguments as above in S.3 to find the optimal functions p and z. Moreover then $M = C(r)$ and if $\alpha = 1$, then

$$\lambda_1 \leq \frac{\max y_0'(x)^2}{\int_0^1 y_0(x)^2dx} = 12 = \lim_{r\to\infty} C(r).$$

5. Let $0 < \alpha < 1$ and p be a function of the class R_α. Put

$$y_0'(x)^2 = p(x)^{\alpha-1}$$

and construct the function y_0 in such a way that it vanishes in the end points, increases monotonically on $(0, b)$ and decreases monotonically on $(b, 1)$. Let $M = \max y_0(x)^2$. It is evident that

$$\int_0^1 y_0'(x)^r dx = \int_0^1 p(x)^\alpha dx = 1.$$

Let $b \geq 1/2$. The measure of points $x \in (0, b)$ such that $y_0'(x)^r \geq 4$, is less than $1/4$. Therefore the supplementary set E on $(0, b)$ has the measure $\geq 1/4$ and at the points x of this set we have

$$y_0'(x) \geq 4^{1/r},$$

because $r < 0$. Put now $z(0) = 0$, $z'(x) = 4^{1/r}$ at the points of E and $z'(x) = 0$ at other points of $(0, b)$. Then

$$\int_0^1 y_0(x)^2dx \geq \int_0^r z(x)^2dx \geq \int_0^{1/4} 4^{2/r}x^2dx = c_0$$

and therefore,

$$M_a \leq \frac{\int_0^1 p(x)y_0'(x)^2dx}{\int_0^1 y_0(x)^2dx} \leq \frac{1}{c_0}.$$

To prove the existence of the optimal functions p and z we need the result of Lemma A14.

Let p be an arbitrary positive function of the class R_α. Then there exists a function $y_0(x)$ such that $p(x) = |y_0'|^{2/(\alpha-1)}$ even with respect to $x = 1/2$, increasing in $(0, 1/2)$ and decreasing in $(1/2, 1)$. Furthermore, $L[p, y_0] = G[y_0]$ and therefore, $M_\alpha \leq m$. On the other hand, we have the equality $L[p_0, y_0] = G[y_0]$, if $p_0(x) = |y_0(x)|^{2/(\alpha-1)}$. By Lemma A14 we have

$$m = \inf_{y \in H_0^2} \frac{\int_0^1 p_0(x) y'(x)^2 dx}{\int_0^1 y(x)^2 dx} = L[p_0, y_0]$$

and the proof is complete.

6. Let now $0 > \alpha > -1/2$. We will use the same function y_0 as above, in the beginning of s.5. Let $r = 2\alpha/(\alpha - 1)$. Then $0 < r < 2/3$. By the Hölder inequality

$$\int_0^1 y_0'(x)^r dx \leq \left(\int_0^1 |y_0'(x)| dx\right)^r \left(\int_0^1 dx\right)^{1-r} \leq \left(\int_0^1 |y_0'(x)| dx\right)^r.$$

Since

$$\int_0^1 y_0'(x)^r dx = 1,$$

it follows that

$$\int_0^1 |y_0'(x)| dx \geq 1$$

and therefore

$$M = (\max y_0(x))^2 = y_0(b)^2 \geq 1/4.$$

We can assume that

$$\int_0^r y_0'(x)^r dx \geq 1/2.$$

Let x_1 be such a point of $(0, b)$ that $y_0(x_1) = 1/4^r$. Then as above we have

$$\int_0^{x_1} |y_0'(x)|^r dx \leq \left(\int_0^{x_1} |y_0'(x)| dx\right)^r = 1/4.$$

Therefore,

$$\int_{x_1}^b |y_0'(x)|^r dx \geq 1/4.$$

On the other hand, by the Hölder inequality

$$1/4 \leq \int_{x_1}^b |y_0'(x)|^r dx = \int_{x_1}^b |y_0'(x)|^r y_0(x)^{2r-2} y_0(x)^{2-2r} dx$$

$$\leq (\int_{x_1}^{b} y_0'(x) y_0(x)^{2-1/r} dx)^r (\int_{x_1}^{r} y_0(x)^2 dx)^{1-r}$$

$$= [(\frac{1}{4})^{3-2/r} - m^{3-2/r}]/(3-2/r)(\int_{x_1}^{b} y_0(x)^2 dx)^{1-r}$$

$$\leq C_1(a)(\int_0^1 y_0(x)^2 dx)^{1-r}.$$

Therefore,

$$\int_0^1 y_0(x)^2 dx \geq (4C_1(\alpha))^{1/(r-1)}$$

and

$$\lambda_1 \leq (4C_1(\alpha))^{1/(1-r)}.$$

As above to prove the attainment of the optimal value we need some Lemmas.

7. Let $\alpha < -1/2$. Let

$$p(x) = \begin{cases} k^{1/\alpha}, & \text{if } 0 < x < \varepsilon, \\ k, & \text{if } \varepsilon < x < 1, \end{cases}$$

where k is such a number that

$$k\varepsilon + k^{\alpha}(1-\varepsilon) = 1$$

so that

$$\int_0^1 p(x)^{\alpha} dx = 1.$$

for any $\varepsilon > 0$ we have

$$\int_0^{\varepsilon} y(x)^2 dx \leq \varepsilon^2 \int_0^1 y'(x)^2 dx.$$

On the other hand,

$$\int_{\varepsilon}^1 y(x)^2 dx \leq (1-\varepsilon)^2 \int_{\varepsilon}^1 y'(x)^2 dx.$$

Therefore,

$$\int_0^1 y(x)^2 dx \leq \varepsilon^2 \int_0^1 y'(x)^2 dx$$

$$+(1-\varepsilon)^2 \int_{\varepsilon}^1 y'(x)^2 dx \leq \delta \int_0^1 p(x) y'(x)^2 dx,$$

where

$$\delta = \max(\varepsilon^2 k^{-1/a},\ (1-\varepsilon)^2 k^{-1}) \le \max(\varepsilon^{2+1/a},\ (1-\varepsilon)^2\varepsilon),$$

so that $\delta \to 0$ as $\varepsilon \to 0$. Hence

$$\lambda_1 \ge \frac{1}{\delta} \to \infty$$

as $\varepsilon \to 0$.

8. Consider at last the case when $\alpha = -1/2$. Put

$$p(x) = \max(x^2/\varepsilon^2,\ \delta^2/\varepsilon^2),$$

where $\delta = \exp(1 - 1/\varepsilon)$. Remark that

$$\int_0^1 p(x)^{-1/2}dx = \delta \cdot \varepsilon/\delta + \int_0^1 \varepsilon/x dx$$
$$= \varepsilon - \varepsilon \ln \delta = 1.$$

On the other hand, from the well-known estimate

$$\int_0^1 y(x)^2 dx \le 4 \int_0^1 x^2 y'(x)^2 dx,$$

valid for all functions $y \in C^1$, vanishing in 0, it follows that

$$\int_0^1 y(x)^2 dx \le 4\delta^2 \int_0^\delta y'(x)^2 dx + 4\int_\delta^1 y(x)^2 dx \le 4\varepsilon^2 \int_0^1 p(x) y'(x)^2 dx.$$

It means that for the choosen function p

$$\lambda_1 = \inf \frac{\int_0^1 p(x) y'(x)^2 dx}{\int_0^1 y(x)^2 dx} \ge \frac{1}{4\varepsilon^2},$$

so that the estimate from above is impossible. □

Corollary 3. *If* $\alpha > -1/2$, $\alpha \ne 0$, *then*

$$\lambda_1 \le C(\alpha)\left(\int_0^1 p(x)^\alpha dx\right)^{1/\alpha};$$

if $\alpha \le -1$, *then*

$$\lambda_1 \ge C(\alpha)\left(\int_0^1 p(x)^\alpha dx\right)^{1/\alpha};$$

where $C(\alpha)$ is a positive number depending on α only.

3. On a more general estimate of the first eigen-value of the Sturm-Liouville operator

In this section the Sturm-Lioville problem

$$(P(x)y')' + \lambda Q(x)y = 0, \qquad 0 < x < 1, \tag{1}$$

$$y(0) = y(1) = 0. \tag{2}$$

is considered. Our aim is to estimate the minimal eigen-value λ_1 of this problem under the condition that the non-negative measurable functions $P(x)$ and $Q(x)$ are such that

$$\int_0^1 P(x)^\alpha dx = 1, \int_0^1 Q(x)^\beta dx = 1, \tag{3}$$

where α and β are non-zero real numbers. The variational principle implies that

$$\lambda_1 = \inf_y \frac{\int_0^1 P(x)y'(x)^2 dx}{\int_0^1 Q(x)y(x)^2 dx},$$

where the greatest lower bound is taken in the class of all non-zero functions from $C_0^1[0,1]$.

Let us put

$$M_{\alpha,\beta} = \sup_{P,Q} \lambda_1, \ m_{\alpha,\beta} = \inf_{P,Q} \lambda_1.$$

The main result of this section is the following

Theorem 4. *If $\alpha > -1/2$, $\beta - \alpha + 2\alpha\beta < 0$, then $M_{\alpha,\beta} \le C(\alpha,\beta)$ and $m_{\alpha,\beta} = 0$.*

If $\alpha \le -1$, $\beta \ge 1$, then $m_{\alpha,\beta} \ge C(\alpha,\beta) > 0$ and $M_{\alpha,\beta} = \infty$.

If $1/\alpha - 1/\beta + 2 \le 0$ and either $\alpha > -1$ or $\beta < 1$, then $m_{\alpha,\beta} = 0$ and $M_{\alpha,\beta} = \infty$.

Proof.

I. Estimate of $M_{\alpha,\beta}$.

a. Let at first $\alpha > 0, \beta > 0$ and $\beta - \alpha + 2\alpha\beta > 0$. We show that $M_{\alpha,\beta} = \infty$. For this we put $P(x) = \varepsilon^{-1/\alpha}$ for $0 < x < \varepsilon$ and $P(x) = 0$ for $\varepsilon < x < 1$; $Q(x) = \varepsilon^{-1/\beta}$ for $0 \le x \le \varepsilon$ and $Q(x) = 0$ for $\varepsilon \le x \le 1$, where ε is a small positive number. Then for $y(x) \in C_0^1(0,1)$ we have

$$\int Q(x)y(x)^2dx = \varepsilon^{-1/\beta}\int y(x)^2dx;$$

$$\int P(x)y'(x)^2dx = \varepsilon^{-1/\alpha}\int y'(x)^2dx.$$

Since

$$\int_0^\varepsilon y(x)^2dx \le \varepsilon^2\int_0^\varepsilon y'(x)^2dx; y(x) \in C_0^1(0,1),$$

we have that

$$\int Q(x)y(x)^2dx \le \varepsilon^{2-1/\beta+1/\alpha}\int P(x)y'(x)^2dx;$$

i.e.

$$\lambda_1 \ge \varepsilon^{-2+1/\beta-1/\alpha},$$

and $\lambda_1 \to \infty$ as $\varepsilon \to 0$. Therefore in this case $M_{\alpha,\beta} = \infty$.

b. If $\alpha < 0, \beta > 0$, we put

$$P(x) = \begin{cases} \varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha} & \text{for } 0 < x < \varepsilon, \\ \varepsilon^{1/\alpha}(1-\varepsilon)^{-1/\alpha} & \text{for } \varepsilon < x < 1; \end{cases}$$

$$Q(x) = \begin{cases} 0 & \text{for } 0 < x < 2\varepsilon, \\ (1-2\varepsilon)^{-1/\beta} & \text{for } 2\varepsilon < x < 1. \end{cases}$$

where $0 < \varepsilon < 1/8$. It is clear that

$$\int P(x)^\alpha dx = 1, \int Q(x)^\beta dx = 1,$$

and

$$(1-2\varepsilon)^{1/\beta}\int_0^1 Q(x)y(x)^2dx = \int_{2\varepsilon}^1 y(x)^2dx$$

$$\le \varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha}\int_{2\varepsilon}^1 P(x)y'(x)^2dx.$$

Hence it follows that $\lambda_1 \ge \varepsilon^{1/\alpha}/2$ and so $M_{\alpha,\beta} = \infty$.

c. Let now $\alpha < 0, \beta < 0$ and $\beta - \alpha + 2\alpha\beta > 0$.

We put

$$P(x) = \begin{cases} \varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha} & \text{for } 0 < x < \varepsilon, \\ \varepsilon^{1/\alpha}(1-\varepsilon)^{-1/\alpha} & \text{for } \varepsilon < x < 1; \end{cases}$$

$$Q(x) = \begin{cases} \varepsilon^{-1/\beta}(1-\rho)^{1/\beta} & \text{for } 0 < x < \varepsilon, \\ \rho^{1/\beta}(1-\varepsilon)^{-1/\beta} & \text{for } \varepsilon < x < 1; \end{cases}$$

where $0 < \varepsilon < 1/8, 0 < \rho < 1/8$. It is clear that

$$\int P(x)^\alpha dx = 1, \int Q(x)^\beta dx = 1,$$

and

$$\int_0^1 Q(x)y(x)^2 dx = \varepsilon^{-1/\beta}(1-\rho)^{1/\beta}\int_0^\varepsilon y(x)^2 + \rho^{1/\beta}(1-\varepsilon)^{-1/\beta}\int_\varepsilon^1 y(x)^2 dx$$

$$\leq \varepsilon^{2-1/\beta+1/\alpha}(1-\rho)^{1/\beta}(1-\varepsilon)^{-1/\alpha}\int_0^\varepsilon P(x)y'(x)^2 dx$$

$$+\rho^{1/\beta}\varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha-1/\beta+2}\int_\varepsilon^1 P(x)y'(x)^2 dx \leq C_1\int_0^1 P(x)y'(x)^2 dx,$$

where

$$C_1 = \max(\varepsilon^{2-1/\beta+1/\alpha}(1-\rho)^{1/\beta}(1-\varepsilon)^{-1/\alpha}, \rho^{1/\beta}\varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha-1/\beta+2}).$$

If we put $\rho = \varepsilon^{\beta/2\alpha}$, then for small ε we have $C_1 \leq 2\varepsilon^\gamma$, where

$$\gamma = \min(2 - 1/\beta + 1/\alpha, \beta/2\alpha) > 0.$$

Since $\lambda_1 \geq C_1^{-1}$, we see that in this case $M_{\alpha,\beta} = \infty$.

d. Now we show that $M_{\alpha,\beta} = \infty$, if

$$\beta - \alpha + 2\alpha\beta = 0, \alpha > -1/2.$$

Remark that for $\alpha \neq 0$, $\beta \neq 0$, $\beta - \alpha + 2\alpha\beta = 0$, $\alpha > -1/2$, α and β have the same signs. Let

$$P(x) = C_1 x^{\varepsilon - 1/\alpha}, \quad Q(x) = C_2 x^{\varepsilon - 1/\beta};\ \varepsilon > 0.$$

The constants C_1, C_2 are chosen so that

$$\int P(x)^\alpha dx = 1, \int Q(x)^\beta dx = 1,$$

i.e. $C_1 = (\alpha\varepsilon)^{1/\alpha}, C_2 = (\beta\varepsilon)^{1/\beta}$. From the Hardy inequality

$$\int x^{\varepsilon-1/\beta} y(x)^2 dx \le C_0 \int x^{\varepsilon-1/\beta+2} y'(x)^2 dx;\ y(x) \in C^1(0,1),$$

where $C_0 = 4/[1+(\varepsilon-1/\beta)^2]$, it follows that

$$\int Q(x)y(x)^2 dx \le C_0 C_2/C_1 \int P(x)y'(x)^2 dx.$$

It remains to note that

$$C_2/C_1 = (\alpha\varepsilon)^{-1/\alpha}(\beta\varepsilon)^{1/\beta} = \alpha^{-1/\alpha}\beta^{1/\beta}\varepsilon^2 \to 0$$

as $\varepsilon \to 0$. It means that

$$\lambda_1 \ge \alpha^{1/\alpha}\beta^{-1/\beta}\varepsilon^{-2},$$

i.e. $M_{\alpha,\beta} = \infty$.

e. Now we show that $M_{\alpha,\beta} < \infty$ if

$$-1/2 < \alpha < 0, \beta - \alpha + 2\alpha\beta < 0.$$

Put $y_0'(x)^2 = P(x)^{\alpha-1}$ and define the function $y_0(x)$ so that it increases on the segment $[0, x_0]$ from 0 to some $m > 0$ and decreases on the segment $[x_0, 1]$ from m to 0. By the Hölder inequality

$$\int y_0'(x)^{2\alpha/(\alpha-1)} dx \le (\int y_0' dx)^{2\alpha/(\alpha-1)}.$$

Therefore,

$$\int_0^1 y_0'(x)dx \ge 1,$$

and so $m \ge 1/2$. Let for definiteness

$$\int_0^{x_0} y_0'(x)^{2\alpha/(\alpha-1)} dx \ge 1/2.$$

Let $x_1 \in (0, x_0)$ be a point such that $y_0(x_1)^{1-\alpha} = 4^\alpha$ .Then

$$\int_0^{x_1} y_0'(x)^{2\alpha/(\alpha-1)} dx \le (\int_0^{x_1} y_0'(x)dx)^{2\alpha/(\alpha-1)} = 1/4.$$

Therefore,

$$\int_{x_1}^{x_0} y_0'(x)^{2\alpha/(\alpha-1)} dx \ge 1/4.$$

On the other hand, by the Hölder inequality

$$\int_{x_1}^{x_0} y_0'(x)^{2\alpha/(\alpha-1)}dx$$

$$\leq \Big(\int_{x_1}^{x_0} y_0'(x)^{2\beta/(\beta-1)}dx\Big)^{(1+\alpha)/(1-\alpha)}\Big(\int_{x_1}^{x_0} y_0' y^{\beta(1+\alpha)/\alpha(\beta-1)}dx\Big)^{2\alpha/(\alpha-1)}$$

and so

$$1/4 \leq \int_{x_1}^{x_0} y_0'(x)^{2\alpha/(\alpha-1)}dx$$

$$\leq \Big(\int_{x_1}^{x_0} y_0^{2\beta/(\beta-1)}dx\Big)^{(1+\alpha)/(1-\alpha)} \cdot [(m^{s+1} - 4^{-s-1})/s]^{2\alpha/(\alpha-1)},$$

where

$$1 + s = \frac{\beta(1+\alpha)}{\alpha(\beta-1)} + 1 = \frac{2\alpha\beta - \alpha + \beta}{\alpha(\beta-1)} < 0.$$

If $\beta < 0$, then

$$\int y(x)^{2\beta/(\beta-1)}dx \leq \Big(\int Q(x)y^2(x)dx\Big)^{\beta/(\beta-1)}\Big(\int Q(x)^\beta dx\Big)^{1/(1-\beta)},$$

and therefore

$$\int Q(x)y(x)^2dx \geq \Big(\int y(x)^{2\beta/(\beta-1)}dx\Big)^{(\beta-1)/\beta}.$$

Therefore, the obtained estimate

$$\int_0^{x_1} y_0^{2\beta/(\beta-1)}dx \geq C_1 > 0,$$

implies that

$$\lambda_0 \leq C_1^{(1-\beta)/\beta}.$$

f. Let now $0 < \alpha < 1, \beta - \alpha + 2\alpha\beta < 0$. We obtain the estimate for λ_0 from above.

