UNFOLDINGS OF MEROMORPHIC CONNECTIONS AND A CONSTRUCTION OF FROBENIUS MANIFOLDS

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Abstract. The existence of universal unfoldings of certain germs of meromorphic connections is established. This is used to prove a general construction theorem for Frobenius manifolds. A particular case is Dubrovin's theorem on semisimple Frobenius manifolds. Another special case starts with variations of Hodge structures. This case is used to compare two constructions of Frobenius manifolds, the one in singularity theory and the Barannikov–Kontsevich construction. For homogeneous polynomials which give Calabi–Yau hypersurfaces certain Frobenius submanifolds in both constructions are isomorphic.

1. Introduction

Let $\widetilde{M}$ be a complex manifold. *A structure of a Frobenius manifold on* $\widetilde{M}$ defined by B. Dubrovin consists of several pieces of data of which the most important are: (a) the choice of a flat structure on $\widetilde{M}$ represented by a subsheaf of flat vector fields $\mathcal{T}^f_{\widetilde{M}}$ of the sheaf of holomorphic vector fields $\mathcal{T}_{\widetilde{M}}$; (b) a commutative and associative $\mathcal{O}_{\widetilde{M}}$-bilinear multiplication $\circ$ on $\mathcal{T}_{\widetilde{M}}$.

Let $\nabla^{(0)}$ be the unique torsionless holomorphic connection on the holomorphic tangent bundle $T\widetilde{M}$ with kernel $\mathcal{T}^f_{\widetilde{M}}$. Then the axiom connecting (a) and (b) is the following requirement: $\nabla_{z,X} := \nabla^{(0)}_X - \frac{1}{z}X\circ$ is a pencil (with parameter z) of flat connections on $\widetilde{M}$, where $X \in \mathcal{T}_{\widetilde{M}}$. It was called the first structure connection in [Man2]. We use here its more sophisticated version: cf. lemma 4.4 and theorem 4.2.

Additional pieces of data include a flat holomorphic metric g, a flat identity vector field e for $\circ$, and an Euler field E: see definition 4.3 below for the complete list of their properties and interrelations.

Three general constructions of Frobenius manifolds are known: K. Saito's Frobenius structures on unfolding spaces of singularities; quantum cohomology (physicists' A-models); and Frobenius structures on the extended moduli spaces of Calabi–Yau manifolds (Barannikov–Kontsevich construction, considerably generalized in [Ba1][Ba2], physicists' B-models).

Some of the important issues of this theory require identification of Frobenius manifolds obtained by different constructions. The famous Mirror conjecture of Candelas et al. belongs to this class; it stimulated much of research.

A general strategy for establishing such an isomorphism consists in showing that two Frobenius manifolds to be compared are determined by a more restricted set of data, and then identifying these data. For example, if $\widetilde{M}$ is endowed with a point 0 at which $\circ$ is semisimple and $E\circ$ has a simple spectrum, then a finite number of numbers suffices in order to reconstruct the whole Frobenius germ $(\widetilde{M}, 0)$. This is a result of Dubrovin [Du, Lecture 3]. Generally, however, this is not true; in particular, A- and B-models fail to satisfy the semisimplicity restriction.

In this case it may happen that for an appropriate submanifold $M \subset \widetilde{M}$ the restriction $H := T\widetilde{M}|_M$ endowed with the restriction of the structure connection and some additional remnants of Frobenius structure ("initial conditions") carry sufficient information in order to reconstruct $\widetilde{M}$ uniquely. A statement of this kind will be especially useful if one can ensure that arbitrary initial conditions of a given type come from a Frobenius structure ("strong reconstruction theorem").

One of the main results of this paper (theorem 4.5) consists in exhibiting such a set of initial data (see definition 4.1 (b)). A special case of this result having direct applications is the theorem 5.6 which shows that one can construct a unique Frobenius manifold from any variation of Hodge filtrations over M and an opposite filtration assuming that a certain condition which we call *H^2-generation* holds (definition 5.3).

A prototype of theorem 5.6 is the first reconstruction theorem in [KM, Theorem 3.1]. The uniqueness statement in 5.6 is essentially that of [KM]. It was already used in [Ba1, Theorem 6.5]. The existence is the new part. The more general theorem 4.5 is applicable to the germs of semisimple Frobenius manifolds $(\widetilde{M}, 0)$ mentioned above as well; in this case M is the marked point, and one recovers Dubrovin's result.

The initial datum on M can be reformulated as a meromorphic connection on a bundle on $\mathbb{P}^1 \times M$ with a logarithmic pole along $\{\infty\} \times M$ and a pole of Poincaré rank 1 along $\{0\} \times M$. So theorem 4.5 gives a recipe for constructing Frobenius manifolds: One has to construct such connections.

The proof of theorem 4.5 relies on the study of unfoldings of germs of such connections in the chapters 2 and 3. The main result there is theorem 2.5. It shows that a germ of a meromorphic connection on a bundle on $(\mathbb{C}, 0) \times (M, 0)$ with a pole of Poincaré rank 1 along $\{0\} \times M$ has a universal unfolding if again a certain generation condition is satisfied. It generalizes the case $M = \{pt\}$ which was treated in [Mal, (4.1)].

With theorem 2.5 one can reduce theorem 4.5 to the case when $M = \widetilde{M}$. This case was already known. It was formulated by Sabbah [Sab1, Theorem (4.3.6)] [Sab2, Théorème VII 3.6], and independently by Barannikov [Ba1][Ba2][Ba3]. A major part of the initial data is equivalent to Barannikov's notion of *a semi-infinite variation of Hodge structures.* Theorem 4.5 in the case $M = \widetilde{M}$ is also implicit in the construction in singularity theory [SK][SM].

Chapter 6 is devoted to a class of variations of Hodge structures to which theorem 5.6 applies, the variations of Hodge structures on the primitive parts of the middle cohomologies of smooth hypersurfaces in $\mathbb{P}^n$ whose degrees divide $n + 1$ (theorem 6.1). The corresponding Frobenius manifolds turn up as submanifolds in two different classes of Frobenius manifolds.

One class arises in singularity theory. The base space of a semiuniversal unfolding of an isolated hypersurface singularity can be equipped with the structure of a Frobenius manifold [SK][SM][He1]. Suppose that the singularity is a homogeneous polynomial in $\mathbb{C}[x_0, ..., x_n]$ of a degree which divides $n + 1$. The submanifold of the base space which parametrizes homogeneous deformations carries a variation of Hodge structures isomorphic to the one of the projective hypersurfaces in $\mathbb{P}^n$. Then a larger submanifold which parametrizes certain semihomogeneous deformations is a Frobenius submanifold. It is the one determined by the variation of Hodge filtrations and an opposite filtration. This is discussed in chapter 7.

The other class of Frobenius manifolds arises from the Barannikov–Kontsevich construction [BK][Ba1][Ba2]. There for any Calabi–Yau manifold a formal germ of a space which extends the moduli space of complex structure deformations is constructed and is equipped with a semi-infinite variation of Hodge structures. Together with the choice of an opposite filtration this induces the structure of a Frobenius super manifold on the extended moduli space. In the case of a Calabi–Yau hypersurface in $\mathbb{P}^n$ it turns out [Ba1, Theorem 6.5] that the whole semi-infinite variation of Hodge structures is determined by the variation of Hodge structures on the moduli space of complex structure deformations. The whole chapter 8 is a discussion and reformulation of this result of Barannikov.

In the case of a homogeneous polynomial of degree $n + 1$, the results in chapters 7 and 8 give for suitable opposite filtrations isomorphic Frobenius submanifolds in the two classes of Frobenius manifolds.

We thank Dennis Borisov for some discussions about these Frobenius manifolds.

Index of notations and definitions. Both basic structures, that of Frobenius manifolds and initial data, admit many useful weakenings and variations. For reader's convenience, we compiled an index of the versions used in this paper and related notions.

2. Unfoldings of meromorphic connections

Theorem 2.5 below generalizes a result of Malgrange [Mal, (4.1)]. We start with the same setting as in [Mal].

Definition 2.1. (a) Consider a germ $(M,0)$ of a complex manifold, a germ $H \to (\mathbb{C},0)\times(M,0)$ of a holomorphic vector bundle, and a flat connection ∇ on the restriction of H to $(\mathbb{C}^*,0)\times(M,0)$. Let z be the coordinate on $(\mathbb{C},0)$. The connection ∇ has a pole of Poincaré rank $r\in\mathbb{Z}_{\geq 0}$ along $\{0\}\times(M,0)$ if

$$\nabla : \mathcal{O}(H) \to \frac{1}{z^r}\left[\mathcal{O}_{\mathbb{C}\times M,0}\cdot\Omega^1_{M,0} + \mathcal{O}_{\mathbb{C}\times M,0}\cdot\frac{\mathrm{d}z}{z}\right]\otimes\mathcal{O}(H). \tag{2.1}$$

Here $\mathcal{O}(H)$ is the $\mathcal{O}_{\mathbb{C}\times M,0}$-module of germs of holomorphic sections of H. A pole of Poincaré rank 0 is called a logarithmic pole.

(b) A (TE)-structure is a tuple $((M,0),H,\nabla)$ as in (a) with a pole of Poincaré rank 1.

Remarks 2.2. (i) The results in [Mal] are formulated also for poles of higher Poincaré rank. We restrict to Poincaré rank 1 because that is the case needed in the later chapters and because a proof of a generalization of theorem 2.5, if possible, would be much more technical.

(ii) The notations (TE), $(TEP(w))$, etc. in the definitions 2.1 and 3.1 are compatible with those in [He2]. Here 'T' stands for Twistor, 'E' for Extension (in z-direction), 'P' for Pairing, and w is an integer.

If $((M,0),H,\nabla)$ is a (TE)-structure and $\varphi:(M',0)\to(M,0)$ a holomorphic map of germs of manifolds then one can pull back H and ∇ with $\mathrm{id}\times\varphi : (\mathbb{C},0)\times(M',0)\to(\mathbb{C},0)\times(M,0)$. One easily sees that $\varphi^*(H,\nabla)$ gives a (TE)-structure on $(M',0)$.

Definition 2.3. Fix a (TE)-structure $((M,0),H,\nabla)$.

(a) An *unfolding* of it is a (TE)-structure $((M\times\mathbb{C}^l,0),\widetilde{H},\widetilde{\nabla})$ together with a fixed isomorphism

$$i : ((M,0),H,\nabla)\to((M\times\mathbb{C}^l,0),\widetilde{H},\widetilde{\nabla})|_{(M\times\{0\},0)}. \tag{2.2}$$

(b) One unfolding $((M\times\mathbb{C}^l,0),\widetilde{H},\widetilde{\nabla},i)$ *induces* a second unfolding $((M\times\mathbb{C}^{l'},0),\widetilde{H}',\widetilde{\nabla}',i')$ if there is an isomorphism j from the second unfolding to the pullback of the first unfolding by a map

$$\varphi : (M\times\mathbb{C}^{l'},0)\to(M\times\mathbb{C}^l,0) \tag{2.3}$$

which is the identity on $(M\times\{0\},0)$, and if

$$i = j|_{(M\times\{0\},0)}\circ i'. \tag{2.4}$$

(Then j is uniquely determined by φ and (2.4).)

(c) An unfolding is *universal* if it induces any unfolding via a unique map φ.

By definition, a universal unfolding of a (TE)-structure is unique up to canonical isomorphism if it exists. For existence we need some conditions which are formulated in terms of the data of the following lemma.

Lemma 2.4. [Sab2][He2, ch. 2] *Let $((M,0),H,\nabla)$ be a (TE)-structure. Define the germ of a holomorphic vector bundle $K := H|_{\{0\}\times(M,0)}$ on $(M,0)$ with $\mathcal{O}_{M,0}$-module $\mathcal{O}(K)$ of germs of holomorphic sections. Define maps*

$$\begin{aligned} C := [z\nabla] \quad : \quad & \mathcal{O}(K) \to \Omega^1_{M,0} \otimes \mathcal{O}(K) \\ & C_X[a] = [z\nabla_X a] \text{ for } X \in \mathcal{T}_{M,0}, a \in \mathcal{O}(H) \end{aligned} \tag{2.5}$$

and

$$\mathcal{U} := [z\nabla_{z\partial_z}] : \mathcal{O}(K) \to \mathcal{O}(K); \tag{2.6}$$

here [ ] means restriction to $\{0\} \times (M,0)$ and $X \in \mathcal{T}_{M,0}$ is lifted canonically to $(\mathbb{C},0) \times (M,0)$.

The maps C_X, $X \in \mathcal{T}_{M,0}$, and $\mathcal{U}$ are commuting $\mathcal{O}_{M,0}$-linear endomorphisms of $\mathcal{O}(K)$. Therefore the map C is a Higgs field on K.

Proof. This follows from the flatness of ∇ and the fact that ∇ has a pole of Poincaré rank 1.

For a general discussion of Higgs fields see e.g. [Sab2, p. 36]. A symmetric Higgs field on the tangent bundle is the same as a commutative and associative $\mathcal{O}_M$-bilinear multiplication. □

Theorem 2.5. *Let $((M,0),H,\nabla)$ be a (TE)-structure with data $(K,C,\mathcal{U})$ as in lemma 2.4. Suppose that there exists a vector $\zeta \in K_0$ in the fiber K_0 of K at 0 such that*

(IC) (injectivity condition) the map

$$C_\bullet\zeta : T_0M \to K_0, \quad X \mapsto C_X\zeta \tag{2.7}$$

is injective and

(GC) (generation condition) ζ and its images under iteration of the maps $\mathcal{U} : K_0 \to K_0$ and $C_X : K_0 \to K_0$, $X \in T_0M$, generate K_0.

Then a universal unfolding of $((M,0),H,\nabla)$ exists. An unfolding $((M \times \mathbb{C}^l,0),\widetilde{H},\widetilde{\nabla},i)$ with data $(\widetilde{K},\widetilde{C},\widetilde{\mathcal{U}})$ as in lemma 2.4 is a universal unfolding if and only if the map

$$\widetilde{C}_\bullet i(\zeta) : T_0(M \times \mathbb{C}^l) \to \widetilde{K}_0, \quad X \mapsto \widetilde{C}_X i(\zeta) \tag{2.8}$$

is an isomorphism.

The rest of this chapter is devoted to the proof of this theorem. In [Mal, (4.1)] the case $M = \{pt\}$ is considered. In this case the generation condition (GC) is satisfied if and only if $\mathcal{U} : K_0 \to K_0$ has for each eigenvalue only one Jordan block. For example, if $\mathcal{U} : K_0 \to K_0$ is semisimple, it must have simple eigenvalues. But in chapter 5 we will use theorem 2.5 in the case $\mathcal{U} = 0$.

The first part of the proof follows [Mal], using an extension of H to a bundle on $\mathbb{P}^1 \times (M,0)$ and using the rigidity properties of logarithmic poles, which are formulated in lemma 2.6.

Lemma 2.6. *(a) (e.g. [Sab2, III.1.20]) Let $(L' \to (\mathbb{C}^*,0) \times (M,0),\nabla)$ be the germ of a holomorphic vector bundle with flat connection ∇, and let $(L^{(0)} \to (\mathbb{C},0) \times \{0\},\nabla)$ be an extension of $(L',\nabla)|_{(\mathbb{C}^*,0)\times\{0\}}$ with a logarithmic pole at*

0. Then an extension $(L \to (\mathbb{C},0)\times(M,0),\nabla)$ *of* (L',∇) *with a logarithmic pole along* $\{0\}\times(M,0)$ *exists with* $(L,\nabla)|_{(\mathbb{C},0)\times\{0\}} = (L^{(0)},\nabla)$. *It is unique up to canonical isomorphism and it is isomorphic to the pullback* $\varphi^*(L^{(0)},\nabla)$ *where* $\varphi:(M,0)\to\{0\}$.

(b) (e.g. [He2, 5.1]*) Let* $(L\to(\mathbb{C},0)\times(M,0),\nabla)$ *be (the germ of) a holomorphic vector bundle with a flat connection* ∇ *over* $(\mathbb{C}^*,0)\times(M,0)$ *and a logarithmic pole along* $\{0\}\times(M,0)$. *The bundle* $L|_{\{0\}\times(M,0)}$ *is equipped with a flat connection* ∇^{res}, *the residual connection, and a* ∇^{res}*-flat endomorphism* $\mathcal{V}^{res}$, *the residue endomorphism. They are defined by*

$$\begin{aligned}\nabla^{res}_X[a] &:= [\nabla_X a] \quad \textit{for } X\in\mathcal{T}_{M,0}, a\in\mathcal{O}(L), && (2.9)\\ \mathcal{V}^{res}[a] &:= [\nabla_{z\partial_z} a] \quad \textit{for } a\in\mathcal{O}(L). && (2.10)\end{aligned}$$

Lemma 2.7. *Let* $(H\to(\mathbb{C},0)\times(M,0),\nabla)$ *be a germ of a holomorphic vector bundle with a flat connection* ∇ *over* $(\mathbb{C}^*,0)\times(M,0)$. *There exists an extension to a trivial bundle* $H^{(gl)}\to\mathbb{P}^1\times(M,0)$ *with flat connection* ∇ *over* $(\mathbb{C}^*-\{1\})\times(M,0)$, *with logarithmic poles along* $\{1\}\times(M,0)$ *and* $\{\infty\}\times(M,0)$, *and with trivial monodromy around* $\{1\}\times(M,0)$.

