Wave Packet Transform in Symplectic Geometry and Asymptotic Quantization

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Abstract

We discuss a universal quantization procedure based on an integral transform that takes functions on the configuration space to functions on the phase space and is closely related to the Bargmann transform. In the leading term this procedure yields Schrödinger's quantization of observables, Maslov's quantization of Lagrangian modules, and Fock's quantization of canonical transforms.

Introduction

This text is an extended version of [1]. We deal with *asymptotic, or semi-classical, quantization*. Let us first explain this notion in some detail.

1. By quantization of classical mechanics physicists mean the assignment of quantum objects to the corresponding classical ones. The main classes of objects are *states* and *observables*. We recall that in classical mechanics the state of a system is determined by a point (q, p) in the phase space (the space of coordinates and momenta), and observables are functions $f(q, p)$ on this space. In quantum mechanics, the states of the system are described by *ψ-functions* (or *wave functions*) $\psi(x)$ and observables are described by linear operators in the state space.

For the Schrödinger quantization, the correspondence between the classical and the quantum *observables* is given by the rule

$$q \mapsto \hat{q} = x, \qquad p \mapsto \hat{p} = -ih\frac{\partial}{\partial x},$$

so that the (pseudodifferential) operator corresponding to an observable $f(q,p)$ has the form

$$\hat{f} = f(\overset{2}{\hat{q}}, \overset{1}{\hat{p}}) \tag{0.1}$$

(the numbers 1 and 2 indicate the order of action of the operators $\hat{q}$ and $\hat{p}$ [2]; different orderings, such as $f(\overset{1}{\hat{q}}, \overset{2}{\hat{p}})$, the Weyl ordering $f(\overset{2}{\hat{q}} + \overset{3}{\hat{q}}, \overset{1}{\hat{p}})$, or the Jordan ordering $(1/2)(f(\overset{1}{\hat{q}}, \overset{2}{\hat{p}}) + f(\overset{2}{\hat{q}}, \overset{1}{\hat{p}}))$ give the same result with the accuracy of $O(h)$).

The correspondence between classical and quantum *states* is not so simple. Although simultaneous measurement of the coordinates and the momenta is impossible, it makes sense to speak of the *joint probability density of the coordinates and the momenta* for a quantum particle in a state $\psi(x)$: the mean value of an arbitrary observable $f(q,p)$ in the state ψ is

$$\langle f \rangle_\psi = (\psi, \hat{f}\psi)_{L_2} = \operatorname{tr}(\hat{f}\hat{P}_\psi) = \int\limits_{\mathbf{R}^{2n}} f(q,p)\rho(q,p)\,dq\,dp,$$

where $\hat{P}_\psi$ is the orthogonal projection on ψ and

$$\rho(q,p) = \psi(q)\widetilde{\psi}(p)e^{ipx/h}, \quad \widetilde{\psi}(p) = (2\pi h)^{-n/2}\int e^{ipx/h}\psi(x)\,dx.$$

The function $\rho(q,p)$ is the desired joint probability density (it is known as the *density function* corresponding to $\psi(x)$). In the semiclassical limit ($h \to 0$) the density function vanishes for some classical states (p,q); if the support of the limit density is a manifold and if a certain additional condition is satisfied, then this manifold is necessary an *isotropic* submanifold of the phase space, that is, a submanifold on which the Cartan form $p\,dq$ is closed.

From the viewpoint of a quantum particle, this submanifold is the *oscillation front* of the ψ-function. Quantization must assign a ψ-function (more precisely, a class of ψ-functions) to a given isotropic manifold in such a way that the oscillation fronts of these functions lie in this isotropic manifold.

2. Let us now present a mathematical treatment of the above physical reasons. A similar discussion can be found in [3]. It will be convenient to use the language of the category theory.

Let us fix a phase space, for example, the cotangent space of a smooth real manifold M with the canonical symplectic structure $dp \wedge dq$.

i) Consider the category $\mathcal{C}$ whose *objects* are the modules $C^\infty(\Lambda)$ of smooth complex functions on compact Lagrangian manifolds Λ over the ring of classical observables. Thus, an object in this category is the abelian group $C^\infty(\Lambda)$ for some Lagrangian manifold Λ with the following action of the ring $C^\infty(T^*M)$ of classical observables:

$$f \cdot \varphi = f(p,q)|_\Lambda \varphi,$$

where the usual pointwise multiplication is used on the right-hand side.

Morphisms in this category are induced by *symplectic transforms* of the phase space. Namely, let Λ_1 and Λ_2 be Lagrangian manifolds. If

$$g : T^*M \to T^*M \tag{0.2}$$

is a symplectic transform such that $\Lambda_2 = g(\Lambda_1)$, then to (0.2) we assign the module homomorphism

$$g^* : C^\infty(\Lambda_2) \to C^\infty(\Lambda_1)$$

over the ring homomorphism

$$g^* : C^\infty(T^*M) \to C^\infty(T^*M).$$

Note that since Λ_1 and Λ_2 are compact, we can always assume that g is well-behaved at infinity (i.e., all derivatives of g are uniformly bounded).

Remark 1 In §3 we also consider a different (though isomorphic) realization of the category $\mathcal{C}$.

ii) Consider the category $\mathcal{Q}$ whose *objects* are the spaces $C_h^\infty(M,\Lambda)$ of smooth functions $\psi(x,h)$ depending on the parameter $h \in (0,1]$ with oscillation fronts in Λ. The space $C_h^\infty(M,\Lambda)$ is viewed as a module over the ring $\mathrm{PSD}(M)$ of quantum observables (that is, pseudodifferential operators). Note that pseudodifferential operators preserve oscillation fronts and hence the module structure is well defined.

Morphisms in this category are given by invertible mappings

$$T : C^\infty(M) \to C^\infty(M)$$

such that

$$T : C_h^\infty(M,\Lambda_1) \to C_h^\infty(M,\Lambda_2) \tag{0.3}$$

and the operator $T\hat{H}T^{-1}$ is a pseudodifferential operator for any pseudodifferential operator $\hat{H}$. Note that (0.3) is a module homomorphism over the ring homomorphism $\hat{H} \to T\hat{H}T^{-1}$.

iii) A *semi-classical quantization* is a projective contravariant functor[1]:

$$\mathcal{F} : \mathcal{C} \to \mathcal{Q} \quad (\mathrm{mod} O(h))$$

such that the module of smooth functions on M with oscillation front in a given Lagrangian manifold is assigned to the module of smooth functions on this Lagrangian manifold together with the $\mathcal{F}$-functorial mappings

$$\mu : C^\infty(T^*M) \to \mathrm{PSD}(M)$$

and

$$K_\sigma : C_0^\infty(\Lambda) \to C_h^\infty(M, \Lambda)$$

for each Lagrangian manifold Λ equipped with a measure σ. (By saying that these mappings are $\mathcal{F}$-functorial we mean that they are naturally included in projectively commutative diagrams involving morphisms in $\mathcal{C}$ and $\mathcal{Q}$ related by $\mathcal{F}$.)

