Max-Planck-Institut für Mathematik
Bonn

Similar relatively hyperbolic actions of a group

by

Victor Gerasimov
Leonid Potyagailo

Max-Planck-Institut für Mathematik
Preprint Series 2014 (34)

Similar relatively hyperbolic actions of a group

Victor Gerasimov
Leonid Potyagailo

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Departamento de Matemática
Universidade Federal de Minas Gerais
Av. Antonio Carlos 6627
Caixa Postal 702
CEP 31270-901 Belo Horizonte
Brasil

UFR de Mathématiques
Université de Lille 1
59655 Villeneuve d'Ascq cedex
France

MPIM 14-34

SIMILAR RELATIVELY HYPERBOLIC ACTIONS OF A GROUP

VICTOR GERASIMOV AND LEONID POTYAGAILO

Abstract. Let a discrete group G possess two convergence actions by homeomorphisms on compacta X and Y. Consider the following question: does there exist a convergence action $G\curvearrowright Z$ on a compactum Z and continuous equivariant maps $X \leftarrow Z \rightarrow Y$? We call the space Z (and action of G on it) *pullback* space (action). In such general setting a negative answer follows from a recent result of O. Baker and T. Riley [BR].

Suppose, in addition, that the initial actions are relatively hyperbolic that is they are non-parabolic and the induced action on the distinct pairs are cocompact. Then the existence of the pullback space if G is finitely generated follows from [Ge2]. The main result of the paper claims that the pullback space exists if and only if the maximal parabolic subgroups of one of the actions are dynamically quasiconvex for the other one.

We provide an example of two relatively hyperbolic actions of the free group G of countable rank for which the pullback action does not exist.

We study an analog of the notion of geodesic flow for relatively hyperbolic groups. Further these results are used to prove the main theorem.

1. Introduction

This paper is a further development of our project of studying convergence group actions including the actions of relatively hyperbolic groups.

An action of a discrete group G by homeomorphisms of a compactum X is said to *have convergence property* if the induced action on the space of distinct triples of X is properly discontinuous. We call such an action 3-*discontinuous*. The complement $\mathbf{\Lambda}_X G$ of the maximal open subset where the action is properly discontinuous is called the *limit set* of the action. The action is said to be *minimal* if $\mathbf{\Lambda}_X G{=}X$.

The goal of the paper is to establish similarity properties between different convergence actions of a fixed group. The first motivation for us was the following question:

Q1 : *Given two minimal* 3*-discontinuous actions of a group G on compacta X, Y does there exists a* 3*-discontinuous action $G \curvearrowright Z$ on a compactum Z and continuous equivariant maps $X \leftarrow Z \rightarrow Y$?*

We call such an action *pullback* action and the space Z *pullback* space.

The answer to this question is negative in general. This follows from a recent result of O. Baker and T. Riley [BR]. They indicated a hyperbolic group G and a free subgroup H of rank three such that the embedding $H \rightarrow G$ does not admit an equivariant continuous extension to the hyperbolic boundaries $\partial_\infty H \rightarrow \partial_\infty G$ (so called *Cannon-Thurston map*). It is an easy consequence of this result that 3-discontinuous actions $H\curvearrowright\partial_\infty H$ and $H\curvearrowright\mathbf{\Lambda}_{\partial_\infty G}H$ do not possess a pullback 3-discontinuous action (see section 4).

The question Q1 has a natural modification. Suppose in addition that our actions $G\curvearrowright X$, $G\curvearrowright Y$ are 2-*cocompact*, that is the quotient of the space of *distinct pairs* of the corresponding space by G is compact. This condition is natural because the class of groups which admit non-trivial 3-discontinuous and 2-cocompact actions (we say 32-*actions*) coincides with the class of *relatively hyperbolic* groups [GePo2, Theorem 3.1].

So the following question is the main subject of the paper.

Q2 : *Given two minimal* 32*-actions of a group G on compacta X, Y does there exist a* 3*-discontinuous action $G\curvearrowright Z$ possessing continuous equivariant maps $X \leftarrow Z \rightarrow Y$?*

We note that if such an action $G \curvearrowright Z$ exists then one can choose it to be 2-cocompact (see Lemma 5.4). So the pullback action is also of type (32).

Recall few standard definitions. An action on a compactum is called *parabolic* if it admits a unique fixed point. For a 3-discontinuous action $G \curvearrowright X$ a point $p \in X$ is called *parabolic* if it is the unique fixed point for its stabilizer

Date: May 27, 2014.

2010 *Mathematics Subject Classification.* Primary 20F65, 20F67; Secondary 57M07, 22D05.

Key words and phrases. Convergence actions, relatively hyperbolic group, equivariant map between actions, pullback space.

$P = \mathrm{St}_G p$ and $\mathrm{St}_G p$ is infinite. The subgroup P is called a *maximal parabolic* subgroup. We denote by Par_X the set of the parabolic points for an action on X. If $G \curvearrowright X$ is a 32-action then the set of all maximal parabolic subgroups is called *peripheral structure* for the action. It consists of finitely many conjugacy classes of maximal parabolic subgroups [Ge1, Main theorem, a].

If G is finitely generated then an affirmative answer to the question Q2 can be easily deduced from [Ge2, Map theorem] (see section 5 below). Furthermore there exists a "universal" pullback space in this case. Namely every 32-action of a finitely generated group G on a compactum X admits an equivariant continuous map from the Floyd boundary $\partial_f G$ of G to X. The space $\partial_f G$ is universal as it does not depend on the action on X (it depends on a scalar function f rescaling the word metric of the Cayley graph and a fixed finite set of generators of G).

However the same method does not work if the group is not finitely generated. One cannot use the Cayley graph since the quotient of the set of its edges by the group is not finite (the action is not *cofinite on edges*) the condition which is needed for the construction of the above map. Replacing the Cayley graph by a relative Cayley graph changes the situation since the latter graph depends on the 32-action of G on a compactum X. Indeed the vertex set of the graph contains the parabolic points for the action $G \curvearrowright X$. This problem turns out to be crucial since the answer to the question Q2 is negative in general. We show in the following theorem that a counter-example exists already in the case of free groups of countable rank.

Theorem I (Proposition 5.5). *The free group F_∞ of countable rank admits two 32-actions not having a pullback space.*

We note that this is a rare example when certain properties of the relatively hyperbolic groups are true for finitely generated groups and are false for non-finitely generated (even countable) groups.

Our next goal is to provide necessary and sufficient conditions for two 32-actions of a group to have a common pullback space.

The following theorem is the main result of the paper.

Theorem II (Theorem 5.6, Theorem B). *Two 32-actions of G on compacta X and Y with peripheral structures $\mathcal{P}$ and $\mathcal{Q}$ admit a pullback space Z if and only if one of the following conditions is satisfied:*

1) $\mathsf{C}(X, Y)$: *every element $P \in \mathcal{P}$ acts 2-cocompactly on its limit set in Y.*
2) $\mathsf{C}(Y, X)$: every element $Q \in \mathcal{Q}$ acts 2-cocompactly on its limit set in X.

Here are several remarks about the theorem. As an immediate corollary we obtain that $\mathsf{C}(X, Y)$ is equivalent to $\mathsf{C}(Y, X)$. This statement seems to be new even in the finitely generated case. It follows from Theorem II that a parabolic subgroup H for a 32-action of a finitely generated group G acts 2-cocompactly on its limit set for every other such action of G. Theorem I implies that this is not true if G is not finitely generated (see Corollary 6.1.f).

The peripheral structure $\mathcal{R}$ for the pullback action on Z is given by the system of subgroups $\mathcal{R} = \{Q \cap P \ : P \in \mathcal{P}, Q \in \mathcal{Q}, |P \cap Q| = \infty\}$. In particular Theorem II provides a criterion when the system $\mathcal{R}$ is a peripheral structure for some relatively hyperbolic action of G (Corollary 6.1.a).

The proof of Theorem II uses several intermediate results which occupy first sections of the paper and which have independent interest. We will now briefly describe them.

In section 3 we study an analog of the geodesic flow introduced by M. Gromov in the case of hyperbolic groups. If the group G admits a 32-action on a compactum X, then there exists a connected graph Γ such that G acts properly and cofinitely on the set of edges Γ^1 of Γ [GePo2, Theorem A]. The set of vertices Γ^0 of Γ is $\mathrm{Par}_X \sqcup G$. The union $\widetilde{X} = X \cup \Gamma^0 = X \sqcup G$ admits a Hausdorff topology whose restriction on X and on G coincide with the initial topology and the discrete topology respectively, and G acts on $\widetilde{X}$ 3-discontinuously [Ge2, Proposition 8.3.1]. The action is also 2-cocompact (Lemma 5.4). We call the space $\widetilde{X}$ *attractor sum* of X and G.

Consider the space of maps $\gamma : \mathbb{Z} \to \widetilde{X}$ for which there exist $m, n \in \mathbb{Z} \cup \{\pm\infty\}$ such that γ is constant on one or both (possibly empty) sets $]-\infty, m], [n, +\infty[$ and is geodesic in Γ^0 outside of these sets. We call such a map *eventual geodesic* and denote by $\mathrm{EG}(\Gamma)$ the space of all eventual geodesics. We prove in section 3 (Proposition 3.3) that $\mathrm{EG}(\Gamma)$ is closed in the space of maps $X^{\mathbb{Z}}$ equipped with the Tikhonov topology. Then we show that the boundary map $\partial : \mathrm{EG}(\Gamma) \to \widetilde{X}^2$ is continuous at every non-constant eventual geodesic (Proposition 3.1). In particular we show that every two distinct points of $\widetilde{X}$ can be joined by a geodesic (Theorem 3.7). This allow us to consider the convex hull $\mathrm{Hull}(B)$ of a subset $B \subset \widetilde{X}$ which the union of the images of all geodesics in $\widetilde{X}$ with the endpoints in B. We prove that $\mathrm{Hull}(B)$ is closed if B is.

We extensively use so called *visibility property* of the uniformity of the topology of $\widetilde{X}$, that is for every two disjoint closed subsets A and B of $\widetilde{X}$ there exists a finite set $F \subset \Gamma^1$ such that every geodesic with one endpoint in A and the other in B contains an edge in F. At the end of the section using the group action we show that the space $\widetilde{X}$ cannot contain geodesic horocycles, i.e. non-trivial geodesics whose endpoints coincide.

In Section 4 we study properties of subgroups of a group acting 3-discontinuously on a compactum X. According to Bowditch [Bo2] a subgroup H of a group G is called *dynamically quasiconvex* if for every neighborhood $\mathbf{u}$ of the diagonal $\mathbf{\Delta} X$ of $X^2 = X{\times}X$ the set $\{g{\in}G : (g\mathbf{\Lambda}_X H)^2 \not\subset \mathbf{u}\}/H$ is finite.

Using the results of Section 3 we obtain here the following theorem.

Theorem III (Theorem A). For a 32-action $G\curvearrowright X$ a subgroup $H < G$ is dynamically quasiconvex if and only if its action on $\mathbf{\Lambda}_X H$ is 2-compact.

The proofs of Theorems I and II are given in Section 5. They use the results of the previous sections.

In the last section we provide a list of corollaries of the main results (Corollary 6.1).

Acknowledgements. During the work on the paper both authors were partially supported by the ANR grant BLAN 2011 *BS*01 013 04 "Facettes des groupes discrets" and by Labex CEMPI (ANR-11-LABX-0007-01). The first author is also thankful to the CNRS for providing a research fellowship for his stay in France. The second author is grateful to the Max-Planck Institute in Bonn where a part of this work was done.

2. Preliminaries

2.1. **Entourages and Cauchy-Samuel completions.** We recall some well-known notions from the general topology. For further references see [Ke].

Let X be a set. We denote by $\mathsf{S}^n X$ the quotient of the product space $\underbrace{X{\times}\ldots{\times}X}_{n \text{ times}}$ by the action of the permutation group on n symbols. We regard the elements of $\mathsf{S}^n X$ as non-ordered n-tuples. Let $\Theta^n X$ be the subset of $\mathsf{S}^n X$ whose elements are non-ordered n-tuples whose components are all distinct. Denote $\Delta^n X = \mathsf{S}^n X \setminus \Theta^n X$.

An *entourage* is a neighborhood of the diagonal $\mathbf{\Delta}^2 X = \{(x, x) \ : \ x \in X\}$ in $\mathsf{S}^2 X$. The set of entourages of X is denoted by EntX. We use the bold font to denote entourages. For $\mathbf{u} \in$ EntX a pair of points $(x, y) \in X^2$ is called $\mathbf{u}$-small if $(x, y) \in \mathbf{u}$. Similarly a set $A \subset X$ is $\mathbf{u}$-small if $\mathsf{S}^2 A \subset \mathbf{u}$. Denote by Small($\mathbf{u}$) the set of all $\mathbf{u}$-small subsets of X.

For an entourage $\mathbf{u}$ we define its power $\mathbf{u}^n$ as follows: $(x, y) \in \mathbf{u}^n$ if there exist $x_i \in X$ such that $(x_{i-1}, x_i) \in \mathbf{u}$ $(x_0 = x, x_n = y, i = 1, ..., n-1)$. We denote by $\sqrt[n]{\mathbf{u}}$ an entourage $\mathbf{v}$ such that $\mathbf{v}^n \subset \mathbf{u}$.

A filter $\mathcal{U}$ on $\mathsf{S}^2 X$ whose elements are entourages is called *uniformity* if

$$\forall\, \mathbf{u} \in \mathcal{U}\ \exists\, \mathbf{v} \in \mathcal{U} : \mathbf{v}^2 \subset \mathbf{u}.$$

A uniformity $\mathcal{U}$ defines the $\mathcal{U}$-topology on X in which every neighborhood of a point has a $\mathbf{u}$-small subset containing the point for some $\mathbf{u} \in \mathcal{U}$. A pair $(X, \mathcal{U})$ of a set X equipped with an uniformity $\mathcal{U}$ is called *uniform* space. A *Cauchy filter* $\mathcal{F}$ on the uniform space $(X, \mathcal{U})$ is a filter such that $\forall\, \mathbf{u} \in \mathcal{U}\ : \mathcal{F} \cap \mathsf{Small}(\mathbf{u}) \neq \emptyset$. A space X is *complete* if every Cauchy filter on X contains all neighborhoods of a point. The uniform space $(X, \mathcal{U})$ admits a completion $(\overline{X}, \overline{\mathcal{U}})$ called *Cauchy-Samuel* completion whose construction is the following. Every point of $\overline{X}$ is the minimal Cauchy filter ξ. For every $\mathbf{u} \in \mathcal{U}$ we define an entourage $\overline{\mathbf{u}}$ on $\overline{X}$ as follows:

$$\overline{\mathbf{u}} = \{(\xi, \eta) \in \mathsf{S}^2 \overline{X} : \xi \cap \eta \cap \mathsf{Small}(\mathbf{u}) \neq \emptyset\}. \tag{1}$$

The uniformity $\overline{\mathcal{U}}$ of $\overline{X}$ is the filter generated by the entourages $\{\overline{\mathbf{u}} : \mathbf{u} \in \mathcal{U}\}$. We note that the completion $(\overline{X}, \overline{\mathcal{U}})$ is *exact* [Bourb, II.3, Théorème 3]:

$$\forall\, a, b \in \overline{X}\ a \neq b\ \exists\, \overline{\mathbf{u}} \in \overline{\mathcal{U}} : (a, b) \notin \overline{\mathbf{u}}.$$

If X is a compactum then the filter of the neighborhoods of the diagonal $\Delta^2 X$ is the unique exact uniformity $\mathcal{U}$ consistent with the topology of X, and X equipped with $\mathcal{U}$ is a complete uniform space [Bourb, II.4, Théorème 1].