The function $P(x)$ is non-negative and $\int P(x)^\alpha dx = 1$. It is sufficient to get the uniform estimate from above for the functions $P(x)$, taking positive values only. Put $y_0'(x)^2 = P(x)^{\alpha-1}$ and define the function $y_0(x)$ so that it increases on the segment $[0, x_0]$ from 0 to some $m > 0$ and decreases on the segment $[x_0, 1]$ from m to 0. We have

$$\int_0^1 P(x)y_0'(x)^2dx = \int_0^1 (y_0'(x))^{2\alpha/(\alpha-1)}dx = \int_0^1 P(x)^\alpha dx = 1.$$

By the Hölder inequality we have for $0 < \beta < 1$

$$\int Q(x)^\beta dx \leq (\int Q(x)y^2(x)dx)^\beta (\int y(x)^{2\beta/(\beta-1)}dx)^{1-\beta},$$

so that

$$\int Q(x)y(x)^2 dx \geq (\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}.$$

Thus the estimate from above follows from the inequality

$$\int y(x)^{2\beta/(\beta-1)}dx \leq C_1,$$

that will be proved.

We note that for $0 < x < x_0$ we have

$$x = \int_0^x dt = \int y_0'(x)^s y_0'(x)^{-s} dt$$

$$\leq (\int y_0'(x)^{sp_1}dt)^{1/p_1} \cdot (\int y_0'(x)^{-sp_2}dt)^{1/p_2} \cdot x^{1/p_3},$$

where

$$1 > s > 0,\ p_1 = \frac{1}{s},\ p_2 = \frac{2\alpha}{s(1-\alpha)},\ p_3 = \frac{1}{(1-s/2-s/2\alpha)}.$$

Since $\int y_0'(x)^{-sp_2}dt = 1$, it follows that

$$x^{s/2+s/2\alpha} \leq y_0(x)^{1/p_1},$$

i.e. $y_0(x) \geq x^{1/2+1/2\alpha}$. Therefore

$$\int_0^{x_0} y_0(x)^{2\beta/(\beta-1)}dx \leq \int_0^{x_0} x^{(\alpha+1)\beta/\alpha(\beta-1)}dx = C_1,$$

since

$$\frac{(\alpha+1)\beta}{\alpha(\beta-1)} > -1$$

in virtue of our conditions. Analogously one can show that $y_0(x) \geq (1-x)^{1/2+1/2\alpha}$ for $x_0 < x < 1$, and thus $y_0(x)^{2\beta/(\beta-1)}dx \leq 2C_1$.

If $\beta < 0$, then

$$\int y(x)^{2\beta/(\beta-1)}dx \leq (\int Q(x)y^2(x)dx)^{\beta/(\beta-1)}(\int Q(x)^\beta dx)^{1/(1-\beta)},$$

and therefore

$$\int Q(x)y(x)^2dx \geq (\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}.$$

Thus in this case it is sufficient to prove that

$$\int y(x)^{2\beta/(\beta-1)}dx \geq C > 0.$$

As in above we have the estimates

$$y_0(x) \geq x^{1/2+1/2\alpha} \text{ for } 0 < x < x_0,$$

$$y_0(x) \geq (1-x)^{1/2+1/2\alpha} \text{ for } x_0 < x < 1$$

and thus

$$\int_0^1 y_0(x)^{2\beta/(\beta-1)}dx$$

$$\geq \int_0^{x_0} x^{(\alpha+1)\beta/\alpha(\beta-1)}dx + \int_{x_0}^1 (1-x)^{(\alpha+1)\beta/\alpha(\beta-1)}dx = C > 0.$$

g. Let $\alpha > 1, \beta - \alpha + 2\alpha\beta < 0$. We show that $\lambda_0 \leq C(\alpha, \beta)$. We have

$$\int P(x)y'(x)^2dx \leq (\int y'(x)^{2\alpha/(\alpha-1)}dx)^{(\alpha-1)/\alpha}(\int P(x)^\alpha dx)^{1/\alpha}$$

and

$$\int Q(x)y(x)^2dx \geq (\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}.$$

Therefore

$$\lambda_1 \leq \inf_y \frac{\int(y'(x)^{2\alpha/(\alpha-1)}dx)^{\alpha-1)/\alpha}}{(\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}} \leq C.$$

Let $y_0(x) = x^\rho$ for $0 \leq x \leq 1/2$ and $y_0(x) = (1-x)^\rho$ for $1/2 \leq x \leq 1$. The number ρ must be such that $2\beta/(\beta-1)\rho > -1$ and $2\alpha(\rho-1)/(\alpha-1) > -1$.

If $\alpha > 1, 0 < \beta < 1$, then such ρ exists if $(1+\alpha)/\alpha < (1-\beta)/\beta$, i.e. for $\beta - \alpha + 2\alpha\beta < 0$. And if $\alpha > 1$, but $\beta < 0$, then as ρ one can take any number, greater than $(1+\alpha)/2\alpha$, since the first condition is satisfied for any $\rho > 0$.

II . Estimates of $m_{\alpha,\beta}$.

a. We prove that $m_{\alpha,\beta} = 0$ for $\beta < 1$. For that we put $P(x) = 1$. If $\beta < 0$, we put $Q(x) = \varepsilon^{-1/\beta}(1-\varepsilon)$ for $x - 1/2 < \varepsilon/2$ and $Q(x) = N$ for

$0 < x < 1/2 - \varepsilon/2$ and for $1/2 + \varepsilon/2 < x < 1$, assuming that $0 < \varepsilon < 1/4$ and N is such a constant that

$$N^\beta(1-\varepsilon) + \varepsilon^{-1}(1-\varepsilon)^\beta \varepsilon = 1,$$

so that $N \to \infty$ as $\varepsilon \to 0$. We note that $b_\varepsilon = \int Q(x)^{1/4}dx \to \infty$ as $\varepsilon \to 0$. Let $q_\varepsilon(x) = Q_\varepsilon(x) b_\varepsilon^{-4}$. Then from the equation

$$y'' + \lambda_0 Q_\varepsilon(x) y = 0$$

it follows that

$$y'' + m q_\varepsilon(x) y = 0, \text{ where } m = \lambda_0 b_\varepsilon^4,$$

and $\int q_\varepsilon^{1/4}dx = 1$. In virtue of the first part of our theorem, if $\alpha = 2, \beta = 1/4$, the first eigen-value is bounded from above, i.e. $m \le C$ and C is independent of ε. But then $\lambda_0 \le C b_\varepsilon^{-4}$, and so λ_1 can take arbitrarily small values.

If $0 < \beta < 1$, then we put $P(x) = 1$ and $Q(x) = \varepsilon^{-1/\beta}(1-\varepsilon)$ for $x - 1/2 < \varepsilon/2$ and $Q(x) = 0$ for $0 < x < 1/2 - \varepsilon/2$ and for $x > 1/2 + \varepsilon/2$, assuming that $0 < \varepsilon < 1/4$. Let $y_0 \in C_0^\infty(0,1)$ and $y_0(x) = 1$ for $1/2 - x < \varepsilon$. Then

$$\lambda_0 = \inf_y \frac{\int_0^1 P(x) y'(x)^2 dx}{\int_0^1 Q(x) y(x)^2 dx} \le \frac{\int_0^1 y_0'(x)^2 dx}{\int_0^1 Q(x) y_0(x)^2 dx} = C\varepsilon^{1/\beta - 1}.$$

Therefore, $\lambda_0 \to 0$ as $\varepsilon \to 0$.

b. Show that $m_{\alpha,\beta} = 0$ for $\alpha > 0$. For this we put $Q(x) \equiv 1$. Since $\alpha > 0$, the function $P(x)$ can vanish on the segment $[0, 1/2]$, when $y(x) = 0$ on the segment $[1/2, 1]$, so that $\lambda_1 = 0$.

c. Let $0 > \alpha > -1$. Let $Q(x) \equiv 1$. Let us put

$$P(x) = \begin{cases} \delta & \text{for } 0 < x < \varepsilon, \\ \varepsilon^{-1} & \text{for } \varepsilon < x < 1 - \varepsilon, \\ \delta & \text{for } 1 - \varepsilon < x < 1, \end{cases}$$

and

$$y(x) = \begin{cases} x/\varepsilon & \text{for } 0 < x < \varepsilon, \\ 1 & \text{for } \varepsilon < x < 1 - \varepsilon, \\ (1-x)/\varepsilon & \text{for } 1 - \varepsilon < x < 1, \end{cases}$$

where δ is such a number that

$$\int P(x)^\alpha dx = 2\varepsilon\delta^\alpha + (1-2\varepsilon)\varepsilon^{-\alpha} = 1,$$

i.e. $\delta \approx \varepsilon^{-1/\alpha}$. It is obvious that

$$\int y^2(x)dx = 1 - 2\varepsilon + 2\varepsilon/3 = 1 - 4\varepsilon/3.$$

On the other hand

$$\int P(x)y_0'(x)^2 dx = 2\delta/\varepsilon \approx 2^{1-1/\alpha}\varepsilon^{-1-1/\alpha}.$$

Therefore,

$$\lambda_0 \le C\varepsilon^{-1-1/\alpha}$$

and since $-1-1/\alpha > 0$ for $0 > \alpha > -1$, the value λ_1 can be arbitrarily small.

d. Let now $\alpha \le -1, \beta \ge 1$. Using the Hölder inequality we get that

$$\int Q(x)y(x)^2dx \le (\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta},$$

$$\int y'(x)^{2\alpha/(\alpha-1)}dx \le \int (P(x)y'(x)^2dx)^{\alpha/(\alpha-1)}.$$

Thus

$$\lambda_1 \ge \inf_y \frac{(\int y'(x)^{2\alpha/(\alpha-1)}dx)^{\alpha-1)/\alpha}}{(\int y(x)^{2\beta/(\beta-1)}dx)^{(\beta-1)/\beta}} \ge 1.$$

The last inequality follows from the estimate

$$(\int y^q(x)dx)^{1/q} \le (\int y'(x)^p dx)^{1/p},$$

where

$$2 < q = 2\beta/(\beta-1) < \infty, \ 1 \le p = 2\alpha/(\alpha-1) < 2.$$

In its turn it is implied by the inequality

$$\max|y(x)| \le \int |y'(x)|dx$$

following from the formula $y(x) = \int_0^x y'(t)dt$. If $\alpha \le -1, \beta = 1$, we can use instead of the Hölder inequality the estimate

$$\int Q(x)y(x)^2dx \le \max y(x)^2$$

and since

$$\max y(x)^2 \leq (\int y'(x)^p dx)^{2/p} \text{ for } 1 \leq p < 2,$$

we can see that $\lambda_0 \geq 1$.

The proof is complete.

The proved theorem can be stated in the following way:

Theorem 5. *Let λ_1 be the first eigen-value of the problem (1)-(2). If $\alpha > -1/2, \beta - \alpha + 2\alpha\beta < 0$, then*

$$\lambda_0 \leq C(\alpha,\beta)\frac{(\int P(x)^\alpha dx)^{1/\alpha}}{(\int Q(x)^\beta dx)^{1/\beta}};$$

if $\alpha \leq -1, \beta \geq 1$, then

$$\lambda_0 \geq C(\alpha,\beta)\frac{(\int P(x)^\alpha dx)^{1/\alpha}}{(\int Q(x)^\beta dx)^{1/\beta}},$$

where $C(\alpha, \beta)$ is a positive constant, depending on α and β only.

4. On estimates of all eigen-values

Once more consider the Sturm-Liouville problem:

$$y'' + \lambda Q(x)y = 0, \ y(0) = 0, \ y(1) = 0$$

under the condition that

$$\int_0^1 Q(x)^\beta dx = 1$$

and estimate the k-th eigen-value λ_k. Our main result is following.

Theorem 6. *If $\beta \geq 1$, then $\lambda_k \geq C_0(\beta)k^2$. If $\beta < \frac{1}{2}$, $\beta \neq 0$, then $\lambda_k \leq C_0(\beta)k^2$. The constant $C_0(\beta)$ here is independent of k.*

Proof. Let $\beta \geq 1$ and y_k be an eigen-function of the Sturm-Liouville problem having the number k. This function has $k-1$ zeroes in the interval $(0,1)$: $\nu_1, \ldots, \nu_{k-1}$. Let $\nu_0 = 0 < \nu_1 < \ldots < \nu_k = 1$ and I be one of the intervals (ν_j, ν_{j+1}), where $j = 0, 1, \ldots, k-1$. Consider the function $y_k(x)$ on the interval I. If $\max_I y_k'(x) = y_k'(\xi_1) = 1$ and $y_k'(\xi) = 0$, $\xi \in I$, then

$$l = \int_\xi^{\xi_1} y_k''(x)dx = \lambda_k \int_\xi^{\xi_1} Q(x)y_k(x)dx.$$

Remark that $|y_k(x)| \leq C(\nu_{j+1} - \nu_j)$ and we can assume that $y_k(x) \geq 0$ in I. Therefore

$$\int_I Q(x)dx \geq (\nu_{j+1} - \nu_j)^{-1}\lambda_k^{-1}.$$

It follows that

$$\int_0^1 Q(x)dx = \sum \int_{I'_j} Qdx \geq \lambda_k^{-1} \sum_1^N (\nu_{j+1} - \nu_j)^{-1} \geq \lambda_k^{-1}k^2$$

and thus

$$\lambda_k \geq k^2 (\int_0^1 Qdx)^{-1} \geq k^2 (\int_0^1 Q^\beta dx)^{-1/\beta} \text{ for } \beta \geq 1.$$

Therefore

$$\int_{\nu_j}^{\nu_{j+1}} Q(x)^\beta dx \geq C_0(\beta)^\beta (\nu_{j+1} - \nu_j)^{1-2\beta} \lambda_k^{-\beta}.$$

Summing these inequalities over j, we obtain that

$$1 \geq C_0(\beta)^\beta \lambda_k^{-\beta} \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1-2\beta}.$$

Remark that $1 - 2\beta \leq -1$ and by Lemma A11

$$\sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1-2\beta} \geq k \cdot (\frac{1}{k})^{1-2\beta} = k^{2\beta}.$$

Therefore,

$$1 \geq C_0(\beta)^\beta \lambda_k^{-\beta} k^{2\beta}$$

and

$$\lambda_k \geq C_0(\beta) k^2.$$

Analogously, if $0 < \beta < 1/2$, then we get the inequality

$$1 \leq C_0(\beta)^\beta \lambda_k^{-\beta} \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1-2\beta}.$$