Such a functor exists and can be explicitly constructed on the basis of a certain intergal transform, which we call the *wave packet transform*. This is *an invertible* transform taking functions $f(x,h)$ determined on the configuration space $\mathbf{R}^n$ to some subspace of functions $\tilde{f}(q,p,h)$ determined on the phase space $T^*\mathbf{R}^n$. By using such a transform one can carry out a unified construction of quantization of all classical objects.

Moreover, this procedure conicides in the leading term with the Schrödinger quantization[2] [5] for observables, with the Fock quantization [6] for canonical (symplectic) transforms and with the Maslov quantization [7] for Lagrangian modules.

Implementing this construction, we obtain $1/h$-pseudodifferential operators as quantization of observables, Fourier integral operators as quantization of symplectic transforms and Maslov's canonical operator as quantization of Lagrangian modules (the reader can find the notions used here, for example, in [8]).

3. We conclude these preliminary considerations with some remarks. To obtain the correspondence between classical and quantum objects, that is, to construct a quantization procedure, we try to decompose any quantum state $\psi(x,h)$ into a sum of elements corresponding to points (q,p) of the phase space $T^*\mathbf{R}^n$, that is, to classical states. Such a decomposition is a *microlocalization procedure*. Different realizations of this procedure were widely presented in the literature (see, e. g. [9]). The realization proposed in this paper leads to a transform providing the direct quantization procedure.

[1]By a *projective* functor we mean a mapping between categories such that the composition of morphisms is preserved up to a unimodular factor; this makes sense if the sets of morphisms have the structure of vector spaces over $\mathbf{C}$.

[2]We actually use the anti-Wick quantization (see, e. g., [4]), which coincides with the Schrödinger quantization in the leading term.

Localization of a function $f(x,h)$ in the phase space can be accomplished by localization along the fibers of $T^*\mathbf{R}^n$ followed by localization along the base. The localization along the base uses the "integral partition of unity" of the form

$$1 = \left(\frac{1}{2\pi h}\right)^{n/2} \int e^{-\frac{1}{2h}(x-x_0)^2}\, dx.$$

Localization along the base is therefore obtained with the help of multiplication by

$$\delta_h(x-x_0) = \left(\frac{1}{2\pi h}\right)^{n/2} e^{-\frac{1}{2h}(x-x_0)^2};$$

note that

$$\delta_h(x-x_0) \to \delta(x-x_0)$$

as $h \to 0$.

Localization along the fibers can be done with the help of the quantum Fourier transform, in other words, by means of p-representation, at the point x_0:

$$F_{x\to p_0}[f] = \left(\frac{1}{2\pi i h}\right)^{n/2} \int e^{-\frac{i}{h}p_0(x-x_0)} f(x)\, dx.$$

By composition of these two localizations we obtain the *microlocal element* corresponding to the function $f(x,h)$ in the form

$$\begin{aligned} f_{(x_0,p_0)} &= F_{x\to p_0}\{\delta_h(x-x_0)\, f(x,h)\} = \left(\frac{1}{2\pi h}\right)^{n/2} \left(\frac{1}{2\pi i h}\right)^{n/2} \times \\ &\times \int \exp\left\{\frac{i}{h}\left[-p_0(x-x_0) + \frac{i}{2}(x-x_0)^2\right]\right\} f(x)\, dx. \end{aligned}$$

The latter formula determines an integral transform, which we call the *wave packet transform* of the function $f(x,h)$. The inverse transform is given by

$$\tilde{f}(x_0,p_0) \mapsto f(x) = \left(\frac{i}{2\pi h}\right)^{n/2} \int e^{\frac{i}{h}[p_0(x-x_0)+\frac{i}{2}(x-x_0)^2]}\, \tilde{f}(x_0,p_0)\, dx_0 dp_0.$$

It is convenient to renormalize the obtained transform is such a way that the Parseval identity takes place. This normalization is used below.

1 Definition and basic properties of the wave packet transform

1. In 1961 V.Bargmann [10] introduced a remarkable integral transform relating the "harmonic oscillator representation" of the creation-annihilation operators for Bose particles in quantum field theory and the Fock representation of these operators [11, 12]. Let us briefly recall these results. In the harmonic oscillator representation, the creation and annihilation operators act in the space $L^2(\mathbf{R}^n_q)$ of square integrable functions of the variables $q = (q_1, \ldots, q_n)$ and have the form[3] ($\hat{p}_i = -i\partial/\partial q_i$):

$$a_i^* = \frac{1}{\sqrt{2}}(q_i - i\hat{p}_i) = \frac{1}{\sqrt{2}}\left(q_i - \frac{\partial}{\partial q_i}\right), \quad i = 1, \ldots, n \quad \text{(creation operators)};$$
$$a_i = \frac{1}{\sqrt{2}}(q_i + i\hat{p}_i) = \frac{1}{\sqrt{2}}\left(q_i + \frac{\partial}{\partial q_i}\right), \quad i = 1, \ldots, n \quad \text{(annihilation operators)}. \tag{1.1}$$

The operators a_i and a_i^* are adjoints of each other with respect to the inner product on $L^2(\mathbf{R}^n_q)$ and satisfy the commutation relations

$$[a_i, a_j] = [a_i^*, a_j^*] = 0, \quad [a_i, a_j^*] = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{1.2}$$

Fock introduced a different solution of the commutation relations (1.2), namely,

$$\tilde{a}_i = \frac{\partial}{\partial z_i}, \quad \tilde{a}_i^* = z_i, \quad i = 1, \ldots, n. \tag{1.3}$$

Here it is required, in analogy with (1.2), that the operators $\tilde{a}_i$ and $\tilde{a}_i^*$ are mutually adjoint in some Hilbert space of functions of $z = (z_1, \ldots, z_n)$. One can achieve this by assuming that the z_i are *complex variables*, $z_i = x_i + iy_i$. Then the operators (1.3) are pairwise adjoint in the Hilbert space $\mathcal{F}_n$ of entire analytic functions $f(z)$ with the scalar product

$$(f, g) = \frac{1}{\pi^n} \int_{\mathbf{C}^n} \bar{f}(z) g(z) e^{-\bar{z}z} dx\, dy = \sum_{|\alpha|=0}^{\infty} \alpha! \bar{f}_\alpha g_\alpha \tag{1.4}$$

(here $dx\, dy$ is the standard Lebesgue measure in $\mathbf{C}^n$,

$$\bar{z}z = \sum_{i=1}^{n} \bar{z}_i z_i, \quad \alpha = (\alpha_1, \ldots, \alpha_n)$$

[3] In the system of units in which $h = 1$.

is a multi-index, and f_α and g_α are the Taylor coefficients of f and g:

$$f = \sum_{|\alpha|=0}^{\infty} f_\alpha z^\alpha,$$

and similarly for g).