Proof. We follow [Mal, ch. 3]. First, one extends H to a bundle $H^{(\mathbb{C})}\to\mathbb{C}\times(M,0)$ with flat connection ∇ over $\mathbb{C}^*\times(M,0)$. Then one can choose an extension of $H^{(\mathbb{C})}$ to a bundle $H^{(\mathbb{P}^1)}\to\mathbb{P}^1\times(M,0)$ with a logarithmic pole along $\{\infty\}\times(M,0)$. This bundle is not necessarily trivial.

One chooses an $\mathcal{O}_{M,0}$-basis $\sigma_1,....,\sigma_{\mathrm{rk}\,H}$ of germs of flat sections of $\mathcal{O}(H^{(\mathbb{P}^1)})_{(1,0)}$ and defines for $r=(r_1,...,r_{\mathrm{rk}\,H})\in\mathbb{Z}^{\mathrm{rk}\,H}$ a bundle $H^{(r)}\to\mathbb{P}^1\times(M,0)$ as follows: on $(\mathbb{P}^1-\{1\})\times(M,0)$ it coincides with $H^{(\mathbb{P}^1)}$; the germ $\mathcal{O}(H^{(r)})_{(1,0)}$ is generated by the sections $(z-1)^{r_i}\sigma_i$. The bundle $H^{(r)}$ has a logarithmic pole along $\{1\}\times(M,0)$. For some r the restriction $H^{(r)}|_{\mathbb{P}^1\times\{0\}}$ is trivial [Mal, (3.2)]. Because being a trivial bundle on $\mathbb{P}^1$ is an open property, the bundle $H^{(gl)}:=H^{(r)}$ for such an r is trivial. □

Remark 2.8. If one applies lemma 2.7 to a (TE)-structure $(H\to(\mathbb{C},0)\times(M,0),\nabla)$, then the extension $(H^{(gl)},\nabla)$ has two distinguished properties:

(i) Because of lemma 2.6 (a), any unfolding $(\widetilde{H}\to(\mathbb{C},0)\times(M\times\mathbb{C}^l,0),\widetilde{\nabla})$ of $(H,\widetilde{\nabla})$ has a unique extension $(\widetilde{H}^{(gl)}\to\mathbb{P}^1\times(M\times\mathbb{C}^l,0),\widetilde{\nabla})$ with all the properties in lemma 2.7 whose restriction to $\mathbb{P}^1\times(M\times\{0\},0)$ coincides with $(H^{(gl)},\nabla)$.

(ii) Denote by ∇^{res} the residual connection on $H|_{\{\infty\}\times(M,0)}$. The space

$$\begin{aligned}V(H^{(gl)}) := \{ &\text{ global hol. sections } v \text{ in } H^{(gl)} \\ &\text{with } \nabla^{res}(v|_{\{\infty\}\times(M,0)})=0\}\end{aligned} \qquad (2.11)$$

is a vector space of dimension $\mathrm{rk}\,H$. A basis of it is an $\mathcal{O}_{(\mathbb{P}^1,z)\times(M,0)}$-basis of $\mathcal{O}(H^{(gl)})_{(z,0)}$ for any $z\in\mathbb{P}^1$. Below we will work with the connection matrix with respect to such a basis. For an extension $(\widetilde{H}^{(gl)},\widetilde{\nabla})$ as in (i) the sections in $V(H^{(gl)})$ extend uniquely to the sections in $V(\widetilde{H}^{(gl)})$.

Using lemma 2.7 and these observations we can control the unfoldings of (TE)-structures as in theorem 2.5. The following lemma is the main step in its proof.

Lemma 2.9. *Let $(H^{(gl)} \to \mathbb{P}^1 \times (M,0), \nabla)$ be a trivial holomorphic vector bundle of rank $n \geq 1$ with a flat connection over $(\mathbb{C}^* - \{1\}) \times (M,0)$, with logarithmic poles along $\{1\} \times (M,0)$ and $\{\infty\} \times (M,0)$ and with a pole of Poincaré rank 1 along $\{0\} \times (M,0)$. Define $K := H^{(gl)}|_{\{0\}\times(M,0)}$, C and $\mathcal{U}$ as in lemma 2.4. Suppose that the generation condition (GC) of theorem 2.5 is satisfied for a vector $\zeta \in K_0$. Choose a basis $v_1, ..., v_n$ of $V(H^{(gl)})$ with $v_1|_{(0,0)} = \zeta$. Choose $l \in \mathbb{N}$ and n functions $f_1, ..., f_n \in \mathcal{O}_{M\times\mathbb{C}^l,0}$ with $f_i|_{(M\times\{0\},0)} = 0$. Let $(t_1, ..., t_m, y_1, ..., y_l) = (t, y)$ be coordinates on $(M \times \mathbb{C}^l, 0)$.*

Then there exists a unique unfolding $(\widetilde{H}^{(gl)} \to \mathbb{P}^1 \times (M \times \mathbb{C}^l, 0), \widetilde{\nabla})$ of $(H^{(gl)}, \nabla)$ with the following properties. $\widetilde{H}^{(gl)}$ is a trivial vector bundle with a flat connection over $(\mathbb{C}^* - \{1\}) \times (M \times \mathbb{C}^l, 0)$, with logarithmic poles along $\{1\} \times (M \times \mathbb{C}^l, 0)$ and $\{\infty\} \times (M \times \mathbb{C}^l, 0)$ and a pole of Poincaré rank 1 along $\{0\} \times (M \times \mathbb{C}^l, 0)$. Its restriction to $\mathbb{P}^1 \times (M \times \{0\}, 0)$ is $(H^{(gl)}, \nabla)$. Let $\widetilde{v}_1, ..., \widetilde{v}_n \in V(\widetilde{H}^{(gl)})$ be the canonical extensions of $v_1, ..., v_n \in V(H^{(gl)})$. Define $\widetilde{K} := \widetilde{H}^{(gl)}|_{\{0\}\times(M\times\mathbb{C}^l,0)}$, $\widetilde{C}$ and $\widetilde{U}$ as in lemma 2.4. Then

$$\widetilde{C}_{\partial/\partial y_\alpha} \widetilde{v}_1 = \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_\alpha} \widetilde{v}_i \quad \textit{for} \quad \alpha = 1, ..., l. \tag{2.12}$$

Proof. Suppose for a moment that $(\widetilde{H}^{(gl)}, \widetilde{\nabla})$ were already constructed. The connection matrix Ω with respect to the basis $\widetilde{v}_1, ..., \widetilde{v}_n$,

$$\widetilde{\nabla}(\widetilde{v}_1, ..., \widetilde{v}_n) = (\widetilde{v}_1, ..., \widetilde{v}_n) \cdot \Omega, \tag{2.13}$$

would take the form

$$\Omega = \frac{1}{z} \sum_{i=1}^{m} C_i \mathrm{d}t_i + \frac{1}{z} \sum_{\alpha=1}^{l} F_\alpha \mathrm{d}y_\alpha + (\frac{1}{z^2} U + \frac{1}{z} V + \frac{1}{z-1} W) \mathrm{d}z \tag{2.14}$$

with matrices

$$C_i, F_\alpha, U, V, W \in M(n \times n, \mathcal{O}_{M\times\mathbb{C}^l,0}). \tag{2.15}$$

This follows from $\widetilde{\nabla}^{res} \widetilde{v}_i|_{\{\infty\}\times(M\times\mathbb{C}^l,0)} = 0$ and from the pole orders of $(\widetilde{H}^{(gl)}, \widetilde{\nabla})$ along $\{0, 1, \infty\} \times (M \times \mathbb{C}^l, 0)$. The flatness condition $\mathrm{d}\Omega + \Omega \wedge \Omega = 0$ could be written as

$$[C_i, C_j] = 0 \tag{2.16}$$

$$[C_i, F_\alpha] = 0, \tag{2.17}$$

$$[F_\alpha, F_\beta] = 0, \tag{2.18}$$

$$\frac{\partial C_i}{\partial t_j} = \frac{\partial C_j}{\partial t_i}, \tag{2.19}$$

$$\frac{\partial C_i}{\partial y_\alpha} = \frac{\partial F_\alpha}{\partial t_i}, \tag{2.20}$$

$$\frac{\partial F_\alpha}{\partial y_\beta} = \frac{\partial F_\beta}{\partial y_\alpha}, \tag{2.21}$$

$$[C_i, U] = 0, \tag{2.22}$$
$$[F_\alpha, U] = 0, \tag{2.23}$$
$$\frac{\partial U}{\partial t_i} = [V, C_i] - C_i, \tag{2.24}$$
$$\frac{\partial U}{\partial y_\alpha} = [V, F_\alpha] - F_\alpha, \tag{2.25}$$
$$\frac{\partial W}{\partial t_i} = [W, C_i], \tag{2.26}$$
$$\frac{\partial W}{\partial y_\alpha} = [W, F_\alpha], \tag{2.27}$$
$$\frac{\partial V}{\partial t_i} = -[W, C_i], \tag{2.28}$$
$$\frac{\partial V}{\partial y_\alpha} = -[W, F_\alpha]. \tag{2.29}$$

The condition (2.12) would mean

$$(F_\alpha)_{i1} = \frac{\partial f_i}{\partial y_\alpha}. \tag{2.30}$$

We will show first inductively uniqueness and existence of matrices C_i, F_α, U, V, W with (2.16) - (2.30) and with coefficients in $\mathcal{O}_{M,0}[[y]] = \mathcal{O}_{M,0}[[y_1, ..., y_l]]$. Then their convergence will be proved with the Cauchy–Kovalevski theorem.

Define for $w \in \mathbb{Z}_{\geq 0}$

$$\mathcal{O}_{M,0}[y]_w := \{\sum_r b_r y^r \mid r \in \mathbb{Z}^l_{\geq 0}, \sum_i r_i = w, b_r \in \mathcal{O}_{M,0}\}, \tag{2.31}$$
$$\mathcal{O}_{M,0}[y]_{>w} := \sum_{k>w} \mathcal{O}_{M,0}[y]_k, \quad \mathcal{O}_{M,0}[y]_{\leq w} := \sum_{k=0}^{w} \mathcal{O}_{M,0}[y]_k, \tag{2.32}$$
$$\mathcal{O}_{M,0}[[y]]_{>w} := \mathcal{O}_{M,0}[[y]] \cdot \mathcal{O}_{M,0}[y]_{>w} \tag{2.33}$$

and

$$M(w) := M(n \times n, \mathcal{O}_{M,0}[y]_w), \tag{2.34}$$
$$M(> w) := M(n \times n, \mathcal{O}_{M,0}[y]_{>w}), \tag{2.35}$$
$$M(\leq w) := M(n \times n, \mathcal{O}_{M,0}[y]_{\leq w}). \tag{2.36}$$

Beginning of the induction for $w = 0$: The connection matrix $\Omega^{(0)}$ of $(H^{(gl)}, \nabla)$ with respect to the basis $v_1, ..., v_n$ takes the form

$$\Omega^{(0)} = \frac{1}{z} \sum_{i=1}^{m} C_i^{(0)} \mathrm{d}t_i + \left(\frac{1}{z^2} U^{(0)} + \frac{1}{z} V^{(0)} + \frac{1}{z-1} W^{(0)} \right) \mathrm{d}z \tag{2.37}$$

with matrices $C_i^{(0)}, U^{(0)}, V^{(0)}, W^{(0)} \in M(0)$. The flatness condition $\mathrm{d}\Omega^{(0)} + \Omega^{(0)} \wedge \Omega^{(0)} = 0$ is equivalent to the equations (2.16), (2.19), (2.22), (2.24), (2.26), (2.28) for the matrices $C_i^{(0)}$, $U^{(0)}$, $V^{(0)}$, $W^{(0)}$ instead of C_i, U, V, W.

Induction hypothesis for $w \in \mathbb{Z}_{\geq 0}$: Unique matrices $C_i^{(k)}, U^{(k)}, V^{(k)}, W^{(k)} \in M(k)$ for $0 \leq k \leq w$ and $F_\alpha^{(k)} \in M(k)$ for $0 \leq k \leq w-1$ are constructed such that the matrices

$$C_i^{(\leq w)} := \sum_{k=0}^{w} C_i^{(k)} \in M(\leq w) \tag{2.38}$$

and the analogously defined matrices $U^{(\leq w)}, V^{(\leq w)}, W^{(\leq w)}, F_\alpha^{(\leq w-1)}$ satisfy (2.16), (2.19), (2.22), (2.24), (2.26), (2.28) modulo $M(> w)$, (2.17), (2.18), (2.20), (2.23), (2.25), (2.27), (2.29) modulo $M(> w-1)$, (2.21) modulo $M(> w-2)$ and (2.30) modulo $\mathcal{O}_{M,0}[[y]]_{>w-1}$.

Induction step from w to $w+1$: It consists of four steps:

(i) Construction of matrices $F_\alpha^{(w)} \in M(w)$ such that the matrices $F_\alpha^{(\leq w)} = F_\alpha^{(\leq w-1)} + F_\alpha^{(w)}$ together with the matrices $C_i^{(\leq w)}, U^{(\leq w)}, V^{(\leq w)}, W^{(\leq w)}$ satisfy (2.17), (2.18), (2.23) modulo $M(> w)$ and (2.30) modulo $\mathcal{O}_{M,0}[[y]]_{>w}$.
(ii) Proof of (2.21) modulo $M(> w-1)$ for these matrices.
(iii) Construction of matrices $C_i^{(w+1)}, U^{(w+1)}, V^{(w+1)}, W^{(w+1)} \in M(w+1)$ such that the matrices $C_i^{(\leq w+1)} = C_i^{(\leq w)} + C_i^{(w+1)}$, the analogously defined matrices $U^{(\leq w+1)}, V^{(\leq w+1)}, W^{(\leq w+1)}$ and the matrices $F_\alpha^{(\leq w)}$ satisfy (2.20), (2.25), (2.27), (2.29) modulo $M(> w)$.
(iv) Proof of (2.16), (2.19),(2.22), (2.24), (2.26), (2.28) modulo $M(> w+1)$ for these matrices.

(i) The matrices $C_i^{(\leq w)}$ and $U^{(\leq w)}$ generate an algebra of commuting matrices in $M(n \times n, \mathcal{O}_{M,0}[y]/\mathcal{O}_{M,0}[y]_{>w})$. The generation condition (GC) shows two things:

(α) This algebra contains for any $i = 1, ..., n$ a matrix $E_i^{(\leq w)} \in M(n \times n, \mathcal{O}_{M,0}[y]_{\leq w})$ with first column

$$(E_i^{(\leq w)})_{j1} = \delta_{ij}. \tag{2.39}$$

(β) Any matrix in $M(n \times n, \mathcal{O}_{M,0}[y]_{\leq w})$ which commutes with the matrices $C_i^{(\leq w)}$ and $U^{(\leq w)}$ modulo $M(> w)$ is modulo $M(> w)$ a linear combination of the matrices $E_i^{(\leq w)}$ with coefficients in $\mathcal{O}_{M,0}[y]_{\leq w}$.

Therefore the matrices $F_\alpha^{(\leq w)} \in M(\leq w)$ which are defined by

$$F_\alpha^{(\leq w)} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_\alpha} \cdot E_i^{(\leq w)} \text{ modulo } M(n \times n, \mathcal{O}_{M,0}[[y]]_{>w}) \tag{2.40}$$

are the unique matrices which satisfy (2.17), (2.23) modulo $M(> w)$, (2.30) modulo $\mathcal{O}_{M,0}[[y]]_{>w}$ and $F_\alpha^{(\leq w)} = F_\alpha^{(\leq w-1)} + F_\alpha^{(w)}$ for some $F_\alpha^{(w)} \in M(w)$. They satisfy (2.18) modulo $M(> w)$.