Since $1 - 2\beta < 1$, we have by Lemma A11, that

$$\sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1-2\beta} \leq k \cdot (\frac{1}{k})^{1-2\beta} = k^{2\beta}.$$

Thus

$$1 \le C_0(\beta)^\beta \lambda_k^{-\beta} k^{2\beta},$$

i.e.

$$\lambda_k \le C_0(\beta) k^2.$$

If $\beta < 0$, then the inequality takes the form

$$1 \ge C_0(\beta)^\beta \lambda_k^{-\beta} \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1-2\beta},$$

and since $1 - 2\beta > 1$, we get

$$1 \ge C_0(\beta)^\beta \lambda_k^{-\beta} k^{2\beta},$$

that gives the inequality

$$\lambda_k \le C_0(\beta) k^2.$$

□

Now consider an other Sturm-Liouville problem:

$$(P(x)y')' + \lambda y = 0 \; y(0) = 0, \; y(1) = 0$$

under the condition

$$\int_0^1 P(x)^\alpha dx = 1.$$

Theorem 7. *If $\alpha > -1/2$, $\alpha \ne 0$, then $\lambda_k \ge C_0(\alpha)k^2$. If $\alpha \le -1$, then $\lambda_k \le C_0(\alpha)k^2$. Here $C_0(\alpha)$ is a positive constant independent of k.*

Proof. Let at first $\alpha > -1/2$, $\alpha \ne 0$. As above, consider the k-th eigenfunction $y_k(x)$, corresponding to the eigen-value λ_k. Let $\nu_0, \ldots, \nu_k$ be the zeroes of $y_k(x)$ and $\nu_0 = 0 < \nu_1 < \ldots < \nu_k = 1$. Let I be one of the intervals (ν_j, ν_{j+1}) with $j = 0, 1, \ldots, k-1$, let

$$l = \nu_{j+1} - \nu_j, \; \rho = \int_{\nu_j}^{\nu_{j+1}} P(x)^\alpha dx.$$

Substituting x by $\nu_j + tl$, $P(x)$ by $p(t)(\rho/l)^{1/\alpha}$ and λ_k by $\rho^{1/\alpha} l^{-2-1/\alpha}$, we obtain that

$$(p(t)y''_{tt})'_t + my = 0, \; 0 \le t \le 1; \; y(0) = 0, \; y(1) = 0;$$

$$\int_0^1 p(t)^\alpha dt = 1.$$

Theorem 2 implies that $m \leq C_0(\alpha)$, so that

$$\lambda_k \leq C_0(\alpha) l^{-2-1/\alpha} \rho^{1/\alpha}.$$

If $\alpha > 0$, then it follows that

$$\lambda_k^\alpha \leq C_0(\alpha)^\alpha l^{-2\alpha-1} \rho$$

or

$$\lambda_k^\alpha (\nu_{j+1} - \nu_j)^{2\alpha+1} \leq C_0(\alpha)^\alpha \int_{\nu_j}^{\nu_{j+1}} P(x)^\alpha dx.$$

Summing over j from 0 to $k-1$ we obtain that

$$\lambda_k^\alpha \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{2\alpha+1} \leq C_0(\alpha)^\alpha.$$

Since $1 + 2\alpha > 1$, we have by lemma A11 that

$$\sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{2\alpha+1} \geq k(1/k^{2\alpha+1}) = k^{-2\alpha}$$

so that

$$\lambda_k \leq C_0 k^2.$$

If $-1/2 < \alpha < 0$, then as above

$$\lambda_k^\alpha \geq C_0(\alpha)^\alpha l^{-2\alpha-1} \rho^2$$

or

$$\lambda_k^\alpha (\nu_{j+1} - \nu_j)^{2\alpha+1} \geq C_0(\alpha)^\alpha \int_{\nu_j}^{\nu_{j+1}} P(x)^\alpha dx.$$

Summing over j between 0 and $k-1$ we see that

$$\lambda_k^\alpha \sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1+2\alpha} \geq C_0(\alpha)^\alpha.$$

since $1 + 2\alpha > 0$, Lemma A11 implies that

$$\sum_{j=0}^{k-1} (\nu_{j+1} - \nu_j)^{1+2\alpha} \leq k(1/k^{1+2\alpha}) = k^{-2\alpha}.$$

Therefore

$$\lambda_k^\alpha k^{-2\alpha} \geq C_0(\alpha)^\alpha$$

and since $\alpha < 0$, we see that

$$\lambda_k \leq C_0(\alpha)k^2.$$

For $\alpha \leq -1$ the same arguments lead to the estimate

$$\lambda_k \geq C_0(\alpha)l^{-2-1/\alpha}\rho^{1/\alpha},$$

i.e.

$$(\nu_{j+1} - \nu_j)^{1+2\alpha}\lambda_k^\alpha \leq C_0(\alpha)^\alpha \int_{\nu_j}^{\nu_{j+1}} P(x)^\alpha dx.$$

Summing over j from 0 to $k-1$ we get the estimate

$$\lambda_k^\alpha \sum_{j=0}^{k-1}(\nu_{j+1} - \nu_j)^{1+2\alpha} \leq C_0(\alpha)^\alpha.$$

Since $2\alpha + 1 \leq -1$, Lemma A11 implies that

$$\sum_{j=0}^{k-1}(\nu_{j+1} - \nu_j)^{1+2\alpha} \leq k(1/k^{1+2\alpha}) = k^{-2\alpha}$$

and hence

$$\lambda_k^\alpha k^{-2\alpha} \leq C_0(\alpha)^\alpha,$$

so that

$$\lambda_k \geq C_0(\alpha)k^2.$$

□

The proved Theorems can be reformulated in the following form.

Theorem 8. *Let λ_k be the k-th eigen-value of the Sturm-Liouville problem considered in Theorem 6. If $\beta \geq 1$ then*

$$\lambda_k \geq C_0(\beta)k^2\left(\int_0^1 Q(x)^\beta dx\right)^{-1/\beta}.$$

If $\beta < \frac{1}{2}$, $\beta \neq 0$, then

$$\lambda_k \leq C_0(\beta)k^2\left(\int_0^1 Q(x)^\beta dx\right)^{-1/\beta}.$$

Theorem 9. *Let λ_k be the k-th eigen-value of the Sturm-Liouville problem considered in Theorem 7. If $\alpha > -1/2$, $\alpha \neq 0$, then*

$$\lambda_k \geq C_0(\alpha)k^2(\int_0^1 P(x)^\alpha dx)^{1/\alpha}.$$

If $\alpha \leq -1$, then

$$\lambda_k \leq C_0(\alpha)k^2(\int_0^1 P(x)^\alpha dx)^{1/\alpha}.$$

5. On estimates of first eigen-value of a Sturm-Liouville problem for operators of higher order

Let us consider the dependence of the first eigen-value λ_1 of the Sturm-Liouville problem

$$(-1)^{n+1}y^{(2n)}(x) + \lambda q(x)y(x) = 0 \quad (4)$$

on the segment $0 \leq x \leq 1$, with the boundary conditions

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = y(1) = y'(1) = \ldots = y^{(n-1)}(1) = 0 \quad (5)$$

on the potential q. Denote R_β the set of real-valued measurable on $(0,1)$ functions q with positive values such that

$$\int_0^1 q(x)^\beta dx = 1,$$

where β is a real number, $\beta \neq 0$. The problem (4),(5) has a discrete spectrum. The variational principle implies that the first eigen-value λ_0 can be found as

$$\lambda_1 = \inf_{y \in C_0^\infty(0,1)} \frac{\int_0^1 y^{(n)}(x)^2 dx}{\int_0^1 q(x)y(x)^2 dx}.$$

It is easy to see that all eigen-functions of the problem (4),(5) are real and positive. We will estimate the values

$$m_\beta = \inf_{q \in R_\beta} \lambda_1, \; M_\beta = \sup_{q \in R_\beta} \lambda_1.$$

Put

$$L[q,y] = \frac{\int_0^1 y^{(n)}(x)^2 dx}{\int_0^1 q(x) y(x)^2 dx}.$$

Theorem 10. *If $\beta \geq 1$, then $m_\beta \geq 1$.*

Proof. If $y(x)$ is an eigen-function, corresponding to λ_1, then by the Rolle theorem each function $y'(x), \ldots, y^{(n-1)}(x)$ has at least one zero on $(0,1)$. Therefore,

$$y^{(n-1)}(x) = \int_\xi^x y^{(n)}(t) dt,$$

where ξ is a zero of $y^{(n-1)}(x)$. So $|y^{(n-1)}(x)| \leq \int_0^1 |y^{(n)}(t)| dt$. Analogously, $|y^{(n-i)}(x)| \leq \int_0^1 |y^{(n-i+1)}(t)| dt$. By induction we get the inequality $|y(x)|^2 \leq (\int_0^1 |y^{(n)}(t)| dt)^2 \leq \int_0^1 |y^{(n)}(t)|^2 dt$. Hence

$$\frac{\int_0^1 |y^{(n)}(t)|^2 dt}{\int_0^1 q(t) y(t)^2 dt} \geq \frac{1}{\int_0^1 q(t) dt} \geq \frac{1}{(\int_0^1 q(t)^\alpha dt)^{1/\alpha}} \geq 1.$$

□

Theorem 11. *If $\beta \geq 1/n$, then $M_\beta = \infty$.*

Proof. Let $q_\varepsilon(x) = C_\varepsilon (x+\varepsilon)^{-n}$, where C_ε is such that $\int_0^1 q_\varepsilon(x)^\beta dx = 1$. It is easy to see that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $y \in W_2^n(0,1)$ and y satisfy the conditions (5). Put $y(x) = 0$ outside of $(0,1)$. The Hardy inequality

$$\int_0^1 y(x)^2 (x+\varepsilon)^{-n} dx \leq C_1 \int_0^1 |y^{(n)}(t)|^2 dt$$

implies the inequality

$$\int_0^1 q_\varepsilon(t) y(t)^2 dt \leq C_\varepsilon C_1 \int_0^1 |y^{(n)}(t)|^2 dt.$$

Therefore

$$L[q_\varepsilon, y] = \frac{\int_0^1 y^{(n)}(t)^2 dt}{\int_0^1 q_\varepsilon(t) y(t)^2 dt} \geq \frac{1}{C_\varepsilon C_1} \to \infty \text{ as } \varepsilon \to 0.$$

Thus $M_\beta = \infty$. □

Theorem 12. *If $\beta < 1/n$, then $M_\beta = C(\beta) < \infty$.*

Proof. Let at first $0 < \beta < 1/n$. Using the Hölder inequality we obtain

$$1 = \int_0^1 q(x)^\beta dx \le (\int_0^1 q(x)y(x)^2dx)^\beta (\int_0^1 |y(x)|^p dx)^{1-\beta},$$

where $p = 2\beta/(\beta - 1)$ so that $0 > p > -2(n-1)$. Therefore

$$L[q,y] \le \frac{\int_0^1 y^{(n)}(x)^2 dx}{(\int_0^1 |y(x)|^p dx)^{2/p}}.$$

Put $y_0(x) = x^{n+\delta-1}(1-x)^{n+\delta-1}$. Then $\int_0^1 y^{(n)}(x)^2dx = c_1$ if $\delta > 1/2$ and $\int_0^1 |y(x)|^p dx = c_2 < \infty$, if $p(n+\delta-1)+1 > 0$, i.e. if

$$\delta < (1 + \beta - 2\beta n)/2\beta.$$

Since $(1+\beta-2\beta n)/2\beta > 1/2$, there exists δ satisfying all the conditions. Thus $\lambda_1 \le L[q, y_0] < c_3$.

Now let $\beta < 0$. Using the Hölder inequality we obtain

$$\int_0^1 |y(x)|^p dx = \int_0^1 q(x)^{p/2}|y(x)|^p q(x)^{-p/2}dx$$

$$\le (\int_0^1 q(x)y(x)^2dx)^{p/2}(\int_0^1 q(x)^\beta dx)^{1/(1-\beta)} = (\int_0^1 q(x)y(x)^2dx)^{p/2}.$$

Therefore,

$$\int_0^1 q(x)y(x)^2dx \ge (\int_0^1 |y(x)|^p dx)^{2/p}.$$

Hence

$$L[q,y] \le \frac{\int_0^1 y^{(n)}(x)^2 dx}{(\int_0^1 |y(x)|^p dx)^{2/p}}.$$

Putting $y_0(x) = x^{n-1}(1-x)^{n-1}$, we see that $L[q,y] \le c$. □

Theorem 13. *If $\beta < 1$, then $m_\beta = 0$.*

Proof. Let at first $\beta < 0$. Put

$$q(x) = \begin{cases} (1-\varepsilon)^{1/\beta}\varepsilon^{-1/\beta}, & \text{if } 0 < x < \varepsilon, \\ (1-\varepsilon)^{-1/\beta}\varepsilon^{1/\beta}, & \text{if } \varepsilon < x < 1, \end{cases}$$

where $\varepsilon > 0$ is a small number. Let $y_0(x) = x^{n-1}(1-x)^{n-1}$. Then

$$\lambda_1 \leq \frac{\int_0^1 y_0^{(n)}(x)^2 dx}{\int_0^1 q(x) y_0(x)^2 dx} \leq C\varepsilon^{-1/\beta}.$$

Therefore λ_1 can be arbitrary small.

Let $0 < \beta < 1$. Put

$$q(x) = \begin{cases} (2\varepsilon)^{-1/\beta}, & \text{if } |x - 1/2| > \varepsilon, \\ 0, & \text{if } |x - 1/2| < \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a small number. Let y_0 be a smooth function, vanishing in the points $x = 0$ and $x = 1$, which is equal to 1 in $(1/3, 2/3)$. Then

$$\lambda_1 \leq \frac{C}{\int_{1/2-\varepsilon}^{1/2+\varepsilon} (2\varepsilon)^{-1/\beta} dx} = C_1 \varepsilon^{1/\beta - 1} \to 0$$

as $\varepsilon \to 0$. Therefore λ_1 can be arbitrary small. □

In the study of the Sturm-Liouville problem for an equation of second order we have obtained the sharp values of the first eigern-value for the operator $y''(x) + \lambda q(x) y = 0$ under the condition $\int_0^1 q(x)^\beta = 1$. We have found also the potentials q, when these sharp estimates are true. In the case of an operator of a higher order one can write down the differential equations for the function q. However, it is an equation of order n and we cannot find the explicit solution.

Consider more in details the following problem

$$-y^{(4)} + \lambda q(x) y = 0,$$

$$y(0) = y'(0) = 0, \; y(1) = y'(1) = 0, \; q(x) \geq 0, \; \int_0^1 q(x)^\beta = 1, \; \beta > 1,$$

q is a bounded measurable function. As we have shown above the least eigenvalue of this problem is bigger than a positive constant, independent of q. Namely

$$\lambda_1 = \inf_{y \in C_0^\infty(0,1)} \frac{\int_0^1 y'(x)^2 dx}{\int_0^1 q(x) y(x)^2 dx} \geq \inf_{y \in C_0^\infty(0,1)} \frac{\int_0^1 y''(x)^2 dx}{(\int_0^1 |y(x)|^p dx)^{2/p}}.$$

Let

$$G[y] = \frac{\int_0^1 y''(x)^2 dx}{(\int_0^1 |y(x)|^p dx)^{2/p}}$$

and $m = \inf_{y \in C_0^\infty(0,1)} G[y]$. Let $\{y_k\}$ be a minimizing sequence. By the homogeneity we can assume that

$$\int_0^1 y''(x)^2 dx = 1.$$

The sequence $\{y_k\}$ contains a subsequence converging to y_0 uniformly and weakly in $W_{2,0}^2(0,1)$. The Euler-Lagrange equation for the functional L has the form

$$y^{(4)} - m|y|^{(\beta+1)/(\beta-1)} sgn\, y = 0,$$

$$y(0) = y'(0) = 0,\ y(1) = y'(1) = 0,\ \int_0^1 y''(x)^2 dx = 1.$$

Put

$$q(x) = \lambda |y_0|^{2/(\beta-1)},$$

where λ is such that $\int_0^1 q(x)^\beta dx = 1$. Then the problem

$$-y^{(4)} + \lambda q(x) y = 0,$$

$$y(0) = y'(0) = 0,\ y(1) = y'(1) = 0$$

has an eigen-value m, to which the eigen-function y corresponds. This eigen-value is minimal for the considered class of the functions q. So the finding of the extremal q and λ_1 is reduced to the boundary value problem for an equation of fourth order. The same is true for other values of β and for equations of order $n > 4$.