2.2. **Properties of (32)-actions of groups.** Let X be a compactum, i.e a compact Hausdorff space, and G be a group acting 3-discontinuously on X (convergence action). Recall that the limit set, denoted by $\mathbf{\Lambda}_X G$ (or $\mathbf{\Lambda} G$ if X is fixed), is the set of accumulation (limit) points of any G-orbit in X.

The action G on X is said to be *minimal* if $X = \mathbf{\Lambda} G$.

The action $G \curvearrowright X$ is *elementary* if $|\mathbf{\Lambda} G| < 3$. If the action is not elementary then $\mathbf{\Lambda} G$ is a perfect set [Tu2]. If G is non-elementary then $\mathbf{\Lambda} G$ is the minimal non-empty closed subset of X invariant under G.

An elementary action of a group G on X is called *parabolic* (or *trivial*) if $\mathbf{\Lambda}_X G$ is a single point. A point $p \in \mathbf{\Lambda}_X G$ is *parabolic* if its stabilizer $\mathrm{St}_G p$ is a maximal parabolic subgroup fixing p. The set of parabolic points for the action on X is denoted by Par_X.

A parabolic fixed point $p \in \mathbf{\Lambda} G$ is called *bounded parabolic* if the quotient space $(\mathbf{\Lambda} G \setminus \{p\})/\mathrm{St}_G p$ is compact.

We will use an equivalent reformulation of the convergence property in terms of *crosses*. A cross $(r, a)^\times \in X \times X$ is the set $r \times X \cup X \times a$ where $(r, a) \in X \times X$. By identifying every $g \in G$ with its graph one can show that G acts 3-discontinuously on X if and only if all the limit points of the closure of G in $X \times X$ are crosses [Ge1, Proposition P]. The points a and r are called respectively *attractive* and *repelling* points (or attractor and repeller).

A point $x \in \mathbf{\Lambda} G$ is *conical* if there is an infinite set $S \subset G$ such that for every $y \in X \setminus \{x\}$ the closure of the set $\{(s(x), s(y)) : s \in S\}$ in X^2 does not intersect the diagonal $\Delta^2 X$.

A group G acting on the space X acts on the set of entourages EntX. For $\mathbf{u} \in \mathrm{EntX}$ we denote by $g\mathbf{u}$ the set $\{(x, y) \in X^2 : g^{-1}(x, y) \in \mathbf{u}\}$ and by $G\mathbf{u}$ the G-orbit of $\mathbf{u}$. We will say that the orbit $G\mathbf{u}$ is *generating* if it generates EntX as filter.

An action $G \curvearrowright X$ is *2-cocompact* if $\Theta^2 X/G$ is compact. Suppose that a group G admits a 3-discontinuous and 2-cocompact non-parabolic minimal action (32-action) on a compactum X. Then every point of X is either a bounded parabolic or conical [Ge1, Main Theorem]. P. Tukia showed that if X is metrisable then the converse statement is true [Tu2, Theorem 1C, (b)].

Let Γ be a graph. We denote by Γ^0 and Γ^1 the set of vertices and edges of Γ respectively. Recall that an action of G on Γ is *proper on edges* if the stabilizer $\mathrm{St}_\Gamma e$ of every edge e in Γ is finite. The action $G \curvearrowright \Gamma$ is called *cofinite* if $|\Gamma^1/G| < \infty$.

According to B. Bowditch [Bo1] a graph Γ is called *fine* if for any two vertices the set of simple arcs of fixed length joining them is finite.

It is shown in [GePo2, Theorem 3.1] that if G admits a non-parabolic 32-action on a compactum X then Bowditch's condition of relatively hyperbolicity is satisfied. This means that there exists a connected fine and hyperbolic graph Γ acted upon by G cofinitely and properly on edges. Every vertex of Γ is either an element of G, or belongs to the set of parabolic points $\mathsf{P}ar_X$.

Consider the union of two topological spaces $\widetilde{X} = X \sqcup G = X \cup \Gamma^0$ where G is equipped with the discrete topology. By [Ge2, Proposition 8.3.1] $\widetilde{X}$ admits a unique compact Hausdorff topology whose restrictions on X and G coincide with the original topologies of X and G, and the action on $\widetilde{X}$ is 3-discontinuous (the description of this topology see in Proposition 5.2)

Following [Ge2] we call the space $\widetilde{X}$ *attractor sum* of X and Γ.

The action on $\widetilde{X}$ is also 2-cocompact. Indeed by assumption the action of $G \curvearrowright \Theta^2 X$ is cocompact. So there exists a compact fundamental set $K \subset \Theta^2 X$. Hence $\widetilde{K} = K \cup (\{1\} \times (\widetilde{X} \setminus \{1\}))$ is a compact fundamental set for the action on $\Theta^2 \widetilde{X}$. Therefore the action $G \curvearrowright \widetilde{X}$ is a 32-action. We summarize all these facts in the following lemma.

Lemma 2.1. [Ge2], [GePo2]. *Let G admits a non-parabolic 32-action on a compactum X. Then there exists a connected, fine and hyperbolic graph Γ acted upon by G properly and cofinitely on edges. Furthermore G acts 3-discontinuously and 2-cocompactly on the attractor sum $\widetilde{X} = X \cup \Gamma^0$ and $\Gamma^0 = G \cup \mathsf{P}ar_X$ is the set of all non-conical points for the action on $\widetilde{X}$.* □

We will consider the entourages $\mathbf{u} \in \mathrm{Ent}\widetilde{\mathrm{X}}$ on the attractor sum $\widetilde{X}$ as well as their restrictions on Γ and on X.

Following Bowditch [Bo1] for a fixed group G a G-invariant set M is called *connected G-set* if there exists a connected graph Γ such that $M = \Gamma^0$ and the action $G \curvearrowright \Gamma^1$ on edges is proper and cofinite.

Recall some more definitions. An entourage $\mathbf{u}$ on a connected G-set M is called *perspective* if for any pair $(a, b) \in M \times M$ the set $\{g \in G \quad g(a, b) \notin \mathbf{u}\}$ is finite.

An entourage $\mathbf{u}$ given on a connected G-set M is called *divider* if there exists a finite set $F \subset G$ such that $(\mathbf{u}_F)^2 \subset \mathbf{u}$ where $\mathbf{u}_F = \cap_{f \in F} f\mathbf{u}$.

We say that uniformity $\mathcal{U}$ of a compactum $\widetilde{X}$ is *generated by an entourage* $\mathbf{u}$ if it is generated as a filter by the orbit $G\mathbf{u}$.

Lemma 2.2. [Ge2, Proposition 8.4.1]. *If a group G acts 3-discontinuously and 2-cocompactly on a compact space X then the uniformity $\mathcal{U}$ on the compactum $\widetilde{X} = X \cap G$ is generated by a perspective divider* $\mathbf{u}$.

The following result describes the opposite way which starts from a perspective divider on a connected G-set $M = \Gamma^0$ and gives a 32-action on the compactum $\widetilde{X} = X \cup \Gamma$ where X is a "boundary" of Γ.

Definition 2.3. *Let $e \in \Gamma^1$ be an edge. A pair of vertices (a, b) of Γ is called $\mathbf{u}_e$-small if there exists a geodesic in Γ with endpoints a and b which does not contain e.*

A uniformity $\mathcal{U}^0$ on $M = \Gamma^0$ has a visibility property if for every entourage $\mathbf{u}^0 \in \mathcal{U}^0$ there exists a finite set of edges $F \subset \Gamma^1$ such that $\mathbf{u}_F = \cap\{\mathbf{u}_e \mid e \in F\} \subset \mathbf{u}^0$.

The following lemma describes the completion $\widetilde{X}$ mentioned above and will be often used in the paper.

Lemma 2.4. [Ge2, Propositions 3.5.1, 4.2.2]. *Suppose that a group G acts on a connected graph Γ properly and cofinitely on edges. Let $\mathcal{W}^0$ be a uniformity on Γ^0 generated by a perspective divider. Then $\mathcal{W}^0$ has the visibility property. Furthermore the Cauchy-Samuel completion $(Z, \mathcal{W})$ of the uniform space $(\Gamma^0, \mathcal{W}^0)$ admits a 32-action of G.*

Let Γ be a connected graph. We now recall the definition of the *Floyd completion (boundary)* of Γ mentioned in the Introduction (see also [F], [Ka], [Ge2]).

A function $f : \mathbb{N} \to \mathbb{R}$ is said to be a (Floyd) *scaling function* if $\sum_{n \geqslant 0} f_n < \infty$ and there exists a positive λ such that $1 \geqslant f_{n+1}/f_n \geqslant \lambda$ for all $n \in \mathbb{N}$.

Let f be a scaling function and let Γ be a connected graph. For each vertex $v \in \Gamma^0$ we define on Γ^0 a path metric $\boldsymbol{\delta}_{v,f}$ for which the length of every edge $e \in \Gamma^1$ is $f(d(v, e))$. We say that $\boldsymbol{\delta}_{v,f}$ is the *Floyd metric* (with respect to the scaling function f) *based* at v.

When f and v are fixed we write $\boldsymbol{\delta}$ instead of $\boldsymbol{\delta}_{v,f}$.

One verifies that $\boldsymbol{\delta}_u/\boldsymbol{\delta}_v \geqslant \lambda^{\mathsf{d}(u,v)}$ for $u, v \in \Gamma^0$. Thus the Cauchy completion $\overline{\Gamma}_f$ of Γ^0 with respect to $\boldsymbol{\delta}_{v,f}$ does not depend on v. The *Floyd boundary* is the space $\partial_f \Gamma = \overline{\Gamma}_f \setminus \Gamma^0$. Every d-isometry of Γ extends to a homeomorphism $\overline{\Gamma}_f \to \overline{\Gamma}_f$. The Floyd metrics extend continuously onto the Floyd completion $\overline{\Gamma}_f$.

In the particular case when Γ is a Cayley graph of G we denote by $\partial_f G$ its Floyd boundary or by ∂G if f is fixed.

3. Geodesic flows on graphs

In this section we study the properties of geodesics on a class of graphs. Let Γ be a connected graph. We will assume that $\Gamma^0 \subset \widetilde{X}$ for a compactum $\widetilde{X}$. Let $\mathcal{U}$ be the uniformity consistent with the topology of $\widetilde{X}$. Since $\widetilde{X}$ is Hausdorff the uniformity $\mathcal{U}$ is exact. In this section we will always admit the following.

Assumption. The uniformity $\mathcal{U}$ has the visibility property on Γ^0.

The most of the material of the section does not relate to any group action. However the only known example when the above assumption is satisfied is the case when a compactum X admits a 32-action of a group G and $\widetilde{X} = X \cup \Gamma$ is the attractor sum (see Lemma 2.4).

A path in Γ is a map $\gamma : \mathbb{Z} \to \Gamma$ such that $\gamma\{n, n+1\}$ is either an edge of Γ or a point $\gamma(n) = \gamma(n+1)$. A path γ can contain a "stop" subpath, i.e. a subset J of consecutive integers such that $\gamma|_J \equiv$ const.

For a finite subset $I \subset \mathbb{Z}$ of consecutive integers we define the boundary $\partial(\gamma|_I)$ to be $\gamma(\partial I)$. We extend naturally the meaning of $\partial\gamma$ over the half-infinite and bi-infinite paths in $\Gamma^0 \subset \widetilde{X}$ in the case if the corresponding half-infinite branches of γ converge to points in $\widetilde{X}$. The latter one means that for every entourage $\mathbf{v} \in \mathcal{U}$ the set $\gamma|_{[n,\infty[}$ is $\mathbf{v}$-small for a sufficiently big n.

Lemma 3.1. *Every half-infinite geodesic ray $\gamma : [0, \infty[\to \Gamma$ converges to a point in $\widetilde{X}$.*

Proof: Fix an entourage $\mathbf{v} \in \mathcal{U}$. By the visibility property there exists a finite set of edges $F \subset \Gamma^1$ such that $\mathbf{u}_F = \bigcap_{e \in F} \mathbf{u}_e \subset \mathbf{v}$. Since γ is a geodesic, the ray $\gamma|_{[n_0,\infty[}$ does not contain F for some $n_0 \in \mathbb{N}$. So $\gamma|_{[n_0,\infty[}$ is $\mathbf{u}_F$-small and therefore $\mathbf{v}$-small. □

Definition 3.2. *A path $\gamma : I \to \Gamma$ is an eventual geodesic if it is either a constant map, or each its maximal stop-path is infinite and outside of its maximal stop-paths γ is a geodesic in Γ^0.*

The set of eventual geodesics in Γ is denoted by $\mathrm{EG}(\Gamma)$.

The image of every eventual geodesic in Γ is either a geodesic (one-ended, or two-ended or finite) or a point.

Proposition 3.3. *The space $\mathrm{EG}(\Gamma)$ is closed in the space of maps $\widetilde{X}^{\mathbb{Z}}$ equipped with the Tikhonov topology.*

Proof: Let α belongs to the closure $\overline{\mathrm{EG}(\Gamma)}$ of $\mathrm{EG}(\Gamma)$ in $\widetilde{X}^{\mathbb{Z}}$. If α is a constant then $\alpha \in \mathrm{EG}(\Gamma)$ and there is nothing to prove. So we assume that $\alpha \in \overline{\mathrm{EG}(\Gamma)}$ is a non-trivial map. The proof follows from the following three lemmas having their own interest.