(ii) Derivatives of (2.17), (2.23) and (2.30) by $\frac{\partial}{\partial y_\beta}$ give equations modulo $M(> w-1)$ respectively $\mathcal{O}_{M,0}[[y]]_{>w-1}$,

$$\left[\frac{\partial C_i^{(\leq w)}}{\partial y_\beta}, F_\alpha^{(\leq w-1)}\right] \equiv -\left[C_i^{(\leq w-1)}, \frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta}\right], \tag{2.41}$$

$$\left[\frac{\partial U^{(\leq w)}}{\partial y_\beta}, F_\alpha^{(\leq w-1)}\right] \equiv -\left[U^{(\leq w-1)}, \frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta}\right], \tag{2.42}$$

$$\frac{\partial^2 f_i}{\partial y_\alpha \partial y_\beta} \equiv \left(\frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta}\right)_{i1}. \tag{2.43}$$

Recall that $F_\alpha^{(\leq w)}$ is uniquely determined by (2.17), (2.23) modulo $M(> w)$, (2.30) modulo $\mathcal{O}_{M,0}[[y]]_{>w}$ and $F_\alpha^{(\leq w)} \equiv F_\alpha^{(\leq w-1)}$ modulo $M(> w-1)$. A comparison of monomials shows that $\frac{\partial F_\alpha^{(w)}}{\partial y_\beta}$ is *uniquely* determined by the parts in $M(w-1)$ respectively in $\mathcal{O}_{M,0}[y]_{w-1}$ of (2.41) - (2.43). Now one calculates modulo $M(> w-1)$ with (2.20) and (2.18)

$$\left[\frac{\partial C_i^{(\leq w)}}{\partial y_\beta}, F_\alpha^{(\leq w-1)}\right] \equiv \left[\frac{\partial F_\beta^{(\leq w-1)}}{\partial t_i}, F_\alpha^{(\leq w-1)}\right] \equiv (\beta \leftrightarrow \alpha) \tag{2.44}$$

and with (2.25) and (2.18)

$$\begin{aligned}\left[\frac{\partial U^{(\leq w)}}{\partial y_\beta}, F_\alpha^{(\leq w-1)}\right] &\equiv \left[[V^{(\leq w-1)}, F_\beta^{(\leq w-1)}] - F_\beta^{(\leq w-1)}, F_\alpha^{(\leq w-1)}\right] \\ &\equiv (\beta \leftrightarrow \alpha).\end{aligned} \tag{2.45}$$

Therefore $\frac{\partial F_\alpha^{(w)}}{\partial y_\beta}$ and $\frac{\partial F_\beta^{(w)}}{\partial y_\alpha}$ are uniquely determined by the same equations. This shows (2.21) modulo $M(> w-1)$ for the matrices $F_\alpha^{(\leq w)}$.

(iii) The equations (2.21), (2.29) and (2.18) modulo $M(> w-1)$ show that modulo $M(> w-1)$

$$\frac{\partial}{\partial y_\beta}\frac{\partial F_\alpha^{(\leq w)}}{\partial t_i} \equiv (\beta \leftrightarrow \alpha), \tag{2.46}$$

$$\begin{aligned}&\frac{\partial}{\partial y_\beta}\left([V^{(\leq w)}, F_\alpha^{(\leq w)}] - F_\alpha^{(\leq w)}\right) \\ \equiv\ & [-[W^{(\leq w-1)}, F_\beta^{(\leq w-1)}], F_\alpha^{(\leq w-1)}] + [V^{(\leq w-1)}, \frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta}] - \frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta} \\ \equiv\ & (\beta \leftrightarrow \alpha),\end{aligned} \tag{2.47}$$

$$\begin{aligned}&\frac{\partial}{\partial y_\beta}[W^{(\leq w)}, F_\alpha^{(\leq w)}] \\ \equiv\ & [[W^{(\leq w-1)}, F_\beta^{(\leq w-1)}], F_\alpha^{(\leq w-1)}] + [W^{(\leq w-1)}, \frac{\partial F_\alpha^{(\leq w)}}{\partial y_\beta}] \equiv (\beta \leftrightarrow \alpha).\end{aligned} \tag{2.48}$$

Therefore there are uniquely defined matrices $C_i^{(w+1)}, U^{(w+1)}, V^{(w+1)}, W^{(w+1)} \in M(w+1)$ such that the matrices $C_i^{(\leq w+1)}, U^{(\leq w+1)}, V^{(\leq w+1)}, W^{(\leq w+1)}$ and $F_\alpha^{(\leq w)}$ satisfy (2.20), (2.25), (2.27) and (2.29) modulo $M(> w)$.

(iv) One checks for any $\alpha = 1, ..., l$ that the derivatives by $\frac{\partial}{\partial y_\alpha}$ of the equations (2.16), (2.19),(2.22), (2.24), (2.26), (2.28) modulo $M(> w+1)$ hold. For this one uses (2.17), (2.19), (2.20), (2.23) - (2.29) modulo $M(> w)$. For example one calculates modulo $M(> w)$

$$\begin{aligned} & \frac{\partial}{\partial y_\alpha}[C_i^{(\leq w+1)}, U^{(\leq w+1)}] \qquad (2.49) \\ \equiv & \left[\frac{\partial F_\alpha^{(\leq w)}}{\partial t_i}, U^{(\leq w)}\right] + [C_i^{(\leq w)}, [V^{(\leq w)}, F_\alpha^{(\leq w)}] - F_\alpha^{(\leq w)}] \\ \equiv & -\left[F_\alpha^{(\leq w)}, \frac{\partial U^{(\leq w)}}{\partial t_i}\right] + [F_\alpha^{(\leq w)}, [V^{(\leq w)}, C_i^{(\leq w)}] - C_i^{(\leq w)}] \\ \equiv & \ 0 \end{aligned}$$

The other calculations are similar or easier. This finishes the proof of the induction step from w to $w+1$. It shows uniqueness and existence of matrices $C_i, F_\alpha, U, V, W \in M(n \times n, \mathcal{O}_{M,0}[[y]])$ with (2.16) - (2.30) and with restrictions $(C_i, U, V, W)|_{y=0} = (C_i^{(0}, U^{(0)}, V^{(0)}, W^{(0)})$.

In order to show holomorphy of these matrices it is sufficient to consider the case $l = 1$, $y_1 = y$. We want to apply the Cauchy–Kovalevski theorem in the following form ([Fo, (1.31), (1.40), (1.41)]; there the setting is real analytic, but proofs and statements hold also in the complex analytic setting): Given $N \in \mathbb{N}$ and matrices $A_i, B \in M(N \times N, \mathbb{C}\{t_1, ..., t_m, y, x_1, ..., x_N\})$ there exists a unique vector $\Phi \in M(N \times 1, \mathbb{C}\{t_1, ..., t_m, y\})$ with

$$\frac{\partial \Phi}{\partial y} = \sum_{i=1}^m A_i(t, y, \Phi) \frac{\partial \Phi}{\partial t_i} + B(t, y, \Phi), \qquad (2.50)$$

$$\Phi(t, 0) = 0. \qquad (2.51)$$

We will construct a system (2.50) - (2.51) with $N = (m+3)n^2$ such that it will be satisfied with the entries of the matrices $C_i - C_i^{(0)}, U - U^{(0)}, V - V^{(0)}, W - W^{(0)}$ as entries of Φ. The system will be built from the equations (2.20), (2.25), (2.27), (2.29) and the following equations (2.52), (2.53) with which one can express the entries of F_1 as functions of the entries of Φ.

The commutative subalgebra of $M(n \times n, \mathcal{O}_{M,0}[[y]])$ which is generated by the matrices $C_1, ..., C_m, U$ is a free $\mathcal{O}_{M,0}[[y]]$-module of rank n. Choose monomials $G^{(j)}$, $j = 1, ..., n$, in the matrices $C_1, ..., C_m, U$ which form an $\mathcal{O}_{M,0}[[y]]$-basis of this module. Then the matrix $(G_{i1}^{(j)})_{ij}$ of the first columns of the matrices $G^{(j)}$ is invertible in $M(n \times n, \mathcal{O}_{M,0}[[y]])$. Equation (2.40) gives

$$F_1 = \sum_{j=1}^n g_j G^{(j)} \qquad (2.52)$$

with coefficients $g_j \in \mathcal{O}_{M,0}[[y]]$ such that

$$\left(\frac{\partial f_1}{\partial y}, ..., \frac{f_n}{\partial y}\right)^{tr} = (G_{i1}^{(j)}) \cdot (g_1, ..., g_n)^{tr}. \tag{2.53}$$

Replacing the entries of the matrices $C_i - C_i^{(0)}, U - U^{(0)}, V - V^{(0)}, W - W^{(0)}$ by indeterminates one obtains from (2.20), (2.25), (2.27), (2.29), (2.52), (2.53) a system (2.50) - (2.51). The theorem of Cauchy–Kovalevski shows $C_i, F_1, U, V, W \in M(n\times n, \mathcal{O}_{M\times\mathbb{C},0})$. This finishes the proof of lemma 2.9. □

Proof of theorem 2.5. Let $(H \to (\mathbb{C}, 0) \times (M, 0), \nabla)$ be a (TE)-structure with $K, C, \mathcal{U}$ and $\zeta \in K_0$ with all the properties in theorem 2.5. We choose an extension to a trivial bundle with connection $(H^{(gl)} \to \mathbb{P}^1 \times (M, 0), \nabla)$ with the properties in lemma 2.7, and we choose a basis $v_1, ..., v_n$ of the vector space $V(H^{(gl)})$ (cf. remark 2.8) with $v_1|_{(0,0)} = \zeta$.

By remark 2.8 (i) an unfolding $(\widetilde{H} \to (\mathbb{C}, 0) \times (M \times \mathbb{C}^l, 0), \widetilde{\nabla}, i)$ of (H, ∇) extends to an unfolding $(\widetilde{H}^{(gl)} \to \mathbb{P}^1 \times (M \times \mathbb{C}^l, 0), \widetilde{\nabla})$ of $(H^{(gl)}, \nabla)$ with the properties in lemma 2.7, and the sections $v_1, ..., v_n$ extend uniquely to sections $\widetilde{v}_1, ..., \widetilde{v}_n$ in the space $V(\widetilde{H}^{(gl)})$.

Consider the connection matrix Ω and the matrices C_i, F_α, U, V, W as in (2.13) - (2.15) for such an unfolding. Because of (2.19) - (2.21) there exists a unique matrix $A \in M(n \times n, \mathcal{O}_{M\times\mathbb{C}^l,0})$ with $A(0) = 0$ and

$$\Omega = \frac{1}{z}\mathrm{d}A + (\frac{1}{z^2}U + \frac{1}{z}V + \frac{1}{z-1}W)\mathrm{d}z. \tag{2.54}$$

The matrices $A(t, 0), U(t, 0), V(t, 0), W(t, 0)$ are determined by $(H^{(gl)}, \nabla, v_1, ..., v_n)$. Lemma 2.9 says that for an *arbitrary* choice of functions $f_i(t, y) = A_{i1}(t, y) - A_{i1}(t, 0)$ a *unique* unfolding $(\widetilde{H}^{(gl)}, \widetilde{\nabla})$ exists. The first columns of A give a map

$$\psi = (A_{11}, ..., A_{n1}) : (M \times \mathbb{C}^l, 0) \to (\mathbb{C}^n, 0). \tag{2.55}$$

The map in (2.8) is an isomorphism if and only if ψ is an isomorphism.

Fix an unfolding $(\widetilde{H}, \widetilde{\nabla})$ such that ψ is an isomorphism. This is possible thanks to the injectivity condition (IC). Consider a second unfolding $(\widetilde{H}' \to (\mathbb{C}, 0) \times (M \times \mathbb{C}^{l'}, 0), \widetilde{\nabla}')$ of (H, ∇) with $\widetilde{H'}^{(gl)}, \widetilde{v}_1', ..., \widetilde{v}_n', \Omega', A', \psi'$ defined analogously. If it is induced from $(\widetilde{H}, \widetilde{\nabla})$ via a map

$$\varphi : (M \times \mathbb{C}^{l'}, 0) \to (M \times \mathbb{C}^l, 0) \tag{2.56}$$

then $(\mathrm{id} \times\varphi)^*\Omega = \Omega'$ and therefore $A \circ \varphi = A'$ and

$$\psi \circ \varphi = \psi'. \tag{2.57}$$

This shows that the inducing map φ is unique.

If one does not yet know that $(\widetilde{H}', \widetilde{\nabla}')$ is induced from $(\widetilde{H}, \widetilde{\nabla})$ one can define φ by (2.57) and compare the unfoldings $(\widetilde{H}', \widetilde{\nabla}')$ and $\varphi^*(\widetilde{H}, \widetilde{\nabla})$. Since the first columns of the matrix A' and the corresponding matrix for $\varphi^*(\widetilde{H}, \widetilde{\nabla})$ coincide, lemma 2.9 says that the unfoldings are isomorphic. This finishes the proof of theorem 2.5.

3. Supplements

For the applications to Frobenius manifolds in chapter 4 we need (TE)-structures (definition 2.1) with an additional ingredient, a pairing. It is also useful to consider weaker structures, (T)-structures and (L)-structures. After giving their definitions we will discuss how the concepts and results of chapter 2 extend.

Definition 3.1. [He2, ch. 2] (a) Fix $w \in \mathbb{Z}$. A $(TEP(w))$-structure is a (TE)-structure $(H \to (\mathbb{C},0) \times (M,0), \nabla)$ together with a ∇-flat, $(-1)^w$-symmetric, nondegenerate pairing

$$P : H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C} \text{ for } (z,t) \in (\mathbb{C}^* \times M, 0) \tag{3.1}$$

on a representative of H such that the pairing extends to a nondegenerate z-sesquilinear pairing

$$P : \mathcal{O}(H) \times \mathcal{O}(H) \to z^w \mathcal{O}_{\mathbb{C}\times M,0}. \tag{3.2}$$

(b) An $(LEP(w))$-structure is a germ of a bundle $H \to (\mathbb{C},0) \times (M,0)$ with a flat connection ∇ on the restriction to $(\mathbb{C}^*,0) \times (M,0)$ with a logarithmic pole along $\{0\} \times (M,0)$ and with a pairing P with the same properties as in (a).

(c) Fix $r \in \mathbb{Z}_{\geq 0}$. Consider a germ of a holomorphic vector bundle $H \to (\mathbb{C},0) \times (M,0)$ with a map

$$\nabla : \mathcal{O}(H) \to \frac{1}{z^r}\mathcal{O}_{\mathbb{C}\times M,0} \cdot \Omega^1_{M,0} \tag{3.3}$$

such that for some representative of H the restrictions $H|_{\{z\}\times(M,0)}$, $z \in (\mathbb{C}^*,0)$, are flat connections. The tuple $((M,0), H, \nabla)$ is called a (T)-structure if $r = 1$, and an (L)-structure if $r = 0$.

(d) A (T)-structure with a pairing P with all properties in (a) is a $(TP(w))$-structure, an (L)-structure with such a pairing is an $LP(w)$-structure.

Lemma 3.2. *Let $(H \to (\mathbb{C},0) \times (M,0), \nabla, P)$ be a $(TEP(w))$-structure with generation condition (GC) (theorem 2.5) and let $(\widetilde{H} \to (\mathbb{C},0) \times (M \times \mathbb{C}^l,0), \widetilde{\nabla})$ be an unfolding of the underlying (TE)-structure. Then P extends to $\mathcal{O}(\widetilde{H})$ and $(\widetilde{H}, \widetilde{\nabla}, P)$ is a $(TEP(w))$-structure.*

Proof. It is sufficient to consider an unfolding in one parameter y. For some representative of $\widetilde{H}$ the pairing P extends to a ∇-flat pairing on the restriction to $(\mathbb{C}^*,0) \times (M \times \mathbb{C},0)$. We have to show that it takes values on $\mathcal{O}(\widetilde{H})$ in $z^w\mathcal{O}_{\mathbb{C}\times M\times\mathbb{C},0}$. A priori the values are in $\mathcal{O}_{\mathbb{C}^*\times M\times\mathbb{C},0}$. Denote $n := \operatorname{rk} H$ and let $(z, t_1, ..., t_m, y)$ be coordinates on $(\mathbb{C} \times M \times \mathbb{C}, 0)$. Choose an $\mathcal{O}_{\mathbb{C}\times M\times\mathbb{C},0}$-basis $\widetilde{v}_1, ..., \widetilde{v}_n$ of $\mathcal{O}(\widetilde{H})$ with connection matrix

$$\Omega = \frac{1}{z}\sum_{i=1}^m C_i \mathrm{d}t_i + \frac{1}{z}F\mathrm{d}y + \frac{1}{z^2}U\mathrm{d}z \tag{3.4}$$

with matrices $C_i, F, U \in M(n \times n, \mathcal{O}_{\mathbb{C}\times M\times\mathbb{C},0})$, and the matrix

$$R := (P(\widetilde{v}_i, \widetilde{v}_j)) \in M(n \times n, \mathcal{O}_{\mathbb{C}^*\times M\times\mathbb{C},0}). \tag{3.5}$$

Flatness and z-sesquilinearity of the pairing give

$$\mathrm{d}R(z,t,y) = \Omega^{tr}(z,t,y)R(z,t,y) + R(z,t,y)\Omega(-z,t,y), \tag{3.6}$$

that means,

$$z\frac{\partial}{\partial z}R(z,t,y) = \frac{1}{z}U^{tr}(z,t,y)R(z,t,y) - \frac{1}{z}R(z,t,y)U(-z,t,y), \tag{3.7}$$

$$\frac{\partial}{\partial t_i}R(z,t,y) = \frac{1}{z}C_i^{tr}(z,t,y)R(z,t,y) - \frac{1}{z}R(z,t,y)C_i(-z,t,y), \tag{3.8}$$

$$\frac{\partial}{\partial y}R(z,t,y) = \frac{1}{z}F^{tr}(z,t,y)R(z,t,y) - \frac{1}{z}R(z,t,y)F(-z,t,y). \tag{3.9}$$

Write R as a power series

$$R(z,t,y) = \sum_{k=0}^{\infty} R^{(k)}(z,t) \text{ with } R^{(k)} \in M(n\times n, \mathcal{O}_{\mathbb{C}^*\times M,0}\cdot y^k) \tag{3.10}$$

and define

$$R^{(\leq k)}(z,t,y) := \sum_{j=0}^{k} R^{(j)}(z,t), \tag{3.11}$$

analogously for C_i, F, U, with $C_i^{(k)}, F^{(k)}, U^{(k)} \in M(n\times n, \mathcal{O}_{\mathbb{C}\times M,0}\cdot y^k)$. Then $R^{(0)} \in M(n\times n, z^w\mathcal{O}_{\mathbb{C}\times M,0})$ because (H,∇,P) is a $(TEP(w))$-structure.