Let us consider the boundary value problem

$$(-1)^{k_s} y^{(2n)}(x) + \lambda q(x) y(x) = 0$$

on the segment $0 \le x \le 1$, with the boundary conditions

$$y^{(i)}(x_j) = 0,\ i = 0, \ldots, k_j,\ 0 = x_1 < x_2 < \ldots < x_s = 1,$$

where

$$s \le 2n - 1, \quad k_1 + \ldots + k_s = 2n - s.$$

We assume that

$$q(x) \geq 0, \int_0^1 q(x)^\beta = 1.$$

We have shown that this problem has positive eigen-values. Let λ_1 be the minimal of them. Let us show that $\lambda_1 \geq 1$. Indeed, the corresponding eigen-function y_1 has at least $2n$ zeroes (taking into account their multiplicity.) The function $y^{(2n-1)}$ has at least one zero ξ. Hence

$$y^{(2n-1)}(x) = \int_\xi^1 y^{(2n)}(x)dx,$$

and therefore

$$|y^{(2n-1)}(x)| \leq \lambda_1 \int_0^1 q(x)dx \cdot \max|y(x)| \leq \lambda_1 \max|y(x)|,$$

Since each function $y'(x), \ldots, y^{(n-1)}(x)$ has at least one zero on $(0,1)$, we have

$$y^{(2n-i)}(x) = \int_{\xi_i}^x y^{(2n-i+1)}(t)dt,$$

where ξ_i is a zero of $y^{(2n-i)}(x)$. Hence

$$|y^{(2n-i)}(x)| \leq \max_t |y^{(2n-i+1)}(t)|.$$

In particular,

$$|y(x)| \leq \max_t |y^{(2n-1)}(t)| \leq \lambda_1 \max|y(x)|$$

and thus $\lambda_1 \geq 1$. □

6. On a Lagrange problem

6.1. Introduction

The considered Lagrange problem consists in the finding of extremal values of the following functional:

$$L[Q,y] = \frac{\int_0^1 Q(x)y''(x)^2dx}{\int_0^1 y'(x)^2dx}$$

under the conditions $y \in H^2(0,1)$, $Q(x)$ is a bounded measurable function,

$$\int_0^1 Q(x)^\alpha dx = 1, \; Q(x) \geq 0, \tag{6}$$

$$y(0) = 0, \; y'(0) = 0, \; y(1) = 0, \; y'(1) = 0, \tag{7}$$

where $\alpha \in \mathbf{R} \setminus \{0\}$. It is easy to see that this problem is equivalent to the variational problem on the extremum of the functional

$$F[Q,y] = \frac{\int_0^1 Q(x)y'(x)^2 dx}{\int_0^1 y(x)^2 dx},$$

under the conditions (6) and

$$y(0) = 0, \; y(1) = 0, \; \int_0^1 y(x)dx = 0.$$

The Euler-Lagrange equation for the functional L has the form

$$(Q(x)y'')'' + \lambda y'' = 0, \; y(0) = 0, \; y'(0) = 0, \; y(1) = 0, \; y'(1) = 0. \tag{8}$$

This problem is very important for applications. For example, it is essential for the finding of the strongest column of a given volume (the most important values are then $\alpha = 1/2$ or $1/3$) and was considered by many authors (see, for example, [1]-[9]). The authors of the articles [4]-[9] used methods of the functional analysis and of the variational calculus, sometimes very complicated. However, the problem has not been solved until now.

Let us reproduce some Keller's arguments. Let us suppose that there exists a function $Q_0(x)$ which maximizes the lowest eigenvalue. Let $Q(x,\varepsilon)$ be a family of functions which depend smoothly on ε and such that $Q(x,0) = Q_0(x)$. Assume that λ and y, the lowest eigen-value and corresponding eigenfunction with $Q = Q(x,\varepsilon)$, also depend smoothly on ε. Then we may differentiate the equation

$$(Q(x,\varepsilon)y''(x,\varepsilon))'' + \lambda(\varepsilon)y''(x,\varepsilon) = 0$$

with respect to ε to obtain the equation

$$(Q_0(x)z'')'' + \lambda z'' + (Q_1(x)y'')'' + \mu y'' = 0,$$

where $Q_1(x) = \partial Q(x,0)/\partial\varepsilon$, $z(x) = \partial y(x,0)/\partial\varepsilon$ and $\mu = \partial\lambda(0)/\partial\varepsilon$. Multiply the first equation by z, the second by y, subtract one from the other and integrate the result over $[0,1]$. In virtue of the boundary conditions we have

$$\int_0^1 ((Q_0(x)y'')''z - (Q_0(x)z'')''y)dx = 0,$$

$$\int_0^1 (y''z - z''y)dx = 0.$$

Therefore

$$\int_0^1 Q_1(x)y''(x)^2dx - \int_0^1 \mu y'(x)^2dx = 0.$$

Since λ is the maximal value of $\lambda(\varepsilon)$, we see that $\mu = 0$ and so

$$\int_0^1 Q_1(x)y''(x)^2dx = 0$$

and this is true for any function $Q_1(x)$ such that

$$\int_0^1 Q_0(x)^{\alpha-1}Q_1(x)dx = 0.$$

Thus we have as a necessary condition for a maximum the relation $y''(x)^2 = CQ_0(x)^{\alpha-1}$. It leads to a non-linear equation for y

$$(|y''|^{2/(\alpha-1)}y'')'' + \lambda y'' = 0,$$

which is integrable. Indeed, if we put $y_0'' = z$, then

$$(|z|^{2/(\alpha-1)}z)'' + \lambda z = 0,$$

and if we put now $z' = P(z)$ we obtain a linear equation of first order for P^2. The weak point of this proof is that the function $\lambda(\varepsilon)$ can be nonregular, because the lowest eigen-value λ can be double. Besides, the existence of the optimal solution was never proved.

The authors of [9] claimed to prove that the result of Keller-Tadjbakhsh [3] is not correct. However, their calculations are erroneous and the value $16\pi^2/3$ found in [3] is optimal.

We propose here another approach, allowing to say that the indicated solution is really optimal and gives the globally extremal value to the functional L. Let us remark that we had used the Sobolev's type spaces $W_p^l(0,1)$ with $l = 1, 2$ and any real values of $p \neq 0$, what is interesting also outside of the frames of the Lagrange problem. Furthermore, we prove the existence of the optimal solution. The obtained results can be extended to the multi-dimensional case also. Close results for functionals depending on y, y' only have been obtained in our works [10]-[12].

The important role in that follows belongs to the functional

$$G[y] = \frac{(\int_0^1 |y'(x)|^p dx)^{2/p}}{\int_0^1 y(x)^2 dx}, \text{ where } p = \frac{2\alpha}{\alpha - 1}.$$

Let R_α be the set of bounded measurable functions Q defined on $[0, 1]$ satisfying the conditions (1).

Let $K_p(a, b)$ for real $p \neq 0$ be the set of non-decreasing real functions y defined on $[a, b]$, absolutely continuous on $[a, b-\varepsilon]$ for any $\varepsilon > 0$ and such that $y(0) \geq 0$,

$$\int_a^b y'(x)^p dx < \infty, \int_a^b y(x)^2 dx < \infty.$$

Let $K_p(a, b, c)$ be the set of real functions y defined on $[a, c]$ and such that $y \in K_p(a, b)$, $y(-x) \in K_p(-c, -b)$, $\int_a^b |y'(x)|^p dx < \infty$ and $\int_b^c |y'(x)|^p dx < \infty$.

Let H be the set of functions y belonging to $H^2(0, 1)$ and satisfying the conditions (7).

Put at last

$$m_\alpha = \inf_{Q \in R_\alpha} \inf_{y \in H} L[Q, y], \quad M_\alpha = \sup_{Q \in R_\alpha} \inf_{y \in H} L[Q, y].$$

Our aim is to find the values of m_α and M_α and the functions Q, y realizing these extremal values.

6.2. Preliminary estimates

Theorem 14. *Let $\alpha \in \mathrm{R} \setminus 0$. Then*

1. *M_α is finite for $\alpha > -1/2, \alpha \neq 0$ and $M_\alpha = \infty$ for $\alpha \leq -1/2$;*

2. *$m_\alpha > 0$ for $\alpha \leq -1$ and $m_\alpha = 0$ for $\alpha > -1$.*

Proof.

1. If $\alpha > 1$, then by the Hölder inequality

$$\int_0^1 Q(x) y''(x)^2 dx \leq (\int_0^1 Q(x)^\alpha dx)^{1/\alpha} (\int_0^1 |y''(x)|^p dx)^{2/p},$$

where $p = 2\alpha/(\alpha - 1)$. Put $y_0(x) = x^2(1-x)^2$. Then

$$\inf_{y\in H} L[Q,y] \le \frac{(\int_0^1 |y_0''(x)|^p dx)^{2/p}}{\int_0^1 y_0'(x)^2 dx} = C.$$

Therefore, $M_\alpha \le C$.

Similarly, for $\alpha = 1$ we have

$$\inf_{y\in H} L[Q,y] \le \frac{\max y_0''(x)^2}{\int_0^1 y_0'(x)^2 dx} = C.$$

2. Let $0 < \alpha < 1$ and Q be a function from the class R_α. According to Lemma A12, we can construct a function $y(x)$ such that

$$y''(x)^2 = [Q(x)+1]^{\alpha-1}, \; y(0) = y'(0) = y(1) = y'(1) = 0.$$

By our construction

$$\int_0^1 |y''(x)|^p dx = \int_0^1 [Q(x)+1]^\alpha dx,$$

where $p = 2\alpha/(\alpha-1)$. Let r be the maximum point of the function $y(x)$. The function y' satisfies the conditions of Lemma A6 on the intervals $[0,r]$ and $[r,1]$. Therefore, $L[Q,y] \le L[Q+1,y] = G[y'] \le C$ and thus $M_\alpha \le C$.

3. Let now $0 > \alpha > -1/2$. We will use the same function $y(x)$ as above, in s.2. Let $p = 2\alpha/(\alpha-1)$. Then $0 < p < 2/3$. Using Lemma A6, as above, we obtain that

$$\int_0^1 y'(x)^2 dx \ge C > 0$$

and therefore, $M_\alpha \le C^{-1}$.

4. Let $\alpha < -1/2$ and $\varepsilon \in (0, 1/10)$. Let

$$Q(x) = \begin{cases} \varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha}, & \text{if } 0 < x < \varepsilon, \\ (1-\varepsilon)^{-1/\alpha}\varepsilon^{1/\alpha}, & \text{if } \varepsilon < x < 1, \end{cases}$$

so that

$$\int_0^1 Q(x)^\alpha dx = 1.$$

Since

$$\int_0^1 y'(x)^2dx \le \varepsilon^2\int_0^\varepsilon y''(x)^2dx + (1-\varepsilon)^2\int_\varepsilon^1 y''(x)^2dx$$
$$= (1-\varepsilon)^{-1/\alpha}\varepsilon^{2+1/\alpha}\int_0^\varepsilon Q(x)y''(x)^2dx$$
$$+(1-\varepsilon)^{2+1/\alpha}\varepsilon^{-1/\alpha}\int_\varepsilon^1 Q(x)y''(x)^2dx$$
$$\le 2\varepsilon^\gamma\int_0^1 Q(x)y''(x)^2dx,$$
where $\gamma = \min(2+1/\alpha, -1/\alpha) > 0$, we obtain that

$$M_\alpha \ge \varepsilon^{-\gamma}/2$$

and therefore $M_\alpha = \infty$.

5. Consider now the case when $\alpha = -1/2$. Put

$$Q(x) = \max(x^2/\varepsilon^2,\ \delta^2/\varepsilon^2),$$

where $\delta = \exp(1 - 1/\varepsilon)$. Remark that

$$\int_0^1 Q(x)^{-1/2}dx = \delta\cdot\varepsilon/\delta + \int_\delta^1 \varepsilon/x dx = \varepsilon - \varepsilon\ln\delta = 1.$$

On the other hand, the well-known estimate

$$\int_0^1 y'(x)^2dx \le 4\int_0^1 x^2y''(x)^2dx,$$

valid for all functions $y' \in W_2^1(0,1)$, vanishing at 0, implies that

$$\int_0^1 y'(x)^2dx \le 4\delta^2\int_0^\delta y''(x)^2dx + 4\int_\delta^1 x^2y''(x)^2dx$$
$$\le 4\varepsilon^2\int_0^1 Q(x)y''(x)^2dx.$$

It means that

$$\inf_{y\in H}\frac{\int_0^1 Q(x)y''(x)^2dx}{\int_0^1 y'(x)^2dx} \ge \frac{1}{4\varepsilon^2},$$

so that $M_\alpha = \infty$.

6. Let $\alpha \le -1$. By the Hölder inequality

$$\int_0^1 |y''(x)|^p dx \le (\int_0^1 Q(x)y''(x)^2dx)^{p/2}(\int_0^1 Q(x)^\alpha dx)^{1/(1-\alpha)},$$

where $p = 2\alpha/(\alpha-1)$. Since $p \ge 1$, the inequality

$$\int_0^1 y'(x)^2dx \le \int_0^1(\int_0^x |y''(t)|dt)^2dx \le (\int_0^1 |y''(t)|dt)^2 \le (\int_0^1 |y''(t)|^p dt)^{2/p}$$

holds for all functions $y \in H$, and thus $m_\alpha \ge 1$.

7. If $\alpha > 0$, then we can take a function y vanishing in $[0, 1/2]$ and such that $\int_0^1 y'^2dx = 1$. Since the function Q can have arbitrarily small values in $[1/2, 1]$, the value of m_α is equal to 0.

8. Let $0 > \alpha > -1$. Let us show that in this case $m_\alpha = 0$. Put for that

$$y'(x) = \begin{cases} 2x, & \text{if } 0 < x < \varepsilon, \\ 2\varepsilon, & \text{if } \varepsilon < x < 1/2 - \varepsilon, \\ (1-2x), & \text{if } 1/2 - \varepsilon < x < 1/2, \end{cases}$$

$$Q(x) = \begin{cases} \varepsilon^{-1/\alpha}(1-\varepsilon)^{1/\alpha}, & \text{if } |x - 1/2| < \varepsilon/4 \\ & \text{or } |x-1/2| > 1/2 - \varepsilon/4, \\ (1-\varepsilon)^{-1/\alpha}\varepsilon^{1/\alpha} & \text{for other } x. \end{cases}$$

If $y(0) = 0$, the function y is defined for $0 \le x \le 1/2$. Let us put now $y(x) = -y(1-x)$ for $x \in (1/2, 1)$. It is easy to see that

$$\int_0^1 Q(x)^\alpha dx = 1; \quad \int_0^1 Q(x)y''(x)^2dx = 4\varepsilon^{1-1/\alpha}(1-\varepsilon)^{1/\alpha},$$

$$\int_0^1 y'(x)dx = 0, \quad \int_0^1 y'(x)^2dx = 16\varepsilon^3/3 + 4\varepsilon^2(1-\varepsilon).$$

Therefore,

$$m_\alpha \le \frac{\int_0^1 Q(x)y''(x)^2dx}{\int_0^1 y'(x)^2dx} \le \varepsilon^{-1-1/\alpha},$$

and since $-1-1/\alpha > 0$, the value of m_α is equal to zero. □

6.3. Precise results

Now we consider the question on the attainability of the extremal values of the functional L.

Theorem 15. *If $\alpha < -1$, then there exist a function $y \in H$ and a function Q satisfying (1), such that*

$$L[Q,y] = m_\alpha = \frac{4(2\alpha+1)}{\alpha}\Big(\frac{\alpha+1}{2\alpha+1}\Big)^{1-1/\alpha} B\Big(\frac{1}{2},\frac{1}{2}+\frac{1}{2\alpha}\Big)^2,$$

where B is the Euler function.

If $1 > \alpha > -1/2$, $\alpha \neq 0$, then there exist a function $y_0 \in H$ and a function Q satisfying (1), such that

$$\inf_y L[Q,y] = L[Q,y_0] = M_\alpha.$$

Furthermore, if $\alpha \geq 1$, then

$$M_\alpha \leq \frac{4(2\alpha+1)}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha} B\Big(\frac{1}{2},\frac{1}{2}+\frac{1}{2\alpha}\Big)^2.$$

If $0 < \alpha < 1$, then

$$M_\alpha = 4\frac{(2\alpha+1)}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha} B\Big(\frac{1}{2},\frac{1}{2}+\frac{1}{2\alpha}\Big)^2.$$

If $-1/2 < \alpha < 0$, then

$$M_\alpha = -4\frac{2\alpha+1}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha}\Big(\int_0^\infty \frac{dt}{(1+t^2)^{1/2-1/2\alpha}}\Big)^2.$$

Remark 16. In particular, in the classical Lagrange problem with $\alpha = 1/2$ the optimal function $Q(x)$ is defined on $(0,1)$ in the parametric form as follows

$$x = (2t + \sin 2t)/4\pi,\ Q(x) = 16\cos^4 t/9;\ 0 \leq t \leq 2\pi.$$

The optimal value $M = 16\pi^2/3$ has been indicated by Keller-Tadjbaksh in [3]. The optimal column has two points at which $Q(x)$ vanishes.

Remark also that we don't know the sharp value of M_α if $\alpha > 1$ and cannot prove that the optimal functions Q, y_0 do exist in this case.