The Bargmann transform A_n acts from $L^2(\mathbf{R}_q^n)$ into $\mathcal{F}_n$ according to the formula

$$(A_n\psi)(z) = \int_{\mathbf{R}_q{}^n} A_n(z,q)\,\psi(q)\,dq, \quad \psi \in L^2(\mathbf{R}_q^n), \tag{1.5}$$

where the kernel has the form

$$A_n(z,q) = \frac{1}{\pi^{n/4}} \exp\left\{-\frac{1}{2}(z^2+q^2) + \sqrt{2}zq\right\}. \tag{1.6}$$

The main properties of the Bargmann transform are given by the following theorem.

Theorem 1 i) *The transform*

$$A_n : L^2(\mathbf{R}_q^n) \to \mathcal{F}_n$$

is an isometric isomorphism (that is, a unitary operator).

ii) *The inverse transform is given by the formula*

$$(A_n^{-1}f)(q) = \lim_{\lambda\to 1} \frac{1}{\pi^n} \int_{\mathbf{C}^n} \overline{A_n(z,q)} f(\lambda z) e^{-\bar{z}z} dx\,dy, \tag{1.7}$$

where $\lambda \to 1$ from below, and the limit is understood in the strong sense in $L^2(\mathbf{R}_q^n)$.

iii) *The transform A_n is an intertwining operator for the representations* (1.1) *and* (1.3) *of the commutation relations* (1.2), *that is,*

$$A_n \cdot a_i = \tilde{a}_i \cdot A_n, \quad A_n \cdot a_i^* = \tilde{a}_i^* \cdot A_n, \quad i = 1, \dots, n. \tag{1.8}$$

The comparison of the formulas

$$a_i^* = \frac{1}{\sqrt{2}}(q_i - i\hat{p}_i) \quad \text{and} \quad \tilde{a}_i^* = z_i$$

suggests that it might be useful to identify the complex space $\mathbf{C}^n$, on which the elements of $\mathcal{F}_n$ are defined, with the phase space $\mathbf{R}_q^n \oplus \mathbf{R}_p^n$ according to the formula

$$z = \frac{1}{\sqrt{2}}(q - ip).$$

In the "exact" theory this identification, as Bargmann noted, is of limited applicability since q_k and $\hat{p}_k$ do not commute. However, it is quite adequate in the asymptotic theory ($h \to 0$), but we need to consider a different transform.

2. Consider the *Gaussian wave packet*

$$G_h(q,p;x) = \exp\left\{\frac{i}{h}(x-q)p - \frac{1}{2h}(x-q)^2\right\}, \quad x \in \mathbf{R}^n, \tag{1.9}$$

where $h > 0$, $q \in \mathbf{R}^n$, and $p \in \mathbf{R}^n$ are parameters. We use the function (1.9) as a kernel to define an integral transform U acting on $L^2(\mathbf{R}^n_x)$ as follows:

$$U[f](q,p) = 2^{-\frac{n}{2}}(\pi h)^{-\frac{3n}{4}} \int\limits_{\mathbf{R}_x{}^n} \overline{G_h(q,p;x)}\, f(x)\, dx, \quad f \in L^2(\mathbf{R}^n_x), \tag{1.10}$$

where the bar denotes complex conjugation and $dx = dx_1 \cdot dx_2 \cdot \ldots \cdot dx_n$ is the standard Lebesgue measure on $\mathbf{R}^n_x$. The integral on the right-hand side in (1.10) is obviously well defined, since $G_h(q,p;x)$ belongs to $L^2(\mathbf{R}^n_x)$ for any fixed h, q, and p.

Definition 1 The integral transform U defined in (1.10) will be called the *wave packet transform.*

Remark 2 In [1] this transform was called the "Fourier–Gauss transform," but we prefer the present name since this transform is the symplectic analog of the wave packet transform considered by Cordoba and Fefferman [13] (see also [14]).

Theorem 2 (i) *The wave packet transform is a bounded operator in the spaces*

$$U : L^2(\mathbf{R}^n_x) \to L^2(\mathbf{R}^{2n}_{q,p}) \tag{1.11}$$

and satisfies the Parseval identity

$$(Uf, Ug)_{L^2} = (f,g)_{L^2}. \tag{1.12}$$

(ii) *The adjoint operator*

$$U^* : L^2(\mathbf{R}^{2n}_{q,p}) \to L^2(\mathbf{R}^{n2}_x) \tag{1.13}$$

is given by the formula

$$U^*[\psi](x) = 2^{-n/2}(\pi h)^{-\frac{3n}{4}} \int\limits_{\mathbf{R}^{2n}_{(q,p)}} G_h(q,p;x)\psi(q,p)\, dq\, dp, \tag{1.14}$$

where the integral on the right-hand side (which is not absolutely convergent at infinity in general) is understood as the limit of the similar integrals with $\psi(q,p)$ replaces by $\psi_k(x,p)$, where $\{\psi_k\}$ is a sequence of compactly supported functions convergent to ψ in $L^2(\mathbf{R}^{2n}_{q,p})$.

(iii) *One has the inversion formula*

$$U^*U = 1 \tag{1.15}$$

(iv) *The range of U is the closed subspace $\mathcal{F}^2(\mathbf{R}^{2n}_{q,p}) \subset L^2(\mathbf{R}^{2n}_{q,p})$ of functions $F(q,p)$ that satisfy the equations*

$$\left[h\frac{\partial}{\partial q_j} - ih\frac{\partial}{\partial p_j} - ip_j\right] F(q,p) = 0, \;\; j = 1, \dots, n. \tag{1.16}$$

Remark 3 Obviously, condition (1.16) is equivalent to saying that $\exp\{p^2/(2h)\}$ $F(q,p)$ is an analytic function of the variables $q - ip = (q_1 - ip_1, \dots, q_n - ip_n)$.