Induction hypothesis for $k\in\mathbb{Z}_{\geq 0}$:

$$R^{(\leq k)} \in M(n\times n, z^w\mathcal{O}_{\mathbb{C}\times M\times\mathbb{C},0}). \tag{3.12}$$

Induction step from k to $k+1$: Recall the definition of $M(>k)$ in (2.35). The equations (3.7) and (3.8) show that one has modulo $M(>k)$

$$\begin{aligned} & C_i^{(\leq k)tr}(0,t,y)[z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0} \\ \equiv\ & [z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0}C_i^{(\leq k)}(0,t,y), \end{aligned} \tag{3.13}$$

$$\begin{aligned} & U^{(\leq k)tr}(0,t,y)[z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0} \\ \equiv\ & [z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0}U^{(\leq k)}(0,t,y). \end{aligned} \tag{3.14}$$

Because of the generation condition (GC) the matrix $F^{(\leq k)}(0,t,y)$ is an element of the commutative subalgebra of $M(n\times n, \mathcal{O}_{M,0}[y])/M(>k)$ which is generated by $C_1^{(\leq k)},...,C_m^{(\leq k)},U^{(\leq k)}$. Therefore modulo $M(>k)$

$$\begin{aligned} & F^{(\leq k)tr}(0,t,y)[z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0} \\ \equiv\ & [z^{-w}R^{(\leq k)}(z,t,y)]|_{z=0}F^{(\leq k)}(0,t,y). \end{aligned} \tag{3.15}$$

This together with (3.9) completes the induction step. □

Remarks 3.3. (i) A bundle $(H\to(\mathbb{C},0)\times(M,0),\nabla)$ with a logarithmic pole along $\{0\}\times(M,0)$ is also called an (LE)-structure.

(ii) A $(TP(w))$-structure is essentially equivalent to Barannikov's notion of a semi-infinite variation of Hodge structures [Ba2][Ba3].

(iii) The notion of an unfolding of a (TE)-structure (definition 2.3 (a)) carries over to structures of type $(TEP(w))$, (L), $(LP(w))$, (LE), $(LEP(w))$, (T), $(TP(w))$.

(iv) Lemma 2.6 (a) says that any unfolding of an (LE)-structure is trivial. The same is true for (L)-structures. Therefore the analogue of lemma 3.2 holds for $(LEP(w))$-structures and $(LP(w))$-structures trivially. An (L)-structure comes equipped with a residual connection ∇^{res} as in lemma 2.6. In fact, an (L)-structure is just a germ of a holomorphic family of flat connections, parametrized by $(\mathbb{C},0)$ with the connection ∇^{res} for the parameter $z=0$.

(v) The analogue of lemma 2.7 for (T)-structures is easy: A (T)-structure $(H \to (\mathbb{C},0)\times(M,0),\nabla)$ can be extended to a trivial bundle $(H^{(gl)} \to \mathbb{P}^1\times(M,0),\nabla)$ with a holomorphic family of flat connections on the restrictions $H^{(gl)}|_{\{z\}\times(M,0)}$ for $z\in\mathbb{P}^1-\{0\}$. To see this, one chooses an $\mathcal{O}_{\mathbb{C},0}$-basis of sections of $\mathcal{O}(H|_{(\mathbb{C},0)\times\{0\}})$; one glues $H|_{(\mathbb{C},0)\times\{0\}}$ to a trivial bundle on $\mathbb{P}^1-\{0\}$, using this basis; one extends the trivial bundle to $(\mathbb{P}^1-\{0\})\times(M,0)$ and glues it to H with ∇.

(vi) A (T)-structure $(H\to(\mathbb{C},0)\times(M,0),\nabla)$ comes equipped with a Higgs field C on $K:=H|_{\{0\}\times(M,0)}$ as in lemma 2.4.

(vii) The analogues of theorem 2.5 and lemma 2.9 hold for (T)-structures. The proofs are the same, of course without the data encoding the part of the connection in z-direction. The generation condition reads:
(GC)' A vector $\zeta\in K_0$ exists which together with its images under iterations of the maps $C_X: K_0\to K_0$, $X\in T_0M$, generates K_0.

(viii) The analogue of lemma 3.2 holds for (T)-structures with the generation condition (GC)'. The proof is the same.

(ix) Lemma 3.4 below holds also for $(TP(w))$-structures and $(LP(w))$-structures, of course except (3.18) and (3.19).

Lemma 3.4. *(a)* [He2, 2.5] *Let* $(H\to(\mathbb{C},0)\times(M,0),\nabla,P)$ *be a* $(TEP(w))$-*structure with* $K,C,\mathcal{U}$ *as in lemma 2.4. Define a pairing* $g:\mathcal{O}(K)\times\mathcal{O}(K)\to\mathcal{O}_{M,0}$ *by*

$$g([a],[b]) := z^{-w}P(a,b) \mod z\mathcal{O}_{\mathbb{C}\times M,0} \text{ for } a,b\in\mathcal{O}(H). \tag{3.16}$$

It is $\mathcal{O}_{M,0}$-*bilinear, symmetric, nondegenerate, and it satisfies*

$$g(C_Xa,b) = g(a,C_Xb) \text{ for } X\in\mathcal{T}_{M,0}, a,b\in\mathcal{O}(K). \tag{3.17}$$

$$g(\mathcal{U}a,b) = g(a,\mathcal{U}b) \text{ for } a,b\in\mathcal{O}(K). \tag{3.18}$$

(b) [He2, 5.1] *Let* $(H\to(\mathbb{C},0)\times(M,0),\nabla,P)$ *be an* $(LEP(w))$-*structure with* $K:=H|_{\{0\}\times(M,0)}$, *residual connection* ∇^{res} *and residue endomorphism* $\mathcal{V}^{res}$ *as in lemma 2.6 (b). Define a pairing* g *as in (a). It is* $\mathcal{O}_{M,0}$-*bilinear, symmetric, nondegenerate,* ∇^{res}-*flat, and it satisfies*

$$g(\mathcal{V}^{res}a,b)+g(a,\mathcal{V}^{res}b) = w\cdot g(a,b) \text{ for } a,b\in\mathcal{O}(K). \tag{3.19}$$

Proof. All statements follow easily from the ∇-flatness of P and its other properties, See [He2, Lemma 2.14 and Lemma 5.3] for details. □

4. Construction theorem for Frobenius manifolds

Associated to a holomorphic Frobenius manifold $\widetilde{M}$ is a series of meromorphic connections, parametrized by $w\in\mathbb{Z}$, the *(first) structure connections* (lemma 4.4). Under certain assumptions theorem 2.5 allows to reconstruct anyone of them from its restriction to a submanifold $M \subset \widetilde{M}$. This restricted connection is considered as initial datum. Definition 4.1 and theorem 4.2 formalize its properties in two equivalent ways. Theorem 4.5 is a construction theorem for Frobenius manifolds, starting from such an initial datum. Definition 4.1 and theorem 4.2 are also discussed (with different notations) in [Sab1, I 1] [Sab2, VI 2].

Definition 4.1. [He2, 5.2] (a) Fix $w \in \mathbb{Z}$. A $(trTLEP(w))$-structure is a tuple $((M,0), H, \nabla, P)$ with the following properties. $(M,0)$ is a germ of a complex manifold; $H \to \mathbb{P}^1 \times (M,0)$ is a trivial holomorphic vector bundle with a flat connection on $H|_{\mathbb{C}^* \times (M,0)}$; and P is a $(-1)^w$-symmetric, nondegenerate, ∇-flat pairing

$$P : H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C} \text{ for } (z,t) \in (\mathbb{C}^*, 0) \times (M,0). \tag{4.1}$$

The restriction of (H, ∇, P) to the germ $(\mathbb{C}, 0) \times (M, 0)$ is a $(TEP(w))$-structure and the restriction to the germ $(\mathbb{P}^1, \infty) \times (M, 0)$ is an $(LEP(-w))$-structure (definition 3.1).

(b) A *Frobenius type structure* is a tuple $((M,0), K, \nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ with the following properties. $(M,0)$ is a germ of a complex manifold; $K \to (M,0)$ is a germ of a holomorphic vector bundle with flat connection ∇^r; the map $C : \mathcal{O}(K) \to \Omega^1_{M,0} \otimes \mathcal{O}(K)$ is a Higgs bundle, i.e., a map such that all the endomorphisms $C_X : \mathcal{O}(K) \to \mathcal{O}(K)$, $X \in \mathcal{T}_{M,0}$, commute; the endomorphism $\mathcal{U} : \mathcal{O}(K) \to \mathcal{O}(K)$ of K satisfies $[C, \mathcal{U}] = 0$; the endomorphism $\mathcal{V} : \mathcal{O}(K) \to \mathcal{O}(K)$ of K is ∇^r-flat; and $g : \mathcal{O}(K) \times \mathcal{O}(K) \to \mathcal{O}_{M,0}$ is a symmetric, nondegenerate, ∇^r-flat pairing. These data satisfy

$$\nabla^r_X(C_Y) - \nabla^r_Y(C_X) - C_{[X,Y]} = 0, \tag{4.2}$$
$$\nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C = 0, \tag{4.3}$$
$$g(C_X a, b) = g(a, C_X b), \tag{4.4}$$
$$g(\mathcal{U}a, b) = g(a, \mathcal{U}b), \tag{4.5}$$
$$g(\mathcal{V}a, b) = -g(a, \mathcal{V}b) \tag{4.6}$$

for $X, Y \in \mathcal{T}_{M,0}$, $a, b \in \mathcal{O}(K)$.

Theorem 4.2. [Sab1, I 1][Sab2, VI 2][He2, 5.2] *Fix $w \in \mathbb{Z}$. There is a one-to-one correspondence between $(trTLEP(w))$-structures and Frobenius type structures on holomorphic vector bundles. It is given by the steps in (a) and (b). They are inverse to one another.*

(a) Let $(K \to (M,0), \nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ be a Frobenius type structure on K. Let $\pi : \mathbb{P}^1 \times (M,0) \to (M,0)$ be the projection. Define $H := \pi^* K$, and let $\psi_z : H_{(z,t)} \to K_t$ for $z \in \mathbb{P}^1$ be the canonical projection. Extend $\nabla^r, C, \mathcal{U}, \mathcal{V}, g$ canonically to H. Define

$$\nabla := \nabla^r + \frac{1}{z} C + (\frac{1}{z}\mathcal{U} - \mathcal{V} + \frac{w}{2}\,\mathrm{id})\frac{\mathrm{d}z}{z}\ . \tag{4.7}$$

Define a pairing

$$\begin{array}{rcl} P : H_{(z,t)} \times H_{(-z,t)} & \to & \mathbb{C} \quad \textit{for } (z,t) \in (\mathbb{C}^*, 0) \times (M,0) \\ (a,b) & \mapsto & z^w g(\psi_z a, \psi_{-z} b)\ . \end{array} \tag{4.8}$$

Then (H,∇,P) is a $(trTLEP(w))$-structure.

(b) Let (H,∇,P) be a $(trTLEP(w))$-structure. Define $K := H|_{\{0\}\times(M,0)}$, C and $\mathcal{U}$ as in lemma 2.4 and g as in lemma 3.4 (a). Let ∇^{res} and $\mathcal{V}^{res}$ be the residual connection and the residue endomorphism on $H|_{\{\infty\}\times(M,0)}$ as in lemma 2.6 (b).

Because H is a trivial bundle, there is a canonical projection $\psi : H \to K$, and the bundles K and $H|_{\{\infty\}\times(M,0)}$ are canonically isomorphic. Structure on $H|_{\{\infty\}\times(M,0)}$ can be shifted to K. Let ∇^r on K be the shift of ∇^{res} and let $\mathcal{V}$ on K be the shift of $\mathcal{V}^{res} + \frac{w}{2}\,\mathrm{id}$. Then $(K \to M, \nabla^r, C, g, \mathcal{U}, \mathcal{V})$ is a Frobenius type structure and (4.7) holds.

Proof. Part of it follows from the lemmas 2.4, 2.6 (b) and 3.4. For the rest and for details see [He2, Theorem 5.7] or [Sab2, VI 2]. □

Frobenius type structures and $(trTLEP(w))$-structures can be restricted to any submanifold of the manifold $(M,0)$ over which they are defined.

Definition 4.3. (Dubrovin) A *Frobenius manifold* $(M,\circ,e,E,g)$ is a complex manifold M of dimension ≥ 1 with a commutative and associative multiplication $\circ$ on the holomorphic tangent bundle TM , a unit field $e \in \mathcal{T}_M$, an *Euler field* $E \in \mathcal{T}_M$, and a symmetric nondegenerate $\mathcal{O}_{M,0}$-bilinear pairing g on TM with the following properties. The metric g is multiplication invariant,

$$g(X\circ Y, Z) = g(Y, X\circ Z) \quad \text{for } X,Y,Z \in \mathcal{T}_M; \tag{4.9}$$

the Levi–Civita connection ∇^g of the metric g is flat; together with the Higgs field $C : \mathcal{T}_M \to \Omega^1_{M,0} \otimes \mathcal{T}_M$ with $C_X Y := -X \circ Y$ it satisfies the potentiality condition

$$\nabla^g_X(C_Y) - \nabla^g_Y(C_X) - C_{[X,Y]} = 0 \quad \text{for } X,Y \in \mathcal{T}_M; \tag{4.10}$$

the unit field e is ∇^g-flat; the Euler field satisfies $\mathrm{Lie}_E(\circ) = \circ$ and $\mathrm{Lie}_E(g) = (2-d)\cdot g$ for some $d \in \mathbb{C}$.

Lemma 4.4. *(Structure connections of a Frobenius manifold, e.g.* [Du, Lecture 3], [Man2, I 2.5.2], [Sab2, VII 1], [He2, Lemma 5.11]*) Let $((M,0),\circ,e,E,g)$ be the germ of a Frobenius manifold with Higgs field C, Levi-Civita connection ∇^g and $d \in \mathbb{C}$ as in definition 4.3. Define the endomorphisms $\mathcal{U} := E\circ : \mathcal{T}_{M,0} \to \mathcal{T}_{M,0}$ and*

$$\mathcal{V} := \mathcal{T}_{M,0} \to \mathcal{T}_{M,0},\ X \mapsto \nabla^g_X E - \frac{2-d}{2} X. \tag{4.11}$$

Then $(TM,\nabla^g,C,\mathcal{U},\mathcal{V},g)$ is a Frobenius type structure on TM. The unit field e satisfies $\nabla^g e = 0$ and $\mathcal{V}e = \frac{d}{2}e$.

For any $w \in \mathbb{Z}$ theorem 4.2 gives a $(trTLEP(w))$-structure $((M,0), H = \pi^* TM, \nabla, P)$, where $\pi : \mathbb{P}^1 \times (M,0) \to (M,0)$ is the projection. These structures are called (first) structure connections of the Frobenius manifold.

One can recover a Frobenius manifold from a structure connection and the unit field. More abstractly, one can construct from a $(trTLEP(w))$-structure and a global section with sufficiently nice properties a unique Frobenius manifold such that the $(trTLEP(w))$-structure and the global section are isomorphic to a structure connection and the unit field [Sab1][Sab2][Ba1][Ba2][Ba3] (cf. remark 4.6 (i)). Theorem 2.5 allows to start under certain assumptions with a $(trTLEP(w))$-structure and a global section over a *smaller* base space, to unfold them universally and then get a Frobenius manifold. This is formulated in theorem 4.5 in terms of Frobenius type structures.

Theorem 4.5. *(Construction theorem for Frobenius manifolds) Let $((M,0),K,\nabla^r,C,\mathcal{U},\mathcal{V},g)$ be a Frobenius type structure and $\zeta\in K_0$ a vector with the following properties:*

(IC) (injectivity condition) the map $C_\bullet\zeta : T_0M\to K_0,\ X\mapsto C_X\zeta$ is injective.