Corollary 17. *If $p = 1$, then $m = 16$.*

Proof. For all $y \in K$, and $p > 1$ we have

$$m \int_0^1 y(x)^2 dx \leq (\int_0^1 |y'(x)|^p dx)^{2/p},$$

and $\lim_{p \to 1+0} m = 16$. Therefore, $m \geq 16$. On the other hand, putting $y(x) = 1$ for $\varepsilon < x < 1/2 - \varepsilon$, $y(x) = -1$ for $1/2 + \varepsilon < x < 1 - \varepsilon$, $y(x) = x/\varepsilon$ for $0 < x < \varepsilon$, $y(x) = (1/2 - x)/\varepsilon$ for $1/2 - \varepsilon < x < 1/2 + \varepsilon$ and $y(x) = (x-1)/\varepsilon$ for $1 - \varepsilon < x < 1$, we can see that

$$\int_0^1 y(x)^2 dx = 1 - 8\varepsilon^2/3, \int_0^1 |y'(x)| dx = 4$$

and so $m = 16$. □

In the same way one can prove the following

Corollary 18. *If $p = \infty$, then $m = \lim_{p\to\infty} m(p) = 48$. The estimate is realized by the function y_1 equal to $1/4 - |x - 1/4|$ for $0 < x < 1/2$ and to $|x - 3/4| - 1/4$ for $1/2 < x < 1$.*

Corollary 19. *Let $-1/2 < \alpha < 1, \alpha \neq 0$ and $z(x) = y_0(x)$ for $0 < x < 1/2$, $z(x) = -y_0(x-1/2)$ for $1/2 < x < 1$, where y_0 is the function found in Lemmas A5 and A6 for $r = 1/2$ and $p = 2\alpha/(\alpha - 1)$. Put $Q(x) = |z'(x)|^{2/(\alpha-1)}$. Then*

$$Q(x) = Q(1-x) \text{ for } 0 < x < 1, \; Q(x) = Q(1/2 - x) \text{ for } 0 < x < 1/2,$$

$$Q(x) = c|x - 1/4|^\gamma [1 + o(1)] \text{ as } x \to 1/4,$$

$$Q(x) = c|x - 3/4|^\gamma [1 + o(1)] \text{ as } x \to 3/4,$$

where $\gamma = 2/(\alpha + 1) \in]1, 2[$ if $\alpha > 0$ and $\gamma = 2$ if $\alpha < 0$.

Lemma 20. *Let $0 < \alpha < 1$ and $Q(x)$ be the function found in Corollary 19. Let*

$$m_1 = \inf_{y \in H_0^2(0,1)} \frac{\int_0^1 Q(x) y''(x)^2 dx}{\int_0^1 y'(x)^2 dx}.$$

Then $m_1 = m$, where m was indicated in Theorem 15. The minimal value is attained on the function y_1 that is equal to $\int_0^x z(t)dt$.

Proof. Consider the minimizing sequence $y_k(x)$ such that $\int_0^1 y_k'(x)^2 dx = 1$, then the integrals $\int_0^{1/4-\varepsilon} y_k''(x)^2 dx$, $\int_{1/4+\varepsilon}^{3/4-\varepsilon} y_k''(x)^2 dx$ and $\int_{3/4+\varepsilon}^1 y_k''(x)^2 dx$ are bounded and one can choose a subsequence converging almost everywhere in $(0, 1/4-\varepsilon)$, $(1/4+\varepsilon, 3/4-\varepsilon)$ and $(3/4+\varepsilon, 1)$ in H^1 and weakly in $H^2(0, 1/4-\varepsilon)$, $H^2(1/4+\varepsilon, 3/4-\varepsilon)$ and $H^2(3/4+\varepsilon, 1)$. Using the diagonalization, one can find a subsequence converging almost everywhere in $(0, 1)$ to $y_1(x)$.

Let us show that

$$\int_{1/4-\varepsilon}^{1/4-\varepsilon} y_k'(x)^2 dx + \int_{3/4-\varepsilon}^{3/4-\varepsilon} y_k'(x)^2 dx \leq \varepsilon^{2-\gamma} \tag{9}$$

with a constant C independent of ε and k. Indeed, using the equality

$$y_k'(1/4-\varepsilon) = \int_0^{1/4-\varepsilon} y_k''(x)dx,$$

we see that

$$y_k'(1/4-\varepsilon)^2 \leq \int_0^{1/4-\varepsilon} Q(x)y_k''(x)^2 dx \int_0^{1/4-\varepsilon} Q(x)^{-1}dx \leq C_1\varepsilon^{1-\gamma}.$$

Since $\int_{1/4+\varepsilon}^{3/4-\varepsilon} y_k'(x)^2 dx < 1$, there exists a $\theta_k \in]1/4+\varepsilon, 3/4-\varepsilon[$ such that $y_k'(\theta_k)^2 \leq 2$ and

$$y_k'(1/4+\varepsilon)^2 \leq 4 + \int_{\theta_k}^{1/4+\varepsilon} Q(x)y_k''(x)^2 dx \int_{\theta_k}^{1/4+\varepsilon} Q(x)^{-1}dx \leq C_2\varepsilon^{1-\gamma}.$$

Analogously, we have

$$y_k'(3/4-\varepsilon)^2 \leq C_1\varepsilon^{1-\gamma},\ y_k'(3/4+\varepsilon)^2 \leq C_2\varepsilon^{1-\gamma}.$$

If $1/4-\varepsilon < x < 1/4$, then

$$y_k'(x)^2 \leq 2y_k'(1/4-\varepsilon)^2$$
$$+ \int_{1/4-\varepsilon}^{x} Q(x)y_k''(x)^2 dx \int_{1/4-\varepsilon}^{x} Q(x)^{-1}dx \leq C_3\varepsilon^{1-\gamma}$$

and therefore,

$$\int_{1/4-\varepsilon}^{1/4} y_k'(x)^2 dx \leq C_3\varepsilon^{1-\gamma}.$$

So (9) is valid. Since we can assume that $y_k'(x)$ converge uniformly to $y_1'(x)$ outside of ε-neighbourhood of the points $1/4$ and $3/4$ we see that

Then

$$\int_0^1 y_1'(x)^2 dx = 1, \ \int_0^1 Q(x) y_1''(x)^2 dx \le m_1.$$

However, m_1 is the minimal possible value of the latter integral. Therefore, $\int_0^1 Q(x) y_1''(x)^2 dx = m_1$. The function y_1 satisfies the equation

$$(Q(x) y_1''(x))'' + m_1 y_1''(x) = 0,$$

$y_1(0) = 0, \ y_1'(0) = 0, \ y_1(1) = 0, \ y_1'(1) = 0$. The function $z(x) = y_1(x) + y_1(1-x)$ is also minimizing, if it does not vanish identically.

If $z(x) \not\equiv 0$, then it is even and

$$Q(x) z''(x) + m_1 z = C.$$

Using Lemma A9 we see that $C = m_1 z(1/4) = m_1 z(3/4)$. Put $u = z - z(1/4)$. Then

$$Q(x) u'' + m_1 u = 0, \ u(1/4) = u(3/4) = 0.$$

On the other hand, if $v = y_0(x) - y_0(1/4)$, then

$$Q(x) v'' + m v = 0, \ v(1/4) = v(3/4) = 0$$

and $v > 0$ in $(1/4, 3/4)$. If u vanishes in $(1/4, 3/4)$, then by the Sturm's theorem, $m_1 > m$, what is impossible. If u does not vanish in $(1/4, 3/4)$, then we obtain using Lemma A9 that

$$(m_1 - m) \int_{1/4}^{3/4} \frac{uv}{Q} dx = (u'v - uv')|_{x=1/4}^{x=3/4} = 0$$

and $m_1 = m$.

If $z(x) \equiv 0$, then y_1 is odd and

$$Q(x) y_1''(x) + m_1 y_1 = A(x - 1/2),$$

where A is a constant. Applying Lemma A9, we see that $A = -4 m_1 y_1(1/4)$. Putting $u(x) = y_1(x) - A(x - 1/2)/m_1$, we obtain that

$$Q u'' + m_1 u = 0.$$

Moreover, $u(1/4) = u(1/2) = u(3/4) = 0$. Applying once again the Sturm's theorem, we see that $m_1 > m$, what is impossible. □

Lemma 21. *Let $-1/2 < \alpha < 0$ and $Q(x)$ be the function found in Corollary 19. Let*

$$m_1 = \inf_{y \in H_0^2(0,1)} \frac{\int_0^1 Q(x) y''(x)^2 dx}{\int_0^1 y'(x)^2 dx}.$$

Then $m_1 = m$, where m was indicated in Theorem 15. The minimal value is attained on the function y_0 that is equal to $\int_0^x z(t)dt$.

Proof. Let $\mathcal{H}$ be the space of the functions y which are absolutely continuous in $[0, 1/4-\varepsilon[,\ ]1/4+\varepsilon, 3/4-\varepsilon[,\ ]3/4+\varepsilon, 1]$ for any $\varepsilon > 0$ and such that

$$y \in L_2(0,1),\ \int_0^{1/4} Q(x) y'(x)^2 dx + \int_{1/4}^{3/4} Q(x) y'(x)^2 dx$$

$$+ \int_{3/4}^1 Q(x) y'(x)^2 dx < \infty,\ \int_0^1 y(x) dx = 0,\ y(0) = y(1) = 0.$$

It is easy to see that $\mathcal{H}$ is a Hilbert space.

Let us show that for $y \in \mathcal{H}$ there exists a sequence $x_k \to 1/4$ such that $Qyy'(x_k) \to 0$. If it is not so, then there exists a constant $c > 0$ such that $Qyy'(x) \geq c$ for $1/4 - \varepsilon < x < 1/4$. Then $|yy'(x)| \geq c_1 (1/4 - x)^{-2}$ and therefore, $|y(x)| \geq c_2 (1/4 - x)^{-1}$, $c_2 > 0$. However, it contradicts to the condition that $y \in L_2(0,1)$. One can also find similar sequences converging to $1/4 + 0, 3/4 - 0, 3/4 + 0$.

The norm in $\mathcal{H}$ can be defined as

$$\|y\|_{\mathcal{H}}^2 = \int_0^{1/4} Q(x) y'(x)^2 dx + \int_{1/4}^{3/4} Q(x) y'(x)^2 dx + \int_{3/4}^1 Q(x) y'(x)^2 dx.$$

Indeed, if $\|y\|_{\mathcal{H}} = 0$, then $y'(x) = 0$ and $y = C$ on each of the intervals $(0, 1/4)$, $(1/4, 3/4)$ and $(3/4, 1)$. Since $y(0) = y(1) = 0$ we see that $y(x) = 0$ in $(0, 1/4)$ and $(3/4, 1)$. Since $\int_0^1 y(x) dx = 0$, there exists a $\theta \in (1/4, 3/4)$ such that $y(\theta) = 0$ so that $y(x) = 0$ in $(0, 1)$.

Let us verify that

$$\|y\|_{L_2} \leq C \|y\|_{\mathcal{H}}. \tag{10}$$

Indeed, by the Hardy inequality we have

$$\int_0^{1/4} y(x)^2 dx \leq 4 \int_0^{1/4} (x - 1/4)^2 y'(x)^2 dx \leq \int_0^{1/4} Q(x) y'(x)^2 dx,$$

$$\int_{3/4}^1 y(x)^2 dx \leq 4 \int_{3/4}^1 (x - 3/4)^2 y'(x)^2 dx \leq \int_{3/4}^1 Q(x) y'(x)^2 dx.$$

If y vanishes at a point $\theta \in (1/4, 3/4)$, then the same inequalities are valid in the intervals $(1/4, \theta)$ and $(\theta, 3/4)$. Otherwise,

$$\int_{1/4}^{3/4} |y(x)|dx \leq \int_0^{1/4} |y(x)|dx + \int_{3/4}^{1} |y(x)|dx$$

$$\leq 1/2[(\int_0^{1/4} |y(x)|^2 dx)^{1/2} + \int_{3/4}^{1} |y(x)|^2 dx)^{1/2}]$$

$$\leq \sqrt{C}/2[(\int_0^{1/4} Q(x)|y'(x)|^2 dx)^{1/2} + \int_{3/4}^{1} Q(x)|y'(x)|^2 dx)^{1/2}]$$

and there is a point $\theta \in (1/4, 3/4)$ such that

$$|y(\theta)|^2 \leq 2C(\int_0^{1/4} Q(x)|y'(x)|^2 dx + \int_{3/4}^{1} Q(x)|y'(x)|^2 dx).$$

Since

$$\int_{1/4}^{3/4} |y(x) - y(\theta)|^2 dx \leq C \int_{1/4}^{3/4} Q(x) y'(x)^2 dx,$$

we obtain that

$$\int_{1/4}^{3/4} |y(x)|^2 dx \leq C_1 \int_0^{1/4} Q(x) y'(x)^2 dx$$

and (10) is proved.

Let us verify that the operator A defined in $\mathcal{H} \cap H^2(0,1)$ as $Ay = -(Q(x)y')'$ is closed in $L_2(0,1)$. Let $y_k \to y, Ay_k \to v$ in $L_2(0,1)$. The equation $Au = v$ has a solution $u \in \mathcal{H}$ since $v \in L_2(0,1)$. It follows from the Riesz theorem and (10).

Applying (10) we see that

$$\|y_k - u\|_{L_2(0,1)} \leq C\|y_k - u\|_{\mathcal{H}} = C\|A(y_k - u)\|_{L_2(0,1)} = C\|Ay_k - v\|_{L_2(0,1)}$$

and therefore, $\|y_k - u\|_{L_2(0,1)} \in 0$ so that $u = y$.

On the other hand, if $Au = 0, u \in \mathcal{H}$, then (10) implies that $u = 0$. Therefore, the operator A is self-adjoint.

If $m_1 < m$, then there exists a function $v \in \mathcal{H}$ such that $(Av, v) < m(v, v)$. However, then there exists an eigen-function u such that

$$Au = \lambda u, \ u \in \mathcal{H}, \ \lambda < m.$$

So we have

$$(Q(x)u')' + \lambda u = 0, \ (Q(x)z')' + mz = 0.$$

Since the mean value of u vanishes, there exists $x_0 \in]0,1[$ such that $u(x_0) = 0$. We may assume that $0 < x_0 \leq 1/2$ and $u(x) > 0$ on $]0, x_0[$. Multiplying the first equation by z, the second by y and integrating the difference over $]0, x_0[$, we see that

$$(Q(x)u'(x)z(x) - Q(x)u(x)z'(x))|_{x=0}^{x=x_0} + (\lambda - m)\int_0^{x_0} u(x)z(x)dx = 0.$$

If $x_0 > 1/4$, we have used here the vanishing of the functions $Qu'z$ and Quz' at $x = 1/4$. On the other hand,

$$z(0) = u(0) = u(x_0) = 0, u'(x_0) < 0, z(x_0) \geq 0,$$

so that

$$(\lambda - m)\int_0^{x_0} u(x)z(x)dx \geq 0.$$

Since $u(x)z(x) > 0$ on $]0, x_0[$, we obtain that $\lambda \geq m$ in contradiction to our assumption. Therefore, $m_1 = m$ and the proof is complete. □

Proof of Theorem 15. Let Q be an arbitrary positive function, satisfying (6).

Let at first $\alpha \leq -1$. Then by the Hölder inequality

$$\int_0^1 Q(x)|y''(x)|^2 dx \geq (\int_0^1 |y''(x)|^p dx)^{2/p}(\int_0^1 Q(x)^\alpha dx)^{1/\alpha},$$

where $p = 2\alpha/(\alpha - 1), 2 > p \geq 1$. Therefore, $L[Q, y] \geq G[y']$. Lemma A13 implies the existence of the function $y_0 \in W_{p,0}^2(0,1)$ satisfying the conditions

$$\int_0^1 |y_0''(x)|^p dx = 1, \ G[y_0'] \leq G[y']$$

for all $y \in W_{2,0}^2(0,1)$ such that $\int_0^1 |y''(x)|^p dx = 1$ and the value of m is also indicated in Lemma A13. Therefore, we have $m = m_\alpha = L[Q, y_0]$, if $Q(x) = |y_0''|^{2/(\alpha-1)}$. If $p = 1$, the optimal functions Q and y do not exist, but the value of m was indicated in Corollary 17.

