Counting ramified coverings and intersection theory on spaces of rational functions I
(Cohomology of Hurwitz spaces)

S. K. Lando[*], D. Zvonkine[†]

May 13, 2003

To Sasha Mednykh
on the occasion
of his 50th birthday

Abstract

The Hurwitz space is a compactification of the space of rational functions of a given degree. The Lyashko–Looijenga map assigns to a rational function the set of its critical values. It is known that the number of ramified coverings of $\mathbb{C}\mathrm{P}^1$ by $\mathbb{C}\mathrm{P}^1$ with prescribed ramification points and ramification types is related to the degree of the Lyashko–Looijenga map on various strata of the Hurwitz space. Here we explain how the degree of the Lyashko–Looijenga map is related to the intersection theory on this space. We describe the cohomology algebra of the Hurwitz space and prove several relations between the homology classes represented by various strata.

1 Introduction

1.1 Statement of the problem

In a series of two papers, the present one and the second one by the second author [15], we continue the study of the Hurwitz problem [7] concerning counting ramified coverings of the 2-sphere. Roughly speaking, this problem can be formulated as follows:

[*]Russian–French Laboratory for Mathematics, Informatics, and Mathematical Physics of the Independent University of Moscow and Institute for System Research, RAS; partly supported by the Russian Foundation for Basic Research (project 01-01-00660), NWO-RFBR (project 047.008.005), and INTAS (project 00259), **e-mail:** lando@lando.mccme.rssi.ru

[†]bât. 425, Université Paris-Sud, 91400 Orsay, France. **e-mail:** dimitri.zvonkine@math.u-psud.fr
Both authors are partially supported by the RFBR grant 02-01-22004.

Given a set of fixed ramification points on the target 2-sphere and a set of ramification types over these points, count the number of non-isomorphic ramified coverings $S \to S^2$ by a 2-surface S, having the prescribed ramification types over the prescribed ramification points.

Two coverings are considered to be isomorphic if there is a homeomorphism of the covering surfaces taking the first covering to the second one. In fact, we count isomorphism classes with the weight equal to the inverse order of the group of automorphisms of the class; in this form the problem is more natural, and admits a lot of applications.

In a more modern setting this problem can be reformulated as the computation of the Gromov–Witten invariants of the complex projective line. A detailed exposition of the history of the problem and a description of various approaches to its solution can be found in [12].

Hurwitz [8] himself gave an explicit answer to the problem in the case, where the covering surface is also a sphere, and there is one ramification point with arbitrary ramification, while all others are points of simple ramification. In spite of the importance of the problem, only recently Hurwitz's original results were extended to more general cases and treatable answers were obtained:

- I. Goulden and D. Jackson [5] solved the combinatorial problem that allows one to give the answer to Hurwitz's problem for polynomials (that is, ramified coverings of the sphere by a sphere having a point with a single preimage and arbitrary ramification over other points); later their results were reestablished in [13] and [14] by absolutely different methods;

- T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein [2] gave an expression for the number of ramified coverings of the sphere by a surface of arbitrary genus with a single point of arbitrary ramification type and all other points of simple ramification, in terms of intersection indices on moduli spaces of curves (when the covering surface is the sphere, the answer given by the formula coincides with Hurwitz's one); another proof of the formula was given in [6].

Here we address the special case of the Hurwitz problem, where the covering surface is also a sphere, but in contrast to the Hurwitz case, ramification points of arbitrary type are allowed. We are far from solving the problem in this generality, but we suggest a general approach, and show how it works in some special cases, thus producing new enumerative results.

To a ramification point (= a critical value) of multiplicity d we assign a partition κ of d, called the *ramification type*. The elements of κ are the multiplicities of the critical points that correspond to our critical value.

Recall the Hurwitz theorem.

Theorem 1 (Hurwitz) *The number h_κ of degree n coverings of the sphere by the sphere, having the ramification type $\kappa = (k_1, \ldots, k_m)$ over one point, $d = k_1 + \ldots + k_m$,* *while all other ramification points are fixed and simple, is*

$$h_\kappa = \frac{(2n-2-d)!}{(n-d-m)!\,|\mathrm{Aut}(\kappa)|} \prod_{i=1}^{m} \frac{(k_i+1)^{k_i+1}}{(k_i+1)!} n^{n-d-3};$$

here $|\mathrm{Aut}(\kappa)|$ *is the order of the automorphism group of the partition* κ*,* $2n-2-d$ *is the number of nondegenerate ramification points, and* $n-d-m$ *is the number of simple preimages of the multiple ramification point.*

As a consequence of our approach we obtain, in the second part of this paper [15], for example, the following result.

Theorem 2 *The number* $h_{2,2}$ *of degree* n *coverings of the sphere by the sphere, having two double critical points and* $2n-6$ *simple critical points, all the* $2n-4$ *critical values being fixed, is*

$$h_{2,2} = \frac{3}{4}\,(27n^2 - 137n + 180)\,\frac{n^{n-6}\,(2n-6)!}{(n-3)!}.$$

As far as we know, this formula, as well as similar formulas for some other ramification types, is new.

1.2 The basis of the approach

First of all we reduce the problem to the calculation of the degree of some map, called the Lyashko–Looijenga map. This step is now standard. The Lyashko–Looijenga map (below, the *LL* map) takes a meromorphic function to the set of its critical values. Its source space can be chosen in a variety of ways; here we define the *LL* map on the Hurwitz space $\mathcal{H}_n$ constructed in [2]. It is the space of stable maps from genus zero complex curves to the projective line, having trivial ramification over infinity. All spaces of functions possessing degenerate ramification are considered as subvarieties in this space. These subvarieties form a stratification of the Hurwitz space.

Theorem 3 *The number* $h_{\{\kappa_1,\ldots,\kappa_c\}}$ *of ramified coverings of the sphere, having the ramification types* $\kappa_1,\ldots,\kappa_c$ *over prescribed ramification points with multiplicities* $d_1,\ldots,d_c$*, is given by the formula*

$$h_{\{\kappa_1,\ldots,\kappa_c\}} = \frac{1}{n!}\,\frac{|\mathrm{Aut}\{\kappa_1,\ldots,\kappa_c\}|}{|\mathrm{Aut}\{d_1,\ldots,d_c\}|}\,\mu_{\{\kappa_1,\ldots,\kappa_c\}}$$

where $\mu_{\{\kappa_1,\ldots,\kappa_c\}}$ *is the degree of the LL map restricted to the stratum* $\Sigma_{\{\kappa_1,\ldots,\kappa_c\}}$ *consisting of functions with these ramification types.*

This is an instance of a general situation, see e.g. [12]. Indeed, if we fix a ramified covering $f: S \to S^2$ and choose a complex structure on the target sphere S^2, then there exists a unique complex structure on the covering surface S making the function f into a meromorphic function. This complex structure is produced by the Riemann construction. Hence, there is a one-to-one correspondence between ramified coverings with fixed ramification points of given types and meromorphic functions with fixed critical values of the same types. The latter number is exactly the degree of the LL map on the corresponding moduli space. The coefficient $1/n!$ results from the fact that in our construction of the space $\mathcal{H}_n$ we choose a numbering of the n poles of each rational function. The factor

$$\frac{|\mathrm{Aut}\{\kappa_1,\dots,\kappa_c\}|}{|\mathrm{Aut}\{d_1,\dots,d_c\}|}$$

is due to the fact that if we permute the ramification types $\kappa_1,\dots,\kappa_c$ preserving the multiplicities $d_1,\dots,d_c$, we obtain a new set of ramified coverings that lie on the same stratum $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ and have the same image under the LL map.

The space $\mathcal{H}_n$ is, in fact, a vector bundle over the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{0;n}$ of stable genus zero curves. The latter space is a smooth projective variety. The multiplicative group $\mathbb{C}^*$ of nonzero complex numbers acts on this bundle fiberwise. This action consists in just multiplying a meromorphic function by a constant. It preserves the stratification of the space $\mathcal{H}_n$ because the multiplication does not change the type of singularities. Deleting the zero section of the bundle and taking the quotient modulo this action we reduce the variety to the projectivization $\mathbb{P}\mathcal{H}_n$ of the vector bundle $\mathcal{H}_n$.

The projectivization $\mathbb{P}\mathcal{H}_n$ carries the tautological line bundle $\mathcal{T}\to\mathbb{P}\mathcal{H}_n$, and a natural cohomology class $\Psi_n\in H^2(\mathbb{P}\mathcal{H}_n)$, $\Psi_n=c_1(\mathcal{T}^\vee)$, the class of a hyperplane section. The degree of the Lyashko–Looijenga map restricted to a subvariety in $\mathcal{H}_n$ is related to the intersection index of the subvariety with the complementary power of the class Ψ_n. Namely, the following statement is true.

Theorem 4 *The degree $\mu_{\{\kappa_1,\dots,\kappa_c\}}$ of the LL map restricted to the stratum $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ is equal to*

$$|\mathrm{Aut}\{d_1,\dots,d_c\}|\,\langle\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}},\Psi_n^d\rangle.$$

Here d is the dimension of the stratum $\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$, d_i is the multiplicity of the ith ramification point (or the sum of the elements of κ_i), and $\langle\cdot,\cdot\rangle$ is the coupling between a homology and a cohomology class.

This theorem is proved in Section 3.2.

1.3 Cohomology of the Hurwitz spaces and Kazarian's principle

Theorem 4 implies that our approach requires the study of the cohomology ring $H^*(\mathbb{P}\mathcal{H}_n)$. All cohomology groups we consider are with complex coefficients, and we do not specify the coefficients explicitly. We need

- a reasonable description of this ring, say, in terms of generators and relations;
- reasonable expressions for the cohomology classes of the strata and the class Ψ_n in terms of the generators.

The solution of the first problem is known since the space $\mathcal{H}_n$ is a vector bundle over the moduli space of stable rational curves, whose cohomology is known. The second problem seems to be much more difficult.