(GC) (generation condition) ζ and its images under iteration of the maps $C_X : K_0\to K_0,\ X\in T_0M,$ and $\mathcal{U}: K_0\to K_0$ generate K_0.

(EC) (eigenvector condition) $\mathcal{V}\zeta=\frac{d}{2}\zeta$ for some $d\in\mathbb{C}$.

Then there exist up to canonical isomorphism unique data $((\widetilde{M},0),\circ,e,E,\widetilde{g},i,j)$ with the following properties. $((\widetilde{M},0),\circ,e,E,\widetilde{g})$ is a germ of a Frobenius manifold, $i:(M,0)\to(\widetilde{M},0)$ is an embedding, $j:K\to T\widetilde{M}|_{i(M)}$ is an isomorphism above i of germs of vector bundles which maps ζ to $e|_0$ and which identifies the Frobenius type structure on K with the natural Frobenius type structure on $T\widetilde{M}|_{i(M)}$ which is induced by that on $T\widetilde{M}$.

Proof. Choose $w\in\mathbb{Z}$. Let $((M,0),H,\nabla,P)$ be the $(trTLEP(w))$-structure which corresponds to the Frobenius type structure on $K\to(M,0)$ by theorem 4.2. Its germ over $(\mathbb{C},0)\times(M,0)$ is a $(TEP(w))$-structure with all the properties in theorem 2.5. Consider a universal unfolding of this germ with base space $(\widetilde{M},0)=(M\times\mathbb{C}^l,0)$. It extends uniquely to a $(trTLEP(w))$-structure $((\widetilde{M},0),\widetilde{H},\widetilde{\nabla},P)$ which unfolds $((M,0),H,\nabla,P)$. This follows from the rigidity of logarithmic poles (lemma 2.6 (a) and remark 3.3 (iii)) and from lemma 3.2.

Let $((\widetilde{M},0),\widetilde{K},\widetilde{\nabla}^r,\widetilde{C},\widetilde{\mathcal{U}},\widetilde{\mathcal{V}},\widetilde{g})$ be the Frobenius type structure which corresponds to this $(trTLEP(w))$-structure by theorem 4.2. There is a canonical isomorphism from the Frobenius type structure on K to the restricted one on $\widetilde{K}|_{(M\times\{0\},0)}$. Let $\widetilde{\zeta}\in\widetilde{K}_0$ be the image of $\zeta\in K_0$. It extends to a unique $\widetilde{\nabla}^r$-flat section $\widetilde{v}_1\in\mathcal{O}(\widetilde{K})$. The map

$$v:\mathcal{T}_{\widetilde{M},0}\to\mathcal{O}(\widetilde{K}),\ X\mapsto -C_X\widetilde{v}_1 \tag{4.12}$$

is an isomorphism. It allows to shift the structure on $\widetilde{K}$ to structure on $T\widetilde{M}$. Define

$$\nabla^v:=v^*\widetilde{\nabla}^r,\ g^v:=v^*\widetilde{g},\ e:=v^{-1}(\widetilde{v}_1),\ E:=v^{-1}(\widetilde{\mathcal{U}}(\widetilde{v}_1)). \tag{4.13}$$

The connection ∇^v on $T\widetilde{M}$ is flat with $\nabla^v g^v=0$ and $\nabla^v e=0$. Applying (4.2) to $\widetilde{v}_1$ shows that ∇^v is torsion free; so it is the Levi–Civita connection of g^v. One defines a commutative and associative multiplication $\circ$ on $T\widetilde{M}$ by

$$v(X\circ Y)=-C_X v(Y)=C_X C_Y\widetilde{v}_1. \tag{4.14}$$

Then e is the unit field. The potentiality condition follows from (4.2). It rests to prove $\mathrm{Lie}_E(\circ) = \circ$ and $\mathrm{Lie}_E(g) = (2-d)\cdot g$. We refer to [He2, Theorem 5.12] for details. The calculations use (4.2), (4.3), (4.6), $\widetilde{\nabla}^r \widetilde{v}_1 = 0$ and $\widetilde{\mathcal{V}}\widetilde{v}_1 = \frac{d}{2}\widetilde{v}_1$. They show especially

$$v(\nabla^v_X E) = (\widetilde{\mathcal{V}} + \frac{2-d}{2}\,\mathrm{id})v(X) \quad \text{for } X \in \mathcal{T}_{M,0}. \tag{4.15}$$

One obtains a germ of a Frobenius manifold $((\widetilde{M},0),\circ,e,E,g^v)$. Each step in its construction is essentially unique. □

Remarks 4.6. (i) Theorem 4.5 is reduced with theorem 2.5 to the case when $M = \widetilde{M}$ and when the map $C_\bullet\zeta : T_0M \to K_0$ is an isomorphism. This case was formulated by Sabbah [Sab1, Theorem (4.3.6)] [Sab2, Théorème VII.3.6], and independently by Barannikov [Ba1][Ba2][Ba3]. He called a major part of the initial data *semi-infinite variation of Hodge structures* (cf. remark 3.3 (ii)). Theorem 4.5 in the case $M = \widetilde{M}$ is also implicit in the construction in singularity theory [SK][SM].

(ii) A Frobenius type structure with $M = \{0\}$ is simply a vector space K with a pairing g and two endomorphisms $\mathcal{U}$ and $\mathcal{V}$ which satisfy (4.5) and (4.6). Then the condition (IC) in theorem (4.5) is empty, (GC) must be satisfied by $\mathcal{U}$ alone, and (GC) and (EC) together are still more restrictive. For example, this situation is satisfied at a point of a Frobenius manifold where the multiplication with the Euler field is semisimple with different eigenvalues. This case was first considered by Dubrovin [Du, Lecture 3].

(iii) The case of a Frobenius type structure with $\mathcal{U} = 0$ can be considered as opposite to the case in (ii). It will be discussed in chapter 5.

(iv) One can define $(trTLP(w))$-structures and Frobenius type structures without operators $\mathcal{U}$ and $\mathcal{V}$ [He2, 5.2]. Omitting the corresponding parts in theorem 4.2, one obtains a one-to-one correspondence between them. One can define Frobenius manifolds without Euler field. The analogues of lemma 4.4 and theorem 4.5 hold. This follows with the remarks 3.3. But now the generation condition (GC) requires $\dim M > 0$.

(v) All the structures in chapters 2 to 4 were convergent with respect to parameters $(t_1, ..., t_m) \in (\mathbb{C}^m, 0) \cong (M, 0)$. One can formulate everything in structures which are formal in these parameters.

(vi) A. Kresch [Kr, Theorem 1] proved a strong reconstruction theorem (existence and uniqueness) for formal germs of Frobenius manifolds without Euler field and with some additional properties typical for quantum cohomology. It strengthens the first reconstruction theorem in [KM, Theorem 3.1], which establishes only uniqueness. It looks as if his result is the special case of the analogue of theorem 4.5 without Euler field and with $(M,0) \subset (\widetilde{M},0)$ being the small quantum cohomology space. The collection of $N(\beta,d)$ in [Kr, Theorem 1] gives the Higgs field C on $T\widetilde{M}|_M$, the conditions on A_1 and A give the generation condition (GC)' and the property $C_XC_Y = C_YC_X$ for $X,Y \in \mathcal{T}_M$ of the Higgs bundle. The other conditions on Frobenius type structures (without $\mathcal{U}$ and $\mathcal{V}$) seem to be built-in.

5. H^2-generated variations of filtrations and Frobenius manifolds

In this chapter the special case of the construction theorem 4.5 for Frobenius manifolds is studied when the Frobenius type structure satisfies $\mathcal{U} = 0$. A Frobenius type structure with $\mathcal{U} = 0$ is equivalent to a variation of filtrations with Griffiths transversality and additional structure (lemma 5.1). Typical cases are variations of polarized Hodge structures with a condition, which is called H^2-generation condition, motivated by quantum cohomology. The definitions 5.3 and 5.4 present the relevant notions, theorem 4.5 reformulates the construction theorem in the case $\mathcal{U} = 0$.

Lemma 5.1. *(a) The structures in* (α) *and* (β) *are equivalent.*

(α) *A Frobenius type structure* $((M,0), K, \nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ *together with an integer* w *such that* $\mathcal{U} = 0$ *and* $\mathcal{V}$ *is semisimple with eigenvalues in* $\frac{w}{2} + \mathbb{Z}$.

(β) *A tuple* $((M,0), K, \nabla, F^\bullet, U_\bullet, w, S)$. *Here* $K \to (M,0)$ *is a germ of a holomorphic vector bundle;* ∇ *is a flat connection on* K*;* $F^\bullet$ *is a decreasing filtration by germs of holomorphic subbundles* $F^p \subset K$, $p \in \mathbb{Z}$, *which satisfies Griffiths transversality*

$$\nabla : \mathcal{O}(F^p) \to \Omega^1_{M,0} \otimes \mathcal{O}(F^{p-1}); \tag{5.1}$$

$U_\bullet$ *is an increasing filtration by flat subbundles* $U_p \subset K$ *such that*

$$H = F^p \oplus U_{p-1} = \bigoplus_q F^q \cap U_q; \tag{5.2}$$

$w \in \mathbb{Z}$, *and* S *is a* ∇*-flat,* $(-1)^w$*-symmetric, nondegenerate pairing on* K *with*

$$S(F^p, F^{w+1-p}) = 0, \tag{5.3}$$

$$S(U_p, U_{w-1-p}) = 0. \tag{5.4}$$

(b) One passes from (α) *to* (β) *by defining*

$$\nabla \;:=\; \nabla^r + C, \tag{5.5}$$

$$F^p \;:=\; \bigoplus_{q \geq p} \ker(\mathcal{V} - (q - \frac{w}{2})\,\mathrm{id} : K \to K), \tag{5.6}$$

$$U_p \;:=\; \bigoplus_{q \leq p} \ker(\mathcal{V} - (q - \frac{w}{2})\,\mathrm{id} : K \to K), \tag{5.7}$$

$$S(a,b) \;:=\; (-1)^p g(a,b) \text{ for } a \in \mathcal{O}(F^p \cap U_p), b \in \mathcal{O}(K). \tag{5.8}$$

Proof. First we prove part (b). The connection ∇^r and the Higgs field C are maps $\mathcal{O}(K) \to \Omega^1_{M,0} \otimes \mathcal{O}(K)$. They can be extended canonically to maps $\Omega^1_{M,0} \otimes \mathcal{O}(K) \to \Omega^2_{M,0} \otimes \mathcal{O}(K)$. Then the flatness of ∇^r means $(\nabla^r)^2 = 0$, the Higgs field C satisfies $C^2 = 0$, and the potentiality condition (4.2) means $\nabla^r(C) := \nabla^r \circ C + C \circ \nabla^r = 0$. Therefore $\nabla^2 = (\nabla^r + C)^2 = 0$, the connection ∇ is flat. The filtrations $F^\bullet$ and $U_\bullet$ obviously satisfy (5.2) and

$$F^p \cap U_p = \ker(\mathcal{V} - (p - \frac{w}{2})\,\mathrm{id} : K \to K). \tag{5.9}$$

The connection ∇^r maps $\mathcal{O}(F^p \cap U_p)$ to itself because $\mathcal{V}$ is ∇^r-flat. Because of $\mathcal{U} = 0$ the condition (4.3) is $[C, \mathcal{V}] = C$. Equivalent is that C_X, $X \in \mathcal{T}_{M,0}$, maps $\mathcal{O}(F^p \cap U_P)$ to $\mathcal{O}(F^{p-1} \cap U_{p-1})$. Therefore $U_\bullet$ is ∇-flat and $F^\bullet$ satisfies Griffiths transversality.

The condition (4.6) says that for $a \in \mathcal{O}(F^p \cap U_p)$, $b \in \mathcal{O}(F^q \cap U_q)$

$$S(a,b) = (-1)^p g(a,b) = 0 \text{ if } p+q \neq w. \tag{5.10}$$

Therefore S is $(-1)^w$-symmetric and satisfies (5.3) and (5.4). It is ∇-flat because for $X \in \mathcal{T}_{M,0}$ and ∇^r-flat sections $a \in \mathcal{O}(F^p \cap U_p)$, $b \in \mathcal{O}(K)$

$$\begin{aligned}(\nabla_X S)(a,b) &= X\, S(a,b) - S(\nabla^r_X a + C_X a, b) - S(a, \nabla^r_X b + C_X b) \\ &= 0 - (-1)^{p+1} g(C_X a, b) - (-1)^p g(a, C_X b) = 0. \end{aligned}\tag{5.11}$$

This shows part (b). One passes from (β) to (α) as follows. One defines the endomorphism $\mathcal{V}$ by (5.9), the pairing g by (5.8), and one decomposes ∇ into ∇^r and C such that ∇^r maps $\mathcal{O}(F^p \cap U_p)$ to itself and C_X for $X \in \mathcal{T}_{M,0}$ maps $\mathcal{O}(F^p \cap U_p)$ to $\mathcal{O}(F^{p-1} \cap U_{p-1})$. One easily checks all conditions of a Frobenius type structure. □

Remarks 5.2. (i) If one adds in lemma 5.1 (a) (β) real structure with suitable conditions then one obtains a germ of a variation of polarized Hodge structures of weight w.

(ii) A Frobenius type structure is equivalent to a $(trTLEP(w))$-structure (theorem 4.2), which is composed of a $(TEP(w))$-structure at $(\mathbb{C},0) \times (M,0)$ and an $(LEP(w))$-structure at $(\mathbb{P}^1,\infty) \times (M,0)$. One can refine lemma 5.1 (a). There is a correspondence between $(TEP(w))$-structures with $\mathcal{U} = 0$ and monodromy $(-1)^w$ id on the one hand and germs of variations of filtrations with Griffiths transversality and a pairing on the other hand [He2, Corollary 7.14]. Under this correspondence an $(LEP(w))$-structure corresponds to a trivial, flat variation of filtrations. Putting these together, a $(trTLEP(w))$-structure corresponds to the structure in lemma 5.1 (a) (β).

But the variation of filtrations $F^\bullet$ which corresponds to a $(TEP(w))$-structure $(H \to (\mathbb{C},0) \times (M,0), \nabla)$ does not live on the bundle $K = H|_{\{0\}\times(M,0)}$. It lives on a flat bundle on $(M,0)$ whose fibers are all isomorphic to the space of global flat manyvalued sections in $H|_{(\mathbb{C}^*,0)\times(M,0)}$. The bundle K is only isomorphic to $\bigoplus_p F^p/F^{p+1}$. Only in the case of a second filtration $U_\bullet$ and a splitting (5.2) one obtains an isomorphism between K and the flat bundle on which $F^\bullet$ lives.

Definition 5.3. (a) A germ of an *H^2-generated variation of filtrations of weight w* is a tuple $((M,0), K, \nabla, F^\bullet, w)$ with the following properties. w is an integer, $K \to (M,0)$ is a germ of a holomorphic vector bundle with flat connection ∇ and variation of filtrations $F^\bullet$ which satisfies Griffiths transversality (5.1) and

$$0 = F^w \subset F^{w-1} \subset \ldots \subset K \tag{5.12}$$

$$\operatorname{rk} F^{w-1} = 1, \quad \operatorname{rk} F^{w-2} = 1 + \dim M \geq 2. \tag{5.13}$$

Griffiths transversality and flatness of ∇ give a Higgs field C on the graded bundle $\bigoplus_p F^p/F^{p+1}$ with commuting endomorphisms

$$C_X = [\nabla_X] : \mathcal{O}(F^p/F^{p+1}) \to \mathcal{O}(F^{p-1}/F^p) \text{ for } X \in \mathcal{T}_{M,0}. \tag{5.14}$$

H^2-generation condition: the whole module $\bigoplus_p \mathcal{O}(F^p/F^{p+1})$ is generated by $\mathcal{O}(F^{w-1})$ and its images under iterations of the maps C_X, $X \in \mathcal{T}_{M,0}$.

(b) A pairing and an opposite filtration for an H^2-generated variation of filtrations $((M,0),K,\nabla,F^\bullet)$ of weight w are a pairing S and a filtration $U_\bullet$ as in lemma 5.1 (a) (β).

Definition 5.4. An H^2-generated germ of a Frobenius manifold of weight $w \in \mathbb{N}_{\geq 3}$ is a germ $((M,0),\circ,e,E,g)$ of a Frobenius manifold with the properties (5.15), (I) and (II).

$$E|_{t=0} = 0. \tag{5.15}$$

Let ∇^g be the Levi–Civita connection of g. The endomorphism $\nabla^g E : \mathcal{T}_{M,0} \to \mathcal{T}_{M,0}$, $X \mapsto \nabla^g_X E$, acts on the space of ∇^g-flat vector fields. In particular $e \in \ker(\nabla^g E - \mathrm{id})$.

(I) It acts semisimply with eigenvalues $\{1, 0, ..., -(w-3)\}$.