Proof. Straightforward computation shows that

$$U[f](q,p) = (2\pi\sqrt{h})^{-n/2} \exp\left(-\frac{p^2}{2h}\right) \left[\exp\left(-\frac{z^2}{2}\right) F(z)\right]\bigg|_{z=\frac{q-ip}{\sqrt{2h}}}, \tag{1.17}$$

where $F(z) = \mathcal{B}[f](z)$ is the Bargmann transform of $f(x\sqrt{h})$. Now we obtain all assertions of Theorem 2 from the corresponding properties of the Bargmann transform (Theorem 1) by routine computations.

The following statement is quite obvious.

Theorem 3 *Set*

$$U^{-1} = U^*|_{\mathcal{F}^2(\mathbf{R}^{2n}_{q,p})}. \tag{1.18}$$

Then

$$U^{-1}U = 1, \;\; UU^{-1} = 1, \tag{1.19}$$

but

$$U^*U = 1, \;\; UU^* = P, \tag{1.20}$$

where P is the operator of orthogonal projection in $L^2(\mathbf{R}^{2n}_{p,q})$ onto $\mathcal{F}^2(\mathbf{R}^{2n}_{p,q})$.

Next, on analogy with the Fourier transform, let us derive some commutation formulas for U.

Theorem 4 *One has the commutation formulas*

$$U \circ x = \left(q + ih\frac{\partial}{\partial p}\right) \circ U, \tag{1.21}$$

$$U \circ \left(-ih\frac{\partial}{\partial x}\right) = \left(-ih\frac{\partial}{\partial q}\right) \circ U \tag{1.22}$$

(as usual, the equality of two unbounded operators implies that their domains coincide).

Remark 4 Since any function F in the range of U satisfies (1.16), we can derive numerous other formulas; for example, replacing $ih\partial/\partial p$ by $h\partial/\partial q - ip$ in (1.21) gives

$$U \circ x = \left(q + h\frac{\partial}{\partial q} - ip\right) \circ U, \tag{1.23}$$

etc. This trick will often be used in the sequel.

Proof. Differentiating the kernel $\bar{G} = \overline{G_h(q,p;x)}$ gives

$$ih\frac{\partial \bar{G}}{\partial p} = (x - q)\bar{G}, \quad \frac{\partial \bar{G}}{\partial q} = -\frac{\partial \bar{G}}{\partial x}, \tag{1.24}$$

which readily yields (1.21) and (1.22) (to prove the latter formula, one also has to integrate by parts once).

We are now in a position to study the action of U and U^* in the following spaces, often used in examining asymptotic expansions as $h \to 0$ (for details, see [8]).

Definition 2 Let $k \in \mathbf{Z}_+ = \{0,1,2,\ldots\}$. By $H_k^{1/h}(\mathbf{R}_x^n)$ we denote the space of functions $f(x)$ with finite norm

$$||f||_{k,n}^2 = \int\limits_{\mathbf{R}^n} \overline{f(x)} \left[\left(1 + x^2 - h^2\frac{\partial^2}{\partial x^2}\right)^k f(x)\right] dx, \quad x^2 = \sum x_j^2, \quad \frac{\partial^2}{\partial x^2} = \sum \frac{\partial^2}{\partial x_j^2}. \tag{1.25}$$

Obviously, $H_k^{1/h}(\mathbf{R}_x^n)$ is a Hilbert space and we have the filtration

$$L^2(\mathbf{R}^n) = H_0^{1/h}(\mathbf{R}^n) \supset H_1^{1/h}(\mathbf{R}^n) \supset \ldots \supset H_k^{1/h}(\mathbf{R}^n) \supset \ldots.$$

Similarly, we introduce the spaces $H_k^{1/h}(\mathbf{R}_{p,q}^{2n})$ equipped with the norms

$$||\psi||_{k,h}^2 = \int\limits_{\mathbf{R}^{2n}} \overline{\psi(p,q)} \left[\left(1 + q^2 + p^2 - h^2\frac{\partial^2}{\partial q^2} - h^2\frac{\partial^2}{\partial p^2}\right)^k \psi(p,q)\right] dp\, dq. \tag{1.26}$$

Next, let us consider functions $f(x, h)$ depending on the parameter $h \in (0, 1]$. We introduce the norm

$$||f||_k = \sup_{h\in(0,1]} ||f||_{k,h} \tag{1.27}$$

and denote by $H_k(\mathbf{R}^n)$ the space of functions with finite norm (1.27). Furthermore, we consider the Fréchet space

$$H_\infty(\mathbf{R}^n) = \bigcap_{k=0}^{\infty} H_k(\mathbf{R}^n) \tag{1.28}$$

with the topology defined by the countable system of seminorms (1.27). The spaces $H_k(\mathbf{R}^{2n})$ and $H_\infty(\mathbf{R}^{2n})$ are defined similarly.

Finally, let $H_{-k}^{1/h}(\mathbf{R}^n)$ be the dual space of $H_k^{1/h}(\mathbf{R}^n)$ with respect to the L_2 inner product. The elements of $H_{-k}^{1/h}$ are naturally interpreted as distributions, and we have the embeddings

$$\supset H_{-k}^{1/h}(\mathbf{R}^n) \supset \ldots \supset H_{-1}^{1/h}(\mathbf{R}^n) \supset H_0^{1/h}(\mathbf{R}^n) = L_2(\mathbf{R}^n) \supset H_1^{1/h}(\mathbf{R}^n) \supset \ldots.$$

The definition of the spaces H_k extends to negative k, and we set

$$H_{-\infty}(\mathbf{R}^n) = \bigcup_{k=\infty}^{-\infty} H_k(\mathbf{R}^n); \tag{1.29}$$

a net $\{\psi_k\}$ is said to be convergent in $H_{-\infty}$ if it converges in some H_k; with this topology, $H_{-\infty}$ is the dual of H_∞.

Theorem 5 (i) *For any $k \in \mathbf{Z}$ the mappings U and U^* are continuous in the spaces*

$$U : H_k(\mathbf{R}_x^n) \to H_k(\mathbf{R}_{p,q}^{2n}), \quad U^* : H_k(\mathbf{R}_{p,q}^{2n}) \to H_k(\mathbf{R}_x^n) \tag{1.30}$$

(for negative k, we extend these mappings from L_2 by continuity).