In fact, we need less than the whole cohomology ring. The symmetric group S_n acts on the space $\mathcal{H}_n$ by permuting the indices, whence it acts on the cohomology space $H^*(\mathbb{P}\mathcal{H}_n)$. All the cohomology classes we are interested in are symmetric with respect to this action. Therefore, we only need to know the S_n-symmetric part of the cohomology algebra $H^*(\mathbb{P}\mathcal{H}_n)$. This is an additional problem, but its solution can lead to a simplification of the decompositions of the strata.

The situation would become even easier if we restrict ourselves to the subalgebra in the cohomology algebra $H^*(\mathbb{P}\mathcal{H}_n)$ generated by the classes of the strata.

Consider two closed codimension 1 subvarieties C_n and M_n in $\mathcal{H}_n$; the first of them is called the *caustic* and it is the closure of the space of functions having a critical point of order 2, the second one is the *Maxwell stratum* and it is the closure of the space of functions having two distinct critical points with coinciding critical values.

Conjecture 1.1 The subalgebra of the cohomology algebra $H^*(\mathbb{P}\mathcal{H}_n)$ generated by the cohomology classes of the strata is generated by the two classes C_n and M_n in $H^2(\mathbb{P}\mathcal{H}_n)$.

Up to now, the basis of this conjecture is not too solid. But it does not contradict our sample calculations, and it has a nonformal justification coming from Kazarian's theory.

Kazarian's theory concerns cohomology classes of multisingulatities of a map $f : M \to N$ of two complex manifolds, $\dim M \leq \dim N$. Its main statement claims that there is a universal way to express these cohomology classes in terms of the characteristic classes of the tangent bundles over the two manifolds M and N, more precisely in terms of the class $c(TM)/f^*c(TN)$, where c is the total Chern class. It is a development of the theory of the Thom polynomial. Up to now, only a preliminary version [9] of the text describing the theory is available, and we are not going to refer directly to statements from it. However, the ideology of the theory seems to be applicable in the situation we are studying, and it leads to a number of conjectures concerning the part of the cohomology ring of the Hurwitz spaces we are interested in.

In our situation, the variety M is the universal curve $\mathcal{U}_n$ over the Hurwitz space $\mathbb{P}\mathcal{H}_n$, while the variety N is the quotient of $\mathcal{H}_n \times \mathbb{C}\mathrm{P}^1$ by the natural $\mathbb{C}^*$ action. The map f is the universal map over $\mathbb{P}\mathcal{H}_n$. It is easy to see that the map f almost identifies the tangent spaces TM and TN; more precisely, one has $c(TM)/f^*c(TN) = (1+a)/(1+b)$, where a and b are first Chern classes of some linear bundles. However, new complications arise, since we are interested in cohomology classes of $\mathbb{P}\mathcal{H}_n$ and not of $\mathcal{U}_n$.

Kazarian's ideology leads to Conjecture 1.1.

If the conjecture is true, then the computation of intersection numbers of the strata in the Hurwitz space with the complementary powers of the class $\Psi_n \in H^2(\mathbb{P}\mathcal{H}_n)$ can be split into two independent stages:

- express the cohomology class of the required stratum as a (homogeneous) polynomial in the classes M_n and C_n;
- find the intersection index of each monomial in M_n and C_n with the complementary degree of the class Ψ_n.

Another cohomology class in $H^2(\mathbb{P}\mathcal{H}_n)$ is also distinguished. This is the class Δ_n dual to the subvariety of functions defined on singular curves. This class also can be expressed in terms of the classes C_n and M_n. This means that the subring in $H^*(\mathbb{P}\mathcal{H}_n)$ generated by the classes C_n and M_n coincides with that generated by Ψ_n and Δ_n. The linear relations relating these four classes are as follows:

$$\begin{array}{rcl} C_n & = & 6(n-1)\Psi_n - 3\Delta_n; \\ M_n & = & 2(n-1)(n-6)\Psi_n + 4\Delta_n. \end{array}$$

The coefficients in these linear relations are polynomial in n, which (together with the informal support of Kazarian's theory) allows us to sharpen Conjecture 1.1 in the following way:

Conjecture 1.2 Each cohomology class dual to a stratum in $\mathbb{P}\mathcal{H}_n$ can be expressed as a homogeneous polynomial in Ψ_n, Δ_n whose coefficients are polynomials in n.

Therefore the calculations must even be simplified.

1.4 Plan of the papers

In Section 2 we give precise definitions of notions mentioned in this introduction. Section 3 is devoted to the description of the Lyashko–Looijenga map. In Section 4 the structure of cohomology of the Hurwitz spaces is analyzed.

1.5 Acknowledgments

We are greatful to A. Zorich who invited us both to Rennes, where the main part of this work was accomplished. The work was completed during our participation in the Mini-Workshop on Hurwitz theory and its ramifications, and we thank its organizers J. H. Kwak and A. D. Mednykh for inviting us. We also thank B. Shapiro and M. Shapiro for their interest and M. Kazarian for explaining the basics of his theory.

2 Main definitions

In this section we briefly recall the definitions of stable curves and stable maps (we will restrict our attention to curves of genus 0), as well as the definition of the Hurwitz space $\mathcal{H}_n$ from [2].

2.1 Moduli spaces of stable curves

Consider the Riemann sphere $\mathbb{C}\mathrm{P}^1$ with $n \geq 3$ distinct numbered marked points on it. Two choices of marked points are considered equivalent if there exists an automorphism of $\mathbb{C}\mathrm{P}^1$ sending one set of marked points to the other, preserving their numbering.

The space of all nonequivalent choices of n numbered marked points on $\mathbb{C}\mathrm{P}^1$ can be endowed with a natural structure of a smooth irreducible (noncompact) complex manifold of dimension $n-3$. We denote this moduli space by $\mathcal{M}_n$.

Example 2.1 The space $\mathcal{M}_3$ is a point, because any triple of points on $\mathbb{C}\mathrm{P}^1$ can be sent to any other triple by a biholomorphic map. Similarly, $\mathcal{M}_4$ is isomorphic to $\mathbb{C}\mathrm{P}^1 - \{0, 1, \infty\}$ (here we suppose that the projective line is endowed with an arbitrary complex coordinate). Indeed, for any four marked points on $\mathbb{C}\mathrm{P}^1$ one can send the first three to 0, 1, and ∞. The fourth marked point will be sent to some uniquely determined point λ, different from 0, 1, and ∞, and this value of λ is the point of $\mathcal{M}_4$.

The spaces $\mathcal{M}_n$ are compactified by adding to them new points corresponding to singular stable curves. All the curves under consideration are compact. The only singularities allowed are simple *nodes*, or *double points* (as at the origin in the plane curve $xy = 0$); such curves are called *nodal*.

Separating the two branches of a nodal curve at each node, we obtain a disjoint union of smooth compact curves, which is called the *normalization* of the original curve. The connected components of the normalization are the *irreducible components* of the curve. Associate to a nodal curve a graph whose vertices correspond to the connected components of the normalization, and two vertices are connected by an edge iff the corresponding components intersect at a node (loops appear if a component intersects itself). This graph is called the *modular graph* of the nodal curve. By definition, the genus of a nodal curve is zero if all connected components of its normalization are projective lines, and the modular graph is a tree.

Definition 2.2 A *stable genus 0 curve* is a nodal curve of genus 0 with n distinct marked and numbered points satisfying the following conditions: (i) the marked points do not coincide with the nodal points; (ii) the number of automorphisms of the curve preserving the marked points is finite.

The second condition is equivalent to the following easily verifiable condition (ii'): the total number of marked and nodal points on every irreducible component of the nodal curve is greater than or equal to 3.

The space of genus zero stable curves with n marked points is endowed with a natural structure of a smooth irreducible compact complex manifold of dimension $n-3$; we denote this space by $\overline{\mathcal{M}}_n$. The space $\mathcal{M}_n$ of smooth curves form a dense subvariety in $\overline{\mathcal{M}}_n$.

Example 2.3 Apart from smooth curves, there are exactly 3 stable genus 0 curves with 4 marked points. They are shown in Fig. 1. If λ from Example 2.1 tends to 0

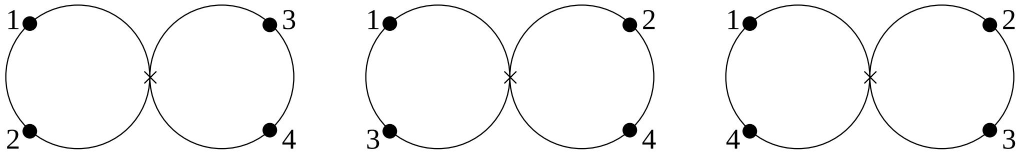

Figure 1: Singular stable curves with 4 marked points.

(respectively, 1 or ∞), the corresponding smooth curve tends to the 3rd (respectively 2nd, or 1st) stable curve in the figure.

Accordingly, $\overline{\mathcal{M}}_4$ is obtained from $\mathcal{M}_4$ by adding three initially punctured points. Actually, $\overline{\mathcal{M}}_4$ is isomorphic to $\mathbb{C}\mathrm{P}^1$.

2.2 Hurwitz spaces

Now consider the space of meromorphic functions on $\mathbb{C}\mathrm{P}^1$ with exactly $n \geq 1$ numbered simple poles. This space has a structure of a smooth (noncompact) complex manifold of dimension $2n - 2$. In the case when $n \geq 3$, if we forget the function itself and only remember the positions of its poles, we obtain a map from the space of functions onto $\mathcal{M}_n$. Because of this, we will often say "marked points" instead of "poles of f".

We are going to construct a compactification of the space of meromorphic functions following [2], similar to that of the moduli space $\mathcal{M}_n$. To do that, we will need to define meromorphic functions on nodal curves. A meromorphic function f on a nodal curve S is simply a meromorphic function defined on each irreducible component of the nodal curve. If two components intersect at a nodal point, the function must take the same value at this point on both components. Two pairs (S_1, f_1) and (S_2, f_2) are equivalent if there exists a biholomorphic isomorphism $\varphi : S_1 \to S_2$ that preserves the numbering of the poles and makes the following diagram commutative:

$$\begin{tikzcd} S_1 \arrow[rr, "\varphi"] \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & \mathbb{C}\mathrm{P}^1 & \end{tikzcd}$$

Definition 2.4 A *stable genus 0 meromorphic function* is a function $f : S \to \mathbb{C}\mathrm{P}^1$ defined on a nodal curve S and satisfying the following conditions. (i) The function f does not have poles at nodal points. (ii) The number of automorphisms of the pair (S, f) is finite.