Lemma 3.3.1 *If $\alpha(n) \neq \alpha(n+1)$ then there exists a neighborhood O of α in $\widetilde{X}^{\mathbb{Z}}$ such that $\gamma(n) = \alpha(n)$, $\gamma(n+1) = \alpha(n+1)$ for every $\gamma \in O \cap \mathrm{EG}(\Gamma)$. In particular $\{\alpha(n), \alpha(n+1)\}$ is an edge of Γ.*

Proof of the Lemma. Since $\mathcal{U}$ is an exact uniformity there exists $\mathbf{u} \in \mathcal{U}$ such that $(\alpha(n), \alpha(n+1)) \notin \mathbf{u} \in \mathcal{U}$ (see Section 2.1). Since $\mathcal{U}$ is a uniformity $\exists\, \mathbf{v} \in \mathcal{U} \ :\ \mathbf{v}^3 \subset \mathbf{u}$ and $\mathbf{v} \in \mathcal{U}$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $\mathbf{u}_F \subset \mathbf{v}$. Let F^0 denote the set of vertices of the edges in F. Let O_n be a $\mathbf{v}$-small neighborhood of $\alpha(n)$ disjoint from $F^0 \backslash \{\alpha(n)\}$ and O_{n+1} be a neighborhood of $\alpha(n+1)$ defined in the same way.

Set $O = \{\gamma \in \widetilde{X}^{\mathbb{Z}} : \gamma(n) \in O_n\}$. If $\gamma \in O$ then $(\gamma(n), \gamma(n+1)) \notin \mathbf{v}$ and $\gamma(n) \neq \gamma(n+1)$. If in addition $\gamma \in \mathrm{EG}(\Gamma)$ then $\gamma|_{[n,n+1]}$ is a geodesic and necessarily $\{\gamma(n), \gamma(n+1)\} \in F$. By the definition of O_n and O_{n+1} we obtain $\gamma(n) = \alpha(n)$, $\gamma(n+1) = \alpha(n+1)$. □

Lemma 3.3.2. *Every maximal stop for α is infinite.*

Proof of Lemma. By contradiction assume that $J \subset \mathbb{Z}$ is a finite maximal stop for α. By Lemma 1 this means that, for $m, n \in \mathbb{Z}$ such that $n - m \geqslant 3$ one has $\alpha(m) \neq \alpha(m+1)$, $\alpha(n-1) \neq \alpha(n)$ and $\alpha(k) = b \in \Gamma^0$ for $m < k < n$. Moreover, by Lemma 1 there exists a neighborhood O of α in $\widetilde{X}^{\mathbb{Z}}$ such that if $\gamma \in O$ then $\gamma(m) = \alpha(m)$, $\gamma(m+1) = \alpha(m+1)$, $\gamma(n-1) = \alpha(n-1)$, $\gamma(n) = \alpha(n)$. Since α belongs to the closure of $\mathrm{EG}(\Gamma)$ such γ does exist. But this implies that $\gamma(m+1) = \gamma(n-1) = b$. As γ is an eventual geodesic the interval $\{k \in \mathbb{Z} : m < k < n\}$ is a stop for γ so either $\alpha(m) = \gamma(m) = b$ or $\alpha(n) = \gamma(n) = b$ contradicting the maximality of J. □

Lemma 3.3.3. *If J is a finite subset of consecutive integers and J does not contain stops for α then $\alpha|_J$ is geodesic.*

Proof of the Lemma. By the hypothesis each two consecutive values of $\alpha|_J$ are distinct. Since J is finite by Lemma 1 there exists a neighborhood O of α in $\widetilde{X}^{\mathbb{Z}}$ such that if $\gamma \in O \cap \mathrm{EG}(\Gamma)$ then $\gamma|_J = \alpha|_J$. Since $\alpha \in \overline{\mathrm{EG}(\Gamma)}$ such γ does exist. □

It follows from lemmas 2 and 3 that $\alpha \in \mathrm{EG}(\Gamma)$. The proposition is proved. □

We stress that the vertex set Γ^0 is not supposed to be locally compact. However we have the following.

Corollary 3.4. *For every $n \geqslant 0$ the ball $\mathsf{B}_n q$ of radius n in Γ^0 centered at any $q \in \Gamma^0$ is closed in $\widetilde{X}$.*

Proof. Let $p \in \overline{\mathsf{B}_n q} \setminus \mathsf{B}_n q$. For a neighborhood O of p in $\widetilde{X}$ let $\gamma_O : \{k \in \mathbb{Z} : 0 \leqslant k \leqslant n\} \to \Gamma^0$ be a geodesic joining q with a point in O. We make each such γ_O an eventual geodesic by extending it by constants. Since $\widetilde{X}^{\mathbb{Z}}$ is compact there is an accumulation point γ for the set of all γ_O. By Proposition 3.3 $\gamma \in \mathrm{EG}(\Gamma)$. Since the projections $\widetilde{X}^{\mathbb{Z}} \to \widetilde{X}$ are continuous and $\widetilde{X}$ is Hausdorff we have $\gamma(0) = q$, $\gamma(n) = p$. Since γ is eventual geodesic we have $p \in \Gamma^0$. □

Corollary 3.5. *For every finite path $l = \{a_1, ..., a_n\} \subset \Gamma$ the set*

$$(\mathrm{EG}(\Gamma))_l = \{\gamma \in \mathrm{EG}(\Gamma) : \gamma(I) = l,\ I \subset Z,\ \gamma(-\infty) \neq a_1,\ \gamma(\infty) \neq a_n\}$$

is open.

Proof: By the proof of Proposition 3.3 γ admits a neighborhood $O \subset \mathrm{EG}(\Gamma)$ such that $\forall\, \lambda \in O\ \gamma|_l = \lambda|_l$. □

By Lemma 3.1 for a half-infinite geodesic ray $\gamma : \mathbb{Z}_{>0} \to X$ $\lim\limits_{t\to+\infty}\gamma(t)$ exists. The following proposition refines this statement.

Proposition 3.6. *The boundary map $\partial : \mathrm{EG}(\Gamma) \to \widetilde{X}^2$ where $\partial : \gamma \to \partial\gamma = \{\lim\limits_{t\to-\infty}\gamma(t), \lim\limits_{t\to+\infty}\gamma(t)\}$ is continuous at every non-constant eventual geodesic.*

Proof: For $\alpha \in \mathrm{EG}(\Gamma)$ we denote by $\alpha_{-\infty}$ and $\alpha_{+\infty}$ the limits $\lim\limits_{t\to-\infty}\alpha(t)$ and $\lim\limits_{t\to+\infty}\alpha(t)$ respectively. We will prove that both coordinate functions $\pi_- : \alpha \to \alpha_{-\infty}$ and $\pi_+ : \alpha \to \alpha_{+\infty}$ are continuous for every non-constant geodesic α.

Fix $\alpha \in \mathrm{EG}(\Gamma)$ and suppose that $a = \alpha_{+\infty}$. We need to prove that for every small neighborhood U of a there exists a neighborhood of $N_\alpha \subsetneqq \mathrm{EG}(\Gamma)$ such that one of the endpoints of every eventual geodesic belonging to N_α is in U.

Case 1. $\alpha(n)$ $(n \geq 0)$ are all distinct.

Let U be a closed neighborhood of a such that $b = \alpha(0) \notin U$. Choose a "smaller" closed neighborhood V of a such that the interior $\mathring{U}$ of U contains V. By the exactness of the uniformity $\mathcal{U}$ there exists an entourage $\mathbf{v} \in \mathcal{U}$ such that $\mathbf{v} \cap (U \times V) = \emptyset$. Then, by the visibility property, we have $\mathbf{u}_F = \cap_{e\in F}\mathbf{u}_e \subset \mathbf{v}$ for some finite set F of edges of Γ. So every eventual geodesic γ passing from $b \in U' = \widetilde{X} \setminus U$ to V contains an edge from F. Denote by d the diameter (in the graph distance) $\mathrm{diam}F = \max\{d(a_i, a_j) : a_i \in F^0\}$ of the set of vertices F^0 of F. Since Γ is connected and F is finite d is finite. Since $\{\alpha(n)\}_n$ converges to a there exists m such that $\alpha(m+i) \in \mathring{V} \cap \Gamma^0$ for $i = 0, ..., d+1$.

Consider the following set:

$$N_\alpha = \{\gamma \in \mathrm{EG}(\Gamma) \ :\ \{\gamma(m), ..., \gamma(m+d+1)\} \subset \mathring{V} \cap \Gamma^0\}. \tag{1}$$

We have $N_\alpha \neq \emptyset$ as $\alpha \in N_\alpha$. Furthermore N_α is open in $\mathrm{EG}(\Gamma)$. Indeed $\mathring{V} \cap \Gamma^0$ is open in Γ^0 so the condition $\gamma(t) \in \mathring{V} \cap \Gamma^0$ defines an open subset of $\mathrm{EG}(\Gamma)$ and N_α is the intersection of finitely many such subsets.

Let $\gamma \in N_\alpha$. We claim that γ cannot quit $U \cap \Gamma$. Indeed if not then γ contains a finite subpath γ' which passes from U to V, then it passes through at least $d+1$ distinct consecutive vertices $\gamma(i) \in \mathring{V}$ $(i = m, ..., m+d+1)$, and after it goes back to $U' = \widetilde{X} \setminus U$. Assuming γ' to be the minimal subpath of γ having these properties, by Lemma 3.3.2 we obtain that γ' a geodesic of length at least $d+1$. Then there is a couple (i, j) of indices such that $i \leq m,\ j \geq m+d+1,\ \{q_i = \gamma'(i), q_j = \gamma'(j)\} \subset F^0$ and $d(q_i, q_j) \geq d+1 > \mathrm{diam}(F)$ what is impossible.

Case 2. $a \in \Gamma^0$.

A similar argument works in this case too. We can assume that $a = \alpha(0)$. Up to re-parametrisation of α we can assume that $\forall t \geq 0 : \alpha(t) = a$ and $\alpha(-1) = b \neq a$. Consider two disjoint closed neighborhoods U and V of a such that $V \subset \mathring{U}$ and $b \notin U$. As above there exists a finite set $F \subset \Gamma^1$ such that every eventual geodesic passing from V to U (or vice versa) contains an edge of F. Let $d = \mathrm{diam}F$.

For $k \geq d+1$ put

$$N_\alpha = \{\gamma \in \mathrm{EG}(\Gamma) \ :\ \gamma(-2) = \alpha(-2), \gamma(-1) = \alpha(-1), \{\gamma(0), ..., \gamma(k)\} \subset \mathring{V}\}. \tag{1'}$$

The set N_α is non-empty as $\alpha \in N_\alpha$. We have $N_\alpha = N_1 \cap N_2$ where $N_1 = \{\gamma \in \mathrm{EG}(\Gamma) : \gamma(-2) = \alpha(-2), \gamma(-1) = \alpha(-1)\}$ and $N_2 = \{\gamma \in \mathrm{EG}(\Gamma) : \{\gamma(0), ..., \gamma(k)\} \subset \mathring{V}\}$. Since $\alpha(-2) \neq \alpha(-1)$ by Corollary 3.5 N_1 is open. The set N_2 is also open so is N_α.

Every $\gamma \in N_\alpha$ passes from the point b to V and admits a geodesic sub-interval in V of length at least $d+1$. So by the argument of Case 1 we have that $\gamma_{+\infty} \in U$.

We have proved that the coordinate functions π_- and π_+ are continuous. Therefore $\partial = (\pi_-, \pi_+)$ is continuous at every non-trivial eventual geodesic. □

Remark. Note that the map ∂ is not continuous at any constant eventual geodesic $\alpha(t) = a \in \Gamma^0 (t \in \mathbb{Z})$. Indeed the eventual geodesics of the form $\{\gamma \in \mathrm{EG}(\Gamma) : \gamma|_{]-\infty,n]} \equiv a,\ \gamma_{+\infty} = b \neq a\}$ converge to α when $n \to \infty$ in the Tikhonov topology but $\pi_+(\gamma) \neq a$.

Theorem 3.7. *For every two distinct points $p, q \in \widetilde{X}$ there exists an eventual geodesic γ such that $\gamma_{-\infty} = p$, $\gamma_{+\infty} = q$.*

Proof: If both p and q belong to Γ^0 then the assertion follows from the connectedness of Γ. So we suppose that at least one of the points does not belong to Γ^0. We first consider the case when both p and q do not belong to Γ^0. Then we explain how to modify the argument when exactly one of the points belongs to Γ^0.

Let P_0, Q_0 be two closed disjoint neighborhoods of p, q respectively. By the exactness of $\mathcal{U}$ there exists an entourage $\mathbf{u}_0 \in \mathcal{U}$ such that $\mathbf{u}_0 \cap P_0 \times Q_0 = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$, such that $\mathbf{u}_F \subset \mathbf{u}_0$. Let $\{(a_i, b_i) : i{=}0, 1, \ldots, m\}$ be the list of **ordered** pairs such that $\{a_i, b_i\} \in F$.

For closed neighborhoods P, Q of p, q contained in P_0, Q_0 respectively let

$$W_{i,P,Q} = \{\gamma \in \mathrm{EG}(\Gamma) : \gamma_{-\infty} \in P, \gamma_\infty \in Q, \gamma(0) = a_i, \gamma(1) = b_i\}. \quad (2)$$

We claim that $W_{i,P,Q}$ is closed. Indeed suppose $\gamma \notin W_{i,P,Q}$. If γ is not a constant then Propositions 3.3 and 3.6 imply that the opposite of every condition in (2) defines an open subset in $\mathrm{EG}(\Gamma)$. Then their finite intersection is open. If $\gamma \equiv c$ is a constant then either $\gamma(0) \neq a_i$ or $\gamma(1) \neq b_i$. So there is an open neighborhood $N_\gamma \subset \mathrm{EG}(\Gamma)$ such that every $\beta \in N_\gamma$ satisfies $\{\beta \in \mathrm{EG}(\Gamma) : \beta(0) \neq a_i\}$ or $\{\beta \in \mathrm{EG}(\Gamma) : \beta(1) \neq b_i\}$ respectively. In each case $N_\gamma \cap W_{i,P,Q} = \emptyset$ and the claim follows.

There exists $i \in \{0, ..., m\}$ such that all $W_{i,P,Q}$ are nonempty. Indeed if not then for every $i \in \{1, ..., m\}$ there exist neighborhoods P_i and Q_i of p and q such that $W_{i,P_i,Q_i} = \emptyset$. Then, there would no geodesic between the closed non-empty subsets $\bigcap_{i=0}^m P_i$ and $\bigcap_{i=0}^m Q_i$ which is impossible. Then for some i, say $i = 0$, the family $W_{0,P,Q} \neq \emptyset$ for all P and Q. So it is a centered family of non-empty closed sets. Since $\mathrm{EG}(\Gamma)$ is compact $\exists\, \gamma \in \mathrm{EG}(\Gamma)$ such that $\gamma \in \bigcap_{P,Q} W_{0,P,Q}$. By definition of $W_{0,P,Q}$ the point $\gamma_{+\infty}$ belongs to any neighborhood of q. Hence $\gamma_{+\infty} = q$, and similarly $\gamma_{-\infty} = p$.