(ii) *Let $H_{-\infty,\mathrm{comp}}(\mathbf{R}_{p,q}^{2n})$ be the subspace of elements $\psi \in H_{-\infty}(\mathbf{R}_{p,q}^{2n})$ such that the support* supp ψ *lies in some ball $B_R = \{(q,p) \in \mathbf{R}^{2n} | q^2 + p^2 \le R\}$. For any finite R, the operator U^* is continuous in the spaces*

$$U^* : H_{-\infty,\mathrm{comp}}(\mathbf{R}_{p,q}^{2n}) \to H_\infty(\mathbf{R}_x^n). \tag{1.31}$$

Consequently, the projection $P = UU^*$ is continuous in the spaces

$$P : H_{-\infty,\mathrm{comp}}(\mathbf{R}_{p,q}^{2n}) \to H_\infty(\mathbf{R}_{p,q}^{2n}) \cap \mathcal{F}_2(\mathbf{R}_{p,q}^{2n}). \tag{1.32}$$

Proof. (i) In suffices to prove that the L^2-norm of

$$q^\alpha p^\beta \left(-ih\frac{\partial}{\partial p}\right)^\gamma \left(-ih\frac{\partial}{\partial q}\right)^\delta U[f],$$

where α, β, γ, and δ are multiindices such that $|\alpha| + |\beta| + |\gamma| + |\delta| \leq k$, $k \in \mathbf{Z}_+$, can be estimated via a linear combination of the L^2-norms of $x^\mu(-ih\partial/\partial x)^\nu f(x)$ with $|\mu| + |\nu| \leq k$. We have

$$\begin{aligned}
&q^\alpha p^\beta \left(-ih\frac{\partial}{\partial p}\right)^\gamma \left(-ih\frac{\partial}{\partial q}\right)^\delta U[f] \\
&= q^\alpha p^\beta \left(q - q - ih\frac{\partial}{\partial p}\right)^\gamma \left(-ih\frac{\partial}{\partial q}\right)^\delta U[f] \\
&= \sum_{\tilde{\gamma}+\mathring{\gamma}=\gamma} \frac{\gamma!}{\tilde{\gamma}!\,\mathring{\gamma}!} q^{\alpha+\tilde{\gamma}\beta}(-1)^{|\mathring{\gamma}|} \left(q + ih\frac{\partial}{\partial p}\right)^{\mathring{\gamma}} \left(-ih\frac{\partial}{\partial q}\right)^\delta U[f] \qquad (1.33)\\
&= \sum_{\tilde{\gamma}+\mathring{\gamma}=\gamma} \frac{(-1)^{|\mathring{\gamma}|}\gamma!}{\tilde{\gamma}!\,\mathring{\gamma}!} q^{\alpha+\tilde{\gamma}} p^\beta U\left[x^{\mathring{\gamma}}\left(-ih\frac{\partial}{\partial x}\right)^\delta f\right]
\end{aligned}$$

by Theorem 4. Next,

$$|q| \leq |q - ip| \text{ and } |p| \leq |q - ip|,$$

and so the L_2-norm of the right-hand side of (1.33) does not exceed

$$\sum_{\tilde{\gamma}+\mathring{\gamma}=\gamma} \frac{\gamma!}{\tilde{\gamma}!\,\mathring{\gamma}!} \left\| (q - ip)^{\alpha+\tilde{\gamma}+\beta} U\left[x^{\mathring{\gamma}}\left(-ih\frac{\partial}{\partial x}\right)^\delta f\right] \right\|_{L_2(\mathbf{R}^{2n}_{q,p})}.$$

Now, by (1.23) and (1.22),

$$(q - ip) \circ U = U \circ x - h\frac{\partial}{\partial q} \circ U = U \circ \left(x - h\frac{\partial}{\partial x}\right), \qquad (1.34)$$

so that the last expression can be replaced by

$$\sum_{\tilde{\gamma}+\mathring{\gamma}=\gamma} \frac{\gamma!}{\tilde{\gamma}!\,\mathring{\gamma}!} \left\| U\left[\left(x - h\frac{\partial}{\partial x}\right)^{\alpha+\tilde{\gamma}+\beta} x^{\mathring{\gamma}}\left(-ih\frac{\partial}{\partial x}\right)^\delta f\right] \right\|_{L_2(\mathbf{R}^{2n}_{q,p})},$$

whence the desired estimate for $k \geq 0$ follows immediately. For $k < 0$, one uses the standard duality argument. As to the estimates for U^*, they are even easier to obtain. We have, from (1.24),

$$-ih\frac{\partial G}{\partial p} = (x-q)\,G, \;\; \frac{\partial G}{\partial q} = -\frac{\partial G}{\partial x}, \tag{1.35}$$

where $G = G_h(q,p;x)$, whence

$$\left(-ih\frac{\partial}{\partial x}\right) \circ U^* = U^* \circ \left(-ih\frac{\partial}{\partial q}\right), \tag{1.36}$$

$$x \circ U^* = U^* \circ \left(q + ih\frac{\partial}{\partial p}\right) \tag{1.37}$$

(integration by parts in used to derive (1.37)), and the desired estimates become obvious.

(ii) Each element $\psi(p,q) \in H_{-\infty}(\mathbf{R}^{2n}_{p,q})$ belongs to $H_{-N}(\mathbf{R}^{2n}_{p,q})$ for some $N \geq 0$ and hence can be represented in the form

$$\psi(p,q) = \left(1 + q^2 + p^2 - h^2\frac{\partial^2}{\partial q^2} - h^2\frac{\partial^2}{\partial p^2}\right)^N \psi_N(p,q), \tag{1.38}$$

where $\psi_N(p,q) \in H_N(\mathbf{R}^{2n}_{p,q})$ and

$$||\psi_N||_N \leq \mathrm{const}||\psi||_{-N}. \tag{1.39}$$

Now suppose that $\psi(p,q) \in H_{-\infty,\mathrm{comp}}$. Take a function $\chi(r) \in C_0^\infty(\mathbf{R}^1)$ such that $\chi(p^2+q^2)\psi = \psi$. We have

$$\begin{aligned}
U^*[\psi](x) &= U^*[\chi(p^2+q^2)\psi(p,q)](x) \\
&= 2^{-n/2}(\pi h)^{-3n/4}\int \exp\left[\frac{i}{h}(x-q)p - \frac{1}{2h}(x-q)^2\right]\chi(p^2+q^2) \quad (1.40)\\
&\quad \times \left[\left(1 + q^2 + p^2 - h^2\frac{\partial^2}{\partial q^2} - h^2\frac{\partial^2}{\partial p^2}\right)^N \psi_N(p,q)\right] dp\,dq \\
&= 2^{-n/2}(\pi h)^{-3n/4}\int \psi_N(p,q)\left\{\left(1 + q^2 + p^2 - h^2\frac{\partial^2}{\partial q^2} - h^2\frac{\partial^2}{\partial p^2}\right)^N\right. \\
&\quad \left. \times \exp\left[\frac{i}{h}(x-q)p - \frac{1}{2^h}(x-q)^2\right]\chi(p^2+q^2)\right\} dp\,dq
\end{aligned}$$