As above, a comment must be made about the second condition. An automorphism of the pair (S, f) is a map φ from S to itself that makes the following diagram commutative:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & S \arrow[dl, "f"] \\ & \mathbb{C}\mathrm{P}^1 & \end{tikzcd}$$

Condition (ii) is again easy to check, because it is equivalent to (ii'): If the function f is constant on some irreducible component of S, then the number of nodal points on this component is greater than or equal to 3. (Note that such a component cannot contain marked points precisely because f is constant on it.)

Definition 2.5 The *Hurwitz space* $\mathcal{H}_n$ (for $n \geq 2$) is the space of all stable genus 0 meromorphic functions with n simple numbered poles.

There is a natural action of $\mathbb{C}^*$ on $\mathcal{H}_n$ defined by simply multiplying a stable function by a constant. The invariant points of this action are stable functions such that the nodal curve has at most one pole on every component and the function vanishes on the components without poles.

These points form the *zero section* of the Hurwitz space. (We will soon see that the Hurwitz space is a vector bundle and the points in question indeed form its zero section.)

Definition 2.6 The *projectivized Hurwitz space* $\mathbb{P}\mathcal{H}_n$ is the quotient of $\mathcal{H}_n$, without the zero section, by $\mathbb{C}^*$.

For $n \geq 3$, $\mathbb{P}\mathcal{H}_n$ has the structure of a smooth compact complex manifold of dimension $2n-3$. $\mathbb{P}\mathcal{H}_2$ is a compact smooth orbifold.

Example 2.7 $\mathbb{P}\mathcal{H}_2$ is isomorphic (as an orbifold) to the weighted projective line with weights 2 and 1, that is to

$$(\mathbb{C}^2 - \{0\})/(z,w) \sim (\lambda^2 z, \lambda w).$$

Indeed, if we are given a meromorphic map with 2 poles on $\mathbb{C}\mathrm{P}^1$, we can move its poles to 0 and ∞ and write the function as $f(z) = az+b+c/z$. The numbers a, b, c are considered up to (simultaneous) multiplication by a scalar factor. Moreover, making a change of variables $z \mapsto \lambda z$, we see that the function $(\lambda a)z+b+(c/\lambda)/z$ is equivalent to f. Thus $[ac : b]$ is a set of homogeneous coordinates in $\mathbb{P}\mathcal{H}_2$. The point $ac = 0$ corresponds to the stable meromorphic function defined on a nodal curve with two irreducible components.

The space $\mathbb{P}\mathcal{H}_3$, as will soon be proved, is isomorphic to $\mathbb{C}\mathrm{P}^3$.

2.3 $\mathcal{H}_n$ is a bundle over $\overline{\mathcal{M}}_n$

Now we are going to prove that for $n \geq 3$, $\mathbb{P}\mathcal{H}_n$ is the projectivization of an $(n+1)$-dimensional holomorphic vector bundle on $\overline{\mathcal{M}}_n$. First let us define several line bundles on $\overline{\mathcal{M}}_n$.

Consider a stable curve S with n marked points. For i from 1 to n, let L_i and $L_i^\vee$ be the complex line respectively cotangent and tangent to S at the ith marked point.

Definition 2.8 $\mathcal{L}_i$ (respectively $\mathcal{L}_i^\vee$) is the line bundle over $\overline{\mathcal{M}}_n$ whose fiber over a point $S \in \overline{\mathcal{M}}_n$ is the line L_i (respectively $L_i^\vee$).

Denote by $\mathbb{C}$ the trivial line bundle and let E_n momentarily denote the following $(n+1)$-dimensional bundle over $\overline{\mathcal{M}}_n$:

$$E_n = \mathcal{L}_1^\vee + \ldots + \mathcal{L}_n^\vee + \mathbb{C}.$$

Proposition 2.9 *For $n \geq 3$, we have $\mathcal{H}_n = E_n$.*

Proof. We will show that giving a stable meromorphic function on a nodal curve is equivalent to giving a point in the total space of the bundle E_n. Considering stable functions up to a scalar factor corresponds to taking the projectivization of E_n.

First of all, note that if a meromorphic function f has a simple pole at some smooth point z of a nodal curve S, then it determines a tangent vector to S at z; more precisely, the principal part of the pole is such a tangent vector. Indeed, if we want to couple this tangent vector with a cotangent vector represented by a 1-form ω, the result will be the residue of $f\omega$ at z.

Now, given a stable meromorphic function on a nodal curve, we can divide its poles into two parts. The poles of the first type lie on the irreducible components of the curve that contain at least 3 marked and double points. The poles of the second type lie on the irreducible components that contain a unique pole and a unique nodal point. (These are the only possible cases.)

If we forget the function itself and retain only the positions of its poles, we obtain a nodal curve with n marked points. This curve is not necessarily stable. However, we can make it stable by contracting every component that contains only 1 nodal and 1 marked point. The former node becomes the marked point. Thus we obtain a map from $\mathcal{H}_n$ to $\overline{\mathcal{M}}_n$.

Further, consider the stable curve that we have obtained. We assign to its marked points of the first type the principal parts of the corresponding poles (they are tangent vectors at the marked points). We assign zero tangent vectors to the marked points of the second type. Finally, we assign to the stable meromorphic function a complex number: the sum of its critical values given by the Lyashko–Looijenga map (see Section 3 below). Thus we have assigned to the stable meromorphic function a point in the total space of E_n.

Conversely, given a point in the total space of E_n, we can easily recover the stable meromorphic function. First we extend the stable curve by adding new irreducible components at each marked point of the second type (those, whose tangent vector is equal to 0). Then, knowing the principal parts of the poles, we can recover the function f uniquely up to an additive constant. Knowing the sum of its critical values allows us to fix the constant. ◇

Remark 2.10 The vector bundle $\mathcal{H}_n$ can be understood more conceptually.

Consider the space $\mathcal{S}_n(\mathbb{C}\mathrm{P}^1)$ of all stable degree n maps from genus 0 curves S to $\mathbb{C}\mathrm{P}^1$. Inside this space there is a suborbifold canonically isomorphic to $\overline{\mathcal{M}}_n/S_n$, where S_n is the symmetric group acting on $\overline{\mathcal{M}}_n$ by renumbering the poles. This subvariety consists of stable maps of the following form. Take a stable genus 0 curve S' with n marked but not numbered points. Attach to S' a new spherical component at each marked point. Instead of the former marked points, mark one point on each of the new spherical components. Thus we have obtained a nodal curve S. Consider the map $f : S \to \mathbb{C}\mathrm{P}^1$ that sends the whole curve S' to 0 and is of degree one on each of the new spherical components, the poles being at the new marked points. Such stable maps (S, f) form an orbifold isomorphic to $\overline{\mathcal{M}}_n/S_n$.

Now, the normal bundle to $\overline{\mathcal{M}}_n/S_n$ in $\mathcal{S}_n(\mathbb{C}\mathrm{P}^1)$ (in the orbifold sense) is canonically identified with the quotient $\mathcal{H}_n/S_n$.

Definition 2.11 We denote by $\mathcal{T}$ the canonical line bundle over $\mathbb{P}\mathcal{H}_n$ and by $\Psi_n = c_1(\mathcal{T}^\vee) \in H^2(\mathbb{P}\mathcal{H}_n)$ the first Chern class of the dual line bundle.

3 The Lyashko–Looijenga map

3.1 The definition of the map

Consider the space of unordered sets of m (not necessarily distinct) complex numbers. This space is isomorphic to the space of monic polynomials of degree m: just take a set of m numbers to the polynomial whose roots are these numbers.

Definition 3.1 The *Lyashko–Looijenga map LL* is the map that assigns to a meromorphic function with n simple poles on $\mathbb{C}\mathrm{P}^1$ the (unordered) set of its $2n-2$ critical values counted with multiplicities.

We recall that a *critical point* of a function f is a point, where $df = 0$, while a *critical value* of f is its value at a critical point. The *multiplicity of a critical point* is the multiplicity of the zero of df at this point. In other words, if the function f has the form $f(z) = z^k + c$ in some local coordinate z, then $z = 0$ is a critical point of multiplicity $k - 1$. The *multiplicity of a critical value* is the sum of multiplicities of the critical points at which the critical value in question is attained. If we consider the function as a ramified covering of $\mathbb{C}\mathrm{P}^1$, then its critical values are exactly those points in the image over which the covering is ramified.

The notion of a *cone* (or of a cone bundle) generalizes that of a vector bundle to the case of singular fibers, see [3]. The essential point is that cones carry a fiberwise action of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers. A cone morphism preserves this action.

Proposition 3.2 ([2]) *The Lyashko–Looijenga map extends to a cone morphism*

$$LL : \mathcal{H}_n \to \mathbb{C}^{2n-2}.$$

The proof of the proposition is based on the extension of the notion of set of critical values to maps defined on nodal curves.

First consider all the irreducible components of S where f is constant. Consider the union of these irreducible components, and let S_0 be a connected component of this union. (In general S_0 consists of several irreducible components intersecting at double points.) Suppose S_0 contains k nodal points at which it intersects other irreducible components (on which f is not constant). Then the value of f on S_0 is considered a critical value of multiplicity $2k-2$.

Second, consider a nodal point lying on two irreducible components of S such that f is not constant on either of them. The value that f takes at such a point is considered a double critical value.

Finally, the critical values of f on all the irreducible components are taken into account as ordinary critical values.

One can prove that in that way we have obtained a set of $2n-2$ critical values, and that this set depends continuously on the point of $\mathcal{H}_n$. See [2] for more details.