Now let $\alpha > -1/2, \alpha \neq 0$. Then by Lemma A12 there exists a function $y_Q(x)$ such that $Q(x) = |y_0''|^{2/(\alpha-1)}$, so that $L[Q, y_Q] = G[y_Q']$ and therefore,

$M_\alpha \le m$. On the other hand, we have the equality $L[Q_0, y_0] = G[y_0']$, if $Q_0(x) = |y_0''(x)|^{2/(\alpha-1)}$ and y_0 is equal to $\int_0^x z(t)dt$, where z is the function indicated in Corollary 19. If $-1/2 < \alpha < 0$ or $0 < \alpha < 1$, then by Lemmas 20 and 21 we have

$$m = \inf_{y \in H_0^2} \frac{\int_0^1 Q_0(x) y''(x)^2 dx}{\int_0^1 y'(x)^2 dx} = L[Q_0, y_0].$$

The proof is complete. □

7. Appendix. Technical Lemmas

Lemma A1. *Let p be a real number, $p < 2/3$, $p \neq 0$. Then the function*

$$F(x,y) = x^{2/p} y^{3-2/p} + (1-x)^{2/p}(1-y)^{3-2/p}, \ 0 \le x \le 1, \ 0 \le y \le 1$$

has the minimal value $F_{\min} = 1/4$ at $x = y = 1/2$.

Proof. Remark that among two exponents $2/p$ and $3 - 2/p$ one is always positive and another is negative. Let for definiteviness $p < 0$. Then for $y \neq 0, 1$

$$\lim_{x \to +0} F(x,y) = +\infty, \ \lim_{x \to 1-0} F(x,y) = +\infty.$$

If $y = 0$, then $F(x,0) = (1-x)^{2/p} \ge 1$ and for $y = 1$ we have $F(x,1) = x^{2/p} \ge 1$. Therefore, the values of F on the boundary of the square are ≥ 1.

If $x < \delta, y < \delta$, then

$$F(x,y) \ge (1-x)^{2/p}(1-y)^{3-2/p} \ge (1-x)^{2/p} \ge (1-\delta)^{2/p} \text{ for } p > 0,$$

$$F(x,y) \ge (1-x)^{2/p}(1-y)^{3-2/p} \ge (1-y)^{3-2/p} \ge (1-\delta)^{3-2/p} \text{ for } p < 0,$$

and therefore $F(x,y) > 3/4$, if δ is small enough. On the other hand, we have $F(1/2, 1/2) = 1/4$.

The same is true for a small neighbourhood of the points $(0,1)$, $(1,0)$ and $(1,1)$.

Therefore, the function F has an inner minimum point (x_0, y_0). We have at this point

$$\partial F(x_0, y_0)/\partial x = 2/p[x_0^{2/p-1} y_0^{3-2/p} - (1-x_0)^{2/p-1}(1-y_0)^{3-2/p}] = 0;$$

$$\partial F(x_0,y_0)/\partial y = (3-2/p)[x_0^{2/p}y_0^{2-2/p} - (1-x_0)^{2/p}(1-y_0)^{2-2/p}] = 0.$$

Then

$$x_0^{2/p-1}y_0^{3-2/p} = (1-x_0)^{2/p-1}(1-y_0)^{3-2/p},$$
$$x_0^{2/p}y_0^{2-2/p} = (1-x_0)^{2/p}(1-y_0)^{2-2/p}. \quad (11)$$

Dividing term by term these equalities we obtain that

$$y_0/x_0 = (1-x_0)/(1-y_0),$$

i.e. $x_0 = y_0$. Then (1) implies that $x_0^2 = (1-x_0)^2$ and thus $x_0 = 1/2$. □

Lemma A2. *Let p be a negative number. Then for all functions $y \in K_p(0,h)$ the following estimate is valid:*

$$\left(\int_0^h y'(x)^p dx\right)^{1/p} \le \left(\frac{h}{4}\right)^{1/p-3/2}\left(\int_0^h y(x)^2 dx\right)^{1/2}.$$

Proof. Let at first $h = 1$, $\int_0^1 y'(x)^p dx = 1$. Then $\int_0^{1/2} y'(x)^p dx < 1$. Let E be the subset of points x in $[0,1/2]$ such that $y'(x) > 4^{1/p}$ and μ its measure. Then obviously

$$1 > 4(1/2-\mu),$$

i.e. $\mu > 1/4$. Therefore,

$$y(1/2) \ge \int_0^{1/2} y'(x)dx > \int_E y'(x)dx > 4^{1/p-1}.$$

Since y is increasing, we have $y(x) > 4^{1/p-1}$ for $x > 1/2$. Then

$$\int_0^1 y(x)dx > \int_{1/2}^1 y(x)dx > 4^{1/p-3/2}.$$

Therefore,

$$\left(\int_0^1 y'(x)^p dx\right)^{1/p} \le 4^{3/2-1/p}\int_0^1 y(x)dx \le 4^{3/2-1/p}\left(\int_0^1 y^2(x)dx\right)^{1/2}.$$

Let $h \ne 1$ and $y \in K_p(0,h)$. Then the function $z(x) = y(hx) \in K_p(0,h)$ and we can apply the proved estimate to the function z,so that

$$\left(\int_0^1 z'(x)^p dx\right)^{1/p} \le 4^{3/2-1/p}\left(\int_0^1 z^2(x)dx\right)^{1/2}.$$

Thus

$$\left(\int_0^h y'(x)^p dx\right)^{1/p} \le (4/h)^{3/2-1/p}\left(\int_0^h y^2(x)dx\right)^{1/2}$$

and the proof is complete. □

Lemma A3. *Let p be a real number, $0 < p < 2/3$. Then there exists a constant $C = C(p)$ independent of y and h such that*

$$\left(\int_0^h y'(x)^p dx\right)^{1/p} \le C(p)h^{1/p-3/2}\left(\int_0^h y(x)^2 dx\right)^{1/2}, \ y \in K_p(0,h).$$

Proof. Let at first $h = 1$, $\int_0^1 y'(x)^p dx = 1$. Then there is a point $t_1 \in (0,1)$ such that

$$\int_0^{t_1} y'(x)^p dx = \int_{t_1}^1 y'(x)^p dx = 1/2.$$

By the Hölder inequality we have

$$1/2 \le \left(\int_0^{t_1} y'(x)dx\right)^p \cdot t_1^{1-p}.$$

Therefore,

$$1/2 \le y(t_1)^p \cdot t_1^{1-p} \le y(t_1)^p$$

and $y(x) \ge 2^{-1/p}$ for $x \ge t_1$. By the Hölder inequality

$$1/2 = \int_{t_1}^1 y'(x)^p y^{2p-2} y^{2-2p} dx$$

$$\le \left(\int_{t_1}^1 y'(x)y(x)^{2-2/p}dx\right)^p \left(\int_{t_1}^1 y(x)^2 dx\right)^{1-p} \le C_p\left(\int_0^1 y(x)^2 dx\right)^{1-p},$$

where $C_p = p2^{(2-3p)/p}/(2-3p)$. Thus for $p < 2/3$ we have

$$\int_0^1 y(x)^2 dx \ge (2C_p)^{1/(p-1)},$$

that gives the result with $C = (2C_p)^{1/(1-p)}$.

If $\int_0^1 y'(x)^p dx = I \ne 1$, then one can take instead of the function $y(x)$ the function $y(x)I^{-1/p}$. If $h \ne 1$, one can apply the obtained inequality to the function $y(xh)$. □

Remark that the constants in the estimates in Lemmas A2 and A3 are not the best possible. The exact constants are indicated in Lemmas A5 and A6 below.

Lemma A4. *Let* $p < 2/3$, $p \neq 0$, $0 < r \leq 1$. *Let*

$$m_1 = \sup_{h\in(0,r)} \sup_{y\in K_p(0,h,r)} G[y],$$

where

$$G[y] = \frac{(\int_0^r |y'(x)|^p dx)^{2/p}}{\int_0^r y(x)^2 dx}.$$

Then there exists a constant C_1 *independent of* r *such that* $m_1 \leq C_1 r^{2/p-3}$.

Proof. Let at first $r = 1$. By Lemmas A2 and A3 we have for $y \in K_p(0,h,r)$ the inequalities

$$h^{3-2/p}(\int_0^h |y'(x)|^p dx)^{2/p} \leq C \int_0^h y(x)^2 dx,$$

$$(1-h)^{3-2/p}(\int_h^1 |y'(x)|^p dx)^{2/p} \leq C \int_h^1 y(x)^2 dx,$$

where the value of C, corresponding to $h = 1$, was found in Lemmas 1 and 2. Let $\int_0^1 y'(x)^p dx = 1$ and $\int_0^h y'(x)^p dx = a$. Then

$$a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p} \leq C \int_0^1 y(x)^2 dx.$$

By Lemma A1 the function $F(a,h) = a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p}$ defined in the square $0 < a < 1$, $0 < h < 1$ has the minimal value $1/4$ at the point $a = h = 1/2$. Therefore,

$$C \int_0^1 y(x)^2 dx \geq 1/4,$$

i.e. $G[y] \leq 4C$ for all admissible y. In order to obtain the result for an arbitrary r it suffices to substitute the function $y(x)$ by $y(xr)$. □

Lemma A5. *Let* $0 < p < 2/3$ *and* $m = \sup_{y\in K_p(0,h)} G[y]$. *Then*

$$m = \left(\frac{2-2p}{2-3p}\right)^{2/p}\left(\frac{2}{p} - 3\right)h^{2/p-3}\left(\int_0^\infty \frac{dt}{(1+t^2)^{1/p}}\right)^2$$

and there exists a function $y_0(x) \in K_p(0,h)$ *such that* $G[y_0] = m$. *Besides, as* $x \to h$ *we have*

$$y_0(x) = c_1(h-x)^{p/(p-2)}[1+o(1)], \; y_0'(x) = c_2(h-x)^{2/(p-2)}[1+o(1)].$$

Proof. Let at first $h = 1$ and $\{y_k\}$ be such a sequence of functions of $K_p(0,1)$ that $\int_0^1 y_k'(x)^p dx = 1$ and $G[y_k] \to m$.

Let us show that we can assume all functions y_k be smooth. Let $y \in K_p(0,1)$. Let us define y on the whole line putting $y_1(x) = 0$ for $x < 0$, $y_1(x) = y(x)$ for $0 < x < 1-\varepsilon$ and $y_1(x) = y(1-\varepsilon)$ for $x > 1-\varepsilon$. Obviously $\int_0^1 |y_1(x) - y(x)|^2 dx \to 0$ as $\varepsilon \to 0$ and $\int_{1-\varepsilon}^1 |y'(x)|^p dx \to 0$, as $\varepsilon \to 0$. It allows to assume that y_k are bounded functions.

Let now $y \in K_p(0,1)$ and $0 \le y(x) \le C$. Then $\int_0^1 |y'(x)| dx = y(h) \le C$. Put $z_k(x) = y_k(x) - y_k(0)$, where y_k is the averaging of y with a positive kernel such that

$$y_k(x) = k \int K(k(x-t))y(t)dt, \int K(t)dt = 1, \; K(t) \ge 0, \; K \in C_0^\infty(-1,1).$$

Remark that $|y_k(0)| \le \delta_k \to 0$ as $k \to \infty$, so that z_k converge to $y(x)$ uniformly, $z_k'(x) \ge 0$ and

$$\int_0^1 |z_k(x) - y(x)|^2 dx \to 0, \quad \int_0^1 |z_k'(x) - y'(x)| dx \to 0.$$

We have

$$\int_0^1 |z_k'(x) - y'(x)|^p dx \le (\int_0^1 |z_k'(x) - y'(x)| dx)^p.$$

The elementary inequality

$$|a^p - b^p| \le |a-b|^p,$$

valid for all a, b such that $a \ge 0, b \ge 0$, implies that $\int_0^1 z_k'(x)^p dx \to \int_0^1 y'(x)^p dx$. This allows us to assume that all y_k are smooth functions.

We will call a function y *convex*, if its derivative is decreasing, and *concave*, if its derivative is increasing.

Let us show that if y_k is convex in an interval (x_1, x_2), where $0 \le x_1 \le x_2 \le 1$, then it is possible to substitute it by the linear function

$$z(x) = y_k(x_1) + \gamma(x - x_1), \text{ where } \gamma = [y_k(x_2) - y_k(x_1)]/(x_2 - x_1)$$

and the value of $G[z]$ is bigger than $G[y_k]$.
Indeed we have $y_k(x_1) = z(x_1), \; y_k(x_2) = z(x_2)$ and

$$\int_{x_1}^{x_2} y_k(x)^2 dx \ge \int_{x_1}^{x_2} z(x)^2 dx;$$

$$\int_{x_1}^{x_2} y_k'(x)^p dx \le (\int_{x_1}^{x_2} y_k'(x)dx)^p(\int_{x_1}^{x_2} dx)^{1-p}$$
$$= \gamma^p(x_2 - x_1) = \int_{x_1}^{x_2} z'(x)^p dx.$$

If z coincides with y_k outside the interval (x_1, x_2), then $G[z] \ge G[y_k]$.

We can do the same for all other intervals where y_k is convex. As the result we shall obtain the function z_k that can be described in the following way. Consider the set of points (x, y) such that $0 < x < 1$ and $y > y_k(x)$, and take the convex hull of this set. The lower boundary of the hull serves as the graph of the function z_k. Let us show that $z_k \in C^1(0,1)$. Indeed, if a point x belongs to an interval of the straight line, then it is obvious that z_k is smooth at this point. The same is true in the case when the point $(x, z_k(x))$ belongs to a part of the graph of the function y_k. If z_k is linear on one side of x and coincides with y_k on the other side, then z_k is regular at x since its graph is lying on one side of the straight line, obtaining by the continuation of the linear function. At last, if the point $(x, z_k(x))$ is a limit point for a sequence of such points, then it is also a limit point for a sequence of points belonging to the graph of y_k and the derivative $z_k'(x)$ exists.

Therefore, if one changes every function y_k by a concave function z_k in the indicated way, then the sequence of new functions will be maximizing. It allows us to consider as maximizing the sequences of increasing concave functions, i.e. to suppose that the functions $y_k(x)$ and their derivatives $y_k'(x)$ are increasing.

For large k we have

$$\int_0^1 y_k(x)^2 dx \le m^{-1} + 1.$$

Let $\varepsilon > 0$ be small enough. There is a point $\theta_k \in (1-\varepsilon, 1)$ such that $|y_k(\theta_k)|^2 \le (m^{-1}+1)\varepsilon^{-1}$. Since the functions y_k are monotone, they are uniformly bounded for $0 \le x \le 1 - \varepsilon$.

Analogously, we can deduce from the equality $\int_0^1 y_k'(x)^p dx = 1$ that the sequence $\{y_k'(x)\}$ is uniformly bounded in $[0, 1-\varepsilon]$. By the Arzela theorem one can choose the uniformly converging subsequence $\{y_{n_k}(x)\}$, and by Helly theorem one can suppose that the subsequence $\{y_{n_k}'(x)\}$ converges everywhere in $[0, 1-\varepsilon]$. Using the diagonalization, one can find a subsequence converging to a function $y_0 \in K_p(0, h)$ such that the sequence of the first derivatives converges almost everywhere in $[0, 1)$ and y_0 satisfies the Lipschitz condition on the interval $[0, 1-\varepsilon]$ for any $\varepsilon > 0$. Using Lemma A3 one can conclude that

$$(\int_{1-\varepsilon}^{1} y_k'(x)^p dx)^{1/p} \leq C\varepsilon^{1/p-3/2},$$

where C is independent of k and since $1/p - 3/2 > 0$, we obtain that

$$\int_0^1 y_0'(x)^p dx = 1.$$

Besides, we have

$$\int_0^1 y_0(x)^2 dx \leq m^{-1}.$$

Since this integral cannot be less than m^{-1}, we see that $\int_0^1 y_0(x)^2 dx = m^{-1}$.

If $y_0'(x_0) > 0$, then $y_0'(x) > 0, y_0(x) > 0$ for all $x > x_0$. Let us assume at first that $y_0'(x) > 0$ for $x > 0$. Then one can consider the values of $G[y_0 + tz]$ for any $z \in H_0^1(0,1)$. These values are minimal for $t = 0$ and hence

$$\frac{d}{dt} G[y_0 + tz] = 0, \text{ if } t = 0.$$

It gives the Euler-Lagrange equation of the form

$$(y_0'^{p-1})' + my_0 = 0,$$

so that

$$(p-1)y_0'^{p-1}y_0'' + my_0y_0' = 0$$

or

$$y_0'^p - m_1 y_0^2 = C,$$

where $m_1 = -mp/[2(p-1)] > 0$. Integrating this equality over $(0,1)$, we obtain that

$$C = 1 - m_1 m^{-1} = 1 + p/2(p-1) > 0.$$

Therefore,

$$\int_0^y \frac{dz}{(C + m_1 z^2)^{1/p}} = x.$$

We have for any $\varphi \in K_p(0,1)$ the equality

$$\int_0^1 ((y_0')^{p-1}(x)\varphi'(x) - my_0(x)\varphi(x))dx = 0.$$

If $\varphi(x) = 1$ for $1-\varepsilon < x < 1$, then the integrating by parts gives the equality

$$(y_0')^{p-1}(1-\varepsilon)\varphi(1-\varepsilon) - m\int_0^1 y_0(x)\varphi(x)dx = 0.$$

Tending ε to zero, we obtain that $y_0(1) = \infty$. The equality $y_0'^p - m_1 y_0^2 = C$ yields that $y_0'(1) = \infty$, too.