The assertion is proved for the case $p \notin \Gamma^0$, $q \notin \Gamma^0$. If one of the vertices, say p, belongs to Γ^0 then we modify the definition of $W_{i,P,Q}$ as follows: $W_{i,P,Q} = \{\gamma \in \mathrm{EG}(\Gamma) : \gamma_{-\infty} = \gamma(-n_i) = p, \gamma_{+\infty} \in Q, \gamma(0) = a_i, \gamma(1) = b_i\}$ where n_i is the distance between p and a_i. Then the above argument works without any change. ☐

Let $B \subset \widetilde{X}$ be a closed set. Define its (eventual) geodesic hull as follows:

$$\mathrm{Hull}(B) = \cup\{\mathrm{Im}\gamma : \partial\gamma \subset B\}. \quad (3)$$

The following lemma and its corollary will be used further.

Lemma 3.8. *The set*

$$\mathcal{C} = \{\gamma \in \mathrm{EG}(\Gamma) \ : \ \partial\gamma \in B^2\} \quad (3)$$

is closed in $\mathrm{EG}(\Gamma)$.

Corollary 3.9. *If* $B \subset \widetilde{X}$ *is closed then* $\mathrm{Hull}(B)$ *is a closed subset.*

Proof of the corollary. The projection $\pi : \gamma \in \mathcal{C} \to \gamma(0)$ is continuous by the definition of the Tikhonov topology. Since $X^{\mathbb{Z}}$ is compact and if $\mathcal{C}$ is closed then $\pi(\mathcal{C})$ is closed. ☐

Proof of the lemma. Let us show that $\mathrm{EG}(\Gamma) \setminus \mathcal{C}$ is open. Let $\alpha \in \mathrm{EG}(\Gamma) \setminus \mathcal{C}$ does not belong to $\mathcal{C}$.

Suppose first that α is not a constant. Then $\partial\alpha \notin B^2$. Since B^2 is closed in $\widetilde{X}^2$ there exists an open neighborhood W of $\partial\alpha$ such that $W \cap B^2 = \emptyset$. By Proposition 3.6 $N_\alpha = \partial^{-1}(W)$ is an open subset of $\mathrm{EG}(\Gamma)$, so we are done in this case.

Suppose now that α is a constant: $\alpha \equiv a \in \widetilde{X} \setminus \mathrm{Hull}(B)$. Choose a closed neighborhood U of a disjoint from B. By the visibility property there exists a finite set $F \subset \Gamma^1$ of a finite diameter d such that every eventual geodesic passing from U to B passes through F.

Suppose by contradiction that every neighborhood N_α of a in $\mathrm{EG}(\Gamma)$ intersects $\mathcal{C}$. Note that every such N_α contains a non-trivial eventual geodesic $\gamma \in \mathrm{EG}(\Gamma)$ since otherwise $U \cap B \neq \emptyset$ which is not possible. So for any $k \in \mathbb{Z}_+$ we can find a non-trivial $\gamma \in \mathcal{C}$ such that $\partial\gamma \in B^2$ and $\gamma(i) \in U$ $(i = 0, 1, 2..., k)$. Then γ contains an edge from F on its way from B to U and it meets F again on its way back from U to B. So there is a geodesic subsegment γ' of γ passing through k consecutive points in U and having its endpoints in F. Therefore for $k \geq d$ it would imply $\mathrm{diam}F \geq k \geq \mathrm{length}(\gamma') > d$ which is a contradiction. ☐

At the end of this section we obtain a description of the endpoints of one-ended and two-ended eventual geodesics. It is the first (and the last) time in the section when we use a group action.

Let a group G admits a non-parabolic 32-action on a compactum X. Then by Lemma 2.1 there exists a connected and fine graph Γ such that the action on Γ is proper and cofinite on the edges. We also suppose that

$\Gamma^0 = G \cup \{\text{the parabolic points } \in \widetilde{X}\} = \{\text{the non-conical points of } \widetilde{X}\}$. The existence of a G-finite G-set Γ^1 making Γ^0 into the vertex set of a connected graph is proved in [GePo2, Theorem 3.1]. We note that it is also shown in [GePo2] that the graph Γ is hyperbolic but we will not use it in this section.

Proposition 3.10. *For every non-trivial eventual geodesic γ one has $\gamma_{-\infty} \neq \gamma_{+\infty}$.*

Remark. In other words the proposition claims that the graph Γ does not have non-trivial eventual geodesic horocycles.

Proof: Since the action $G \curvearrowright X$ is 3-discontinuous and 2-cocompact every limit point is either conical or bounded parabolic [Ge1, Main theorem]. We can assume that the action on X is minimal so $X = \mathbf{\Lambda}_X G$.

Suppose first that $p = \gamma_{-\infty} = \gamma_{+\infty}$ is conical. Then there exist closed disjoint sets $A, B \subset X$ and infinite subset S of G such that for every closed set $C \subset \widetilde{X} \setminus \{p\}$ and every closed neighborhood $\widetilde{B}$ of B in $\widetilde{X}$ disjoint from A there exists a subset $S' \subset S$ such that $|S \setminus S'| < \infty$, $S'(C) \subset \widetilde{B}$ and $S'(p) \in A$. For an arbitrary finite set $J \subset \mathbb{Z}$ we have $s\gamma(J) \subset \widetilde{B}$ for all but finitely many $s \in S$. By Lemma 3.8 the set $\partial^{-1}(\widetilde{B}^2)$ is closed. So every accumulation point of the set $S\gamma$ belongs to $\partial^{-1}\widetilde{B}^2$. On the other hand every eventual geodesic $s\gamma$ belongs to the closed set $\partial^{-1}A^2$ as $\partial(s\gamma) = s(p) \in A$. So $\partial^{-1}\widetilde{B}^2 \cap \partial^{-1}A^2 \neq \emptyset$ which is impossible as $A^2 \cap \widetilde{B}^2 = \emptyset$.

Let $p = \gamma_{-\infty} = \gamma_{+\infty} \in \Gamma^0$. In this case γ cannot be of finite length so p is a limit point for the action $G \curvearrowright \widetilde{X}$. Hence p is bounded parabolic. Let K be a compact fundamental set for the action $\mathsf{St}_G p \curvearrowright \widetilde{X} \setminus \{p\}$. Let $\mathbf{u}$ be an entourage such that $\mathbf{u} \cap (\{p\} \times K) = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that every eventual geodesic from p to K contains an edge in F. For every $n \in \mathbb{Z}$ there exists $h \in \mathsf{St}_G p$ such that $h\gamma(n) \in K$. Since $\partial(h\gamma) = (p, p)$, the geodesic $h\gamma$ has edges in F in both its parts: $h\gamma(]-\infty, n])$ and $h\gamma([n, +\infty[)$. Hence $\mathsf{dist}_\Gamma(p, \gamma(n)) = \mathsf{dist}_\Gamma(p, h\gamma(n)) \leqslant \mathsf{dist}_\Gamma(p, F^0) + \mathsf{diam}_\Gamma F^0 < \infty$ $(n \in \mathbb{Z})$. So $\gamma(\mathbb{Z})$ is bounded and can not be an infinite geodesic. □

Proposition 3.11. *Let $\gamma : [0, +\infty) \to \Gamma^0$ be an infinite geodesic ray then $\gamma_{+\infty} \notin \Gamma^0$ and it is a conical point.*

Proof: If the assertion is not true then the point $p = \gamma_{+\infty}$ is bounded parabolic. Let K be a compact fundamental set for the action $\mathsf{St}_G p \curvearrowright \widetilde{X} \setminus \{p\}$. Since the geodesic ray is infinite the graph distance $d(\gamma(0), \gamma(n))$ tend to infinity. Let h_n be an element of $\mathsf{St}_G p$ such that $h_n(\gamma(n)) \in K (n \in \mathbb{Z}_+)$. By applying the "flow map" $\gamma(n) \to \gamma(n-1)$ to the geodesic $h_n\gamma$ we obtain a geodesic γ_n such that $\gamma_n(0) \in K$. The sequence of their other endpoints $\{h_n(\gamma(-n))\}$ converges to p as $n \to +\infty$. Each accumulation point α of the set $\{\gamma_n : n \geqslant 0\}$ in $\mathrm{EG}(\Gamma)$ is an eventual geodesic whose both endpoints are equal to p. Furthermore it cannot be a constant since it has a value in K. We have a non-trivial $\alpha \in \mathrm{EG}(\Gamma)$ such that $\alpha_{-\infty} = \alpha_{+\infty}$ which contradicts Proposition 3.10. □

In the following Proposition we show that the absence of the horocycles is equivalent to the hyperbolicity of the graph Γ.

Proposition 3.12. *The uniformity $\mathcal{U}^0 = \mathcal{U}|_{\Gamma^0}$ is generated by the collection $\{\mathbf{u}_e : e \in \Gamma^1\}$.*

Proof. Let us prove that $\mathbf{u}_e$ is an entourage. By Proposition 3.6 the boundary map ∂ is continuous on the closed set $K_{0,e} = \{\gamma \in \mathrm{EG}(\Gamma) : e = \{\gamma(0), \gamma(1)\}\}$. Hence $\partial K_{0,e}$ is closed. Its complement in $\widetilde{X}^2$ is exactly $\mathbf{u}_e$. By Proposition 3.10 the open set $\mathbf{u}_e$ contains all diagonal pairs (p, p), $p \in \widetilde{X}$, and so is an entourage.

By the visibility property for every $\mathbf{v}^0 \in \mathcal{U}^0$ there exists a finite set F of edges for which $\mathbf{u}_F = \cap_{e \in F} \mathbf{u}_e \subset \mathbf{v}^0$. So $\mathcal{U}^0$ is generated by the set $\{\mathbf{u}_e : e \in \Gamma^1\}$ as a filter. □

Remark. By [Ge2, subection 5.1] this corollary implies that Γ is hyperbolic. Namely each side of every geodesic triangle $\triangle$ is contained in the metric δ-neighborhood of the union of the other sides. The constant δ is determined as follows. Let $E_\#$ be a finite set of edges intersecting each G-orbit in Γ^1. For every $e \in E_\#$, since $\mathbf{u}_e$ is an entourage, there exists a finite set $F \subset \Gamma^1$ such that $\mathbf{u}_F^2 \subset \mathbf{u}_e$ (this property is called *alt-hyperbolicity* in [Ge2]). It follows directly from the definition of $\mathbf{u}_e$ that one can choose $\delta = 1 + \mathsf{max}\{\mathsf{dist}_\Gamma(e, F(e)^0) : e \in E_\#\}$.

This gives an independent proof of Yaman theorem without metrisability and cardinality restrictions. Note that it uses [GePo2] where a connected graph Γ was constructed. It also uses [Ge2] where the visibility property was proved.

4. Dynamical quasiconvexity and 2-cocompactness condition.

4.1. **The statement of the result.** Let X be a compactum. We first restate the definition of the dynamical quasiconvexity in terms of entourages [GePo3].

Definition 4.1. *Let G be a discrete group acting 3-discontinuously on a compactum X. A subgroup H of G is said to be dynamically quasiconvex if for every entourage $\mathbf{u}$ of X the set $G_{\mathbf{u}} = \{gG : g(\mathbf{\Lambda}H) \notin \mathsf{Small}(\mathbf{u})\}/H$ is finite.*

The aim of this Section is the following theorem giving a characterization of the dynamical quasiconvexity.

Theorem A. *Let G be a group which admits 3-discontinuous and 2-cocompact non-trivial action on a compactum X. Let H be a subgroup of G. The following conditions are equivalent.*

1. *The action $H \curvearrowright \mathbf{\Lambda}_X H$ is 2-cocompact.*
2. *H is dynamically quasiconvex.*

Remark. Since every parabolic subgroup is always dynamically quasiconvex we will regard every group action on a one-point set as 2-cocompact.

4.2. **Proof of the implication** 2) $\Rightarrow$ 1)**.** The action $G \curvearrowright X$ is 2-cocompact. So there exists a compact fundamental set K for the action of G on $\Theta^2 X$. Denote by $\mathbf{u}$ the entourage $X^2 \setminus K$. For every two distinct points p and q in X there exists $g \in G$ such that $g(p,q) \in K$. So $(p,q) \notin \mathbf{u}_1$ where $\mathbf{u}_1 = g^{-1}\mathbf{u}$. This means that the orbit $G\mathbf{u}$ has separation property.

Let us first show that the index of H in the stabilizer $\mathrm{St}_G(\mathbf{\Lambda}H) = \{g \in G \ : g(\mathbf{\Lambda}H) = \mathbf{\Lambda}H\}$ of $\mathbf{\Lambda}H$ is finite. Indeed for fixed two distinct points $\{p,q\} \subset \mathbf{\Lambda}H$ by the exactness there exists $\mathbf{u}_1 \in \mathcal{U}$ such that $(p,q) \notin \mathbf{u}_1$. So for every $h \in \mathrm{St}_G(\mathbf{\Lambda}H)$ we have $\{p,q\} \subset h(\mathbf{\Lambda}H)$ and $h(\mathbf{\Lambda}H) \notin \mathsf{Small}(\mathbf{u}_1)$. By the dynamical quasiconvexity applied to $\mathbf{u}_1$ there are at most finitely many such elements $h \in \mathrm{St}_G(\mathbf{\Lambda}H)$ distinct modulo H. So $\mathrm{St}_X(\mathbf{\Lambda}H) = \cup_{j\in J} k_j H$ $(|J| < \infty)$.

Consider now the orbit $G(\mathbf{\Lambda}H)$. Applying the dynamical quasiconvexity to $\mathbf{u}$ we obtain a finite set $\{g_i \in G \ : i \in I\}$ such that once $g(\mathbf{\Lambda}H)$ is not $\mathbf{u}$-small for some $g \in G$ then $g(\mathbf{\Lambda}H) = g_i(\mathbf{\Lambda}H)$ for some g_i.