3.2 A stratification of the Hurwitz space

Consider a degree n ramified covering $f : S \to S^2$ of the sphere. Let $t \in S^2$ be a ramification point of f of multiplicity d. To this ramification point we can assign a partition κ of d. This partition consists of the multiplicities of all critical points of f in S whose image under f is equal to t. We will write such a partition in the multiplicative form, $\kappa = 1^{m_1} \dots d^{m_d}$, where m_i, $i = 1, \dots, d$ is the number of critical points in the preimage of t having multiplicity i, $1 \cdot m_1 + \dots + d \cdot m_d = d(\kappa) = d$. If the covering has c ramification points, $t_1, \dots, t_c$, then c partitions $\kappa_1, \dots, \kappa_c$ are associated to it.

Sets of partitions of n that can appear as sets of ramification partitions of a ramified covering are subject to a number of constraints. If the covering surface S also is the sphere, the Riemann–Hurwitz formula implies that the total degeneracy of the set of ramification partitions of a meromorphic function must be $2n-2$,

$$d(\kappa_1) + \dots + d(\kappa_c) = 2n - 2.$$

However, this property alone does not guarantee the existence of a ramified covering with the prescribed ramification type.

Definition 3.3 The set of all rational functions on $\mathbb{C}\mathrm{P}^1$ with n simple numbered poles such that the set of partitions assigned to their critical values is $\{\kappa_1, \dots, \kappa_c\}$ is called an *open stratum*. Its closure in $\mathcal{H}_n$ is called just a *stratum* and denoted by $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$.

The projectivizations of these strata (lying in $\mathbb{P}\mathcal{H}_n$) are defined in the obvious way. They will be denoted by $\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$.

Sometimes we will omit reference to simple partitions $\kappa = 1$ in the index, omitting in this case the braces as well.

In the sequel we will often describe strata by specifying the set of functions that belong to their open parts and omitting the words "closure of". The reader should bear in mind that we always consider closed strata, unless explicitly stated otherwise.

Example 3.4 There are two strata of complex codimension 1. They are the caustic $C_n \subset \mathcal{H}_n$ and the Maxwell stratum $M_n \subset \mathcal{H}_n$. The *caustic* $C_n = \Sigma_2$ is (the closure of) the set of functions with a double critical point. The *Maxwell stratum* $M_n = \Sigma_{1^2}$ is (the closure of) the set of functions with the same critical value taken at two distinct critical points.

Definition 3.5 The *degree* of the LL map on a stratum $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ is the number of the preimages in $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ of a generic point in $LL(\Sigma_{\{\kappa_1,\dots,\kappa_c\}})$. The degree is denoted by $\mu_{\{\kappa_1,\dots,\kappa_c\}}$.

Example 3.6 Since the space $\overline{\mathcal{M}}_3$ is a point, the space $\mathcal{H}_3$ is isomorphic to $\mathbb{C}^4 = L_1^\vee + L_2^\vee + L_3^\vee + \mathbb{C}$, where the L_i are the cotangent lines at the marked points on $\mathbb{C}\mathrm{P}^1$. It can be identified with the space of rational functions of the form

$$az - \frac{b}{z} - \frac{c}{z-1} + d.$$

In Figure 2 we represent the stratification of the projectivized Hurwitz space $\mathbb{P}\mathcal{H}_3$. The figure shows the real part of this space with a slight simplification: Instead of a 3-dimensional projective space we have represented a 2-dimensional projective space, dropping the parameter d.

The interesting subvarieties in $\mathbb{P}\mathcal{H}_3$ are the following ones. First, the projectivized caustic $\mathbb{P}C_3$, which is a singular cubic with one self-intersection point. It has the homogeneous equation

$$a^3 + b^3 + c^3 + 3ab^2 + 3ac^2 + 3ba^2 + 3bc^2 + 3ca^2 + 3cb^2 - 21abc = 0.$$

Since our picture shows the real part of $\mathbb{P}\mathcal{H}_3$, the self-intersection point, which has the homogeneous coordinates $(1:1:1)$, appears to lie apart from the rest of the cubic. Second, the three projective lines $a=0$, $b=0$, $c=0$, which correspond to reducible curves with one pole lying on one component and the other two poles on the other one. The Maxwell stratum is empty. The cubic intersects each of the three lines at one point, but the intersection has multiplicity three. These three points lie on the line $a+b+c=0$.

The boxes around the picture of $\mathbb{P}\mathcal{H}_3$ show, for each of the subvarieties, what are the corresponding stable rational functions. Short traits represent poles, crosses represent critical points, while crosses with double lines represent double critical points. The numbers written near the arrows are the multiplicities of the LL map on the corresponding subvariety.

Now we are able to prove Theorem 4 from the introduction. We restate it here.

Consider a stratum $\Sigma_{\{\kappa_1,\dots,\kappa_c\}} \subset \mathcal{H}_n$ (see Definitions 3.3 and 3.5) determined by partitions $\kappa_1,\dots,\kappa_c$. Let $d_1 = d(\kappa_1),\dots,d_c = d(\kappa_c)$ be the multiplicities of the critical values. Let $|\mathrm{Aut}\{d_1,\dots,d_c\}|$ be the number of automorphisms of the set $\{d_1,\dots,d_c\}$, i.e., the number of permutations σ of c elements such that $d_i = d_{\sigma(i)}$ for all i.

Theorem 4 *The degree $\mu_{\{\kappa_1,\dots,\kappa_c\}}$ of the Lyashko–Looijenga map on $\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ equals*

$$|\mathrm{Aut}\{d_1,\dots,d_c\}|\,\langle \mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}, \Psi_n^d\rangle,$$

where d is the dimension of the stratum $\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ and $\langle\cdot,\cdot\rangle$ is the coupling between a homology and a cohomology class.

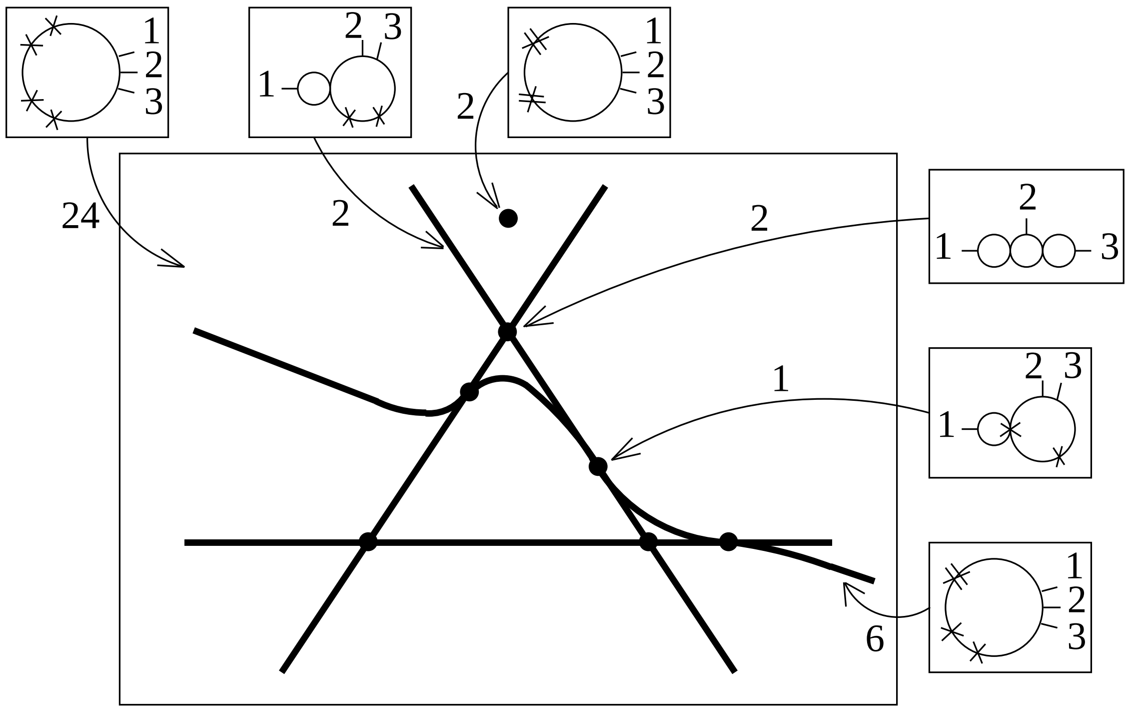

Figure 2: The stratification of the "simplified" projectivized space $\mathbb{P}\mathcal{H}_3$.

Proof of Theorem 4. To prove the theorem we consider a $(2n-2)!$-sheeted ramified covering $\widetilde{\mathbb{P}\mathcal{H}_n}$ of $\mathbb{P}\mathcal{H}_n$. By definition, a point of $\widetilde{\mathbb{P}\mathcal{H}_n}$ corresponds to a stable meromorphic function *with numbered critical points*.

More precisely, consider the Zariski open subset X of $\mathbb{P}\mathcal{H}_n$ constituted by functions f defined on $\mathbb{C}\mathrm{P}^1$, with $2n-2$ simple critical points. Over this open subset we can construct a $(2n-2)!$-sheeted non-ramified covering $\widetilde{X}$, whose points correspond to functions with numbered critical points. It is a connected complex manifold. Then this covering is uniquely extended to a ramified covering of $\mathbb{P}\mathcal{H}_n$ in the following way.

For simplicity we start by describing the set of preimages of every point $f_0 \in \mathbb{P}\mathcal{H}_n$ in the covering $\widetilde{\mathbb{P}\mathcal{H}_n}$. Consider an $f_0 \in \mathbb{P}\mathcal{H}_n$ and a small ball B surrounding f_0. A point $f \in B \cap X$ has $(2n-2)!$ preimages in the covering. The local monodromy group $\pi_1(X \cap B)$ acts by permutations on these preimages. The preimages of f_0 in the covering are the orbits of this action. Now let us give an algebro-geometric definition.