Therefore

$$\int_0^\infty \frac{dz}{(C+m_1z^2)^{1/p}} = 1.$$

Let $z = (C/m_1)^{1/2}t$ so that

$$\int_0^\infty \frac{dt}{(1+t^2)^{1/p}} = C^{1/p-1/2}m_1^{1/2} = \frac{(2-3p)^{1/p-1/2}p^{1/2}m^{1/2}}{(2-2p)^{1/p}}$$

and therefore

$$m^{1/2} = \frac{(2-2p)^{1/p}}{p^{1/2}(2-3p)^{1/p-1/2}}\int_0^\infty \frac{dt}{(1+t^2)^{1/p}}.$$

If $y_0(x) \equiv 0$ for $0 < x < x_0$ and $y_0'(x) > 0$ for $x > x_0$, then $m = C_1(1-x_0)^{1/p-3/2}$, where C_1 does not depend on x_0 and therefore the optimal value of x_0 is equal to 0. Remark that

$$\int_{y_0}^\infty \frac{dz}{(C+m_1z^2)^{1/p}} = 1-x.$$

Therefore for big values of y_0 we have

$$y_0^{1-2/p}(1+o(1)) = C_1(1-x),$$

that implies that $y_0(x) = A(1-x)^\gamma[1+o(1)]$ with $\gamma = p/(p-2) < 0$.

In order to find the value m for an arbitrary $h > 0$ it suffices to substutute in the obtained estimate the function $y_0(xh)$. □

Lemma A6. *Let $p < 0$ and $m = \sup_{y\in K_p(0,h)} G[y]$. Then*

$$m = \frac{1}{4}\Big(\frac{2-2p}{2-3p}\Big)^{2/p}(3-\frac{2}{p})h^{2/p-3}B(1/2, 1-1/p)^2$$

and there exists a function $y_0 \in K_p(0,h)$ *such that* $G[y_0] = m$. *Moreover, as* $x \to h$ *we have*

$$y_0(x) = y_0(h) + c_1(h-x)^{p/(p-1)}[1+o(1)], \; y_0'(x) = c_2(h-x)^{1/(p-1)}[1+o(1)].$$

Proof. Let at first $h = 1$ and $\{y_k\}$ be such a sequence of functions in K that $\int_0^1 y_k'(x)^p dx = 1$ and $G[y_k] \to m$. In virtue of Lemma A2 the value of m is finite and positive.

Let us show that we can assume all functions y_k be smooth. Let $y \in K_p(0,1)$. Let us define y on the whole line putting $y_1(x) = 0$ for $x < 0$, $y_1(x) = y(x)$ for $0 < x < 1-\varepsilon$ and $y_1(x) = y(1-\varepsilon) + \varepsilon^{-1/2p}(x-1+\varepsilon)$ for $x > 1-\varepsilon$, and put $u_k(x) = y_1(x) + \varepsilon_k x$, where $\varepsilon_k \to 0$ as $k \to \infty$. Obviously $\int_0^1 |u_k(x) - y(x)|^2 dx \to 0$ and by Lebesgue theorem, $\int_0^1 |u_k'(x)|^p dx \to \int_0^1 |y'(x)|^p dx$. So we can assume that $y_k'(x) > \varepsilon_k > 0$.

Let now $y \in K_p(0,1)$ and $y'(x) > \varepsilon > 0$. Put $z_k(x) = y_k(x) - y_h(0)$, where y_k is the averaging of y with a positive kernel so that

$$y_k(x) = k\int K(k(x-t))y(t)dt, \; \int K(t)dt = 1, \; K(t) \geq 0, \; K \in C_0^\infty(-1,1).$$

Remark that $|y_k(0)| \leq \delta_k \to 0$ as $k \to \infty$ so that z_k converge to $y(x)$ uniformly and

$$\int_0^1 |z_k(x) - y(x)|^2 dx \to 0, \quad \int_0^1 |z_k'(x) - y'(x)| dx \to 0.$$

Therefore, a subsequence z_{n_k}' converges to $y'(x)$ almost everywhere and $z_{n_k}'(x)^p$ converges to $y'(x)^p$ almost everywhere. Since $0 \leq y(x) \leq C$ for $0 \leq x \leq 1$ and $y'(x) > 0$, we see that $\int_0^1 |y'(x)| dx \leq C$. Since $|z_{n_k}'|^p \leq \varepsilon^p$, we have by the Lebesgue theorem $\int_0^1 z_{n_k}'(x)^p dx \to \int_0^1 y'(x)^p dx$. This allows us to assume that all y_k are smooth functions.

If y_k is convex in an interval (x_1, x_2), where $0 \leq x_1 \leq x_2 \leq 1$, then it is possible to substitute it by the linear function

$$z(x) = y_k(x_1) + \gamma(x - x_1), \text{ where } \gamma = [y_k(x_2) - y_k(x_1)]/(x_2 - x_1)$$

so that the value of the functional G will increase.

Then $y_k(x_1) = z(x_1),\ y_k(x_2) = z(x_2)$ and

$$\int_{x_1}^{x_2} y_k(x)^2 dx \geq \int_{x_1}^{x_2} z(x)^2 dx.$$

On the other hand, by the Hölder inequality

$$x_2 - x_1 = \int_{x_1}^{x_2} y_k'(x)^{p/(p-1)} y_k'(x)^{p/(1-p)} dx$$
$$\leq \Big(\int_{x_1}^{x_2} y_k'(x)dx\Big)^{p/(p-1)} \Big(\int_{x_1}^{x_2} y_k'(x)^p dx\Big)^{1/(1-p)}$$

and therefore,

$$\int_{x_1}^{x_2} y_k'(x)^p dx \geq \Big(\int_{x_1}^{x_2} y_k'(x)dx\Big)^p (x_2 - x_1)^{1-p}$$
$$= [y_k(x_2) - y_k(x_1)]^p (x_2 - x_1)^{1-p}.$$

Since

$$\int_{x_1}^{x_2} z'(x)^p dx = \gamma^p (x_2 - x_1) = [y_k(x_2) - y_k(x_1)]^p (x_2 - x_1)^{1-p},$$

we see that $G[z] \geq G[y_k]$, if z coincides with y_k outside the interval (x_1, x_2).

We can do the same for all other intervals where y_k is convex. As the result we shall obtain the function z_k that can be described in the following way. Consider the set of points (x, y) such that $0 < x < 1$ and $y > y_k(x)$, and take the convex hull of this set. The lower boundary of the hull serves as the graph of the function z_k. Let us show that $z_k \in C^1(0,1)$. Indeed, if a point x belongs to an interval of the straight line, then it is obvious that z_k is smooth at this point. The same is true in the case when the point $(x, z_k(x))$ belongs to a part of the graph of the function y_k. If z_k is linear on one side of x and coincides with y_k on the other side, then z_k is regular at x since its graph is lying on one side of the straight line, obtaining by the continuation of the linear function. At last, if the point $(x, z_k(x))$ is a limit point for a sequence of such points, then it is also a limit point for a sequence of points belonging to the graph of y_k and the derivative $z_k'(x)$ exists.

Therefore, if one changes every function y_k by a concave function z_k in the indicated way, then the sequence of these new functions z_k will be maximizing, too. It allows us to consider the maximizing sequence as a sequence of monotone concave functions, i.e. to suppose that the functions $y_k(x)$ and their derivatives $y_k'(x)$ are increasing.

For large k we have

$$\int_0^1 y_k(x)^2 dx \le m^{-1} + 1.$$

Let $\varepsilon > 0$ be small enough. There is a point $\theta_k \in (1-\varepsilon, 1)$ such that $|y_k(\theta_k)|^2 \le (m^{-1}+1)\varepsilon^{-1}$. Since the functions y_k are monotone, they are uniformly bounded for $0 \le x \le 1-\varepsilon$. It yields

$$\int_{1-2\varepsilon}^{1-\varepsilon} y_k'(x)dx \le C$$

with a constant C independent of k. Therefore, there exists a $\tilde{\theta}_k$ such that $1-2\varepsilon < \tilde{\theta}_k < 1-\varepsilon$ and $y_k'(\theta_k) \le C/\varepsilon$. Therefore the sequence $\{y_k'(x)\}$ is uniformly bounded in $[0, 1-2\varepsilon]$. Since $\int_0^1 y_k'(x)^p dx = 1$ there exists a $\theta_k' \in (0, \varepsilon)$ such that $|y_k'(x)^p(\theta_k')| \le 1/\varepsilon$ and therefore $|y_k'(x)^p| \le 1/\varepsilon$ for $\varepsilon \le x \le 1-2\varepsilon$.

By the Arzela theorem one can choose the uniformly in $[\varepsilon, 1-2\varepsilon]$ converging subsequence $\{y_{n_k}(x)\}$, and by Helly theorem one can suppose that the sequence $\{y_{n_k}'(x)\}$ converges almost everywhere in $[\varepsilon, 1-2\varepsilon]$. Using the diagonalization, one can find a subsequence converging in $(0,1)$ to a function $y_0 \in K_p(0,1)$ such that

$$y_{n_k} \to y_0, \; y_{n_k}'^p \to y_0'^p$$

everywhere in $(0,1)$. By the Fatou theorem we have

$$\int_0^1 y_0(x)^2 dx \le m^{-1}, \; \int_0^1 y_0'(x)^p dx \le 1.$$

Therefore,

$$\left(\int_0^1 y_0'(x)^p dx\right)^{2/p} \ge 1,$$

so that $G[y_0] \ge m$. However, m is the maximal possible value of G, so that $\int_0^1 y_0(x)^2 dx = m^{-1}$ and $\int_0^1 y_0'(x)^p dx = 1$.

The Euler-Lagrange equation has the form

$$(y_0'^{p-1})' + my_0 = 0, \; y_0(0) = 0$$

so that

$$(p-1)y_0'^{p-1}y_0'' + my_0y_0' = 0$$

or

$$y_0'^p + m_1 y_0^2 = C,$$

where $m_1 = mp/[2(p-1)] > 0$. Integrating this equality over $(0,1)$, we obtain that

$$C = 1 + m_1 m^{-1} = 1 + p/2(p-1) > 0.$$

Therefore,

$$\int_0^{y_0} \frac{dz}{(C - m_1 z^2)^{1/p}} = x.$$

Repeating the same arguments as in the proof of the preceding Lemma, we obtain that $y_0'(1) = \infty$. Let $M = \max y_0(x) = y_0(1)$. Then $M^2 m_1 = C$, i.e.

$$M = \Big(\frac{C}{m_1}\Big)^{1/2} = \Big(\frac{3p-2}{mp}\Big)^{1/2}$$

and

$$\int_0^{M} \frac{dz}{(C - m_1 z^2)^{1/p}} = 1.$$

Let $z = (C/m_1)^{1/2} t$ so that

$$\int_0^1 \frac{dt}{(1-t^2)^{1/p}} = C^{1/p-1/2} m_1^{1/2} = \frac{(2-3p)^{1/p-1/2}(-pm)^{1/2}}{(2-2p)^{1/p}},$$

and therefore

$$m^{1/2} = \frac{(2-2p)^{1/p}}{2(2-3p)^{1/p-1/2}(-p)^{1/2}} B\Big(\frac{1}{2}, 1 - \frac{1}{p}\Big).$$

Since

$$\int_{y_0}^{M} \frac{dz}{(C - m_1 z^2)^{1/p}} = 1 - x,$$

we can see that as $x \to 1$ that $y_0(x) = M + c_1(x-1)^{\gamma}[1 + o(1)]$, where $\gamma = p/(p-1)$.

In order to find the value m for any $h > 0$ it suffices to substutute in the obtained estimate the function $y_0(xh)$. □

Lemma A7. *Let $p < 2/3$, $p \neq 0$, $0 < r \leq 1$ and $m_1 = \sup_{y \in K_p(0,h,r)} G[y]$. Then there is a function $y_0 \in K_p(0,h,r)$ such that $y_0(x) = y_0(r-x)$, $G[y_0] =$*

m_1 and $m_1 = 4mr^{2/p-3}$, where the value of m was indicated in Lemmas A4 and A6 for $h = 1$.

Proof. Let at first $r = 1$. The existence of the extremal function for any fixed $h \in [0, 1]$ follows from Lemmas A4 and A6. Furthermore, we can suppose that $y(x)$ is monotone and concave in $[0, h]$ and in $[h, 1]$.

By Lemmas A4 and A6 we have for $y \in K_p(0, h, 1)$ the inequalities

$$h^{3-2/p}(\int_0^h |y'(x)|^p dx)^{2/p} \leq m \int_0^h y(x)^2 dx,$$

$$(1-h)^{3-2/p}(\int_h^1 |y'(x)|^p dx)^{2/p} \leq m \int_h^1 y(x)^2 dx,$$

where the value of m, corresponding to $h = 1$, was found in Lemmas A4 and A6. Let $\int_0^1 |y'(x)|^p dx = 1$ and $\int_0^h |y'(x)|^p dx = a$. Then

$$a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p} \leq m \int_0^1 y(x)^2 dx.$$

By Lemma A1 the function $F(a, h) = a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p}$ defined in the square $0 < a < 1$, $0 < h < 1$ has the minimal value $1/4$ at the point $a = h = 1/2$. Therefore,

$$\int_0^1 y(x)^2 dx \geq 1/4m,$$

i.e. $G[y] \leq 4m$ for all admissible y. On the other hand, if $a = h = 1/2$ and if the function y coincides on $(0, 1/2)$ with the function y_0, found in Lemmas 31 and A6 for $h = 1/2$, and is odd with respect to the point $x = 1/2$, then $G[y] = 4m$.

In order to obtain the result for an arbitrary r it suffices to substitute the function $y(x)$ by $y(xr)$. □

Lemma A8. *Let $p(x)$ be a smooth positive function on $[0, d[$, such that*

$$\lim_{x \to d} p(x)(x-d)^{-\gamma} = a, \; p'(x) = O((x-d)^{\gamma-1}),$$

where $1 < \gamma \leq 2$. Let $y(x)$ be a solution of the equation

$$(p(x)y')' + my(x) = 0, \; 0 < x < d,$$

such that

$$\int_0^d p(x)y'(x)^2dx < \infty.$$

Then

$$\lim_{x\to d} p(x)y(x)y'(x) = 0.$$

Moreover, we have as $x \to d-$

$$y(x) = 1 + o(1)),$$

$$y(x) = (d-x)^{1-\gamma}(C + o(1)),$$

if $1 < \gamma < 2$ *and*

$$y(x) = (d-x)^{\rho}(C + o(1)),$$

$$y'(x) = (d-x)^{\rho-1}(\rho C + o(1)),$$

if $\gamma = 2$ *with* $\rho > -1/2$.

Proof. Put $I[y] = \int_0^d p(x)y'(x)^2dx$.

If $1 < \gamma < 2$, then by solving the Cauchy problem we can find two linearly independent solutions $y(x)$ and $z(x)$ such that as $x \to d-$

$$y(x) = 1 + c_1(d-x)^{2-\gamma} + \dots,\ c_1 = \frac{m}{a(2-\gamma)},$$

$$z(x) = (d-x)^{1-\gamma} + c_2(d-x)^{3-2\gamma} + \dots,\ c_2 = \frac{m}{a(3-2\gamma)(2-\gamma)}.$$

However $z'(x) = (d-x)^{1-\gamma}(m/a + o(1))$ and the integral $I[z] = \int_0^d Q(x)z'(x)^2dx$ is divergent. So the solution with a finite value of I is proportional to y and pyy' vanishes at $x = d$.

If $\gamma = 2$, the corresponding solutions have the form

$$y(x) = (d-x)^{\kappa_1}(1 + o(1)),\ z(x) = (d-x)^{\kappa_2}(1 + o(1)),$$

where κ_j are the different roots of the characteristic equation

$$a\kappa(\kappa + 1) + m = 0$$

so that

$$\kappa_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m}{a}}.$$

If $4m > a$, then the both roots are complex and $Re\ \kappa_{1,2} = -\frac{1}{2}$. The integrals $I[y]$ and $I[z]$ are divergent. If $4m < a$, then the both roots are real, the integral $I[y]$ is finite but $I[z]$ is divergent. We have $pyy'(d) = 0$.

At last, if $\kappa_1 = \kappa_2$, i.e. $a = 4m$, the solutions have the form

$$y(x) = (d-x)^{-1/2}[1+o(1)],\ z(x) = (d-x)^{-1/2}ln(d-x)[1+o(1)],$$

so that the both integrals $I[y]$ and $I[z]$ are divergent.

Therefore, if $\gamma = 2$ and $I[y]$ is finite then $4m < a$ and the function pyy' vanishes at $x = d$. □.