It turns out that then the multiplication on the algebra T_0M respects the grading

$$T_0M = \bigoplus_{p=0}^{w-2} \ker(\nabla^g E - (1-p)\,\mathrm{id} : T_0M \to T_0M) \tag{5.16}$$

and that $\mathrm{Lie}_E(g) = (4-w)\cdot g$.

(II) H^2-generation condition: The algebra T_0M is generated by $\ker(\nabla^g E : T_0M \to T_0M)$.

Remarks 5.5. (i) Properties (I) and (5.15) hold for even-dimensional quantum cohomology of Calabi–Yau manifolds of complex dimension $w-2$. This follows from the vanishing of the canonical class and the standard explicit formulas for the Euler field in quantum cohomology. Generally, our terminology involving "H^2-generation" is motivated by quantum cohomology, for which $\ker \nabla^g E = H^2$. For a more extended discussion of special properties of quantum cohomology Frobenius manifolds, see [Man1, 1.3 and 1.4, in particular Definition 1.4.1].

(ii) The uniqueness statement in theorem 5.6 that a structure as in (γ) is determined by a structure as in (β) is essentially a special case of that in [KM, Theorem 3.1 and 3.1.1 a) and b)]. A refined version of it (cf. lemma 8.2) was used already in the proof of [Ba1, Theorem 6.5]. But the existence statement that any structure as in (β) gives rise to a structure as in (γ) is new.

Theorem 5.6. *There is a one-to-one correspondence between the structures in (α), (β) and (γ).*

(α) *A Frobenius type structure $((M,0),K,\nabla^r,C,\mathcal{U},\mathcal{V},g)$ with $\mathcal{U}=0$ and with a fixed vector $\zeta \in K_0$ which satisfies the conditions (IC), (GC) and (EC) in theorem 4.5.*

(β) *A germ of an H^2-generated variation of filtrations $((M,0),K,\nabla,F^\bullet,w,S,U_\bullet)$ of weight $w \in \mathbb{N}_{\geq 3}$ with pairing and opposite filtration and with a fixed generator $\zeta \in (F^{w-1})_0 \subset K_0$.*

(γ) *An H^2-generated germ of a Frobenius manifold $((\widetilde{M},0),\circ,e,E,\widetilde{g})$ of weight $w \in \mathbb{N}_{\geq 3}$.*

One passes from (α) to (β) by lemma 5.1 (b) and from (α) to (γ) by theorem 4.5. One passes from (γ) to (α) by defining

$$M := \{t \in \widetilde{M} \mid E|_t = 0\}, \tag{5.17}$$

$K := T\widetilde{M}|_{(M,0)}$ *with the canonical Frobenius type structure, and* $\zeta := e|_0$. *The eigenvector condition (EC) is* $\mathcal{V}\zeta = \frac{w-2}{2}\zeta$.

Proof. Let us start with (α). We have to show that for some $w \in \mathbb{N}_{\geq 3}$ the endomorphism $\mathcal{V} + \frac{w}{2}\,\mathrm{id}$ is semisimple with eigenvalues in $\{1, 2, ..., w-1\}$ and with $\mathcal{V}\zeta = \frac{w-2}{2}\zeta$.

The eigenvector condition (EC) says $\mathcal{V}\zeta = \frac{d}{2}\zeta$ for some $d \in \mathbb{C}$. Condition (4.3) reads as $[C, \mathcal{V}] = C$. This together with the generation condition (GC) shows that $\mathcal{V}$ acts semisimply on K_0 with eigenvalues in $\frac{d}{2} + \mathbb{Z}_{\leq 0}$ and that $\ker(\mathcal{V} - \frac{d}{2}\,\mathrm{id}) = \mathbb{C}\cdot\zeta$. The injectivity condition (IC) implies $\dim\ker(\mathcal{V} - (\frac{d}{2} - 1)\,\mathrm{id}) = \dim M > 0$. Condition (4.6) tells that the eigenvalues of $\mathcal{V}$ are in $(\frac{d}{2} + \mathbb{Z}_{\leq 0}) \cap -(\frac{d}{2} + \mathbb{Z}_{\leq 0})$. Therefore $d \in \mathbb{N}$. Define $w := d + 2$.

The conditions (IC) and (GC) show that one passes from (α) to (β) by lemma 5.1 (b). It is also clear that one can pass back.

Theorem 4.5 applied to (α) gives a Frobenius manifold $((\widetilde{M}, 0), \circ, e, E, \widetilde{g})$, an embedding $i : (M, 0) \to (\widetilde{M}, 0)$ and an isomorphism $j : K \to T\widetilde{M}|_{i(M)}$ of Frobenius type structures. It maps $\mathcal{V}$ to the restriction of $\nabla^g E + \frac{2-d}{2}\,\mathrm{id}$ on $T\widetilde{M}|_{i(M)}$ (4.15). Therefore one obtains an H^2-generated germ of a Frobenius manifold of weight w. In suitable flat coordinates the Euler field of this Frobenius manifold takes the form

$$E = \sum_{i=1}^{\dim \widetilde{M}} d_i t_i \frac{\partial}{\partial t_i} \tag{5.18}$$

with $d_i \in \{1, 0, ..., -(w-3)\}$ and

$$\sharp(i \mid d_i = 0) = \dim\ker(\nabla^g E) = \dim\ker(\mathcal{V} - (\frac{d}{2} - 1)\,\mathrm{id}) = \dim M. \tag{5.19}$$

Therefore $i(M) = \{t \in \widetilde{M} \mid E|_t = 0\}$. One sees also easily that one passes from (γ) to (α) as described in the theorem. $\square$

A distinguished class of H^2-generated variations of filtrations are variations of Hodge filtrations associated to certain families of homogeneous polynomials. We will discuss them and the corresponding H^2-generated Frobenius manifolds in the chapters 6, 7 and 8. The following example shows that it is easy to construct abstract H^2-generated variations of filtrations. Therefore one has a lot of freedom in constructing H^2-generated Frobenius manifolds.

Example 5.7. Consider $(M, 0) := (\mathbb{C}, 0)$ with coordinate t, the trivial bundle $H := \mathbb{C}^{w-1} \times (M, 0) \to (M, 0)$ for some $w \in \mathbb{N}_{\geq 3}$ with standard basis $v_1, ..., v_{w-1}$ of sections, a pairing S with $S(v_p, v_q) := (-1)^p \delta_{p,w-q}$, and filtrations $F^\bullet$ and $U_\bullet$ with

$$\mathcal{O}(F^p) := \bigoplus_{q \geq p} \mathcal{O}_{\mathbb{C},0} \cdot v_{w-q}, \quad \mathcal{O}(U_p) := \bigoplus_{q \leq p} \mathcal{O}_{\mathbb{C},0} \cdot v_{w-q}. \tag{5.20}$$

Choose *any* invertible functions $b_2, ..., b_{[\frac{w-1}{2}]} \in \mathcal{O}^*_{\mathbb{C},0}$, define

$$b_1 := 1,\ b_{w-1} := 0,\ b_k := b_{w-1-k} \text{ for } k = \left[\frac{w-1}{2}\right] + 1, ..., w-2 \qquad (5.21)$$

and define a connection ∇ on H by

$$\nabla_{\partial/\partial t} v_i := b_i v_{i+1}. \qquad (5.22)$$

Then $((M,0), K, \nabla, F^\bullet, w, S, U_\bullet)$ is a germ of an H^2-generated variation of filtrations of weight w with pairing and opposite filtration. Moreover, one can see that a second tuple $(b'_2, ..., b'_{[\frac{w-1}{2}]})$ of functions yields isomorphic data only if b_i and b'_i coincide up to multiplication by a constant. The big freedom in constructing H^2-generated Frobenius manifolds of dimension $w-1$ is in striking difference to the semisimple case, where one has only finitely many parameters [Man2, II 3.4.3].

6. Hypersurfaces in $\mathbb{P}^n$

In the case of certain families of smooth hypersurfaces in $\mathbb{P}^n$ one obtains variations of Hodge structures which are the prototype of H^2-generated variations of filtrations (definition 5.3). This is a simple consequence of Griffiths' description of the Hodge filtration on the primitive part of the middle cohomology of a smooth hypersurface in $\mathbb{P}^n$ in terms of rational differential forms on $\mathbb{P}^n$ [Gr].

Fix a degree $d \in \mathbb{N}$ and denote by $\mathbb{C}[x]^{(q)}$ for $q \in \frac{1}{d}\mathbb{Z}_{\geq 0}$ the space of homogeneous polynomials in $\mathbb{C}[x_0, ..., x_n] = \mathbb{C}[x]$ of degree $d \cdot q$. Consider a polynomial $f \in \mathbb{C}[x]^{(1)}$ with isolated singularity at 0. The grading on $\mathbb{C}[x]$ induces a grading on the Jacobi algebra

$$R_f := \mathbb{C}[x] / \left(\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_n}\right) \qquad (6.1)$$

of f with subspaces $R_f^{(q)}$, $q \in \frac{1}{d}\mathbb{Z}_{\geq 0}$. The primitive part of the middle cohomology $H^{n-1}(X, \mathbb{C})$ of the smooth hypersurface $X := \overline{f^{-1}(0)} \subset \mathbb{P}^n$ is denoted by $H^{n-1}_{prim}(X)$, its Hodge filtration by $F^\bullet_{prim} \subset H^{n-1}_{prim}(X)$. The primitive cohomology also comes equipped with a polarizing form S.

Now consider a family of polynomials $F_t \in \mathbb{C}[x]^{(1)}$, $t \in M_0$, with isolated singularities at 0, where M_0 is a smooth parameter space with coordinates (t_α) such that for each $t \in M_0$ the map

$$\mathbf{a} : T_t M_0 \to R^{(1)}_{F_t}, \quad \frac{\partial}{\partial t_\alpha} \mapsto \left[\frac{\partial F_t}{\partial t_\alpha}\right] \qquad (6.2)$$

is an isomorphism. The bundle $H := \bigcup_{t \in M_0} H^{n-1}_{prim}(X_t)$ comes equipped with a real subbundle, a flat connection ∇, the flat pairing S and a variation of Hodge filtrations $F^\bullet_{prim}$. Together they form a variation of polarized Hodge structures of weight $n-1$.

Theorem 6.1. *If $\frac{n+1}{d} \in \mathbb{N}$ then the tuple $(M_0, H, \nabla, S, \widetilde{F}^\bullet := F_{prim}^{\bullet - 2 + (n+1)/d})$ is an H^2-generated variation of filtrations of weight $n + 3 - 2(n+1)/d$ with pairing.*

Proof. The space $\Omega^{n+1}_{alg} = \mathbb{C}[x]\mathrm{d}x_0 ... \mathrm{d}x_n$ of algebraic differential forms on $\mathbb{C}^{n+1}$ is graded with subspaces $(\Omega^{n+1}_{alg})^{(q)} = \mathbb{C}[x]^{(q-(n+1)/d)}\mathrm{d}x_0...\mathrm{d}x_n$ for $q \in \frac{1}{d}\mathbb{Z}_{\geq n+1}$. For $f \in \mathbb{C}[x]^{(1)}$ with isolated singularity the quotient

$$\Omega_f := \Omega^{n+1}_{alg}/\mathrm{d}f \wedge \Omega^n_{alg} \tag{6.3}$$

carries an induced grading with subspaces $\Omega_f^{(q)}$. It is a graded module of the graded algebra R_f, and it is a free module of rank 1 of R_f.

Following Griffiths [Gr], one obtains for $q \in \mathbb{Z}_{\geq (n+1)/d}$ a canonical isomorphism

$$\mathbf{r}_q : \Omega_f^{(q)} \to F^{n-q}_{prim}/F^{n-q+1}_{prim} \tag{6.4}$$

in the following way. Let $\mathcal{E} := \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$ be the Euler field on $\mathbb{C}^{n+1}$. For $\omega \in (\Omega^{n+1}_{alg})^{(q)}$, $q \in \mathbb{Z}_{\geq (n+1)/d}$, consider the form $i_\mathcal{E}(\frac{\omega}{f^q})$, which one obtains from $\frac{\omega}{f^q}$ by contraction with the Euler field. It extends to a rational form on $\mathbb{P}^{n+1} = \mathbb{C}^{n+1} \cup \mathbb{P}^n$ with a pole of order $\leq q$ along $f^{-1}(0) \cup X$. We denote the restriction of this form to $\mathbb{P}^n$ by $\overline{i_\mathcal{E}(\frac{\omega}{f^q})}$. It induces a class $\left[\overline{i_\mathcal{E}(\frac{\omega}{f^q})}\right] \in H^n(\mathbb{P}^n - X, \mathbb{C})$. There is a residue map

$$\mathrm{Res} : H^n(\mathbb{P}^n - X, \mathbb{C}) \to H^{n-1}(X, \mathbb{C}), \tag{6.5}$$

dual to a tube map in the homologies. One defines

$$\begin{aligned} \rho_q : (\Omega^{n+1}_{alg})^{(q)} &\to H^{n-1}(X, \mathbb{C}), \\ \omega &\mapsto \mathrm{Res}\left[\overline{i_\mathcal{E}\left(\frac{\omega}{f^q}\right)}\right] \cdot \frac{(q-1)!}{d}. \end{aligned} \tag{6.6}$$

By [Gr], the image of ρ_q is $F^{n-q}_{prim} \subset H^{n-1}_{prim}(X)$ and the preimage of F^{n-q+1}_{prim} is $(\mathrm{d}f \wedge \Omega^n_{alg})^{(q)}$. This gives the isomorphism $\mathbf{r}_q$.

Now for a family of polynomials $F_t \in \mathbb{C}[x]^{(1)}$, $t \in M_0$, with isolated singularities at 0 the infinitesimal variation of Hodge structures can be calculated. For each $t \in M_0$ it is a set of commuting linear maps

$$C_{\partial/\partial t_\alpha} = [\nabla_{\partial/\partial t_\alpha}] : F^p_{prim}/F^{p+1}_{prim} \to F^{p-1}_{prim}/F^p_{prim}. \tag{6.7}$$

The calculation (cf. [Do, 2.2])

$$\frac{\partial}{\partial t_\alpha}\frac{\omega}{F_t^q} = (-q)\frac{\frac{\partial F_t}{\partial t_\alpha}\cdot\omega}{F_t^{q+1}} \tag{6.8}$$

shows for $[\omega] \in \Omega_f^{(q)}$

$$C_{\partial/\partial t_\alpha}\mathbf{r}_q([\omega]) = \mathbf{r}_{q+1}\left(-\mathbf{a}(\frac{\partial}{\partial t_\alpha}) \cdot [\omega]\right). \tag{6.9}$$

Now one observes two facts: (i) The subring $\bigoplus_{q\in\mathbb{Z}_{\geq 0}} R_{F_t}^{(q)}$ of R_{F_t} is multiplicatively generated by $R_{F_t}^{(1)}$, because any monomial in $\mathbb{C}[x]^{(q)}$ for $q \in \mathbb{Z}_{\geq 1}$ is a product of monomials in $\mathbb{C}[x]^{(1)}$.
(ii) If $\frac{n+1}{d} \in \mathbb{N}$ then $\bigoplus_{q\in\mathbb{Z}} \Omega_{F_t}^{(q)}$ is a free module of rank 1 of the ring $\bigoplus_{q\in\mathbb{Z}_{\geq 0}} R_{F_t}^{(q)}$.

This shows that for a family of polynomials with (6.2) and $\frac{n+1}{d}\in\mathbb{N}$ the H^2-generation condition in definition 5.3 is satisfied. $\square$

Via theorem 5.6 the variations of Hodge filtrations in theorem 6.1 give rise to Frobenius manifolds. All of them can be identified with submanifolds of Frobenius manifolds which arise in singularity theory, see chapter 7. Those with $d=n+1$ can be identified with submanifolds of Frobenius (super) manifolds in the Barannikov–Kontsevich construction, see chapter 8. The hypersurfaces $X=\overline{f^{-1}(0)}\subset\mathbb{P}^n$ are Calabi–Yau if and only if $d=n+1$.

Much of the preceding discussion generalizes to the case of quasihomogeneous singularities, but not all. Lemma 6.2 gives an example where the H^2-generation condition fails to hold.

A weight system $(w_0,...,w_n)$ with $w_i\in\mathbb{Q}\cap(0,\frac{1}{2}]$ induces a new grading on $\mathbb{C}[x_0,...,x_n]$ whose subspaces we also denote by $\mathbb{C}[x]^{(q)}$. A monomial $x_0^{i_0}...x_n^{i_n}$ is in $\mathbb{C}[x]^{(q)}$ for $q\in\mathbb{Q}_{\geq 0}$ if $\sum i_jw_j=q$. The Jacobi algebra R_f of $f\in\mathbb{C}[x]^{(1)}$ is defined as above and inherits a grading with subspaces $R_f^{(q)}$.