Consider the following entourage of $\mathbf{\Lambda}H$:

$$\mathbf{v} = \bigcap_{i\in I, j\in J} (k_j^{-1} g_i^{-1} \mathbf{u} \cap \mathbf{\Lambda}^2 H).$$

Let $(x,y) \in \Theta^2(\mathbf{\Lambda}H)$. Since the orbit $G\mathbf{u}$ has separation property there exists $g \in G$ such that $g(x,y) \notin \mathbf{u}$ and hence $g(\mathbf{\Lambda}H) \notin \mathsf{Small}(\mathbf{u})$. We have $g(\mathbf{\Lambda}H) = g_i(\mathbf{\Lambda}H)$ for some $i \in I$. Hence $g_i^{-1}g(\mathbf{\Lambda}H) = \mathbf{\Lambda}H$ and $g_i^{-1}g = k_j h$ for some $j \in J$ and $h \in H$. So $g(x,y) = g_i k_j h(x,y) \notin \mathbf{u}$. Consequently $h(x,y) \notin \mathbf{v}$. We have proved that for every $(x,y) \in \Theta^2(\mathbf{\Lambda}H)$ there exists $h \in H$ such that $(x,y) \notin h^{-1}\mathbf{v}$. This means that the set $\Theta^2(\mathbf{\Lambda}H) \setminus \mathbf{v}$ is a compact fundamental set for the action $H \curvearrowright \Theta^2(\mathbf{\Lambda}H)$. □

4.3. **Proof of the implication** 1) $\Rightarrow$ 2)**.** We fix a group G and a 32-action of G on a compactum X. By Lemma 2.1 G acts on the attractor sum $\widetilde{X} = X \cup \Gamma$ where Γ is a connected, fine, hyperbolic graph and the action $G \curvearrowright \Gamma^1$ is proper and cofinite. The canonical uniformity $\mathcal{U}$ of $\widetilde{X}$ is generated by an orbit $G\mathbf{u}$. By Lemma 2.2 the restriction $\mathbf{u}|_{\Gamma^0}$ is a perspective divider.

Let $H < G$ be a subgroup. Denote by $\mathbf{\Lambda}H \subset \widetilde{X}$ its limit set for the action on $\widetilde{X}$. By Corollary 3.9 the set $C = \mathrm{Hull}(\mathbf{\Lambda}H)$ is closed in $\widetilde{X}$.

Lemma 4.2. *Every point v in $C^0 = C \cap \Gamma^0$ is either parabolic for the action $H \curvearrowright \mathbf{\Lambda}H$ or the number of edges incident to v in the graph C is finite (i.e. the degree of v in C is finite).*

Proof: Let $v \in C^0 \setminus \mathbf{\Lambda}H$. The set $\mathbf{\Lambda}H \subset X$ is compact. So by the exactness of $\mathcal{U}$ there exists an entourage $\mathbf{w} \in \mathcal{U}$ such that $\{v\} \times \mathbf{\Lambda}H \cap \mathbf{w} = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $\mathbf{u}_F \subset \mathbf{w}$. Hence every eventual geodesic γ with endpoints $a = \gamma_{-\infty}, b = \gamma_{+\infty} \in \mathbf{\Lambda}H$ and containing v passes through e_- and e_+, where e_- and e_+ are edges of F belonging to the geodesic rays joining $\gamma_{-\infty}$ with v and v with $\gamma_{+\infty}$ respectively. Thus every arc γ in C^1 has a simple subarc l between e_- and e_+ which also contains v. Since γ has no intermediate stops l is a geodesic. By the finess property of Γ the number of such geodesic subarcs is finite. Hence the number of edges incident to v is finite.

Suppose now that $v \in C^0 \cap \mathbf{\Lambda}H$. Then it is a parabolic point for G. Since H acts 2-cocompactly on $\mathbf{\Lambda}H$ every point of $\mathbf{\Lambda}H$ is either conical or bounded parabolic [Ge1, Main theorem, b]. If p is conical in $\mathbf{\Lambda}H$ then by 3-discontinuity of the action $G \curvearrowright X$ it is also conical for $G \curvearrowright X$ which is impossible by [Tu2, Theorem 3A]. □

Lemma 4.3. *Let $C = \mathrm{Hull}(\mathbf{\Lambda}H)$. If $|C^1/H| < \infty$ then H is dynamically quasiconvex.*

Proof: We extend the visibility property (see Definition 2.3) from Γ^0 to $\widetilde{X}$. By Theorem 3.7 for every $(x, y) \in \widetilde{X}^2$ there exists a geodesic $\gamma \in \mathrm{EG}(\Gamma)$ whose endpoints are x and y. So for an edge $e \in \Gamma^1$ and $(x, y) \in \widetilde{X}^2$ put $(x, y) \in \mathbf{u}_e$ if there exists such γ which does not contain the edge e.

Let $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v}^3 \subset \mathbf{u}$. Since the graph Γ has the visibility property there exists a finite set $E \subset \Gamma^1$ such that $\mathbf{u}_E = \cap_{e\in E}\mathbf{u}_e \subset \mathbf{v}|_{\Gamma^0}$. Let $(x, y) \notin \mathbf{u}$ and let γ be an eventual geodesic joining x with y. Choose $\{x', y'\} \subset \gamma$ such that $(x, x') \in \mathbf{v}$ and $(y, y') \in \mathbf{v}$. Since $(x, y) \notin \mathbf{u}$ we have $(x', y') \notin \mathbf{v}|_{\Gamma^0}$. Hence $(x', y') \notin \mathbf{u}_E$. So the piece of γ between x' and y' contains an edge from E. Hence $(x, y) \notin \mathbf{u}_e$. We have proved the inclusion $\mathbf{u}_E \subset \mathbf{u}$ on $\widetilde{X}^2$.

If now H is not dynamically quasiconvex then by Definition 4.1 the set $G_{\mathbf{u}} = \{g \in G : g(\mathbf{\Lambda}H) \notin \mathsf{Small}(\mathbf{u})\}/H$ is infinite for some $\mathbf{u} \in \mathcal{U}$. By the above argument there exists a finite $E \subset \Gamma^1$ such that $\mathbf{u}_E \subset \mathbf{u}$ on $\widetilde{X}$. Since $|\Gamma^1/G| < \infty$ there exists an edge $e \in E$ for which the set $\{g \in G : g(\mathbf{\Lambda}H) \notin \mathsf{Small}(\mathbf{u}_e)\}/H$ is infinite. Therefore the set $\{g \in G \ : \ e \in g(C^1)\}/H = \{g \in G \ : \ g^{-1}(e) \in C^1\}/H$ is infinite too. □

Suppose that the action $H \curvearrowright \mathbf{\Lambda}H$ is 2-cocompact. By Lemma 4.3 it is enough to prove that $|C^1/H| < \infty$ where $C = \mathrm{Hull}(\mathbf{\Lambda}H)$.

Let K be a compact fundamental set for the action $H \curvearrowright \Theta^2(\mathbf{\Lambda}H)$. So $HK = \Theta^2(\mathbf{\Lambda}H)$. Let $\mathbf{u} \in \mathcal{U}$ be an entourage such that $\mathbf{u}^3 \cap K = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $\mathbf{u}_F \subset \mathbf{u}$. Thus $\mathbf{u}_F^3 \cap K = \emptyset$. Up to adding a finite number of edges to F we can assume that F is the edge set of a finite connected subgraph of Γ.

We call the edges of C^1 which belong to HF *red* edges. The other edges of C^1 are *white*. Similarly we declare parabolic points of H *red* and other vertices of C are *white.*

Lemma 4.4. *Every infinite ray $\rho : [0, \infty[\to C$ contains at least one red edge. Furthermore every geodesic between two red vertices contains a red egde.*

Proof of the lemma. By Lemma 3.1 the ray ρ converges to a point $x = \rho(\infty) \in \mathbf{\Lambda}H$. Since the action $H \curvearrowright \mathbf{\Lambda}H$ is 2-cocompact by [Ge1] every point of $\mathbf{\Lambda}H$ is either conical or bounded parabolic. By Proposition 3.11 x is conical for the action $H \curvearrowright \mathbf{\Lambda}H$.

Therefore there exists an infinite set $S \subset H$ and two distinct points a and b in $\mathbf{\Lambda}H$ such that a and b are limit points for the sets $S(\rho(\infty))$ and $S(\rho(0))$ respectively. Let U_a and U_b be disjoint $\mathbf{u}$-small neighborhoods of a and b for $\mathbf{u}$ defined before the Lemma. Thus $\exists\, s \in S \ : \ s(\rho(\infty)) \in U_a,\ s(\rho(0)) \in U_b$. There exists $h \in H$ such that $h(a, b) \in K$. Hence $h(a, b) \notin \mathbf{u}^3$. Since $h(s\rho(\infty), a) \in \mathbf{u}$ and $h(s\rho(0), b) \in \mathbf{u}$ we obtain $\partial(hs(\rho)) \notin \mathbf{u}$. Thus $\partial(hs(\rho)) \notin \mathbf{u}_F$ for the finite set F defined therein. It follows that $hs(\rho)$ contains a red edge and so is ρ.

Let now γ be a geodesic between two red points in C. Then $\exists\, h \in H : h(\partial\gamma) \in K$ so the pair $h(\partial\gamma)$ is not $\mathbf{u}_F$-small. Thus every geodesic γ connecting two red points contains at least one red edge. The Lemma is proved. □

It remains to show that the set of white edges of C^1 is H-finite.

Let us say that a segment of an eventual geodesic in $\mathcal{C}$ is *white* if all its edges and vertices are white. Denote by $\mathcal{F}$ the subgraph of C^1 obtained by adding to the set F^1 all adjacent white segments. Since F is connected $\mathcal{F}$ is also connected. By the first statement of Lemma 4.4 every geodesic interval containing only white edges has finite length. Furthermore by Lemma 4.2 the degree of every white vertex is finite. Thus by König Lemma the connected subgraph $\mathcal{F}$ is finite.

We claim that $H\mathcal{F}^1 = C^1$. Indeed if $e = (a, b) \in C^1$ is a white edge then by the second statement of 4.4 one of its vertices, say a, is white. Consider a maximal a white segment l_1 of C starting from a and not containing e. It has a finite length and ends either at a red vertex c or at a red edge. Our aim is to prove that the second case does happen for one of such segments. Suppose it is not true for l_1. Then the other vertex b of the edge e cannot be red. Indeed if b is red, then $l_1 \cup e$ has two red ends c and b, and by Lemma 4.4 l must contains a red edge which is impossible. So b is white. Since $e \in C^1$ there exists another maximal white segment l_2 starting from b. If it ends up at a red vertex d then applying again 4.4 we obtain that $l_1 \cup l_2 \cup e$ contains a red edge. So there exists a white eventual geodesic segment l starting from e and terminating at a red edge e_1. Thus there exists $h \in H \ : \ h(l \cup e) \subset \mathcal{F}$. The Theorem is proved. $\square$

The proof of the above Theorem gives rise to another condition of the dynamical quasiconvexity.

Corollary 4.5. *The following conditions are equivalent:*

a) *H satisfies one of the conditions 1) or 2) of Theorem A;*

b) $|C^1/H| < \infty$ *where* $C = \mathrm{Hull}(\mathbf{\Lambda} H)$.

Proof: By Lemma 4.3 it remains to prove that a) $\Rightarrow$ b). By the statement 2) $\Rightarrow$ 1) of Theorem A the dynamical quasiconvexity implies 2-cocompactness of the action $H \curvearrowright \mathbf{\Lambda}_X H$. We have proved above that the latter one implies that $|C^1/H| < \infty$. $\square$

5. Pullback space for 32-actions

In [Ge1, page 142] the following problem was formulated. Let a group G admit convergence actions on two compacta T_i does there exist a convergence action on a compactum Z and two G-equivariant continuous mappings $\pi_i : Z \to T_i$ $(i = 0, 1)$?

$$\begin{array}{ccc} & Z & \\ {}^{\pi_0}\swarrow & & \searrow^{\pi_1} \\ T_0 & & T_1 \end{array} \tag{1}$$

Definition 5.1. *We call the space Z and the action $G \curvearrowright Z$ pullback space and pullback action respectively.*

Answering a question of M. Mitra [M] O. Baker and T. Riley constructed in [BR] a hyperbolic group G containing a free subgroup H of rank 3 such that the embedding does not induce an equivariant continuous map (called "Cannon-Thurston map") $\partial H \to \partial G$ where ∂ is the boundary of a hyperbolic group. Denote $T_0 = \partial H$, and let $T_1 = \mathbf{\Lambda}_{\partial G} H$ be the limit set for the action of $H \curvearrowright \partial G$. The following proposition shows that Baker-Riley's example is also a contre-example to the pullback problem.

Proposition 5.2. *The compacta T_i $(i = 0, 1)$ do not admit a pullback space on which H acts 3-discontinuously.*

Proof: Suppose by contradiction that the diagram (1) exists. Consider the spaces $\widetilde{Z} = Z \cup H$, $\widetilde{T_0} = T_0 \cup H$, $\widetilde{T_1} = T_1 \cup H$ equipped with the following topology (which we illustrate only for $\widetilde{T_0}$ and is defined similarly in the other cases). A set F is closed in $\widetilde{T_0}$ if

1) $F \cap T_0 \in \mathsf{Closed}(T_0)$;

2) $F \cap H \in \mathsf{Closed}(H)$;

3) $\partial_1(F \cap H) \subset F$ where ∂_1 denotes the set of attractive limit points.

The topology axioms are easily checked. Since H is discrete, its points are isolated in $\widetilde{T_0}$ and the condition 2) is automatically satisfied.

By [Ge2, Proposition 8.3.1] the actions $G \curvearrowright \widetilde{T_i}$ and $G \curvearrowright \widetilde{Z}$ are 3-discontinuous. By the following lemma the maps π_i can be extended to the continuous maps $\widetilde{\pi}_0 : \widetilde{Z} \to \widetilde{T_0}$ and $\widetilde{\pi}_1 : \widetilde{Z} \to \widetilde{T_1}$ where $\widetilde{\pi}_i|_Z = \pi_i$ and $\widetilde{\pi}_i|_H = \mathrm{id}$ $(i = 0, 1)$.

Lemma 5.3. *Let G be a group acting 3-discontinuously on two compacta X and Y. Denote $\widetilde{X}$ and $\widetilde{Y}$ the spaces $X \cup G$ and $Y \cup G$ respectively equipped with the above topologies. Suppose that the action on Y is minimal and $|Y| > 2$. If $f : X \to Y$ is a continuous G-equivariant map then the map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $\widetilde{f}|_X = f$ and $\widetilde{f}|_G \equiv \mathrm{id}$ is continuous.*

Assuming the lemma for the moment let us finish the argument. By hypothesis $H \curvearrowright Z$ is 3-discontinuous. The map π_0 is equivariant and continuous and the action $H \curvearrowright T_0$ is minimal. So π_0 is surjective. Since H is hyperbolic all points of T_0 are conical [Bo3]. By [Ge2, Proposition 7.5.2] the map π_0 is a homeomorphism. So we have the equivariant continuous map $\pi = \pi_1 \circ \pi_0^{-1} : T_0 \to T_1$. By Lemma 5.3 it extends equivariantly to the map $\widetilde{\pi} : \widetilde{T_0} \to \widetilde{T_1}$ where $\widetilde{T_0} = H \cup \partial H$ and $\widetilde{T_1} = G \cup \partial_\infty G$. This is a Cannon-Thurston map. A contradiction with the result of Baker-Riley. The Proposition is proved modulo the following.