Consider an affine chart U in $\mathbb{P}\mathcal{H}_n$ and let A be the algebra of algebraic functions on U, $U = \mathrm{spec}\ A$. Let $F \in A$, be a function whose set of zeroes is the ramification divisor. Then the localization A_F is the algebra of algebraic functions on the complement $U \cap X$ to the ramification divisor. The map $A \hookrightarrow A_F$ is an injection, because A has no zero divisors. Let B be the algebra of algebraic functions on the lifting $\widetilde{U \cap X}$ of $U \cap X$ on $\widetilde{X}$. Then $A_F \hookrightarrow B$ is again an injection. The composition of the two injections gives an injection $A \hookrightarrow B$. We denote by C the *integral closure* of A in B. By definition, the ramified covering over U is then the variety Spec C. Such ramified coverings of affine charts can then be glued into a covering of the total space $\mathbb{P}\mathcal{H}_n$.

Lemma 3.7 *The variety $\widetilde{\mathbb{P}\mathcal{H}_n}$ is normal.*

Proof. Recall that a normal algebraic variety is a variety such that each of its points possesses an affine neighborhood whose algebra of functions is integrally closed in its field of fractions (see [4]). On normal varieties Chern classes of vector bundles and local intersection indices of subvarieties are well-defined. Normality is a local property, therefore we can restrict our attention to an affine chart U as above.

Let us prove that C is integrally closed. First of all, B is integrally closed, because it is an algebra of functions on a smooth variety. Thus an element x of the field of fractions of C integral over C belongs to B. But C is integrally closed in B (being the integral closure of A), thus x necessarily belongs to C. $\diamond$

In general, $\widetilde{\mathbb{P}\mathcal{H}_n}$ is not smooth. For example, the 2-sheeted ramified covering of $\mathbb{C}^2$ ramified over the axes $x = 0$ and $y = 0$ is the cone given by the equation $z^2 = xy$.

Our description of the preimage of a point in $\mathbb{P}\mathcal{H}_n$ under the projection $\widetilde{\mathbb{P}\mathcal{H}_n} \to \mathbb{P}\mathcal{H}_n$ follows from the fact that a normal complex algebraic variety is locally irreducible as a complex manifold ([4], Theorem 6.6).

We denote by $\widetilde{LL}$ the map

$$\widetilde{LL} : \widetilde{\mathbb{P}\mathcal{H}_n} \to \mathbb{C}\mathrm{P}^{2n-3}$$

that assigns to a meromorphic function with numbered critical points in $\widetilde{\mathbb{P}\mathcal{H}_n}$ the set of its numbered critical values. (Recall that both the function and the set of its critical values are considered up to a common scalar factor.) Denote by $\mathcal{O}(-1)$ the tautological line bundle over the target space $\mathbb{C}\mathrm{P}^{2n-3}$.

Recall that $\mathcal{T}$ is the tautological line bundle over $\mathbb{P}\mathcal{H}_n$. Denote by $\widetilde{\mathcal{T}}$ the pull-back of $\mathcal{T}$ under the projection $\widetilde{\mathbb{P}\mathcal{H}_n} \to \mathbb{P}\mathcal{H}_n$. Then $\widetilde{\mathcal{T}}$ coincides with the pull-back of $\mathcal{O}(-1)$ under $\widetilde{LL}$. Indeed, consider a point of the total space of the bundle $\widetilde{\mathcal{T}}$, outside its zero section. It corresponds to a nonvanishing meromorphic function with numbered critical points (and, this time, not considered up to a scalar factor). Similarly, the total space of the bundle $\mathcal{O}(-1)$ without the zero section is $\mathbb{C}^{2n-2} - \{0\}$, and it corresponds to sets of $2n-2$ numbered critical values, not all of which are equal to 0. It follows that the map $\widetilde{LL}$ can be lifted to the total spaces of the bundles $\widetilde{\mathcal{T}}$ and $\mathcal{O}(-1)$: it suffices to assign to a meromorphic function the set of its critical values. This lifting identifies the fibers of the two bundles.

It follows that for any subvariety $\widetilde{\mathbb{P}\Sigma} \subset \widetilde{\mathbb{P}\mathcal{H}_n}$ of pure dimension d such that its image $\widetilde{LL}(\widetilde{\Sigma})$ is irreducible and also of dimension d, we have

$$\left\langle \widetilde{\mathbb{P}\Sigma},\, (c_1(\widetilde{\mathcal{T}}^\vee))^d \right\rangle = \mu(\widetilde{LL}|_{\widetilde{\mathbb{P}\Sigma}}) \deg \widetilde{LL}(\widetilde{\mathbb{P}\Sigma}).$$

Indeed, we have

$$\deg \widetilde{LL}(\widetilde{\mathbb{P}\Sigma}) = \widetilde{LL}(\widetilde{\mathbb{P}\Sigma}) \cap P,$$

where $P \subset \mathbb{C}\mathrm{P}^{2n-3}$ is a generic projective subspace of dimension complementary to that of $\widetilde{LL}(\widetilde{\mathbb{P}\Sigma})$. Now, P is the intersection of d projective hyperplanes. A hyperplane is a section of $\mathcal{O}(1)$ (the line bundle dual to the tautological line bundle $\mathcal{O}(-1)$).

Let us count in two different ways the number of preimages in $\widetilde{\mathbb{P}\Sigma}$ of the points of intersection between $\widetilde{LL}(\widetilde{\mathbb{P}\Sigma})$ and P. First, since each intersection point has $\mu(\widetilde{LL}|_{\widetilde{\mathbb{P}\Sigma}})$ preimages, the number of preimages equals

$$\mu(\widetilde{LL}|_{\widetilde{\mathbb{P}\Sigma}}) \deg \widetilde{LL}(\widetilde{\mathbb{P}\Sigma}).$$

Second, the pull-back to $\widetilde{\mathbb{P}\mathcal{H}}_n$ of a generic hyperplane is a generic section of the line bundle $\widetilde{\mathcal{T}}^\vee$ dual to $\widetilde{\mathcal{T}}$. Taking the intersection between d sections like that and the subvariety $\widetilde{\mathbb{P}\Sigma}$, we obtain

$$\left\langle \widetilde{\mathbb{P}\Sigma},\, (c_1(\widetilde{\mathcal{T}}^\vee))^d \right\rangle.$$

Hence, these two numbers are equal.

The same equality holds if $\widetilde{LL}(\widetilde{\mathbb{P}\Sigma})$ is not irreducible, but all generic points of $\widetilde{LL}(\widetilde{\mathbb{P}\Sigma})$ have the same number of preimages.

Now let $\widetilde{\mathbb{P}\Sigma}$ be the lifting to $\widetilde{\mathbb{P}\mathcal{H}}_n$ of a stratum $\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ in $\mathbb{P}\mathcal{H}_n$. Denote by d their common dimension. For a partition κ denote by $\kappa!$ the product of the factorials of its elements. It is easy to see that a generic point in $\mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}$ has

$$\frac{(2n-2)!}{\kappa_1!\dots\kappa_c!}$$

liftings to $\widetilde{\Sigma}$. Further, the image of $\widetilde{\mathbb{P}\Sigma}$ under the lifted Lyashko–Looijenga map $\widetilde{LL}$ is a union of

$$\frac{(2n-2)!}{d_1!\dots d_c!\,|\mathrm{Aut}\{d_1,\dots,d_c\}|}$$

projective subspaces of dimension d.

Finally, denote by

$$\tilde{\mu}_{\{\kappa_1,\dots,\kappa_c\}} = \mu(\widetilde{LL}|_{\widetilde{\mathbb{P}\Sigma}_{\{\kappa_1,\dots,\kappa_c\}}})$$

the degree of $\widetilde{LL}$ on $\widetilde{\mathbb{P}\Sigma}_{\{\kappa_1,\dots,\kappa_c\}}$, which is the number of preimages in $\widetilde{\mathbb{P}\Sigma}_{\{\kappa_1,\dots,\kappa_c\}}$ of a generic point in the image of $\widetilde{\mathbb{P}\Sigma}_{\{\kappa_1,\dots,\kappa_c\}}$. Simple combinatorial considerations show that

$$\tilde{\mu}_{\{\kappa_1,\dots,\kappa_c\}} = \prod_{i=1}^{c} \frac{d_i!}{\kappa_i!} \cdot \mu_{\{\kappa_1,\dots,\kappa_c\}}.$$

The factor $\prod \frac{d_i!}{\kappa_i!}$ is the number of ways to number the critical points once the critical values are numbered.

Putting everything together we see that

$$\begin{aligned}
\mu_{\{\kappa_1,\dots,\kappa_c\}} &= \prod_{i=1}^{c} \frac{\kappa_i!}{d_i!} \cdot \tilde{\mu}_{\{\kappa_1,\dots,\kappa_c\}} \\
&= \frac{\prod \kappa_i!}{\prod d_i!} \cdot \frac{\prod d_i!\,|\mathrm{Aut}\{d_1,\dots,d_c\}|}{(2n-2)!} \cdot \langle \widetilde{\mathbb{P}\Sigma}, (c_1(\widetilde{\mathcal{T}}^\vee))^d \rangle \\
&= |\mathrm{Aut}\{d_1,\dots,d_c\}| \ \left\langle \mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}\,,\, (c_1(\mathcal{T}^\vee))^d \right\rangle \\
&= |\mathrm{Aut}\{d_1,\dots,d_c\}| \ \left\langle \mathbb{P}\Sigma_{\{\kappa_1,\dots,\kappa_c\}}\,,\, \Psi_n^d \right\rangle.
\end{aligned}$$

This proves the theorem. $\diamond$

4 The cohomology of the Hurwitz space

In this section we describe the cohomology algebra of the moduli spaces and of the Hurwitz spaces. We also prove several cohomological identities which will be used in the subsequent paper [15] to derive recurrence relations on Hurwitz numbers.

4.1 Keel's description of $H^*(\overline{\mathcal{M}}_n)$

The description of the cohomology algebra of the space $\overline{\mathcal{M}}_n$ given in this section is due to Keel [10].

Proposition 4.1 ([10]) *The cohomology ring $H^*(\overline{\mathcal{M}}_n)$ is generated by $H^2(\overline{\mathcal{M}}_n)$.*

Now we will describe a set of 2-cohomology classes that span $H^2(\overline{\mathcal{M}}_n)$. (More precisely, we will describe cycles that represent their Poincaré dual homology classes.)