Lemma 6.2. *Consider $n=5$ and $(w_0,...,w_5)=\frac{1}{9}(1,1,1,2,2,2)$ and any weighted homogeneous polynomial $f\in\mathbb{C}[x]^{(1)}$ with isolated singularity at 0. The subspace of $R_f^{(2)}$ which is generated by the set $R_f^{(1)}\cdot R_f^{(1)}$ has codimension 1 in $R_f^{(2)}$.*

Proof. A monomial in $\mathbb{C}[x]^{(2)}$ is a product of monomials in $\mathbb{C}[x]^{(1)}$ if and only if it contains x_0,x_1 or x_2. Therefore it is sufficient to show that the space

$$\mathbb{C}[x_3,x_4,x_5]^{(2)}\cap\left(\frac{\partial f}{\partial x_0},...,\frac{\partial f}{\partial x_n}\right)\tag{6.10}$$

has codimension 1 in $\mathbb{C}[x_3,x_4,x_5]^{(2)}$. Notice $\frac{\partial f}{\partial x_i}|_{\{x_0=x_1=x_2=0\}}=0$ for $i=3,4,5$ and define $\widetilde{f}_i(x_3,x_4,x_5):=\frac{\partial f}{\partial x_i}|_{\{x_0=x_1=x_2=0\}}$ for $i=0,1,2$. The ideal $(\widetilde{f}_0,\widetilde{f}_1,\widetilde{f}_2)\subset\mathbb{C}[x_3,x_4,x_5]$ has an isolated zero at 0. The dimension of $\left(\mathbb{C}[x_3,x_4,x_5]/(\widetilde{f}_0,\widetilde{f}_1,\widetilde{f}_2)\right)^{(2)}$ is independent of the choice of $\widetilde{f}_0,\widetilde{f}_1,\widetilde{f}_2$ as long as the ideal $(\widetilde{f}_0,\widetilde{f}_1,\widetilde{f}_2)$ has an isolated zero at 0. The choice $\widetilde{f}_0=x_3^4$, $\widetilde{f}_1=x_4^4$, $\widetilde{f}_2=x_5^4$ shows that the dimension is 1. $\square$

7. Frobenius manifolds for hypersurface singularities

In theorem 7.3 the Frobenius manifolds which one obtains from theorem 6.1 combined with theorem 5.5 will be identified with submanifolds of Frobenius manifolds in singularity theory. For each holomorphic function germ $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ with an isolated singularity at 0 the base space $M_\mu\subset\mathbb{C}^\mu$ of a semiuniversal unfolding can be equipped with the structure of a Frobenius manifold [SK][SM]. A detailed account is given in [He1]. In [He2, ch. 8] the construction is recasted as the construction of a $(trTLEP(w))$-structure on M_μ and subsequent application of a special case of theorem 4.5. We restrict now to quasihomogeneous singularities and recall some facts from [He1].

Let $(w_0, ..., w_n)$ be a weight system with $w_i \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and $f \in \mathbb{C}[x]^{(1)}$ (notation from the end of chapter 6) a weighted homogeneous polynomial with isolated singularity at 0 and Milnor number μ. From the grading of the Jacobi algebra R_f one obtains the exponents $\alpha_1, ..., \alpha_\mu$, rational numbers with

$$\alpha_1 \leq ... \leq \alpha_\mu, \quad \alpha_1 = \sum w_i, \quad \sharp(i \mid \alpha_i = \alpha) = \dim R_f^{(\alpha - \alpha_1)}. \tag{7.1}$$

Choose polynomials $m_i \in \mathbb{C}[x]^{(\alpha_i - \alpha_1)}$ which represent a basis of the Jacobi algebra and such that $m_1 = 1$. The function

$$F(x_0, ..., x_n, t_1, ..., t_\mu) = f + \sum_{i=1}^{\mu} t_i m_i \tag{7.2}$$

is a semiuniversal unfolding of f. It should be seen as a family of functions F_t with parameter $t = (t_1, ..., t_\mu) \in M_\mu \subset \mathbb{C}^\mu$. Here M_μ is a suitable open neighborhood of 0. The manifold M_μ comes equipped with the unit field $e = \frac{\partial}{\partial t_1}$, the Euler field $E = \sum_{i=1}^{\mu}(1 + \alpha_1 - \alpha_i) t_i \frac{\partial}{\partial t_i}$, and a multiplication $\circ$ on the holomorphic tangent bundle. The multiplication is induced from a canonical isomorphism of $T_t M_\mu$ with the direct sum of Jacobi algebras of the singularities of the function F_t for $t \in M_\mu$. The tuple $(M_\mu, \circ, e, E)$ is an F-manifold [Man2, I §5][He1, I].

With the Gauß–Manin connection one can construct a metric g on M_μ such that $(M_\mu, \circ, e, E, g)$ is a Frobenius manifold [SK][SM]. In general the metric depends on a choice. For the construction we refer to [He1]. Here we merely explain the choice and state the result.

Let H^∞ be the space of global flat multivalued sections of the flat cohomology bundle $\bigcup_{z \in \mathbb{C}^*} H^n(f^{-1}(z), \mathbb{C})$. It comes equipped with a real subbundle $H^\infty_{\mathbb{R}}$, a semisimple monodromy operator $h : H^\infty \to H^\infty$, a monodromy invariant Hodge filtration $F^\bullet$ and a monodromy invariant pairing S^∞ [He1, ch. 10]. Define $H^\infty_\lambda := \ker(h - \lambda\, \mathrm{id}) \subset H^\infty$ and $H^\infty_{\neq 1} := \bigcup_{\lambda \neq 1} H^\infty_\lambda$.

Then $(H^\infty_{\neq 1}, H^\infty_{\mathbb{R}} \cap H^\infty_{\neq 1}, F^\bullet, S^\infty)$ and $(H^\infty_1, H^\infty_{\mathbb{R}} \cap H^\infty_1, F^\bullet, S^\infty)$ are polarized Hodge structures of weight n and $n+1$. An increasing monodromy invariant filtration $U_\bullet$ on H^∞ is called *opposite to* $F^\bullet$ if

$$H^\infty = \bigoplus_p F^p \cap U_p, \tag{7.3}$$

$$S^\infty(H^\infty_{\neq 1} \cap F^p \cap U_p, H^\infty_{\neq 1} \cap F^q \cap U_q) = 0 \text{ for } p + q \neq n, \tag{7.4}$$

$$S^\infty(H^\infty_1 \cap F^p \cap U_p, H^\infty_1 \cap F^q \cap U_q) = 0 \text{ for } p + q \neq n + 1. \tag{7.5}$$

Theorem 7.1. [He1, Theorem 11.1] *Any choice of an opposite filtration $U_\bullet$ induces an up to a scalar unique metric g on M_μ such that $(M_\mu, \circ, e, E, g)$ is a Frobenius manifold. The opposite filtration is uniquely determined by the metric.*

Consider the submanifold

$$M := \{t \in M_\mu \mid t_i = 0 \text{ if } \alpha_i - \alpha_1 \notin \mathbb{Z}\} \subset M_\mu. \tag{7.6}$$

It does not depend on the choice of the coordinates t_i; that means, any choice with $Et_i = (1 + \alpha_1 - \alpha_i)t_i$ gives the same submanifold M. It parametrizes the semiquasihomogeneous deformations of f by polynomials of integer degree.

Unit field e and Euler field E are tangent to M. One can show that the multiplication $\circ$ on TM_μ restricts to a multiplication on TM. If $\alpha_1 = \sum w_i \in \frac{1}{2}\mathbb{N}$ then a much stronger result holds.

Theorem 7.2. [Man2, III 8.7.1] *Suppose that $\alpha_1 = \sum w_i \in \frac{1}{2}\mathbb{N}$. For any metric g as in theorem 7.1, the submanifold M with induced multiplication, metric, Euler field E and unit field e is a Frobenius manifold.*

Going through the construction in [He1, 11.1] one can even see that the metric on M depends only on $U_\bullet \cap H^\infty_{(-1)^{\alpha_1}}$ and that it determines $U_\bullet \cap H^\infty_{(-1)^{\alpha_1}}$ uniquely.

Finally we restrict to the case $(w_0, ..., w_n) = \frac{1}{d}(1, ..., 1)$ for some $d \in \mathbb{N}$ with $\alpha_1 = \frac{n+1}{d} \in \mathbb{N}$. We use the notations in chapter 5. The manifold

$$M_0 := \{t \in M_\mu \mid t_i = 0 \text{ if } \alpha_i - \alpha_1 \neq 1\} \subset M \tag{7.7}$$

parametrizes the homogeneous polynomials F_t in the unfolding F. The map $\mathbf{a}$ in (6.2) is an isomorphism. Theorem 6.1 applies to the family of functions F_t, $t \in M_0$. As in chapter 6, $X_t := \overline{F_t^{-1}(0)} \subset \mathbb{P}^n$ is the hypersurface in $\mathbb{P}^n$ defined by F_t.

Theorem 7.3. *(a) There is a canonical isomorphism*

$$\psi : \bigcup_{t\in M_0} H^\infty_1(F_t) \to \bigcup_{t\in M_0} H^{n-1}_{prim}(X_t) \tag{7.8}$$

of flat bundles with pairings S^∞ and S. For each $t \in M_0$ it is a $(-1,-1)$ morphism of Hodge structures.

(b) The Frobenius manifold structures on M from theorem 7.2 and the Frobenius manifolds constructed in theorem 5.6 from the variation of filtrations $((M_0, 0), H, \nabla, S, \widetilde{F}^\bullet, \widetilde{U}_\bullet)$ in theorem 6.1 with opposite filtration $\widetilde{U}_\bullet$ on $H = \bigcup_{t\in M_0} H^{n-1}_{prim}(X_t)$ are pairwise isomorphic. Two are isomorphic up to multiplication of the metric by a scalar if and only if the opposite filtrations $U_\bullet \cap H^\infty_1$ on H^∞_1 and $\widetilde{U}_\bullet$ on H satisfy

$$\psi(U_{\bullet -1+(n+1)/d} \cap H^\infty_1) = \widetilde{U}_\bullet. \tag{7.9}$$

Proof. (a) This is essentially well known. The following explanations may be helpful. Consider for some $z \in \mathbb{C}^*$ a fiber $F_t^{-1}(z) \subset \mathbb{C}^{n+1}$. The hypersurface $X_t \subset \mathbb{P}^n$ is the part in $\mathbb{P}^n$ of the closure of $F_t^{-1}(z) \subset \mathbb{C}^{n+1}$ in $\mathbb{P}^{n+1} = \mathbb{C}^{n+1} \cup \mathbb{P}^n$. Consider a tubular neighborhood $T(F_t^{-1}(z))$ of X_t in $F_t^{-1}(z)$. There are canonical isomorphisms

$$H^\infty_1 \leftarrow H^n(F_t^{-1}(z), \mathbb{C})_1 \to H^n(T(F_t^{-1}(z)), \mathbb{C}) \to H^{n-1}_{prim}(F_t) \tag{7.10}$$

The first is the extension to flat sections, the last is a residue map and is the dual of a tube map.

Consider a form $\omega \in (\Omega^{n+1}_{alg})^{(q)}$ for $q \in \mathbb{Z}_{\geq (n+1)/d}$ (notation from chapter 6). The restriction of the form $\frac{1}{d} \cdot i_\mathcal{E} \frac{\omega}{f^q}$ to $F_t^{-1}(z)$ is equal to $\frac{1}{z^{q-1}} \frac{\omega}{\mathrm{d}F_t}$, because $\mathrm{d}F_t \wedge i_\mathcal{E}\omega = d \cdot F_t \cdot \omega$. The section $z \mapsto \left[\frac{1}{z^{q-1}} \frac{\omega}{\mathrm{d}F_t}\right]$ of the bundle $\bigcup_{z\in\mathbb{C}^*} H^n(F_t^{-1}(z), \mathbb{C})_1$ is flat.

Now Varchenko's [Va] description of Steenbrink's Hodge filtration $F^\bullet$ on H_1^∞ reduces here to the following: The space $F^{n+1-q} \subset H_1^\infty$ is generated by such flat sections.

In $\mathbb{P}^{n+1} - (F_t^{-1}(z) \cup X_t)$ the set $T(F_t^{-1}(z))$ can be deformed to a tubular neighborhood of X_t in $\mathbb{P}^n - X_t$. The bundle $\bigcup_{z\in\mathbb{C}^*} H^n(F_t^{-1}(z), \mathbb{C})_1$ and the space $H^n(\mathbb{P}^n - X_t)$ glue to a flat bundle on $\mathbb{P}^1 - \{0\}$. The value in $H^n(\mathbb{P}^n - X_t)$ of a flat section as above is just $\left[\frac{1}{d} \cdot \overline{i_{\mathcal{E}} \frac{\omega}{f^q}}\right]$; compare the proof of theorem 6.1. This reduces Varchenko's to Griffiths' description and shows (a).

(b) This follows from part (a), theorem 5.6 and from the following nontrivial fact: the Frobenius type structure on $TM|_{M_0}$ for a Frobenius manifold M in theorem 7.2 corresponds by lemma 5.1 to a variation of filtrations and an opposite filtration which are up to the shift in (7.9) the variation of Hodge structures and an opposite filtration on $\bigcup_{t\in M_0} H_1^\infty(F_t)$.

This fact is a consequence of the construction of Frobenius manifolds in [He1, Theorem 11.1]. □

8. Barannikov–Kontsevich construction

The Barannikov–Kontsevich construction was initiated in [BK] and further developed in [Ba1][Ba2]. It yields for any Calabi–Yau manifold a family of formal germs of Frobenius submanifolds. A central part of it is the construction of a semi-infinite variation of Hodge structures (defined in [Ba2]). This contains the variation of Hodge structures for complex structure deformations of the given Calabi–Yau manifold. Therefore it is not surprising that in the case of a Calabi–Yau hypersurface in $\mathbb{P}^n$ certain submanifolds in the Barannikov–Kontsevich construction coincide with germs of the Frobenius manifolds which one obtains from theorem 6.1 with theorem 5.6. This will be made precise in theorem 8.1. All the results in this chapter are reformulations of results of Barannikov [Ba1][Ba2].

Let X be a compact Calabi–Yau manifold of dimension $n-1$. Let us fix a holomorphic volume form Ω. Barannikov [Ba1][Ba2] constructed a family of formal germs $(M, 0)$ of Frobenius supermanifolds of dimension $m := \dim H^*(X, \mathbb{C})$. For each of them the tangent space T_0M at 0 is isomorphic to

$$H^*(X, \bigwedge^* \mathcal{T}_X) = \bigoplus_{p,q} H^q(X, \bigwedge^p \mathcal{T}_X) \tag{8.1}$$

with the canonical multiplication and with the $\mathbb{Z}_2$-grading $(p+q) \mod 2$. If $t_1, ..., t_m$ are flat coordinates centered at 0 with $\frac{\partial}{\partial t_i}|_0 \in H^{q_i}(X, \bigwedge^{p_i} \mathcal{T}_X)$ then

$$E = \sum_{i=1}^m \frac{2-p_i-q_i}{2} t_i \frac{\partial}{\partial t_i} \tag{8.2}$$

is an Euler field of the Frobenius supermanifold [Ba1, 5.7]. Euler field and flat metric g satisfy $\mathrm{Lie}_E(g) = (3-n)g$ and $g(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}) = 0$ for $(p_i + p_j, q_i + q_j) \neq (n-1, n-1)$.

This family of Frobenius supermanifolds is parametrized by the set of *opposite filtrations* $W_{\leq \bullet}$ to a Hodge filtration $F^{\geq \bullet}$ on $H^*(X, \mathbb{C})$. These filtrations are defined as follows [Ba2, ch. 4 and 6]. Both respect the splitting

$$H^*(X,\mathbb{C}) = H^{even}(X,\mathbb{C}) \oplus H^{odd}(X,\mathbb{C}) \tag{8.3}$$

and are indexed by half integers. The Hodge filtration is given by

$$F^{\geq r} := \bigoplus_{p,q:p-q\geq 2r} H^{p,q}(X) \quad \text{for } r \in \frac{1}{2}\mathbb{Z}. \tag{8.4}$$

A filtration $W_{\leq \bullet}$ is *opposite* if

$$H^*(X,\mathbb{C}) = \bigoplus_{r\in\frac{1}{2}\mathbb{Z}} F^{\geq r}\cap W_{\leq r+1} \quad \text{and} \tag{8.5}$$

$$(F^{\geq r}\cap W_{\leq r+1}, F^{\geq \widetilde{r}}\cap W_{\leq \widetilde{r}+1}) = 0 \quad \text{for } r+\widetilde{r}\neq 0. \tag{8.6}$$

Here $(,)$ is the Poincaré pairing on $H^*(X,\mathbb{C})$. Let us denote these Frobenius supermanifolds for a moment by $M_{Bar}(X,W_{\leq\bullet})$.