Proof of Lemma 5.3. Let $F \subset \widetilde{Y}$ be a closed set. Denote $F_Y = F \cap Y$ and $F_G = F \cap G$. We need to check that the set $\widetilde{f}^{-1}(F) = f^{-1}(F_Y) \cup F_G$ is closed. The conditions 1) and 2) are obvious for $\widetilde{f}^{-1}(F) \cap X$ and for $\widetilde{f}^{-1}(F) \cap G$ respectively.

Let $z^\times = r \times X \cup X \times a$ be a limit cross for F_G on X. To check condition 3) for the set $f^{-1}(F)$ we need to show that $b = f(a) \in F_Y$. Suppose not, and $b \notin F_Y$ and let B be a closed neighborhood of b such that $B \cap F_Y = \emptyset$. Let $\mathbf{v} \in \mathrm{EntY}$ be an entourage such that $B\mathbf{v} \cap F_Y = \emptyset$ where $B\mathbf{v} = \{y \in Y : (y, b_1) \in \mathbf{v},\ b_1 \in B\}$. Set $A = f^{-1}(B) \ni a$. For a neighborhood R of the repelling point $r \in X$ the set $F_0 = \{g \in F_G : g(X \setminus R) \subset A\}$ is infinite.

Let $w^\times = p \times Y \cup Y \times q$ be a limit cross for F_0 on Y, and $P \times Y \cup Y \times Q$ be its neighborhood. Since $F_Y \subset Y$ is closed by condition 3) we have $q \in F_Y$. Suppose that Q is $\mathbf{v}$-small. By the hypothesis there exist three distinct points $y_i \in Y$ $(i = 1, 2, 3)$. Since the set Y is minimal and f is equivariant one has $f^{-1}(y_i) = X_i \neq \emptyset$ and X_i are mutually disjoint $(i = 1, 2, 3)$.

Let us now put some restrictions on R. Suppose that $R \cap X_i = \emptyset$ for at least two indices $i \in \{k, j\} \subset \{1, 2, 3\}$ and for one of them, say k, we have $y_k \notin P$.

If $g \in G$ is close to $w^\times$ we have $g(Y \setminus P) \subset Q$ and $g(y_k) \in Q$. From the other hand $g(X_k) \subset A$ since $X_k \cap R = \emptyset$. Thus $g(y_k) \in Q \cap B$ and so $(q, g(y_k)) \in \mathbf{v}$. Hence $q \in B\mathbf{v}$ and $q \notin F_Y$. A contradiction. The lemma is proved. □

Since the answer to the pullback problem for general convergence actions is negative, it seems to be rather intriguing to study the pullback problem in a more restrictive case of 2-cocompact actions. The rest of the section is devoted to a discussion of this problem.

If G is a finitely generated group acting 3-discontinuously and 2-cocompactly on compacta X_1 and X_2 then by the Mapping theorem [Ge2, Proposition 3.4.6] there exist equivariant maps $F_i : \partial G \to X_i$ $(i = 1, 2)$ from the Floyd boundary ∂G of G. By [Ka] the action on ∂G is 3-discontinuous. So ∂G is a pullback space for any two 32-actions of G.

If G is not finitely generated this argument does not work as the Mapping theorem requires the cofiniteness on edges of a graph on which the group acts and which is not true for the Cayley graphs in this case. An action of such a group on a relative graph depends on the system of non-finitely generated parabolic subgroups [GePo2, Proposition 3.43]. Furthermore the action on the closure of the diagonal image of the group in the product space may not be 3-discontinuous. However if there is a pullback action for two 3-discontinuous actions of G and both of them are 2-cocompact then as shows the following lemma a quotient of the pullback space also admits a 32-action.

Lemma 5.4. *Suppose that G acts 3-discontinuously and 2-cocompactly on two compacta X_i $(i = 1, 2)$. Let X be a pullback space for X_i and $\pi_i : X \to X_i$ be the corresponding equivariant continuous maps. Then the action on the quotient space $T = \pi(X) = \{(\pi_1(x), \pi_2(x)) \ : \ x \in X\}$ is of type* (32).

Furthermore the action of G on the attractor sum $\widetilde{T} = T \sqcup G$ is also of type (32).

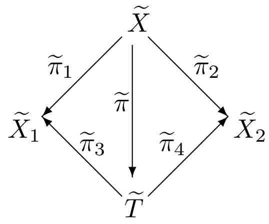
Proof: We will argue in terms of the attractor sums to obtain the more stronger last statement. By lemma 5.3 the maps π_i extend to the continuous equivariant maps $\widetilde{\pi}_i : \widetilde{X} \to \widetilde{X}_i$ where $\widetilde{X} = X \sqcup G$ and $\widetilde{X}_i = X_i \sqcup G$, $\widetilde{\pi}_i|_G = \mathrm{id},\ \widetilde{\pi}_i|X_i = \pi_i$.

By Lemma 2.1 the actions on X_i extends to 32-actions on the attractor sums $\widetilde{X}_i$. Since the action $G \curvearrowright \widetilde{X}_i$ is 2-cocompact there exists an entourage $\mathbf{u}_i$ of $\widetilde{X}$ such that the uniformity $\mathcal{U}_i$ on $\widetilde{X}_i$ is generated as a filter by the orbit $G\mathbf{u}_i$ $(i = 1, 2)$ [Ge1, Proposition E, 7.1].

Let $\widetilde{\mathbf{u}}_i$ denotes the entourage $\widetilde{\pi}_i^{-1}(\mathbf{u}_i)$ on $\widetilde{X}(i = 1, 2)$. Their G-orbits generate the lifted uniformities $\widetilde{\mathcal{U}_i}$. Then $\widetilde{\mathbf{w}} = \widetilde{\mathbf{u}}_0 \cap \widetilde{\mathbf{u}}_1$ is an entourage of $\widetilde{X}$ whose orbit $G\widetilde{\mathbf{w}}$ generates a uniformity $\mathcal{W}$ on $\widetilde{X}$. Note that $\mathcal{W}$ is not a priori exact. Indeed there could exists 2 points in X such that $\pi_i(x) = \pi_i(y)$ and there is no way to separate them using the uniformities $\mathcal{U}_i$ $(i = 1, 2)$. So we consider the following quotient spaces:

$$\widetilde{T} = \widetilde{\pi}(\widetilde{X}) = \{(\widetilde{\pi}_1(x), \widetilde{\pi}_2(x)) \in \widetilde{X}_0 \times \widetilde{X}_1 \ : \ x \in \widetilde{X}\}, T = \pi(X) \text{ where } \pi = \widetilde{\pi}|_X.$$

Since $\widetilde{\pi}_i$ $(i = 0, 1)$ are equivariant the map $\widetilde{\pi}$ is equivariant too. Denoting by $\widetilde{\pi}_i : \widetilde{T} \to \widetilde{X}_{i-2}$ $(i = 3, 4)$ the projections on the factors we obtain the following commutative diagram.



(3)

Since the map π is continuous and surjective the action $G \curvearrowright \widetilde{T}$ is 3-discontinuous too [GePo1, Proposition 3.1]. It remains to prove that it is 2-cocompact in the quotient topology. Let $\widetilde{\mathbf{v}} = \pi(\widetilde{\mathbf{w}})$ and consider the uniformity $\widetilde{\mathcal{V}}$ on $\widetilde{T}$ generated by the orbit $G\widetilde{\mathbf{v}}$. To show that $G \curvearrowright \widetilde{T}$ is 2-cocompact it is enough to prove that $\widetilde{\mathcal{V}}$ is exact [Ge1, Proposion E, 7.1].

So let x, y be two distinct points of $\widetilde{T}$. Then either $\widetilde{\pi}_3(x) \neq \pi_3(y)$ or $\widetilde{\pi}_4(x) \neq \widetilde{\pi}_4(y)$. For example in the first case by the exactness of $\widetilde{\mathcal{U}_0}$ there exists $g \in G$ such that $g(\widetilde{\pi}_3(x), \widetilde{\pi}_3(y)) \notin \widetilde{\mathbf{u}}_0$. By definition of $\widetilde{\mathbf{w}}$ we obtain that $(\widetilde{\pi}(x), \pi(y)) \notin \widetilde{\mathbf{v}}$. So $\widetilde{\mathcal{V}}$ is exact. □

The aim of the following Proposition is to show that two 32-actions may not have a pullback. We note that it is one of the rare cases when a fact known for finitely generated relatively hyperbolic groups is not in general true for non-finitely generated ones.

Proposition 5.5. *The free group F_∞ of countable rank admits two 32-actions not having a pullback space.*

Proof: Let $G =< x_1, ..., x_n, y_1, ..., y_m, ... >$ $(n \geq 2)$ be a group freely generated by the union of a finite set $X = \{x_1, ..., x_n\}$ and an infinite set $Y = \{y_1, ...\}$. Let $A =< X >$ be a subgroup generated by X, and let H be a subgroup of A freely generated by an infinite set $W = \{w_i \ : \ i \in \mathbb{N}\}$.

Set $Z = \{z_m = y_m w_m \ : \ m \in \mathbb{N}\}$, $P =< Y >$ and $Q =< Z >$. The set $X \cup Z$ is obtained by Nielsen transformations from $X \cup Y$ [LS]. So $X \cup Z$ is also a free basis for G, and the map $\varphi \ : \ x_i \to x_i, y_k \to z_k$ $(i = 1, ..., n; k \in \mathbb{N})$ extends to an automorphism of G. We have two splittings of G :

$$G = A * P, \quad \text{and} \quad G = A * Q. \tag{1}$$

Each splittings in (1) gives rise to an action of G on a simplicial tree whose vertex groups are conjugates of either A or P (respectively Q). We now replace the vertices stabilized by A and its conjugates by their Cayley trees. Denote the obtained simplicial G-trees by $\mathcal{T}_i$ $(i = 1, 2)$. Their edge stabilizers are trivial and vertex stabilizers are non-trivial if only if they are conjugate to P (respectively to Q). The vertices of $\mathcal{T}_1$ (respectively $\mathcal{T}_2$) are the elements of G and the parabolic vertices corresponding to conjugates of P (respectively Q). The graph $\mathcal{T}_i$ is a connected fine hyperbolic graph such that the action of G on edges are proper and cofinite. Hence the actions satisfy Bowditch's criterion of relative hyperbolicity [Bo1]. By [Ge2] both actions on the trees extend to 32-actions on compacta R_i which are the limit sets for the actions $G \curvearrowright \mathcal{T}_i$ $(i = 1, 2)$.

We claim that $P \cap g^{-1}Qg = \{1\}$ for all $g \in G$. Indeed consider the endomorphism f such that $f(x_i) = x_i,\ f(y_j) = 1$ $(i = 1, ..., n, j = 1, ...)$. The map f is injective on Q as well as on every conjugate $g^{-1}Qg$. From the other hand $Y \subset \text{Ker} f$. So $P < \text{Ker} f$. We have proved that

$$\forall g \in G : P \cap g^{-1}Qg = \{1\}. \tag{2}$$

Arguing now by contradiction assume that there exists a pullback space R and equivariant projections π_i : $R \to R_i$ $(i = 1, 2)$. By Lemma 5.4 the action on the quotient space:

$$T = \pi(R) = \{(\pi_1(r), \pi_2(r)) \mid r \in R\}$$

is 3-discontinuous and 2-cocompact. Note that the action $G \curvearrowright T$ is minimal because $G \curvearrowright R$ is minimal.

By [Ge1, Main theorem, b] all points of T are either conical or bounded parabolic. If $p \in T$ is parabolic then $\pi_{i+2}(p)$ are parabolic points in both R_i for the map $\pi_{i+2} = \widetilde{\pi}_{i+2}|_T$ (see the diagram in Lemma 5.4). Indeed the preimage of a conical point by an equivariant map is conical [Ge2, Proposition 7.5.2]. So p must be fixed by the intersection of some parabolic subgroup $g_1 P g_1^{-1}$ of the first action and a parabolic subgroup $g_2 Q g_2^{-1}$ of the second $(g_i \in G)$. However by (2) this intersection is trivial. Thus there are no parabolic points for the 32-action $G \curvearrowright T$.

By [Bo3, Theorem 8.1] (see also [GePo2, Corollary 3.40]) the group G is hyperbolic and so finitely generated. This is a contradiction. □

Definition. The set of the stabilizers of the parabolic points for a 32-action on a compactum X is called *peripheral structure* on G.

The following theorem provides a sufficient condition for the existence of pullback space for two 32-actions of a group.

Theorem 5.6. *Let G be a group which admits 32-actions on compacta X and Y. Let $\mathcal{P}$ be the peripheral structure corresponding to the action on X. Suppose that every $P \in \mathcal{P}$ acts 2-cocompactly on $\mathbf{\Lambda}_Y P$. Then there exists a compactum Z equipped with a 32-pullback action of G with respect to its actions on X and Y.*

Remark. Using Theorem A one can reformulate the hypotheses above by requiring that the action of each subgroup $P \in \mathcal{P}$ is dynamically quasiconvex on their limit sets in Y.

Proof of Theorem 5.6. Denote

$$\mathcal{R} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}, |P \cap Q| = \infty\}. \tag{4}$$

We will indicate a compactum Z acted upon by G 3-discontinuously and 2-cocompactly whose peripheral structure is $\mathcal{R}$. Denote by $\mathrm{Par}(Y, P)$ the set of parabolic points for the action of $P \in \mathcal{P}$ on Y. We will need the following lemma.

Lemma 5.7. *Let $G, X, Y, \mathcal{P}, \mathcal{Q}$ be as in Theorem B. The following properties of a subgroup $H \subset G$ are equivalent:*

- a: $H \in \mathcal{R}$;
- b: *there exist $P \in \mathcal{P}$ and $q \in \mathrm{Par}(Y, P)$ such that $H = \mathsf{St}_P q$.*

Proof of the Lemma. b⇒a). If the action of $P \in \mathcal{P}$ on Y admits a parabolic point q then its stabiliser H is an infinite subgroup of P. By [Tu2, Theorem 3.A] the point q is parabolic for the action $G \curvearrowright Y$. We note that the assumption of [Tu2] that the space is metrisable can be omitted by a small modification of the argument. Let $Q = \mathrm{St}_G q \in \mathcal{Q}$ be the stabilizer of q. We obtain $H = P \cap Q \in \mathcal{R}$.

a⇒b). Let $H = P \cap Q$ for $(P, Q) \in \mathcal{P} \times \mathcal{Q}$. We may assume that $P = \mathsf{St}_G p$ and $Q = \mathsf{St}_G q$ for $p \in \mathrm{Par}(X, G)$, $q \in \mathrm{Par}(Y, G)$. Since H is an infinite subset of Q we have $\mathbf{\Lambda}_Y H = \{q\}$. Since $H \subset P$ we have $q \in \mathbf{\Lambda}_Y P$. If q is conical for $P \curvearrowright \mathbf{\Lambda}_Y P$ then it is also conical for $G \curvearrowright Y$ contradicting by [Tu2, Theorem 3.A] the fact that $q \in \mathrm{Par}(Y, G)$. The lemma is proved.