(we have used integration by parts). Now

$$\left(1+q^2+p^2-h^2\frac{\partial^2}{\partial q^2}-h^2\frac{\partial^2}{\partial p^2}\right)^N \exp\left[\frac{i}{h}(x-q)p-\frac{1}{2h}(x-q)^2\right]$$
$$\times\chi(p^2+q^2)=\exp\left[\frac{i}{h}(x-q)p-\frac{1}{2h}(x-q)^2\right] \tag{1.41}$$
$$\times\ \left[1+q^2+p^2+\left(-ih\frac{\partial}{\partial q}-p+i(q-x)\right)^2+\left(-ih\frac{\partial}{\partial p}+x-q\right)^2\right]^N$$
$$\times\chi(p^2+q^2)=\exp\left[\frac{i}{h}(x-q)p-\frac{1}{2h}(x-q)^2\right]\sum_{|\alpha|=0}^{N}(x-q)^\alpha a_\alpha(p,q),$$

where the $a_\alpha(p,q)$ are smooth compactly supported functions bounded with all derivatives unifomly with respect to h (the sum is from $|\alpha|=0$ to N rather than to $2N$ since the terms $i(q-x)^2$ and $(x-q)^2$ cancel each other out). The expression obtained can be rewritten in the form

$$\sum_{|\alpha|=0}^{N} a_\alpha(p,q)\left\{\left(-ih\frac{\partial}{\partial p}\right)^\alpha \exp\left[\frac{i}{h}(x-q)p-\frac{1}{2h}(x-q)^2\right]\right\},$$

and by integrating by parts once more, we get

$$U^*[\psi](x)=\sum_{|\alpha|=0}^{N}U^*\left[\left(-ih\frac{\partial}{\partial p}\right)^\alpha(a_\alpha\psi_N)\right]. \tag{1.42}$$

Since $\psi_N\in H_N(\mathbf{R}^{2n})$ and a_α, as well as their derivatives, are uniformly bounded, it follows that

$$||U^*[\psi]||_0\le \text{const}||\psi_N||_N\le\text{const}||\psi||_{-N}. \tag{1.43}$$

A slight modification of this argument permits one to estimate $||U^*[\psi]||_k$ for any k.

3. Let us summarize the preceding in a somewhat different form. Let $M=\mathbf{R}^{2n}_{q,p}$ be the $2n$-dimensional space equipped with the standard Lebesgue measure $dp\,dq$. In $L^2(\mathbf{R}^n_x)$ consider the system of vectors

$$e_{(q,p)}(x)=2^{-n/2}(\pi h)^{-3n/4}G_h(q,p;x) \tag{1.44}$$

The system (1.44) is *complete* in $L^2(\mathbf{R}^n_x)$ in the sense that

$$(f,f)_{L^2(\mathbf{R}^n_x)}=\int|(f,e_{(q,p)})|^2dq\,dp. \tag{1.45}$$

This is just the Parseval identity (1.12); note that the transform $U[f]$ in these terms is given by

$$U[f](q,p) = (f, e_{(q,p)}) \tag{1.46}$$

and defines an isometric embedding

$$L^2(\mathbf{R}^n_x) \to L^2(\mathbf{R}^{2n}_{q,p}),$$
$$f \mapsto U[f].$$

We have

$$f = \int (f, e_{(q,p)}) e_{(q,p)} dq\, dp \tag{1.47}$$

(the integral is understood in the weak sense); this is just the inversion formula (1.15).

Furthermore, there is an orthogonal projection

$$P = UU^* \ : \ L^2(\mathbf{R}^{2n}_{q,p}) \to L^2(\mathbf{R}^n_x)$$

(identified with its image $\mathcal{F}^2(\mathbf{R}^{2n}_{q,p}) = U(L^2(\mathbf{R}^n_x))$, and the operator P can be extended to a wider set including distributions that belong to $H_{-\infty,R}(\mathbf{R}^{2n})$ for some R. In particular, this set includes the delta functions (more precisely, the functions

$$\varphi_{q_0,p_0} = h^{n/2}\delta(q-q_0)\delta(p-p_0), \tag{1.48}$$

where $\delta(y)$ is the Dirac delta function).

Thus, we are in the situation of the papers [4, 15], which permits us to consider operators with co- and contravariant symbols (or Wick and anti-Wick symbols); this will be used in the next section.

2 Quantization of observables

In this section we use the wave packet transform to study h^{-1}-pseudodifferential operators in the scale $\{H_k(\mathbf{R}^n_x)\}$; as a by-product, we obtain some more properties of wave packet transforms.

We use the symbol class $S^\infty(\mathbf{R}^{2n}_{q,p})$ consisting of smooth functions $H(q,p,h)$, $h \in [0,1]$, such that

$$\left| \frac{\partial^{|\alpha|+|\beta|+k} H(q,p)}{\partial q^\alpha \partial p^\beta \partial h^k} \right| \le C_{\alpha\beta k}(1+|q|+|p|)^m, \ \ |\alpha|, |\beta| = 0,1,2,\ldots, \tag{2.1}$$

where m is independent of k, α and β (but depends on H). For the detailed definition of functions of operators, we refer the reader to [2] and the textbook [16].

1. The idea of quantization of observables, that is, of constructing the correspondence "symbols $\to$ operators", is to use the conjugation of the symbol by U. Since $U^{-1} \neq U^*$ (recall that $U^{-1} = U^*|_{\mathcal{F}_2(\mathbf{R}^{2n})}$), there are two different candidates:

$$H(q,p) \quad \mapsto \quad \hat{H} = U^{-1} \circ H(q,p) \circ U, \tag{2.2}$$

$$H(q,p) \quad \mapsto \quad \hat{H} = U^* \circ H(q,p) \circ U, \tag{2.3}$$

where $H(q,p)$ on the right-hand side stands for the multiplication by $H(q,p)$ in both cases. After a brief study, we see that (2.2) must be rejected, since the multiplication by $H(q,p)$ need not preserve the set of solutions to (1.16) unless $H(q,p)$ is an analytic function of $q - ip$, and hence the subsequent application of U^{-1} is merely undefined. So we shall use (2.3), but first let us note that although (2.2) is meaningless "as is," the idea itself is not so absurduous. Namely, from the commutation relations (1.21), (1.22), which mean that U is an intertwining operator for the representations

$$\left(x, -ih\frac{\partial}{\partial x}\right) \quad \text{and} \quad \left(q + ih\frac{\partial}{\partial p}, -ih\frac{\partial}{\partial q}\right)$$

of the Heisenberg algebra, we can obviously derive the following theorem.