Denote by $D = A \sqcup B$ an unordered partition of the set $\{1, \ldots, n\}$ of marked points into two disjoint parts (a partition $A \sqcup B$ coincides with $B \sqcup A$). Each part must contain at least 2 points.

Consider all stable curves with exactly two irreducible components such that one component contains the marked points of the set A and the other one of the set B. Such stable curves form a (complex) codimension 1 subvariety of $\overline{\mathcal{M}}_n$. Its closure is a smooth closed subvariety of $\overline{\mathcal{M}}_n$ of complex codimension 1. Therefore it represents a 2-cohomology class which we will denote by $[D]$.

Proposition 4.2 ([10]) *The classes $[D]$ span $H^2(\overline{\mathcal{M}}_n)$.*

The generators $[D]$ are not linearly independent. For example, for $n = 4$ all the three generators corresponding to the partitions

$$D_1 = \{1, 2\} \sqcup \{3, 4\} , \quad D_2 = \{1, 3\} \sqcup \{2, 4\} , \quad D_3 = \{1, 4\} \sqcup \{2, 3\}$$

represent the same cohomology class, (Poincaré dual to) a point.

More generally, let us fix four distinct numbers i, j, k, l between 1 and n. Denote by $[ij\mathcal{D}kl]$ the sum of all the generators $[D] \in H^2(\overline{\mathcal{M}}_n)$ corresponding to partitions, where i and j belong to one of the two parts, while k and l belong to the other part. Then the generators $[D]$ satisfy the relations

$$R_{ijkl}: \quad [ij\mathcal{D}kl] = [ik\mathcal{D}jl] = [il\mathcal{D}jk].$$

These relations can be deduced using the forgetful map $\overline{\mathcal{M}}_n \to \overline{\mathcal{M}}_4$ which forgets all the marked points except for i, j, k, l and contracts all the components of the curve that have become unstable.

Proposition 4.3 ([10]) *The relations R_{ijkl} span all the linear relations on the generators $[D]$.*

Finally, we say that two partitions D_1 and D_2 are *compatible* if the set $\{1, \ldots, n\}$ can be divided into a disjoint union of three sets A, B, C, in such a way that

$$D_1 = (A \sqcup B) \sqcup C, \quad D_2 = A \sqcup (B \sqcup C).$$

It is easy to see that if two partitions D_1 and D_2 are not compatible, then the geometric intersection of the corresponding cycles has codimension greater than 2. Therefore, the intersection of two such generators in the cohomology algebra vanishes.

Theorem 5 ([10]) *The cohomology algebra $H^*(\overline{\mathcal{M}}_n)$ is generated by the generators $[D]$ modulo two kinds of relations:* (i) *the linear relations R_{ijkl},* (ii) *the multiplicative relations $[D_1][D_2] = 0$ for incompatible partitions D_1 and D_2.*

This theorem gives a complete description of the cohomology algebra of $\overline{\mathcal{M}}_n$, although in practice it leads to rather heavy computations.

Using this theorem, Kontsevich and Manin [11] found linear generators and relations of $H^k(\overline{\mathcal{M}}_n)$ for all k.

4.2 A description of the cohomology of Hurwitz spaces

In what follows, we will require particular 2-cohomology classes $\psi_i \in H^2(\overline{\mathcal{M}}_n)$. Recall that $\mathcal{L}_i$ denotes the line bundle over $\overline{\mathcal{M}}_n$ whose fiber coincides with the cotangent line at the ith marked point.

Definition 4.4 We denote by $\psi_i = c_1(\mathcal{L}_i)$ the first Chern class of the line bundle $\mathcal{L}_i$.

For $n \geq 6$ the classes ψ_i *do not* span $H^2(\overline{\mathcal{M}}_n)$. The following proposition expresses them in terms of the classes $[D]$.

Let $i, j, k \in \{1, \ldots, n\}$. We denote by $[i * \mathcal{D} jk]$ the sum of all the classes $[D]$ corresponding to partitions containing i in one part, and j, k in the other part. The star in the notation reminds that an irreducible component with a single node cannot contain a single marked point.

Proposition 4.5 *For each i and any distinct j, k different from i we have $\psi_i = [i * \mathcal{D} jk]$.*

Proof. We construct a holomorphic section of the line bundle $\mathcal{L}_i$ in the following way. Consider a stable curve such that the marked points i, j, k are on the same irreducible component. On this component there exists a unique meromorphic differential with simple poles at the points j and k, whose residues at these poles are equal to 1 and -1 respectively. The value of this differential at the point i is an element of the line L_i. Thus we obtain a section of the bundle $\mathcal{L}_i$, and it extends in a unique way to a meromorphic section over the entire base $\overline{\mathcal{M}}_n$. It is easy to see that this section has no poles (thus the section is, actually, holomorphic). Its zeroes are precisely the classes that add up to $i * \mathcal{D} jk$, and it intersects the zero section of $\mathcal{L}_i$ transversally. $\diamond$

Since $\mathcal{H}_n$ is a vector bundle over $\overline{\mathcal{M}}_n$, the cohomology algebra of $\overline{\mathcal{M}}_n$ can be canonically seen as a subalgebra of the cohomology algebra of $\mathbb{P}\mathcal{H}_n$. We will therefore use the same notation for cohomology classes on $\overline{\mathcal{M}}_n$ and their pull-backs on $\mathbb{P}\mathcal{H}_n$.

Now we express the cohomology algebra of $\mathbb{P}\mathcal{H}_n$ in terms of that of $\overline{\mathcal{M}}_n$.

Theorem 6

$$H^*(\mathbb{P}\mathcal{H}_n) = H^*(\overline{\mathcal{M}}_n)[\Psi_n] \ / \ (\Psi_n - \psi_1) \dots (\Psi_n - \psi_n)\Psi_n.$$

Proof. The assertion follows immediately from the description of $\mathcal{H}_n$ as a vector bundle over $\overline{\mathcal{M}}_n$ (Section 2.3) and from the following well-known fact (see, for example, [3]). For any rank n complex vector bundle E over a complex manifold M we have

$$H^*(\mathbb{P}E) = H^*(M)[\Psi] \ / \ \Big(\Psi^n + c_1(E)\Psi^{n-1} + \dots + c_n(E) \Big),$$

that is, the cohomology algebra of $\mathbb{P}E$ is isomorphic to the algebra of polynomials in one variable Ψ with coefficients in $H^*(M)$, modulo the Chern polynomial in Ψ. Here the new variable Ψ can be identified with the first Chern class of the tautological line bundle over $\mathbb{P}E$. $\diamond$

This gives an explicit description of the cohomology algebra of $\mathbb{P}\mathcal{H}_n$ for $n \geq 3$.

Now, let $D = A \sqcup B$ be a partition of the set $\{1, \dots, n\}$ into a disjoint union of two nonempty subsets. The subsets are no longer required to have at least 2 elements. Consider the set of stable meromorphic functions defined on nodal curves with 2 irreducible components, the first component containing the marked points of the subset A and the second one of the subset B. This set is an open submanifold of $\mathcal{H}_n$ of complex codimension 1, so its closure is a cycle Poincaré dual to a cohomology class in $H^2(\mathbb{P}\mathcal{H}_n)$. This class will be denoted by $[D]$.

If A and B both contain at least two elements, then the corresponding class $[D]$ is a pull-back from a 2-cohomology class on $\overline{\mathcal{M}}_n$.

Let $i, j, k \in \{1, \dots, n\}$. Denote by $[i\mathcal{D}jk] \in H^2(\mathbb{P}\mathcal{H}_n)$ the sum of all the classes $[D]$ corresponding to partitions such that i belongs to one part and j and k belong to the other one. (There is no star after i, because the singleton $\{i\}$ is now an allowed subset.)

Proposition 4.6 *For any pairwise distinct $i, j, k \in \{1, \dots, n\}$, we have $\Psi_n = [i\mathcal{D}jk]$.*

Proof. Recall that Ψ_n is the first Chern class of the tautological line bundle on $\mathbb{P}\mathcal{H}_n$. A section of the vector bundle $\mathcal{H}_n^\vee$ dual to $\mathcal{H}_n$ will automatically provide a section of $\mathcal{O}_s(-1)$ (because a linear form on each fiber of $\mathcal{H}_n$ can be restricted to any line contained in that fiber). But a section of the bundle $\mathcal{H}_n^\vee$ can be obtained from a section of any of the line bundles $\mathcal{L}_i$ by taking its direct sum with zero sections of each of the other line bundles $\mathcal{L}_j$ for $j \neq i$. Thus the class Ψ_n is equal to the sum of the class $\psi_i = c_1(\mathcal{L}_i)$ and of the class Poincaré dual to the projectivization of the subbundle

$$\mathcal{L}_1^\vee + \dots + \mathcal{L}_{i-1}^\vee + \mathcal{L}_{i+1}^\vee + \dots + \mathcal{L}_n^\vee + \mathbb{C}.$$

The latter 2-cohomology class is exactly the class that corresponds to the partition

$$D = \{i\} \sqcup \{1, \ldots, i-1, i+1, \ldots n\}.$$

Using the expression for ψ_i given in Proposition 4.5 we obtain $\Psi_n = [i\mathcal{D}jk]$. $\diamond$

Proposition 4.7 *For $n \geq 2$, the classes $[D]$ span $H^2(\mathbb{P}\mathcal{H}_n)$ and generate $H^*(\mathbb{P}\mathcal{H}_n)$ as an algebra.*

Proof. The first assertion clearly follows from the second one. Let us prove the second assertion. For $n = 2$, $\mathbb{P}\mathcal{H}_2$ is topologically a sphere, and the class $[D] = [\{1\} \sqcup \{2\}]$ is a point; thus the proposition is true. For $n \geq 3$, the classes $[D]$ corresponding to partitions with at least two elements in each part generate the cohomology algebra of $\overline{\mathcal{M}}_n$, and those corresponding to partitions with a single element in one part produce the class Ψ_n. Therefore, they generate the cohomology algebra of $\mathbb{P}\mathcal{H}_n$. $\diamond$

4.3 Cohomological identities on the Hurwitz spaces

In this section we study various relations between the cohomology classes Poincaré dual to strata in $\mathbb{P}\mathcal{H}_n$. Note that the symmetric group S_n acts on $\mathcal{H}_n$ by permuting the poles. Therefore, it also acts on the cohomology algebra of $\mathbb{P}\mathcal{H}_n$. Strata of the stratification, as well as most of other cohomology classes that we consider, are invariant under this action. Therefore, their cohomology classes belong to the subalgebra of S_n-invariant cohomology classes.