The peripheral structure $\mathcal{P}$ consists of finitely many G-conjugacy classes [Ge1, Main Theorem, $\mathfrak{a}$]. Since for every $P \in \mathcal{P}$ the action $P \curvearrowright \mathbf{\Lambda}_Y P$ is 2-cocompact there are finitely many P-conjugacy classes of maximal parabolic subgroups in P. So it follows from Lemma 5.7 that $\mathcal{R}$ consists of finitely many G-conjugacy classes. Since the subgroups in $\mathcal{R}$ are infinite, each of them is contained in exactly one $P \in \mathcal{P}$ and in exactly one $Q \in \mathcal{Q}$. So the inclusions induce well-defined maps $\mathcal{P} \overset{\pi}{\longleftarrow} \mathcal{R} \overset{\sigma}{\longrightarrow} \mathcal{Q}$ equivariant by conjugation.

We now extend the maps π, σ identically over the sets $\widetilde{\mathcal{P}} = G \sqcup \mathcal{P}$, $\widetilde{\mathcal{Q}} = G \sqcup \mathcal{Q}$, $\widetilde{R} = G \sqcup \mathcal{R}$. Denoting the extensions by the same symbols we have G-equivariant maps $\widetilde{\mathcal{P}} \overset{\pi}{\longleftarrow} \widetilde{\mathcal{R}} \overset{\sigma}{\longrightarrow} \widetilde{\mathcal{Q}}$. By Lemma 2.1 the set $\widetilde{P}$ is the vertex set of a connected fine graph Δ such that the action on edges $G \curvearrowright \Delta^1$ is cofinite and proper.

We will construct a connected graph Γ whose vertex set is $\Gamma^0 = \widetilde{\mathcal{R}}$ and the action on edges is cofinite and proper. The set of edges Γ^1 will be obtained by replacing the parabolic vertices $\mathcal{P}$ of Δ by connected graphs coming from the action of the groups $P \in \mathcal{P}$ on Y. We do it in the following four steps.

Step 1. *Definition of Γ^1_1.*

Choose a set $\mathcal{R}_\# \subset \mathcal{R}$ that intersects each conjugacy class by a single element. For every $R \in \mathcal{R}_\#$ we join the vertex R with each element of $R \subset G$ and denote by E_R this set of edges. Then put

$$\Gamma^1_1 = \bigcup \{gE_R \ : g \in G, \ R \in R_\#\}.$$

The set Γ^1_1 corresponds to the well-known coned-off construction over every coset gR where $g \in G, R \in \mathcal{R}_\#$ [Fa], [Bo1].

Step 2. *Definitions of Γ^1_2 and Γ^1_3.*

Choose a set $\mathcal{P}_\# \subset \mathcal{P}$ that intersects each conjugacy class of P by a single element. For each $P \in \mathcal{P}_\#$ we add to Γ^1 a connected G-finite set of pairs according to one of the following ways.

Case 2.1. (*hyperbolic case*) $P \in \mathcal{P}_\# \setminus \mathsf{Im}\pi$ (or equivalently $\pi^{-1}(P) \cap \mathcal{R} = \emptyset$).

Then P acts on Y either as an elementary loxodromic 2-ended subgroup, or the action $P \curvearrowright \mathbf{\Lambda}_Y P$ is a non-elementary 32-action without parabolics [Tu1, Theorem 3A]. In both cases every point of $\mathbf{\Lambda}_Y P$ is conical [Ge1, Main Theorem, $\mathfrak{b}$] and G is a hyperbolic group [Bo3, Theorem 8.1] (for another proof of this fact see [GePo1, Appendix]). There exists a P-finite set Γ^1_P of pairs of elements of P such that the graph (P, Γ^1_P) is connected.

Put $\Gamma^1_2 = \bigcup \left\{ g\Gamma^1_P \ : \ g \in G, \ P \in \mathcal{P}_\# \setminus \mathsf{Im}\pi \right\}$.

Case 2.2 (*non-hyperbolic case*) $P \in \mathcal{P}_\# \cap \mathsf{Im}\pi$.
There is a canonical bijection $\tau_P : \mathrm{Par}(Y, P) \to \pi^{-1}P$. Let

$$M_P = \{g \in G : g \text{ is joined by a } \Gamma^1_1\text{-edge with some } R \in \pi^{-1}P\}.$$

The set M_P is P-invariant and P-finite by the construction. Let Γ^1_P be the P-finite set of pairs of the elements of the P-invariant set $M_P \cup \pi^{-1}P$ such that (P, Γ^1_P) is connected. The latter one exists as the graph corresponding to the 32-action $P \curvearrowright \mathbf{\Lambda}_Y P$ is connected by Lemma 2.1. Put

$$\Gamma^1_3 = \bigcup \left\{ g\Gamma^1_P \ : \ g \in G, \ P \in \mathcal{P}_\# \cap \mathsf{Im}\pi \right\}.$$

Step 3. *Definition of* Γ^1_4. Consider in the graph Δ the set of all its "horospherical" edges $\Delta^1_0 = \{(P, g) \ : \ P \in \mathcal{P}, g \in M_P\}$. Let Γ^1_4 be the set of all non-horospherical edges of Δ:

$$\Gamma^1_4 = \Delta^1 \setminus \Delta^1_0.$$

Step 4. *Definition of* Γ^1. Let

$$\Gamma^1 = \Gamma^1_1 \cup \Gamma^1_2 \cup \Gamma^1_3 \cup \Gamma^1_4. \tag{5}$$

The set Γ^1 is obviously G-finite.

Lemma 5.8. *The graph* $\Gamma = (\Gamma^0, \Gamma^1)$ *is connected.*

Proof of the lemma. Since every vertex of Γ is either an element of G or is joined with an element of G it suffices to verify that every two elements of G can be joined by a path in Γ. We initially join them by a path γ in the connected graph Δ. We transform this path as follows. If all vertices of γ belong to G then γ is also a path in Γ. It γ passes through a point $P \in \mathcal{P}$ then it has a subpath of the form $g_0 - P - g_1$ where $g_0, g_1 \in M_P$. The graph of P corresponding to the action on Y is connected. So we can replace this subpath by a subpath with the same endpoints all whose vertices are contained in M_P (in the "hyperbolic" case) or in $M_P \cup \pi^{-1}P$ (in the "non-hyperbolic" case). In both cases the edges of this new subpath belong to Γ^1. The lemma is proved.

End of the proof of Theorem 5.6. By Lemma 2.2 the sets $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ admit perspective dividers $\mathbf{u} \subset \widetilde{\mathcal{P}}^2$, $\mathbf{v} \subset \widetilde{\mathcal{Q}}^2$. Since the projection π and σ commute with the group action the lifts $\pi^{-1}\mathbf{u}$ and $\sigma^{-1}\mathbf{v}$ are perspective dividers on $\Gamma^0 = \widetilde{\mathcal{R}}$.

It is a direct verification that $\mathbf{w} = \pi^{-1}\mathbf{u} \cap \sigma^{-1}\mathbf{v}$ is a perspective divider on Γ^0. Indeed if $g(a, b) \notin \mathbf{w}^0$ then $g(\pi(a), \pi(b)) \notin (\mathbf{u}^0 = \mathbf{u}|_{\mathcal{P}})$ or $g(\sigma(a), \sigma(b)) \notin (\mathbf{v}^0 = \mathbf{v}|_{\mathcal{Q}})$. So there exist at most finitely many such elements $g \in G$ as $\mathbf{u}^0$ and $\mathbf{v}^0$ are both perspective. Similarly $\mathbf{w}^0$ is a divider on Γ^0 as if $(\cap F_1\{\mathbf{u}^0\})^2 \subset \mathbf{u}^0$ and $(\cap F_2\{\mathbf{v}^0\})^2 \subset \mathbf{v}^0$ for some finite $F_i \subset G$ $(i = 1, 2)$ then $(\cap F\{\mathbf{w}^0\})^2 \subset \mathbf{w}^0$ where $F = F_1 \cap F_2$.

It follows that the projections $\pi : (\Gamma^0, \mathbf{w}) \to (\widetilde{\mathcal{P}}, \mathbf{u})$ and $\sigma : (\Gamma^0, \mathbf{w}) \to (\widetilde{\mathcal{Q}}, \mathbf{v})$ are uniformly continuous with respect to the uniformities generated by the divider orbits.

By Lemma 2.4 the action of G on the Cauchy-Samuel completion $\widetilde{Z}$ of $(\Gamma^0, \mathbf{w})$ is a 32-action. By [Bourb, II.23, Proposition 13] the completion $\widetilde{Z}$ coincides with the closure $\mathrm{Cl}_{\widetilde{X} \times \widetilde{Y}}(\Gamma^0)$ of Γ^0 embedded diagonally in $\widetilde{X} \times \widetilde{Y}$ where $\widetilde{X} = X \sqcup G$ and $\widetilde{Y} = Y \sqcup G$. So the projections π and σ extend continuously to the equivariant maps $\widetilde{\pi} : \widetilde{Z} \to \widetilde{X}$ and $\widetilde{\sigma} : \widetilde{Z} \to \widetilde{Y}$ whose restrictions to G is the identity. We have proved that $Z = \mathbf{\Lambda}_{\widetilde{Z}} G$ is a pullback space. The Theorem is proved. □

To prove the statement converse to Theorem 5.6 we need the following direct generalization of the argument of [MOY, Lemma 2.3, (4)] avoiding the metrisability assumption.

Lemma 5.9. *Let a group G admits two non-trivial 3-discontinuous actions on compacta X and Y, and let $f : X \to Y$ be an equivariant continuous map. Let H be a subgroup of G such that $\mathbf{\Lambda}_Y H \subsetneq Y$. Suppose that H acts cocompactly on $Y \setminus \mathbf{\Lambda}_Y H$. Suppose that for every infinite set $B \subset G \setminus H$ there exist an infinite subset $B_0 \subset B$ and at least two distinct points $r_i \in f^{-1}(\mathbf{\Lambda}_Y H)$ such that $\forall\, g \in B_0 : g(r_i) \notin f^{-1}(\mathbf{\Lambda}_Y H)$ $(i = 1, 2)$. Then $f^{-1}(\mathbf{\Lambda}_Y H) = \mathbf{\Lambda}_X H$.*

Corollary 5.10. *If p is a bounded parabolic point for the action of G on Y then $f^{-1}(p)$ is the limit set $\mathbf{\Lambda}_X(\mathrm{St}_Y p)$ of $\mathrm{St}_G p$ for the action on X.*

Proof of the Corollary. By the equivariance and continuity of f we have $\mathbf{\Lambda}_X H \subset f^{-1}(\mathbf{\Lambda}_Y H)$ where $H = \mathrm{St}_G p$. So if $f^{-1}(p)$ is a single point then the statement is trivially true. If $f^{-1}(p)$ contains at least two distinct points r_i $(i = 1, 2)$ then we have $\forall\, g \in G \setminus H \ :\ g(r_i) \notin f^{-1}(\mathbf{\Lambda}_Y H)$ as $g(p) \neq p$. and we apply Lemma 5.9. □

Proof of the Lemma. The statement is trivial if H is finite, so we assume that H is infinite. Suppose first that the set $f^{-1}(\mathbf{\Lambda}_Y H)$ is finite. Since $f(\mathbf{\Lambda}_X H) \subset \mathbf{\Lambda}_Y H$ then $f^{-1}(\mathbf{\Lambda}_Y H)$ is pointwise fixed under a finite index subgroup of H. So $f^{-1}(\mathbf{\Lambda}_Y H) = \mathbf{\Lambda}_X H$ in this case.

Suppose that $f^{-1}(\mathbf{\Lambda}_Y H)$ is infinite. Suppose by contradiction that there exists a point $s \in f^{-1}(\mathbf{\Lambda}_Y H) \setminus \mathbf{\Lambda}_X H$. Then there exist an infinite set $B \subset G \setminus H$ converging to the cross whose attractive limit point is s. By our assumption there exists an infinite subset $B_0 \subset B$ and distinct points $r_i \in f^{-1}(\mathbf{\Lambda}_Y H)$ such that $\forall\, g \in B_0$: $g(r_i) \notin f^{-1}(\mathbf{\Lambda}_Y H)$ $(i = 1, 2)$. Then one of them $z \in \{r_1, r_2\}$ is not repulsive for a limit cross of B_0. So for every open neighborhood U_s of s there exists an infinite subset $B_0' \subset B_0$ such that $\forall\, g \in B_0' \ :\ g(z) \in U_s \setminus f^{-1}(\mathbf{\Lambda}_Y H)$.

Let K be a compact fundamental set for the action $H \curvearrowright (Y \setminus \mathbf{\Lambda}_Y H)$. Since X is compact and f is equivariant the set $f^{-1}(K) = K_1$ is a compact fundamental set for the action of H on $X \setminus f^{-1}(\mathbf{\Lambda}_Y H)$. Therefore for every $g \in B$ there exists $h \in H$ such that $hg(z) \in K_1$. The set

$$A_s = \{h \in H : h(K_1) \cap U_s \neq \emptyset\}$$

is infinite for every open neighborhood U_s. Indeed if it is not true for some U_s then by the argument above the orbit $A_s(K_1)$ intersects every neighborhood U_s^* of s such that $U_s^* \subset U_s$. Then by compactness of K_1 we would have $h^{-1}(s) \in K_1$ for some $h \in H$, implying that $f(s) \in h(K)$. This is impossible as $\mathbf{\Lambda}_Y(H) \cap h(K) = \emptyset$ for any $h \in H$. Therefore there exists infinitely many $h \in H$ such that $h(K_1) \cap U_s \neq \emptyset$ for every neighborhood U_s of s. Thus $s \in \mathbf{\Lambda}_X H$. A contradiction. □

The main result of the paper is the following.

Theorem B. *Two 32-actions of G on compacta X and Y with peripheral structures $\mathcal{P}$ and $\mathcal{Q}$ admit a pullback space Z if and only if one of the following two conditions is satisfied (and so both of them):*

1. *$\forall\, P \in \mathcal{P}$ acts 2-cocompactly on $\mathbf{\Lambda}_Y P$*
2. *$\forall\, Q \in \mathcal{Q}$ acts 2-cocompactly on $\mathbf{\Lambda}_X Q$.*

Proof: After Theorem 5.6 we need only to show that if the pullback space Z exists then every $Q \in \mathcal{Q}$ acts 2-cocompactly on $\mathbf{\Lambda}_X Q$. Suppose that G admit a pullback action $G \curvearrowright Z$ for two 32-actions on X and Y and let $X \overset{f_1}{\leftarrow} Z \overset{f_2}{\rightarrow} Y$ be the equivariant continuous maps.