Theorem 6 *For any symbol $H(q,p) \in S^\infty(\mathbf{R}^{2n})$ one has*

$$U^{-1} H\left(\overset{2}{\overline{q + ih\frac{\partial}{\partial p}}}, -\overset{1}{ih\frac{\partial}{\partial q}}\right) U = H\left(\overset{2}{x}, -\overset{1}{ih\frac{\partial}{\partial x}}\right). \tag{2.4}$$

Remark 5 Note that the left-hand side of (2.4) is well defined. Indeed,

$$\left[h\frac{\partial}{\partial q} - ih\frac{\partial}{\partial p} - ip, q + ih\frac{\partial}{\partial p}\right] = \left[h\frac{\partial}{\partial q} - ih\frac{\partial}{\partial p} - ip, -ih\frac{\partial}{\partial q}\right] = 0,$$

whence it follows that the operator

$$H\left(\overset{2}{\overline{q + ih\frac{\partial}{\partial p}}}, -\overset{1}{\overline{ih\frac{\partial}{\partial q}}}\right)$$

preserves the set of solutions to (1.16).

Let us now leave this topic and return to formula (2.3).

Theorem 7 *The operator*

$$\hat{H} = U^* \circ H(q,p) \circ U \tag{2.5}$$

is the operator with anti-Wick symbol $H(q,p)$.

The operator with anti-Wick symbol $H(x,p)$ is a special case of the general construction of operators with $\mathcal{A}$-symbols suggested in [17]. Namely, let

$$\mathcal{A} = \begin{pmatrix} A & {}^tB \\ B & C \end{pmatrix} \tag{2.6}$$

be a symmetric $2n \times 2n$ matrix with $n \times n$ blocks $A = {}^tA$, ${}^tB$, B, and $C = {}^tC$ and with nonpositive imaginary part (in [17], only real matrices $\mathcal{A}$ were considered). Then the $1/h$-pseudodifferential operator

$$H_{\mathcal{A}} = H_{\mathcal{A}}(x, -i\frac{\partial}{\partial x})$$

with $\mathcal{A}$-symbol $H(q,p)$ is defined by the formula

$$\begin{aligned} H_{\mathcal{A}}\left(x, -ih\frac{\partial}{\partial x}\right) u(x)(2\pi h)^{-n} \int [F_{q\to\xi}F^{-1}_{p\to y}H](\xi, y) \exp\left[\frac{i}{h}\xi x\right] \\ u(x-y)\exp\left\{\frac{-i}{2h}\left\langle \begin{pmatrix} y \\ \xi \end{pmatrix}, \mathcal{A}\begin{pmatrix} y \\ \xi \end{pmatrix}\right\rangle\right\} dy\, d\xi, \end{aligned} \tag{2.7}$$

where $< \cdot, \cdot >$ is the inner product in $\mathbf{R}^{2n}$,

$$[F_{q\to\xi}f](\xi) = \left(\frac{i}{2\pi h}\right)^{n/2} \int e^{-i\xi q/h} f(q)\, dq \tag{2.8}$$

is the $1/h$-Fourier transform, and $F^{-1}_{p\to y}$ is the inverse transform. In particular, for $\mathcal{A} = 0$ we obtain the quantization $\overset{2}{x}, -\, ih\overset{1}{\partial}/\partial x$:

$$H_{\mathcal{A}}(x, -ih\frac{\partial}{\partial x}) = H\left(\overset{2}{x}, -\, ih\frac{\overset{1}{\partial}}{\partial x}\right),$$

the case

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{1}{2}E \\ \frac{1}{2}E & 0 \end{pmatrix} \tag{2.9}$$

corresponds to the Weyl quantization: and for

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} -iE & E \\ E & -iE \end{pmatrix} \tag{2.10}$$

we obtain the anti-Wick quantization:

$$H_{\mathcal{A}}(x, -ih\frac{\partial}{\partial x}) = \hat{H}$$

is the operator with anti-Wick symbol $H(q,p)$). The last assertion can be proved by straightforward computation: one first substitutes the explicit expressions for U^* and U into (2.5) and then reduces the resultant integral to (2.7) with the matrix $\mathcal{A}$ given by (2.10). Using formula (2.7), it is easy to prove the following theorem:

Theorem 8 *The operator $\hat{H}$ given by (2.5) has a $\left(\overset{2}{x}, -ih\overset{1}{\partial}/\partial x\right)$-symbol. More precisely,*

$$\hat{H} = \tilde{H}\left(\overset{2}{x}, -ih\frac{\overset{1}{\partial}}{\partial x}, h\right), \tag{2.11}$$

where

$$\tilde{H}(q,p,h) = \exp\left\{\frac{h}{4}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right) - \frac{ih}{2}\frac{\partial^2}{\partial q\,\partial p}\right\} H(q,p). \tag{2.12}$$

Here

$$\frac{\partial^2}{\partial q\,\partial p} = \sum_{j=1}^{n} \frac{\partial^2}{\partial q_j\,\partial p_j}, \quad \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} = \sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2} + \frac{\partial^2}{\partial p_j^2}; \tag{2.13}$$

note that the exponential is well defined since both operators in (2.13) are self-adjoint and nonpositive with respect to the L^2 inner product.

It is easy to obtain the expansion in powers of h of the symbol $\tilde{H}(q,p,h)$:

$$\tilde{H}(q,p,h) = H(q,p) + \frac{h}{4}\{H_{qq}(q,p) + H_{pp}(q,p) - 2iH_{qp}\} + \dots. \tag{2.14}$$

We see that in the leading term $\tilde{H}$ coincides with H and that the supports of $\tilde{H}$ and H are the same if for some N we neglect functions that are $O(h^N)$.

Formally, we can rewrite (2.12) as

$$H(q,p) = \exp\left\{-\frac{h}{4}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right) + \frac{ih}{2}\frac{\partial^2}{\partial q\,\partial p}\right\} \tilde{H}(q,p,h). \tag{2.15}$$

In other words, to reconstruct the anti-Wick symbol H from the usual symbol $\tilde{H}$, we have to solve the reverse heat equation, which is impossible if, say, $\tilde{H}$ is not real-analytic. However, if we neglect symbols that are $O(h^N)$, we can find an anti-Wick symbol that gives an operator close to the pseudodifferential operator with a given usual symbol. To this end, one must expand the exponential in (2.15) in the Taylor series and retain finitely many terms.