From now on we suppose that $n\geq 4$ and that X is a Calabi–Yau hypersurface in $\mathbb{P}^n$, that is, $X=\overline{f^{-1}(0)}\subset\mathbb{P}^n$ where $f\in\mathbb{C}[x_0,...,x_n]$ is homogeneous of degree $n+1$ with an isolated singularity at 0. Then the cohomology $H^*(X,\mathbb{C})$ splits into two orthogonal pieces,

$$H^*(X,\mathbb{C}) = H^*_{Lef}(X)\oplus H^{n-1}_{prim}(X). \tag{8.7}$$

As in chapter 5 the second piece $H^{n-1}_{prim}(X)\subset H^{n-1}(X)$ is the primitive part of the middle cohomology; with L as standard Lefschetz operator the first piece is

$$H^*_{Lef}(X) := \bigoplus_{k=0}^{n-1} L^k H^0(X,\mathbb{C}). \tag{8.8}$$

If $n-1$ is odd then the splittings (8.3) and (8.7) coincide. Then an opposite filtration $W_{\leq\bullet}$ for $F^{\geq\bullet}$ induces an opposite filtration $\widetilde{U}_\bullet$ for $\widetilde{F}^\bullet := F^{\bullet-1}H^{n-1}_{prim}(X)$ by

$$\widetilde{U}_p := W_{\leq p-\frac{n-1}{2}}\cap H^{n-1}_{prim}(X), \tag{8.9}$$

and this gives a 1–1 correspondence between the opposite filtrations $\widetilde{U}_\bullet$ and the opposite filtrations $W_{\leq\bullet}$. If $n-1$ is even then $H^*(X,\mathbb{C}) = H^{even}(X,\mathbb{C})$. Then formula (8.9) gives a 1–1 correspondence between the opposite filtrations $\widetilde{U}_\bullet$ and those opposite filtrations $W_{\leq\bullet}$ for $F^{\geq\bullet}$ which respect the splitting (8.7).

Let us fix for a moment an opposite filtration $\widetilde{U}_\bullet$ for $\widetilde{F}^\bullet$ on $H^{n-1}_{prim}(X)$. Theorem 6.1 and theorem 5.6 yield a germ $(M',0)$ of a holomorphic Frobenius manifold of dimension $m' := \dim H^{n-1}_{prim}(X)$, which is unique up to multiplication of the metric by a scalar. Let E' be its Euler field, $\mathcal{F}'\in\mathcal{O}_{M',0}$ its potential, and $e'=\frac{\partial}{\partial t_1}$ its unit field for suitable coordinates $t_1,...,t_{m'}$.

The Frobenius manifold can be extended in the following trivial way to a Frobenius (super)manifold of dimension m. Consider $\mathbb{C}^{m-m'}$ with coordinates $(\tau_1,...,\tau_{m-m'})$ of degree $(n-1) \mod 2$. Then the germ $(M',0)\times(\mathbb{C}^{m-m'},0)$

with potential

$$\mathcal{F} := \mathcal{F}' + \frac{1}{2}\sum_{i=1}^{m-m'} t_1\tau_i\tau_{m-m'+1-i}, \tag{8.10}$$

and Euler field

$$E := E' + \sum_{i=1}^{m-m'} \frac{3-n}{2}\tau_i\frac{\partial}{\partial\tau_i} \tag{8.11}$$

is a Frobenius supermanifold for $n-1$ odd and a Frobenius manifold for $n-1$ even. We call it $M_{VHS}(X,\widetilde{U}_\bullet)$. It contains the germ $(M',0)\times\{0\}$ as a Frobenius submanifold. For $n-1$ odd this is the Frobenius submanifold in [Man2, Theorem 8.7.1]. The following theorem is essentially contained in [Ba1, Theorem 6.5].

Theorem 8.1. *Let $X\subset\mathbb{P}^n$ be a Calabi–Yau hypersurface with $n\geq 4$.*

Consider an opposite filtration $\widetilde{U}_\bullet$ for $\widetilde{F}^\bullet := F^{\bullet-1}H^{n-1}_{prim}(X)$ on $H^{n-1}_{prim}(X)$ and an opposite filtration $W_{\leq\bullet}$ for $F^{\geq\bullet}$ on $H^*(X,\mathbb{C})$ which respects the splitting (8.7) *if $n-1$ is even. Then*

$$M_{Bar}(X,W_{\leq\bullet})\cong M_{VHS}(X,\widetilde{U}_\bullet) \tag{8.12}$$

if and only if (8.9) *holds.*

Proof. For $n-1$ odd $M_{Bar}(X,W_{\leq\bullet})$ is a formal germ of a Frobenius supermanifold with odd part corresponding to $H^*_{Lef}(X)$ and even part corresponding to $H^{n-1}_{prim}(X)$. For $n-1$ even everything is even. This follows from the discussion at the beginning of this chapter and from the isomorphisms $\bigwedge^p\mathcal{T}_X\cong\Omega_X^{n-1-p}$ and $H^q(X,\bigwedge^p\mathcal{T}_X)\cong H^{n-1-p,q}(X)$, which one obtains by contraction of the holomorphic volume form Ω on X with holomorphic vector fields.

The condition $n\geq 4$ asserts that $H^{n-2,1}(X)\subset H^{n-1}_{prim}(X)$. Therefore a miniversal family X_t, $t\in M_0$, of complex structure deformations of $X_0=X$ is given by a family of homogeneous polynomials f_t, $t\in M_0$, as in chapter 6 with condition (6.2) and $f_0=f$.

By construction of $M_{Bar}(X,W_{\leq\bullet})$ there is a natural inclusion $(M_0,0)\subset M_{Bar}(X,W_{\leq\bullet})$ of formal germs (the formal germ $(M_0,0)$ is called M^{cx} in [Ba2]). By lemma 4.4 and lemma 5.1 one obtains on $TM_{Bar}(X,W_{\leq\bullet})|_{(M_0,0)}$ a formal germ of a variation of filtrations with pairing and opposite filtration.

The following fact is crucial: This structure is isomorphic to the restriction to the formal germ $(M_0,0)$ of the variation of filtrations $F^{\geq\bullet}(X_t)$ on the bundle $\bigcup_{t\in M_0}H^*(X_t,\mathbb{C})$ and the opposite filtration $W_{\leq\bullet}$.

This fact is at the heart of Barannikov's construction of semi-infinite variations of Hodge structures [Ba2, Theorem 4.2] and Frobenius manifolds.

Now the case $n-1$ odd is easy. The variation of filtrations $F^{\geq\bullet}(X_t)$ and the opposite filtration $W_{\leq\bullet}$ on $\bigcup_{t\in M_0}H^{n-1}_{prim}(X_t)$ correspond to the formal germ of the even Frobenius submanifold. By theorem 5.6 this germ must coincide with the germ $(M',0)$ of the Frobenius manifold before theorem 8.1 for filtrations $\widetilde{U}_\bullet$ and $W_{\leq\bullet}$ with (8.9). Because of the degrees of the flat coordinates the extension with the odd part to a Frobenius supermanifold is rigid and is given by (8.10) and (8.11).

The case $n-1$ even is more difficult because the H^2-generation condition does not hold for the variation of filtrations on $\bigcup_{t\in M_0} H^*(X_t,\mathbb{C})$. But a slightly weaker condition holds so that lemma 8.2 below applies. It shows that the variation of filtrations $F^{\geq\bullet}(X_t)$ on $\bigcup_{t\in M_0} H^*(X_t,\mathbb{C})$ and *any* opposite filtration $W_{\leq\bullet}$ determine a unique holomorphic germ of a Frobenius manifold, whether $W_{\leq\bullet}$ respects the splitting (8.7) or not. It remains to see that precisely the filtrations $W_{\leq\bullet}$ which respect the splitting (8.7) lead to the Frobenius manifolds $M_{VHS}(X,\widetilde{U}_\bullet)$.

One can go into the proof of lemma 8.2 and start there with vector fields $\delta_1,...,\delta_m$ and an index set $I\subset\{1,...,m\}$ such that the isomorphism $T_0M\to H^*(X,\mathbb{C})$ maps δ_i into $H^*_{Lef}(X)$ for $i\in I$ and into $H^{n-1}_{prim}(X)$ for $i\notin I$. Then one has to check that in the case of a filtration $W_{\leq\bullet}$ which respects the splitting (8.7)

$$a_{ij}^k = 0 \text{ for } d_i = 0,\ j\in I \tag{8.13}$$

holds. Going through the induction in the proof of lemma 8.2 one finds that (8.13) holds for all $i\notin I\cup\{1\}$, $j\in I$ and that a_{ij}^k is constant for $i,j\in I$. Then the flat submanifold $\{t\in M\mid t_i=0 \text{ for } i\in I\}$ is a Frobenius submanifold, and the whole Frobenius manifold is $M_{VHS}(\widetilde{U}_\bullet)$. □

One can generalize theorem 5.6 by relaxing the H^2-generation condition slightly. The next lemma formulates the part of this generalization which is needed in the proof of theorem 8.1. It was already used in the proof of [Ba1, Theorem 6.5].

Lemma 8.2. *Consider a germ of a Frobenius manifold $((M,0),\circ,e,E,g)$ with a weight $w\in\mathbb{N}_{\geq 3}$ and all properties in definition 5.4 except the H^2-generation condition, which is replaced by the weaker condition (II)' on the graded algebra*

$$\begin{aligned} T_0M &= \bigoplus_{p=0}^{n-2}(T_0M)_p \quad \textit{with} \\ (T_0M)_p &:= \ker(\nabla^g E-(1-p)\,\mathrm{id}: T_0M\to T_0M). \end{aligned} \tag{8.14}$$

(II)' $(T_0M)_1$ generates multiplicatively a subspace of T_0M which contains $\bigoplus_{p<(w-2)/2}(T_0M)_p$.

Define $M_0:=\{t\in M\mid E|_t=0\}$.

The germ of a Frobenius manifold is uniquely determined by the induced Frobenius type structure (see lemma 4.4) on $TM|_{M_0}$ together with $e|_0\in T_0M$.

Proof. Choose flat coordinates $t_1,...,t_m$ on $(M,0)$ with vector fields $\delta_i:=\frac{\partial}{\partial t_i}$ such that

$$E=\sum_{i=1}^m(-d_i)t_i\delta_i \tag{8.15}$$

with $d_1=-1$, $\delta_1=e$, $d_2=...=d_{m_0+1}=0$, $d_i>0$ for $i>m_0+1$. Then $M_0=\{t\in M\mid t_i=0 \text{ for } d_i\neq 0\}$. Define matrices $A_i=(a_{ij}^k)\in M(m\times m,\mathcal{O}_{M,0})$ for

$i = 1, ..., m$ by

$$\delta_i \circ \delta_j = \sum_k a_{ij}^k \cdot \delta_k. \tag{8.16}$$

Then

$$\begin{aligned} A_i A_j &= A_j A_i, & (8.17)\\ a_{i1}^k &= \delta_{ik}, & (8.18)\\ a_{ij}^k &= a_{ji}^k, & (8.19) \end{aligned}$$

and the potentiality condition is equivalent to

$$\delta_i A_j = \delta_j A_i. \tag{8.20}$$

Especially $\delta_1 A_j = 0$ for all i. Denote for $w \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} \mathcal{O}(M)_w &:= \{f \in \mathcal{O}_{M_0,0}[t_i \mid d_i > 0] \mid Ef = -wf\}, & (8.21)\\ M(w) &:= M(m \times m, \mathcal{O}(M)_w), & (8.22)\\ M(> w) &:= \bigoplus_{k>w} M(k). & (8.23) \end{aligned}$$

Then $t_i \in \mathcal{O}(M)_{d_i}$ for $i \geq 2$. The condition $\mathrm{Lie}_E(\circ) = \circ$ together with (8.15) shows

$$a_{ij}^k \in \mathcal{O}(M)_{d_k - d_i - d_j - 1}. \tag{8.24}$$

It is not hard to see that the Frobenius type structure on $TM|_{M_0}$ and $e|_0 \in T_0 M$ provide after some choice the matrices $A_i \mod M(> 0)$ with $d_i = 0$ and the coefficients $g(\delta_i, \delta_j) \in \mathbb{C}$ of the metric for all i, j. As in lemma 2.9 one has to recover all the matrices A_j in order to uniquely determine the Frobenius manifold. Again this will be done inductively.

Induction hypothesis for $w \in \mathbb{Z}$: one has determined the matrices $A_i \mod M(> w)$ for $d_i = 0$ and $A_j \mod M(> w-1)$ for $d_j > 0$.

Induction step from w to $w+1$: It consists of two steps.

(i) Determine the matrices $A_j \mod M(> w)$ for $d_j > 0$.
(ii) Determine the matrices $A_i \mod M(> w+1)$ for $d_i = 0$.

(i) The weakened generation condition (II)' together with (8.24) shows that one obtains from the matrices $A_i \mod M(> w)$ with $d_i = 0$ and from the matrices $A_j \mod M(> w-1)$ the matrices $A_k \mod M(> w)$ for k with $k < \frac{w-2}{2}$. Because of (8.19) the only unknown coefficients of the matrices $A_k \mod M(> w)$ for $k \geq \frac{w-2}{2}$ are those coefficients a_{kl}^r with $d_l \geq \frac{w-2}{2}$. Because of (8.24), in the case $d_k + d_l = w - 2$ they are constant and determined by $g(\delta_k, \delta_l)$, in the case $d_k + d_l > w - 2$ they vanish.

(ii) Similarly to step (iii) in the proof of lemma 2.9 one uses (8.20) in the form

$$\delta_j (A_i \mod M(> w+1)) = \delta_i (A_j \mod M(> w+1-d_j)) \tag{8.25}$$

for $d_i = 0, d_j > 0$. □

Remark 8.3. The proof of theorem 8.1 contains the three statements: for $n-1$ even any Frobenius manifold $M_{Bar}(X, W_{\leq\bullet})$ is a holomorphic germ of a Frobenius manifold; it is uniquely determined up to a scalar of the metric by the variation of filtrations $F^{\geq\bullet}$ and the opposite filtration $W_{\leq\bullet}$ on $\bigcup_{t\in M_0} H^*(X_t, \mathbb{C})$; it contains a Frobenius submanifold with tangent space at 0 isomorphic to $H^{n-1}_{prim} \subset H^*(X_t, \mathbb{C}) \cong T_0M$ precisely if the filtration $W_{\leq\bullet}$ respects the splitting (8.7)

References

[Ba1] S. Barannikov: Generalized periods and mirror symmetry in dimensions $n > 3$. preprint, math.AG/9903124.

[Ba2] S. Barannikov: Quantum periods – I. Semi-infinite variations of Hodge structures. Int. Math. Res. Notices **2001–23**: 1243–1264.

[Ba3] S. Barannikov: Semi-infinite variations of Hodge structures and integrable hierarchies of KdV-type. preprint, math.AG/0108148.

[BK] S. Barannikov, M. Kontsevich: Frobenius manifolds and formality of Lie algebras of polyvector fields. Int. Math. Res. Notices **1998-14**, 201–215.

[Do] R. Donagi: Generic Torelli for projective hypersurfaces. Comp. Math **50** (1983), 325–353.

[Du] B. Dubrovin: Geometry of 2D topological field theories. In: Integrable systems and quantum groups. Montecatini, Terme 1993 (M. Francoviglia, S. Greco, eds.). Lecture Notes in Math. 1620, Springer Verlag 1996, pp. 120–348.

[Fo] G.B. Folland: Introduction to partial differential equations. Princeton University Press, 2nd ed. 1995.

[Gr] P. Griffiths: On the periods of certain rational integrals I. Annals of Math. **90** (1969), 460–495.

[He1] C. Hertling: Frobenius manifolds and moduli spaces for singularities. To appear in Cambridge Tracts in Mathematics, Cambridge University Press, August 2002.

[He2] C. Hertling: tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities. math.AG/0203054, 81 pages.

[KM] M. Kontsevich, Yu. Manin: Gromov–Witten classes, quantum cohomology, and enumerative geometry. Commun. Math. Phys. **164** (1994), 525–562.

[Kr] A. Kresch: Associativity relations in quantum cohomology. Advances in Mathematics **142** (1999), 151–169.

[Mal] B. Malgrange: Deformations of differential systems, II. J. Ramanujan Math. Soc. **1** (1986), 3–15.

[Man1] Yu. Manin: Three constructions of Frobenius manifolds: a comparative study. Asian J. Math. **3** (1999), 179–220.

[Man2] Yu. Manin: Frobenius manifolds, quantum cohomology, and moduli spaces. American Math. Society, Colloquium Publ. v. **47**, 1999.

[Sab1] C. Sabbah: Frobenius manifolds: isomonodromic deformations and period mappings. Expo. Math. **16** (1998), 1–58.

[Sab2] C. Sabbah: Déformations isomonodromiques et variétés de Frobenius, une introduction. Savoirs Actuels, EDP Sciences/CNRS Éditions, Paris, 2002.

[SK] K. Saito: Period mapping associated to a primitive form. Publ. RIMS, Kyoto Univ. **19** (1983), 1231–1264.

[SM] M. Saito: On the structure of Brieskorn lattices. Ann. Inst. Fourier Grenoble **39** (1989), 27–72.

[Va] A.N. Varchenko: The asymptotics of holomorphic forms determine a mixed Hodge structure. Sov. Math. Dokl. **22** (1980), 772–775.

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