By Lemma 5.4 we may assume that the action $G \curvearrowright Z$ is 2-cocompact. Let $q \in Y$ be a parabolic point and $Q = \mathrm{St}_G q \in \mathcal{Q}$.

By Corollary 5.10 $f_2^{-1}(q)$ is the limit set $\mathbf{\Lambda}_Z Q$. Since $(Y \setminus \{q\})/Q$ is compact, Z is compact and f_2 is equivariant and continuous, Q acts cocompactly on $Z \setminus f_2^{-1}(q) = Z \setminus \mathbf{\Lambda}_Z Q$.

The set $f_1(\mathbf{\Lambda}_Z Q)$ is a closed Q-invariant subset of X. So $\mathbf{\Lambda}_X Q \subset f_1(\mathbf{\Lambda}_Z Q)$. Since f_1 is continuous and equivariant we have $\mathbf{\Lambda}_X Q = f_1(\mathbf{\Lambda}_Z Q)$ and the action $Q \curvearrowright X \setminus \mathbf{\Lambda}_X Q$ is cocompact.

By Lemma 2.1 there exists a connected, fine, hyperbolic graph Γ_1 corresponding to the 32-action $G \curvearrowright X$ such that the action $G \curvearrowright \Gamma_1^1$ is proper and cofinite.

In the following lemma we will use the notion of a dynamical bounded subgroup introduced in [GePo3, section 9.1]. Recall the topological version of this definition: a subgroup Q of G is said to be *dynamically bounded* for the action on X if there exist finitely many proper closed subsets F_i of X such that $\forall\, g \in G\ \exists\, i \ :\ g(\mathbf{\Lambda}_X Q) \subset F_i$.

Considering the action of Q on $\widetilde{X} = X \cup \Gamma_1$ we have.

Lemma 5.11. *If a subgroup Q of G acts cocompactly on $X \setminus \mathbf{\Lambda}_X Q$ then it acts cocompactly on $\widetilde{X} \setminus \mathbf{\Lambda}_X Q$.*

Proof of the lemma. The parabolic subgroup Q is obviously dynamically bounded for the action on Y. Indeed since Y is compact there are finitely many closed proper subsets R_i of Y such that $\forall g \in G\ \exists i \in \{1, ..., m\}$: $g(q) \in R_i$. Since $F_i = f_2 f_1^{-1}(R_i)$ is closed in X and f_i is surjective and equivariant, we have $X = \bigcup\limits_{i\in\{1,\dots,m\}} F_i$ and $g(\mathbf{\Lambda}_X Q) = g(f_2 f_1^{-1}(\mathbf{\Lambda}_Y Q)) \subset F_i$ for some $i \in \{1, ..., m\}$. So Q is dynamically bounded for the action on X.

The proof of [GePo3, Proposition 9.1.3] implies that if Q is dynamically bounded on X and acts cocompactly on $X \setminus \mathbf{\Lambda}_X Q$ then it acts cocompactly on $\widetilde{X} \setminus \mathbf{\Lambda}_X Q$. □

Remark. The metrisability assumption stated in [GePo3, Proposition 9.1.3] was only used to satisfy another (metric) definition of the dynamical boundness which we do not use here.

End of the proof of Theorem B. By Lemma 5.11 the action $Q \curvearrowright (\widetilde{X} \setminus \mathbf{\Lambda}_X Q)$ is cocompact. Let $K \subset (\widetilde{X} \setminus \mathbf{\Lambda}_X Q)$ be a compact fundamental set for this action. By Corollary 4.5 it is enough to prove that $|C^1/Q| < \infty$ where C^1 is the set of edges of $C = \mathrm{Hull}_X(\mathbf{\Lambda} Q)$. Let $e = (a, b) \in C^1$. Then one of its vertices, say a, is not in $\mathbf{\Lambda}_X Q$. By definition of C there exists an infinite eventual geodesic γ such that $e \subset \gamma(\mathbb{Z})$ and $\gamma(\{-\infty, +\infty\} \subset \mathbf{\Lambda}_X Q$. So there exists $g \in Q$ such that $g(a) \in K \cap C$ and $ge \subset g\gamma(\mathbb{Z})$.

We have $\mathbf{\Lambda}_X Q \cap K = \emptyset$. By the exactness of the uniformity $\mathcal{U}$ of the topology $\widetilde{X}$ there exists an entourage $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{u} \cap (K \times \mathbf{\Lambda}_X Q) = \emptyset$. By the visibility property there exists a finite set $F \subset \Gamma^1$ such that $\mathbf{u}_F \subset \mathbf{u}$. So every geodesic from K to $\mathbf{\Lambda}_X Q$ contains an edge from F. Hence $g\gamma(\mathbb{Z})$ contains a finite simple geodesic subarc l such that $g(a) \in l^0$ and $\partial l \subset F^0$. Since the graph Γ is fine there are finitely many geodesic simple arcs joining the vertices of F^0. So the set E of the edges of these arcs is finite. We have proved that $Q(E) = C^1$ and so $|C^1/Q| < \infty$. Theorem B is proved. □

6. Corollaries

The goal of this Section is the following list of corollaries.

Corollary 6.1. *Let a group G acts on compacta X and Y 3-discontinuously and 2-cocompactly. Let $\mathcal{P}$ and $\mathcal{Q}$ be the peripheral structures for the actions on X and Y respectively. Then the following statements are true.*

a) *Suppose that one of the conditions 1) or 2) of Theorem 5.6 is satisfied then G is relatively hyperbolic with respect to the system $\mathcal{R} = \{P \cap Q\ :\ P \in \mathcal{P}, Q \in \mathcal{Q}, |P \cap Q| = \infty\}$.*

b) *Every $P \in \mathcal{P}$ acts 2-cocompactly on $\mathbf{\Lambda}_Y P$ if and only if every $Q \in \mathcal{Q}$ acts 2-cocompactly on $\mathbf{\Lambda}_X P$.*

c) *Assume that $\forall\, P \in \mathcal{P}\ \exists\, Q \in \mathcal{Q} : P < Q$. Then the induced action of every $Q \in \mathcal{Q}$ on $\mathbf{\Lambda}_X Q$ is 2-cocompact.*

d) *Suppose that for every $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P < Q$. Then there exists an equivariant continuous map $f : X \to Y$. Furthermore the induced action of every $Q \in \mathcal{Q}$ on $\mathbf{\Lambda}_X Q$ is 2-cocompact.*

e) *If $\mathcal{P} = \mathcal{Q}$ then the actions are equivariantly homeomorphic.*

f) *Let $H \in \mathcal{P}$. Then we have:*
   - f1) *If G is finitely generated then the subgroup H is dynamically quasiconvex for the action on Y.*
   - f2) *If G is not finitely generated the statement f1) is not true in general.*

Proof: a) directly follows from Theorem 5.6.

b) By Theorem 5.6 the pullback space Z exists. Then by Theorem B every $Q \in \mathcal{Q}$ acts 2-cocompactly on $\mathbf{\Lambda}_X Q$.

c) By the assumptions the elements of $\mathcal{P}$ act parabolically on Y. So they all act 2-cocompactly on their limit sets on Y. Then by b) the elements of $\mathcal{Q}$ act 2-cocompactly on their limit sets on X.

d) By the assumptions the elements of $\mathcal{P}$ act parabolically on Y. So they act 2-cocompactly on their limit sets on Y. By Theorem 5.6 there exists a 32-action $G \curvearrowright Z$ which is a pullback action for the actions on X and Y. We have two equivariant continuous maps $\pi : Z \to X$ and $\sigma : Z \to Y$.

We claim that π is injective. Indeed every point x of X is either conical or bounded parabolic [Ge1, Main Theorem, b]. If $x \in X$ is conical then $\pi^{-1}(x)$ contains is a single point [Ge2, Proposition 7.5.2].

If $p \in X$ is a bounded parabolic and $P = \mathrm{St}_G p$ then by Corollary 5.10 $\pi^{-1}(p) = \mathbf{\Lambda}_Z(P)$. By Theorem 5.6 the peripheral structure for the action $G \curvearrowright Z$ is $\mathcal{R} = \mathcal{P} \cap \mathcal{Q} = \mathcal{P}$. By the assumption we have $P \in \mathcal{R}$ is parabolic for the action on Z, so $\pi^{-1}(p)$ is a single point. Thus π is injective and so is a homeomorphism. Hence map $f = \sigma \circ \pi^{-1} : X \to Y$ is equivariant and continuous.

e) follows from d).

f1) Indeed if G is finitely generated the Floyd boundary $\partial_f G$ is universal pullback space for any two 32-actions of G on X and Y [Ge1, Map theorem]. So if H is a maximal parabolic subgroup for the action on X by the necessary condition of Theorem B it acts 2-cocompactly on $\mathbf{\Lambda}_Y H$. Then by Theorem A it is dynamically quasiconvex.

f2) Proposition 5.5 provides a contre-example. Indeed there is no pullback action for two 32-actions of the free group F_∞ on two spaces. By Theorem B there exists a parabolic subgroup of one of the actions which does not act 2-cocompactly on its limit set for the other one. Again by Theorem A the sugroup is not dynamically quasiconvex for the second action.

The Corollary is proved. □

Remarks 6.2. The statements d) and e) give rise to more restrictive similarity properties of 32-actions given by equivariant maps.

The statement d) was already known in several partial cases. If first, G is finitely generated then it follows from the universality of the Floyd boundary. Indeed by [Ge2] there exist continuous equivariant (Floyd) maps $F_1 : \partial G \to X$ and $F_2 : \partial G \to Y$ where ∂G is the Floyd boundary of the Cayley graph of G (with respect to some admissible scalar function). By [GePo1, Theorem A] for a parabolic point $p \in \mathbf{\Lambda}_X G$ the set $F_1^{-1}(p)$ is the limit set $\mathbf{\Lambda}_{\partial G} P$ of the stabilizer $P = \mathrm{St}_G p$ for the action $G \curvearrowright \partial G$. Since F_2 is equivariant the set $F_2(\mathbf{\Lambda}_{\partial G} P)$ is contained in the limit set $\mathbf{\Lambda}_Y Q = \{q\}$. So the map $f = F_2 F_1^{-1}$ is well-defined on the set of parabolic points of $\mathbf{\Lambda}_X G$. Furthermore f is 1−to−1 at every conical point of $\mathbf{\Lambda}_X G$ [Ge2, Proposition 7.5.2]. Since all spaces are compacta the map f is continuous. It is also equivariant as F_i are equivariant. So the map f satisfies the claim in this case.

The statement of d) with the additional assumptions that G is countable and X and Y are metrisable was proved in [MOY]. Their proof uses the condition e) which was assumed to be known in this case.

The statement e) generalizes the last part of the main result of [Ya] to the case of non-finitely generated groups. It follows from e) that for a 32-action of G on X whose set of the parabolic points is Par_X, there exists an equivariant homeomorphism from X to the Bowditch's boundary of the graph Γ whose vertex set is $G \cup \mathrm{Par}_X$.

References

[BR] O. Baker and T. Riley, *Cannon-Thurston maps do not always exist*, arXiv:1206.0505, to appear in Forum of Math., Sigma, 1, e3, 2013.

[Bo1] B. H. Bowditch, *Relatively hyperbolic groups,* Internat. J. Algebra Comput. 22 (2012), no. 3, 1250–1316.

[Bo2] B. H. Bowditch, *Convergence groups and configuration spaces*, in "Group theory down under" (ed. J.Cossey, C.F.Miller, W.D.Neumann, M.Shapiro), de Gruyter (1999) 23–54.

[Bo3] B. H. Bowditch, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998), no. 3 no. 3 643–667.

[Bourb] N. Bourbaki, *Topologie Générale ch. 1–4.* Diffusion, Paris, 1971.

[Fa] B. Farb, *Relatively hyperbolic groups,* GAFA, 8(5), 1998, 810-840.

[F] W. J. Floyd, *Group completions and limit sets of Kleinian groups*, Inventiones Math. **57** (1980), 205–218.

[Ge1] V. Gerasimov, *Expansive Convergence Groups are Relatively Hyperbolic*, GAFA **19** (2009) 137–169.

[Ge2] V. Gerasimov, *Floyd maps to the boundaries of relatively hyperbolic groups*, GAFA **22** (2012) 1361–1399.

[GePo1] V. Gerasimov, L. Potyagailo, *Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. **15** (2013), no. 6, 2115-2137.

[GePo2] V. Gerasimov and L. Potyagailo, *Non-finitely generated relatively hyperbolic groups and Floyd quasiconvexity*, arXiv: 1008.3470 [math.GR], 2010, to appear in Groups, Geometry and Dynamics.

[GePo3] V. Gerasimov, L. Potyagailo, *Quasiconvexity in relatively hyperbolic groups*, arXiv:1103.1211 [math.GR], 2011.

[Gr] M. Gromov, *Hyperbolic groups*, in: "Essays in Group Theory" (ed. S. M. Gersten) M.S.R.I. Publications No. 8, Springer-Verlag (1987) 75–263.

[Gr1] M. Gromov, *Asymptotic Invariants of Infinite Groups*, "Geometric Group Theory II" LMS Lecture notes **182**, Cambridge University Press (1993)

[LS] R.C. Lyndon,P.E.Schupp, *Combinatorial Group Theory*, Springer Verlag, 1977.

[Ka] A. Karlsson *Free subgroups of groups with non-trivial Floyd boundary*, Comm. Algebra, 31, (2003), 5361–5376.

[Ke] J.L. Kelly, General Topology, Graduate Texts in Mathematics 27, Springer-Verlag, New York, 1975

[MOY] Y. Matsuda, A. Oguni and S. Yamagata, *Blowing up and down compacta with geometrically finite convergence actions of a group*, preprint, arXiv:1201.6104.

[M] M. Mitra, *Cannon-Thurston maps for hyperbolic group extensions*, Topology, 37(3), 527-538, 1998.

[Os] D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties and algorithmic problems*, Mem. AMS **179** (2006) no. 843 vi+100pp.

[Tu1] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces* : New Zealand J. Math. **23** (1994) 157–187.
[Tu2] P. Tukia, *Conical limit points and uniform convergence groups*, J. Reine. Angew. Math. **501** (1998) 71–98.
[W] A. Weil, Sur les espace à structure uniforme et sur la topologie générale, Paris, 1937.
[Ya] A. Yaman, *A topological characterisation of relatively hyperbolic groups*, J. reine ang. Math. **566** (2004), 41–89.

Victor Guerassimov, Departamento de Matemática, Universidade Federal de Minas Gerais, Av. Antonio Carlos, 6627/ Caixa Postal 702, CEP 31270-901 Belo Horizonte, MG, Brasil
E-mail address: victor@mat.ufmg.br

Leonid Potyagailo, UFR de Mathématiques, Université de Lille 1, 59655 Villeneuve d'Ascq cedex, France
E-mail address: potyag@math.univ-lille1.fr