Recall that C_n (respectively M_n) denotes the 2-cohomology class Poincaré dual to the caustic (respectively the Maxwell stratum) in $\mathbb{P}\mathcal{H}_n$ (see Example 3.4).

Let p and q be two positive integers, $p, q \geq 1$, $p + q = n$. We have introduced the cohomology class $[A \sqcup B]$ assigned to a partition $A \sqcup B$ of the set $\{1, \ldots, n\}$ into two nonempty parts. Denote by $\Delta_{p,q}$ the sum of the classes $[A \sqcup B]$ for all partitions such that $|A| = p$, $|B| = q$. (If $p = q$ the partitions $A \sqcup B$ and $B \sqcup A$ are two different terms of the sum, although they determine the same cohomology class.)

Definition 4.8 Denote by Δ_n the 2-cohomology class

$$\Delta_n = \frac{1}{2} \sum_{p+q=n} \Delta_{p,q}$$

and call it the *boundary stratum.*

The classes introduced above satisfy the following linear relations.

Proposition 4.9 *We have*

$$\begin{aligned}
\Psi_n &= \frac{1}{2n(n-1)} \sum_{p+q=n} pq\, \Delta_{p,q}; \\
\Psi_n &= \frac{1}{(2n-2)(2n-3)} (3C_n + 2M_n + \Delta_n).
\end{aligned}$$

Proposition 4.10

$$
\begin{aligned}
C_n &= 6(n-1)\Psi_n - 3\Delta_n. \\
M_n &= 2(n-1)(n-6)\Psi_n + 4\Delta_n. \\
C_n &= 3\sum_{p+q=n}\left(\frac{1}{n}pq - \frac{1}{2}\right)\Delta_{p,q}\ , \\
M_n &= \sum_{p+q=n}\left(\frac{n-6}{n}pq + 2\right)\Delta_{p,q}\ .
\end{aligned}
$$

Proof of Proposition 4.9. The first identity follows from the identity of Proposition 4.6,

$$\Psi_n = [i\mathcal{D}jk],$$

by summing it over all triples (i,j,k) and regrouping the terms.

In order to prove the second identity we consider once again the covering $\widetilde{\mathbb{P}\mathcal{H}_n}$ of $\mathbb{P}\mathcal{H}_n$ introduced in the proof of Theorem 4. Denote by $\widetilde{\Psi}_n \in H^2(\widetilde{\mathbb{P}\mathcal{H}_n})$ the pull-back of the class Ψ_n on $\mathbb{P}\mathcal{H}_n$. In the proof of Theorem 4 we established that

$$c_1\left(\widetilde{LL}^*(\mathcal{O}(-1))\right) = \widetilde{\Psi}_n.$$

(The pull-back under the lifted Lyashko–Looijenga map of the class of a hyperplane in $\mathbb{C}\mathrm{P}^{2n-3}$ equals $\widetilde{\Psi}_n$.) Consider a particular union of hyperplanes in the image space $\mathbb{C}\mathrm{P}^{2n-3}$ of $\widetilde{LL}$. Namely, the union of all hyperplanes where the ith and the jth critical values of f (i.e., the ith and the jth coordinates in $\mathbb{C}\mathrm{P}^{2n-3}$) are equal (for $1 \le i < j \le 2n-2$). This union represents $(2n-2)(2n-3)/2$ times the class of a hyperplane. Now consider the preimage of this union under the map $\widetilde{LL}$. It consists of the union of the strata $\widetilde{C}_n$, $\widetilde{M}_n$, and $\widetilde{\Delta}_n$ (the liftings to $\widetilde{\mathbb{P}\mathcal{H}_n}$ of the caustic C_n, the Maxwell stratum M_n, and the boundary stratum Δ_n). Moreover, $\widetilde{M}_n$, $\widetilde{\Delta}_n$ are simple preimages, but $\widetilde{C}_n$ is a triple preimage. (This means that if we take a generic point $f \in \widetilde{C}_n$ and its image in $\mathbb{C}\mathrm{P}^{2n-3}$, and then take a generic point close to the image, then this point will have three preimages close to f.) Now, the projection of $\widetilde{\mathbb{P}\mathcal{H}_n}$ onto $\mathbb{P}\mathcal{H}_n$ is ramified over C_n and Δ_n, and the order of ramification is 2 in both cases. On the contrary, it is not ramified over M_n. This implies that the geometrical liftings of the strata C_n and Δ_n to $\widetilde{\mathbb{P}\mathcal{H}_n}$ represent only one half of the pull-backs of the corresponding homology classes. On the other hand, the homology class of the geometrical lifting of M_n is equal to the lifting of the homology class of M_n. Putting everything together we obtain

$$\frac{(2n-2)(2n-3)}{2}\ \widetilde{\Psi}_n = 3\widetilde{C}_n + \widetilde{M}_n + \widetilde{\Delta}_n,$$

whence

$$\Psi_n = \frac{2}{(2n-2)(2n-3)}\left(\frac{1}{2}(3C_n)+M_n+\frac{1}{2}\Delta_n\right) = \frac{1}{(2n-2)(2n-3)}\ (3C_n+2M_n+\Delta_n).$$

$\diamond$

Proof of Proposition 4.10. The proof of this proposition is surprisingly difficult and uses some results from the subsequent paper [15].

First of all, note that it suffices to prove the first of the four identities $C_n = 6(n-1)\Psi_n - 3\Delta_n$. The other three identities follow from this one and from Proposition 4.9. Moreover, the 2-cohomology classes in the left- and the right-hand sides of the identity are *symmetric*, i.e., invariant under the action of the symmetric group S_n by renumbering the poles. The symmetric part of $H^2(\mathbb{P}\mathcal{H}_n)$ is spanned by the classes $\Delta_{p,q}$ (see Section 4.2). Therefore, we must construct a set of 2-homology classes of $\mathbb{P}\mathcal{H}_n$ such that their couplings with the classes $\Delta_{p,q}$ allow one to distinguish all linear combinations of the classes $\Delta_{p,q}$. It is then enough to show that their couplings with C_n and $6(n-1)\Psi_n - 3\Delta_n$ coincide. To construct such a set of 2-homology classes, we consider several embeddings of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{P}\mathcal{H}_n$.

For any positive integers p, q such that $p + q = n$ we are going to consider the following embedding of $\mathbb{C}\mathrm{P}^1$ in $\mathbb{P}\mathcal{H}_n$. Let $[u : v]$ be the homogeneous coordinates on $\mathbb{C}\mathrm{P}^1$. Consider the following family of rational functions in the variable z:

$$f_{u,v}(z) = \frac{1}{uz - a_1} + \ldots + \frac{1}{uz - a_p} + \frac{1}{vz - b_1} + \ldots + \frac{1}{vz - b_q},$$

where the a_i and the b_j are generically chosen complex numbers. The rational functions of this family do not belong to $\mathcal{H}_n$ for $u = 0$, $v = 0$ or $a_i/u = b_j/v$ (since they have less than n poles); however, there is a unique way to extend the family to a well-defined map from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{P}\mathcal{H}_n$, because $\mathbb{P}\mathcal{H}_n$ is compact. We denote by $\sigma_{p,q}$ the 2-homology class of the image of this map. We are going to study its couplings with the 2-cohomology classes C_n, Δ_n, and $\Delta_{p,q}$. Note that these classes are defined as Poincaré dual classes to particular codimension 1 subvarieties of $\mathbb{P}\mathcal{H}_n$. Therefore we will usually speak of their *intersection indices* (rather than couplings) with the families $\Delta_{p,q}$.

The three following lemmas describe completely the intersection indices of the family $\sigma_{p,q}$ with the classes $\Delta_{p',q'}$ and C_n. For shortness, we use the following convention. If in a lemma we give the intersection indices $\sigma_{p,q} \cap \Delta_{p',q'}$ and $\sigma_{p,q} \cap \Delta_{p'',q''}$, then in the particular case $p' = p''$, $q' = q''$, the two intersection indices must be added. For example, when we write $\sigma_{p,q} \cap \Delta_{p,q} = \sigma_{p,q} \cap \Delta_{q,p} = 1$, in the particular case $p = q$ the corresponding intersection index is equal to 2.

Lemma 4.11 *At each of the points $u = 0$ and $v = 0$ we have the following intersection indices:*

$$\sigma_{p,q} \cap \Delta_{p,q} = \sigma_{p,q} \cap \Delta_{q,p} = 1;$$

$$\sigma_{p,q} \cap \Delta_{p',q'} = 0$$

for other pairs (p', q');

$$\sigma_{p,q} \cap C_n = 0.$$

Lemma 4.12 *At each of the points $u/v = a_i/b_j$ we have the following intersection indices:*

$$\sigma_{p,q} \cap \Delta_{1,n-1} = \sigma_{p,q} \cap \Delta_{n-1,1} = n - 2;$$

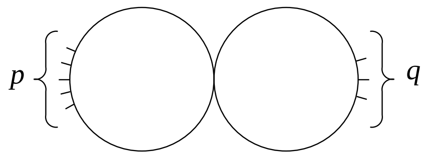

Figure 3: When $u = 0$ or $v = 0$ the function is defined on a 2-component curve with p poles on one component and q on the other one.

$$\sigma_{p,q} \cap \Delta_{2,n-2} = \sigma_{p,q} \cap \Delta_{n-2,2} = 1;$$

$$\sigma_{p,q} \cap \Delta_{p',q'} = 0$$

for other pairs (p', q')*;*

$$\sigma_{p,q} \cap C_n = 3(n-2).$$

Lemma 4.13 *There are* $6(pq-1)$ *more points of simple intersection of the family* $\sigma_{p,q}$ *with* C_n *corresponding to rational functions with a double critical point, defined on a one-component curve.*

Before proving the lemmas, let us make use of them to compute the total intersection indices. (For Ψ_n we use the expression from Proposition 4.9.) We have

$$\sigma_{p,q} \cap \Delta_{1,n-1} = \sigma_{p,q} \cap \Delta_{n-1,1} = pq(n-2);$$

$$\sigma_{p,q} \cap \Delta_{2,n-2} = \sigma_{p,q} \cap \Delta_{n-2,2} = pq;$$

$$\sigma_{p,q} \cap \Delta_{p,q} = \sigma_{p,q} \cap \Delta_{q,p} = 2;$$

$$\sigma_{p,q} \cap \Delta_{p',q'} = 0$$

for other pairs (p', q'). And

$$\sigma_{p,q} \cap \Delta_n = pq(n-1) + 2, \qquad \sigma_{p,q} \cap \Psi_n = pq, \qquad \sigma_{p,q} \cap C_n = 3pq(n-1) - 6.$$

This is consistent with the identity $C_n = 6(n-1)\Psi_n - 3\Delta_n$ that we must prove.