Lemma A9. *Let $Q(x)$ be a smooth positive function on $[0,d[$, such that*

$$\lim_{x\to d} Q(x)(d-x)^{-\gamma} = a \neq 0,$$

where $1 < \gamma \leq 2$. Let $y(x)$ be a solution of the equation

$$Q(x)y''(x) + my(x) = 0,\ 0 < x < d,$$

such that

$$\int_0^d Q(x)y''(x)^2 dx < \infty.$$

Then

$$\lim_{x\to d} y(x) = 0,\ \lim_{x\to d} y(x)y'(x) = 0.$$

Moreover, we have as $x \to d-$

$$y(x) = C[d - x + c_1(d-x)^{3-\gamma}(1+o(1))],$$

$$y'(x) = C[-1 + c_1(3-\gamma)(d-x)^{2-\gamma}(1+o(1))],$$
$$y''(x) = Cc_1(3-\gamma)(2-\gamma)(d-x)^{1-\gamma}(1+o(1))$$

if $1 < \gamma < 2$ and

$$y(x) = (d-x)^{\rho}(C+o(1))],$$
$$y'(x) = (d-x)^{\rho-1}(\rho C + o(1)),$$
$$y''(x) = (d-x)^{\rho-2}(\rho(\rho-1)C + o(1)),$$

if $\gamma = 2$ with $\rho > 1/2$.

Proof. Put $I[y] = \int_0^d Q(x)y''(x)^2 dx$.

If $1 < \gamma < 2$, then by solving the Cauchy problem we can find two linearly independent solutions $y(x)$ and $z(x)$ such that as $x \to d-$

$$y(x) = d - x + c_1(d-x)^{3-\gamma} + \dots, \; c_1 = \frac{m}{a(3-\gamma)(2-\gamma)};$$

$$z(x) = 1 + c_2(d-x)^{2-\gamma} + \dots, \; c_2 = \frac{m}{a(2-\gamma)(1-\gamma)}.$$

However $z''(x) = (d-x)^{-\gamma}(m/a+o(1))$ and the integral $I[z] = \int_0^d Q(x)z''(x)^2dx$ is divergent. So the solution with a finite value of I is proportional to y and vanishes at $x = d$.

If $\gamma = 2$, the corresponding solutions have the form

$$y(x) = (d-x)^{\kappa_1}(1+o(1)), \; z(x) = (d-x)^{\kappa_2}(1+o(1)),$$

where κ_j are the different roots of the characteristic equation

$$a\kappa(\kappa-1) + m = 0$$

so that

$$\kappa_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m}{a}}.$$

If $4m > a$, then the both roots are complex and $Re\,\kappa_{1,2} = \frac{1}{2}$. The integrals $I[y]$ and $I[z]$ are divergent. If $4m < a$, then the both roots are real, the integral $I[y]$ is finite but $I[z]$ is divergent. We have $y(d) = 0$.

At last, if $\kappa_1 = \kappa_2$, i.e. $a = 4m$, the solutions have the form

$$y(x) = (d-x)^{1/2}[1+o(1)], \; z(x) = (d-x)^{1/2}ln(d-x)[1+o(1)],$$

so that the both integrals $I[y]$ and $I[z]$ are divergent.

Therefore, if $\gamma = 2$ and $I[y]$ is finite then $4m < a$ and the solution vanishes at $x = d$. □.

Lemma A10. *If $y \in H_0^1(0,1)$, then*

$$\max_{x\in(0,1)} y(x)^2 \le \frac{1}{4}\int_0^1 y'(x)^2dx.$$

The equality is attained on the function $y_0(x) = 1/2 - |x - 1/2|$.

Proof. Without the loss of generality we can assume that $y(x) \geq 0$ for $0 < x < 1$. Let $M \equiv \max y(x)^2 = y(b)^2$. Then by the Hölder inequality

$$y(b)^2 \leq b \int_0^b y'(x)^2 dx, \; y(b)^2 \leq (1-b) \int_b^1 y'(x)^2 dx,$$

hence

$$(\frac{1}{b} + \frac{1}{1-b}) y(b)^2 \leq (1-b) \int_0^1 y'(x)^2 dx.$$

Since $1/b + 1/(1-b) \geq 4$ for $0 < b < 1$, the proof is complete. □

Lemma A11. *Let $x_1, \ldots, x_k$ be positive numbers and $x_1 + \ldots + x_k = 1$. Then if $0 \leq \gamma \leq 1$ the inequality*

$$x_1^\gamma + \ldots + x_k^\gamma \leq k^{1-\gamma}$$

holds.

If $\gamma \geq 1$ or $\gamma \leq 0$, then

$$x_1^\gamma + \ldots + x_k^\gamma \geq k^{1-\gamma},$$

i.e. the extremum of the function $x_1^\gamma + \ldots + x_k^\gamma$ is attained at the point $x_1 = \ldots = x_k = 1/k$.

Proof. The proof is rather elementary and we leave it to the reader. □

Lemma A12. *Let a function f be summable on $(0,1)$, $f(x) \geq 0$. Then there are two points a and b such that $0 < a < b < 1$ and a function $y \in C^1[0,1]$ such that $y'(x)$ is absolutely continuous,*

$$y(0) = y(1) = y'(0) = y'(1) = 0,$$

$y''(x) = f(x)$ if $0 < x < a$ or $b < x < 1$ and $y''(x) = -f(x)$ if $a < x < b$.

Proof. Let at first $f(x) > 0$. Put

$$F(x) = \int_0^x f(t)dt, \; G(x) = \int_0^x F(t)dt$$

and

$$y'(x) = \begin{cases} F(x), & \text{if } 0 < x < a, \\ 2F(a) - F(x), & \text{if } a < x < b, \\ F(x) - F(1), & \text{if } b < x < 1, \end{cases}$$

the values of a and b will be indicated in what follows. This function is continuous at the point $x = a$ and at the point $x = b$, if

$$2F(a) + F(1) = 2F(b). \tag{12}$$

Let

$$y(x) = \begin{cases} G(x), & \text{if } 0 < x < a, \\ 2F(a)(x-a) - G(x) + 2G(a), & \text{if } a < x < b, \\ G(x) - G(1) - (x-1)F(1), & \text{if } b < x < 1. \end{cases}$$

This function is continuous at the points $x = a$ and $x = b$, if

$$2F(a)(b-a) - G(b) + 2G(a) = G(b) - G(1) - (b-1)F(1). \tag{13}$$

Let

$$H(x) = xF(x) - G(x).$$

Then

$$H'(x) = xf(x) \geq 0, \ H(0) = 0,$$

and the conditions (11), (12) imply that

$$2H(a) + H(1) = 2H(b).$$

The points a and b can be found in the following way. Put

$$K(x) = F(x) - F(1)H(x)/H(1) = H(x)[\frac{F(x)}{H(x)} - \frac{F(1)}{H(1)}].$$

Since

$$(F(x)/H(x))' = -F'(x)G(x)/H(x)^2,$$

we see that $K(x) \geq 0, \ K(0) = 0, \ K(1) = 0$ and

$$K(a) - K(b) = -1/2K(1) = 0.$$

Let the function $\rho(t)$ be defined by the equality

$$F(\rho(t)) - 1/2F(1) = F(t).$$

Then $\rho(0) = \xi$, where ξ is such a point that $F(\xi) = F(1)/2, \ 0 < \xi < 1$ and $\rho(\xi) = 1$. The function

$$S(t) = K(t) - K(\rho(t))$$

is such that $S(0) = K(0) - K(\xi) \le 0$ and $S(\xi) = K(\xi) - K(1) \ge 0$. Therefore, there is a point $a \in (0,\xi)$ such that $S(a) = 0$. If we put $b = \rho(a)$, then we obtain (4) and the equality $K(a) = K(b)$ will be satisfied also.

If $f(x) \ge 0$, then we can construct the function Y_ε corresponding to the function $f(x) + \varepsilon$ and pass to limit, what is easy. □

Lemma A13. *Let $p > 1$ and K be the class of functions y of the space $W^1_{p,0}(0,1)$ such that $\int_0^1 y(x)dx = 0$. Let*

$$m = \inf_{y \in K} G[y],$$

where

$$G[y] = \frac{(\int_0^1 |y'(x)|^p dx)^{2/p}}{\int_0^1 y(x)^2 dx}.$$

Then

$$m = 4\Big(\frac{2p-2}{3p-2}\Big)^{2/p}\Big(3 - \frac{2}{p}\Big)B\Big(\frac{1}{2}, 1 - \frac{1}{p}\Big)^2,$$

where B is the Euler function, and there exists a function $y_0 \in K$ such that $G[y_0] = m$.

Proof. It is evident that the number m is finite and is not greater than, for example, $G[y_1]$, where $y_1(x) = 1/4 - |x - 1/4|$ for $x \in (0, 1/2)$ and $y_1(x) = |x - 3/4| - 1/4$ for $x \in (1/2, 1)$.

Let $\{y_k\}$ be a minimizing sequence such that

$$\int_0^1 y_k(x)^2 dx = 1, \int_0^1 y_k(x)dx = 0 \text{ and } \int_0^1 |y_k'(x)|^p dx \to m^{p/2}.$$

This sequence is compact in $L_2(0,1)$ and weakly compact in $W^1_{p,0}(0,1)$ so that there is a subsequence $\{y_{n_k}\}$ converging in $L_2(0,1)$ to $y_0(x)$ and

$$\int_0^1 y_0(x)^2 dx = 1, \ \int_0^1 |y_0'(x)|^p dx \le m^{p/2}, \ \int_0^1 y_0(x)dx = 0.$$

Since $m^{p/2}$ is the minimal value of integrals $\int_0^1 |y'(x)|^p dx$, we have in fact the equality: $\int_0^1 |y_0'(x)|^p dx = m^{p/2}$, and the function y_0 is extremal. Since the integral of y_0 vanishes, this function has at least one zero in $(0,1)$. We can reconstruct the function y_0 without changing the values of the integrals $\int_0^1 y_0(x)^2 dx$, $\int_0^1 y_0'(x)^2 dx$ and $\int_0^1 y_0(x)dx$ in such a way that it will be positive on $(0, x_0)$ and negative on $(x_0, 1)$, where $0 < x_0 < 1$. To do it we are shifting all intervals on which $y(x) > 0$ to left not changing the values of y on them.

The functional $G[y]$ is differentiable since $p > 1$ and therefore the function y_0 satisfies the Euler-Lagrange equation

$$(|y_0'|^{p-2}y_0')' + m^{p/2}y_0 + m_1 = 0, \ y_0(0) = y_0(1) = 0,$$

which implies that

$$(p-1)|y_0'|^{p-1}y_0'' + m^{p/2}y_0y_0' + m_1y_0' = 0.$$

Therefore,

$$(p-1)|y_0'|^p/p + m^{p/2}y_0^2/2 + m_1y_0 = C.$$

Integrating this equality over $(0, 1)$, we obtain that $C = m^{p/2}[(p-1)/p + 1/2]$.

Moreover,

$$y_0'(0) = -y_0'(x_0) = y_0'(1) = (Cp/(p-1))^{1/p}$$

and $y_0(0) = y_0(x_0) = y_0(1) = 0$. Therefore, $y_0(x) = y_0(x_0 - x)$ for $0 \le x \le x_0$ and $y_0(x) = y_0(1 + x_0 - x)$. Now put

$$z(x) = m^{p/2}y_0 + m_1.$$

Then

$$(|z'|^{p-2}z')' + m^{p(p-1)/2}z = 0.$$

All solutions of this equation are oscillating periodic functions, with the distance between zeroes, equal to the half of the period, and odd with respect to each its zero. Therefore, its mean value in the period is equal to zero. Since

$$z(0) = z(1), \ z'(0) = z'(1),$$

the mean value of z on $(0, 1)$ is equal to zero. However, then we have

$$m_1 = \int_0^1 z(x)dx - m^{p/2}\int_0^1 y_0(x)dx = 0.$$

Therefore, $x_0 = 1/2$ and $y_0(x) = -y_0(1-x)$. The maximal value M of y_0 is defined from the equation

$$(p-1)|y_0'|^p/p + m^{p/2}y_0^2/2 = m^{p/2}[(p-1)/p + 1/2], \tag{14}$$

so that

$$M^2 = 3 - 2/p.$$

Integrating we obtain that for $0 \le x \le 1/4$

$$\int_0^{y_0} \frac{dz}{[3p-2-pz^2]^{1/p}} = \frac{xm^{1/2}}{[2(p-1)]^{1/p}}.$$

In particular, $y_0(1/4) = M$ and

$$\int_0^{M} \frac{dz}{[3p-2-pz^2]^{1/p}} = \frac{m^{1/2}}{4[2(p-1)]^{1/p}}.$$

Changing variable z to $t(3-2/p)^{1/2}$, we see that

$$\int_0^1 \frac{dt}{(1-t^2)^{1/p}} = \frac{m^{1/2}}{4}\Big(\frac{3p-2}{2p-2}\Big)^{1/p}\Big(\frac{p}{3p-2}\Big)^{1/2},$$

or

$$m^{1/2} = 4\Big(\frac{2p-2}{3p-2}\Big)^{1/p}\Big(3-\frac{2}{p}\Big)^{1/2}\int_0^1 \frac{dt}{(1-t^2)^{1/p}}.$$

Remark that

$$\int_0^1 \frac{dt}{(1-t^2)^{1/p}} = \frac{1}{2}B(\frac{1}{2}, 1-\frac{1}{p}),$$

where B is the Euler function. Thus

$$m^{1/2} = 2\Big(\frac{2p-2}{3p-2}\Big)^{1/p}(3-\frac{2}{p})^{1/2}B(\frac{1}{2}, 1-\frac{1}{p}).$$

On the opther hand

$$\int_{y_0}^{M} \frac{dz}{[3p-2-pz^2]^{1/p}} = (x-1/4)\frac{m^{1/2}}{[2(p-1)]^{1/p}},$$

so that $y_0(x) = M + A(1/4-x)^\gamma[1+o(1)]$ as $x \to 1/4-0$ with $\gamma = p/(p-1)$.
□

Lemma A14. *Let $-1/2 < \alpha < 1$, $\alpha \neq 0$ and $y_0(x)$ be the function found in Lemma A7. Let $p_0(x) = |y_0'(x)|^{2/(\alpha-1)}$. Let*

$$m_1 = \inf_{y \in H_0^1(0,1)} \frac{\int_0^1 p(x) y'(x)^2 dx}{\int_0^1 y(x)^2 dx}.$$

Then

$$m_1 = \Big(\frac{2-2p}{2-3p}\Big)^{2/p} \big(3 - \frac{2}{p}\big) h^{2/p-3} B(1/2, 1-1/p)^2.$$

The minimal value is attained on the function y_0.

Proof. Consider a minimizing sequence $y_k(x)$ such that $\int_0^1 y_k(x)^2 dx = 1$. The integrals $\int_0^{1/2-\varepsilon} y_k'(x)^2 dx$ and $\int_{1/2+\varepsilon}^1 y_k'(x)^2 dx$ are bounded and one can choose a subsequence converging almost everywhere in $(0, 1/2-\varepsilon)$ and $(1/2+\varepsilon, 1)$, in $L_2(0,1)$ and weakly in $H^1(0, 1/2-\varepsilon)$ and $H^1(1/2+\varepsilon, 1)$. Using the diagonalization, one can find a subsequence converging almost everywhere in $(0,1)$ to $y_1(x)$. Then

$$\int_0^1 y_1(x)^2 dx = 1, \ \int_0^1 p(x) y_1'(x)^2 dx \leq m_1.$$

However, m_1 is the minimal possible value of the latter integral. Therefore, $\int_0^1 p(x) y_1'(x)^2 dx = m_1$. The function y_1 satisfies the equation

$$(p(x) y_1(x)')' + m_1 y_1(x) = 0,$$

$y_1(0) = 0$, $y_1(1) = 0$. The function $z(x) = y_1(x) + y_1(1-x)$ is also minimizing, if it does not vanish identically.

If $z(x) \not\equiv 0$, then it is even and

$$(p(x) z')'(x) + m_1 z = 0.$$

On the other hand,

$$(p(x) y_0')' + m y_0 = 0, \ y_0(0) = y_0(1) = 0$$

and $y_0 > 0$ in $(0,1)$. We have by Lemma A8

$$(m - m_1) \int_0^1 z y_0 dx = p(x)(z'(x) y_0(x) - z(x) y_0'(x))|_{x=0}^{x=1} = 0$$

and $m_1 = m$.

If $z(x) \equiv 0$, then y_1 is odd and

$$(p(x)y_1'(x))' + m_1 y_1 = 0.$$

We have $y_1(0) = y_1(1/2) = 0$. Let x_0 be the first zero of y_1, so that $y_1(x_0) = 0, y_1(x) > 0$ for $0 < x < x_0$. We have by Lemma A8

$$(m - m_1) \int_0^{x_0} y_1 y_0 dx = p(x)(y_1'(x)y_0(x) - y_1(x)y_0'(x))|_{x=0}^{x=x_0} = 0.$$

Therefore, $m_1 = m$ and $y_1 = y_0$. □

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