Moreover, looking at the intersection indices with the classes $\Delta_{p,q}$ we conclude immediately that the families $\sigma_{p,q}$ for different p and q allow one to distinguish any linear combinations of the classes $\Delta_{p,q}$. Therefore, the identity follows from the three lemmas, that we will now prove.

Proof of Lemma 4.11. Consider the case $u = 0$ (the case $v = 0$ is similar). When $u = 0$ the function f is defined on a 2-component curve shown in Figure 3.

Such a stable function belongs to the stratum $\Delta_{p,q} = \Delta_{q,p}$, but not to other strata $\Delta_{p',q'}$, nor to the caustic C_n. It is easy to check that the intersection with $\Delta_{p,q} = \Delta_{q,p}$ is simple (unless $p = q$, in which case it is double by the definition of $\Delta_{p,p}$). $\diamond$

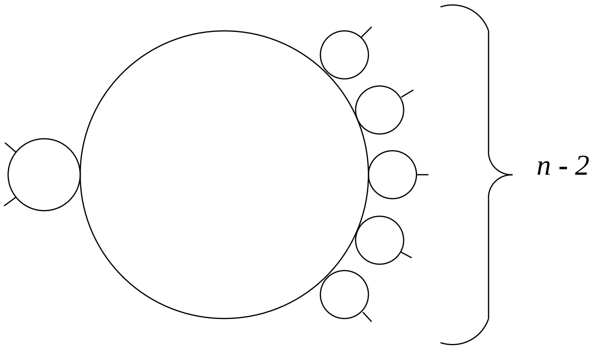

Figure 4: When $u/v = a_i/b_j$ we must multiply the family by $ub_j - va_i$ in order to get a finite limit in $\mathcal{H}_n$ (the limit in $\mathbb{P}\mathcal{H}_n$ does not change). Thus the principle part of $n-2$ poles tend simultaneously to zero, which explains that there appear $n-2$ small spheres attached to the central component.

Proof of Lemma 4.12. When $u/v = a_i/b_j$ the function f is defined on a curve with n irreducible components shown in Figure 4. Such a stable function belongs to $n-2$ different irreducible components of $\Delta_{1,n-1} = \Delta_{n-1,1}$ and to one irreducible component of $\Delta_{2,n-2} = \Delta_{n-2,2}$. It is easy to check that the intersection with each component is simple.

Now we must find the index of intersection of the family $\sigma_{p,q}$ with the caustic C_n at the point that we are considering. This is more difficult, because this point does not belong to the smooth part of the caustic. In order to find the intersection index we have to use a result of the subsequent paper [15], Theorem 3.9. In the particular case under consideration the assertion of this theorem is as follows. Consider the subvariety of $\mathbb{P}\mathcal{H}_n$ obtained from the stable function f by moving in all possible ways the $n-1$ points of intersection of the peripheral components with the central component of the curve in Figure 4, without changing the restrictions of f to the peripheral components. The closure of this subvariety is isomorphic to $\overline{\mathcal{M}}_{n-1}$. Now, the theorem says that the neighborhood of this subvariety in $\mathbb{P}\mathcal{H}_n$ is isomorphic to the neighborhood of $\overline{\mathcal{M}}_{n-1} \times \{0\}$ in $\mathcal{H}_{n-1} \times \mathbb{C}^2$. Moreover, the intersection of the caustic C_n with this neighborhood is isomorphic to $C_{n-1} \times \mathbb{C}^2$, where C_{n-1} is the (non-projectivized) caustic in the smaller Hurwitz space. The intersection of the family $\sigma_{p,q}$ with the neighborhood is a generic complex curve that intersects the subvariety $\overline{\mathcal{M}}_{n-1}$ transversally and is not contained in $C_{n-1} \times \mathbb{C}^2$. Now we claim that the intersection of the caustic C_{n-1} with a generic fiber of $\mathcal{H}_{n-1}$ over $\overline{\mathcal{M}}_{n-1}$ is a hypersurface of degree $3(n-3)$, which is therefore the index of intersection between $\sigma_{p,q}$ and C_n at the point under consideration.

It remains to check that $3(n-3)$ is indeed the degree of the intersection of C_{n-1} with a generic fiber of $\mathcal{H}_{n-1}$ over $\overline{\mathcal{M}}_{n-1}$. To do that it suffices to find the degree of

$$\frac{1}{c_1 \dots c_{n-1}} \operatorname{discrim}_z \left(\operatorname{numer} \left(\frac{d}{dz} \left(\frac{c_1}{z - z_1} + \dots + \frac{c_{n-1}}{z - z_{n-1}} \right) \right) \right)$$

as a polynomial in variables $c_1, \dots, c_{n-1}$ (discrim means discriminant, numer means numerator). Indeed, this polynomial is precisely the equation of the caustic C_{n-1} in the fiber, where the poles are fixed at the points $z_1, \ldots, z_{n-1}$. The division by $c_1 \ldots c_{n-1}$ is needed because, as one can easily check, the hyperplanes $c_i = 0$ are simple zeroes of the discriminant, but do not belong to the caustic. The degree of the above polynomial is indeed equal to $3(n-3)$ for homogeneity reasons. $\diamond$

Proof of Lemma 4.13. In order to calculate the intersection of C_n with the family $\sigma_{p,q}$ outside the points where some poles get glued together, we will consider the discriminant of the derivative of the function f of our family

$$\mathrm{discrim}_z \left(\mathrm{numer} \left(\frac{d}{dz} \left(\frac{1}{uz-a_1} + \ldots + \frac{1}{uz-a_p} + \frac{1}{vz-b_1} + \ldots + \frac{1}{vz-b_q} \right) \right) \right).$$

The degree of this discriminant as a homogeneous polynomial in u and v equals $2n(2n-3)$. One can check that this polynomial has double zeroes at the points $u/v = a_i/b_j$. Moreover, it has a zero of multiplicity $4p^2-6p+3$ at $u=0$ and a zero of multiplicity $4q^2-6q+3$ at $v=0$. Subtracting the multiplicities of all these zeroes, we obtain $6(pq-1)$ zeroes outside the points $u=0, v=0, u/v = a_i/b_j$. $\diamond$

Thus the three lemmas are proved, which completes the proof of Proposition 4.10.

References

[1] **P. Deligne, D. Mumford.** *The irreducibility of the space of curves of given genus,* Inst. Hautes Études Sci. Publ. Math., No. 36, (1969), p. 75–109.

[2] **T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein.** *Hurwitz numbers and intersections on moduli spaces of curves,* Inventiones Mathematicae, vol. **106**, (2001), p. 297–327. `arXiv: math.AG/0004096`

[3] **W. Fulton.** *Intersection theory,* 2nd edition, Springer, (1998).

[4] **S. Greco.** *Normal varieties,* Academic Press, London and New York, (1978).

[5] **I. P. Goulden, D. M. Jackson.** *The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group,* Europ. J. Combinatorics vol. **12**, (1992), p. 357–365.

[6] **T. Graber, R. Vakil.** *Hodge integrals and Hurwitz numbers via virtual localization,* `arXiv: math.AG/0003028`, (2000).

[7] **A. Hurwitz.** *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten,* Math. Ann., vol. **39**, (1891), p. 1–61.

[8] **A. Hurwitz.** *Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten,* Math. Ann., vol. **55**, (1902), p. 53–66.

[9] **M. Kazarian.** *Characteristic classes of multisingularities, cobordisms, and enumerative geometry,* preprint, (2002).

[10] **S. Keel.** *Intersection theory of moduli space of stable n-pointed curves of genus zero,* Transaction of the American Mathematical Society, vol. **330**, (1992), No. 2, p. 545–574.

[11] **M. Kontsevich, Yu. I. Manin.** *Gromov–Witten classes, quantum cohomology, and enumerative geometry,* Commun. Math. Phys., vol. **164** (1994), No. 3, p. 525–562.

[12] **S. K. Lando.** *Ramified coverings of the two-dimensional sphere and intersection theory on moduli spaces of meromorphic functions on algebraic curves,* Russ. Math. Surv., vol. **57**, (2002), No. 3, p. 29–98.

[13] **S. K. Lando, D. Zvonkine.** *On multiplicities of the Lyashko–Looijenga map on strata of the discriminant,* Functional Analysis and its Applications, vol. **33**, (1999), No. 3, p. 178–188.

[14] **D. Panov, D. Zvonkine.** *Counting Meromorphic Functions with Critical Points of Large Multiplicities,* Notes of the Scientific Seminar of the Mathematical Department of St-Petersburg Steklov Institute (POMI), vol. **292**, (2002), Representation Theory, Dynamical Systems, Combinatorics, and Algorithmic Methods, No. 7, p. 92–119. `arXiv: math.CO/0209013`.

[15] **D. Zvonkine.** *Counting ramified coverings and intersection theory on spaces of rational functions II (Local structure of Hurwitz spaces and combinatorial results),* – Preprint, (2003).