The representation (2.5) combined with Theorem 8 permits one to prove boundedness theorems for pseudodifferential operators easily; however, we do not dwell on this topic.

Using (2.12), (2.15) and the usual composition formula for pseudodifferential operators, we arrive at the following theorem.

Theorem 9 *Let $\hat{H}$ and $\hat{G}$ be the operators with anti-Wick symbols $H(q,p)$ and $G(q,p)$, respectively. Then the product $\hat{H} \circ \hat{G}$ has the form*

$$\hat{H} \circ \hat{G} = \hat{W} + O(h^\infty), \tag{2.16}$$

where the operator W has the anti-Wick symbol $W(q,p)$ with the following asymptotic expansion as $h \to 0$:

$$W(q,p) \cong \sum_{|\alpha|=0}^{\infty} \frac{(2h)^\alpha}{\alpha!} \frac{\partial^\alpha H}{\partial z^\alpha} \frac{\partial^\alpha G}{\partial \bar{z}^\alpha}, \tag{2.17}$$

where $z = q - ip$, $\bar{z} = q + ip$, and, accordingly,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p}\right). \tag{2.18}$$

2. Now we shall apply this definition of $1/h$-pseudodifferential operators to study the behavior under U of *fronts of oscillations*. Let us recall this, well-known in semi-classical theory, notion.

We start from the definition of the support of oscillations.

Definition 3 Let $\psi \in H_\infty(\mathbf{R}^n)$. We say that a point $x_0 \in \mathbf{R}^n$ belongs to the *oscillation support* of ψ,

$$x_0 \in \operatorname{osc\,supp} \psi,$$

if for any function $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ independent of h and satisfying

$$\varphi\psi = O(h^\infty)$$

we have $\varphi(x_0) = 0$.

By replacing $O(h^\infty)$ with $O(h^{k+1})$, we obtain the definition of $\mathrm{osc}_k\mathrm{supp}(\psi)$. Obviously, osc supp ψ, as well as each $\mathrm{osc}_k\mathrm{supp}(\psi)$, is a closed subset of $\mathbf{R}^n$.

Definition 4 Let $\psi \in H_\infty(\mathbf{R}^n)$. We say that a point (q_0, p_0) of the phase space $\mathbf{R}^n_q \oplus \mathbf{R}^n_p$ belongs to the *oscillation front* of ψ,

$$(q_0, p_0) \in OF(\psi),$$

if for any symbol $H(q,p) \in C_0^\infty(\mathbf{R}^{2n})$ independent of h and satisfying

$$H\left(\overset{2}{x}, -ih\overset{1}{\partial}/\partial x\right)\psi = O(h^\infty)$$

we have $H(q_0, p_0) = 0$. The definition of $OF_k(\psi)$ is obtained by replacing $O(h^\infty)$ with $O(h^{k+1})$. Obviously, $OF(\psi)$ and all $OF_k(\psi)$ are closed.

The sets $OF(\psi)$ and osc supp ψ satisfy properties closely resembling those of $WF(\psi)$ and sing supp ψ (see [18]). Some of those properties are collected in the following theorem.

Theorem 10 *Let* $\psi \in H_\infty(\mathbf{R}^n)$. *Then*

(i) osc supp $\psi = \pi(OF(\psi))$,

where $\pi : \mathbf{R}^{2n}_{q,p} \to \mathbf{R}^n_x$, $(q,p) \mapsto x = q$, *is the natural projection.*

(ii) osc supp $\hat{H}\psi \subset$ osc supp ψ *and* $OF(\hat{H}\psi) \subset OF(\psi)$

for any pseudodifferential operator $\hat{H}$.

(iii) *If* $H(q,p) = 0$ *in a neighborhood of some point* (q_0,p_0), *then* $(q_0,p_0) \notin OF(\hat{H}\psi)$.

(iv) *Let*

$$\psi(x,h) = e^{\frac{i}{h}S(x)}\varphi(x), \tag{2.19}$$

where $S(x)$ *and* $\varphi(x)$ *are smooth functions and* $\operatorname{Im} S \geq 0$. *Then*

$$OF(\psi) = \overline{\left\{(q,p) \mid \varphi(q) \neq 0, \operatorname{Im} S(q) = 0, \text{ and } p = \frac{\partial S(q)}{\partial q}\right\}}. \tag{2.20}$$

Corollary 1 *The oscillation front of the Gaussian wave packet* $G_h(q,p;x)$, *considered as a function of* x, *has the form*

$$OF(G_h(q,p,\cdot)) = \{(q,p)\}, \tag{2.21}$$

that is, consists of the single point (q,p).

Theorem 11 *For any $\psi \in H(\mathbf{R}^n_x)$ one has*

$$OF[\psi] = \operatorname{osc\,supp}[U[\psi]]. \tag{2.22}$$

This assertion readily follows from the fact that in the leading term the application of a pseudodifferential operator amounts to the multiplication of the wave packet transform by the principal symbol.

3 Quantization of states

Informally, quantization of states is a procedure that assigns ψ-functions (or classes of ψ-functions) on the configuration space to "objects" in (or on) the phase space $\mathbf{R}^{2n}_{q,p}$. In a sense, the simplest quantization rule is delivered by the Gaussian wave packets themselves: to each point $(p,q) \in \mathbf{R}^{2n}_{q,p}$ we assign the Gaussian wave packet

$$\psi(x,h) = G_h(p,q,x)$$

that has the oscillation front $OF[\psi]$ consisting of that very point. If we intend to obtain ψ-functions with oscillation fronts that do not amount to a single point but are some more general closed subsets (say, manifolds) of the phase space, then one of the possible approaches is to integrate the Gaussian packets with respect to the parameters (p,q) with some density. Naively, this density would be supported on the desired oscillation front; however, we shall see that this is not always the case.

The integration can be interpreted twofold: we apply either U^{-1} or U^* to the density. More precisely, we set either

$$\psi = U^*[f], \tag{3.1}$$

where $f \in H_{-\infty}(\mathbf{R}^{2n})$ and is compactly supported[4], or

$$\psi = U^{-1}[\tilde{f}], \tag{3.2}$$

where $\tilde{f} \in \mathcal{F}_2(\mathbf{R}^{2n}_{q,p})$. (Note that we can always pass from (3.1) to (3.2) by setting $\tilde{f} = Pf$, but each description has its own geometric and analytical advantages).

1. First, we briefly discuss formula (3.1), which can be reduced to a construction well-known in literature. Our exposition mainly follows [1].

Suppose that a submanifold $\Lambda \subset (\mathbf{R}^{2n}_{q,p})$ is given, and we intend to construct functions $\psi \in H_\infty(\mathbf{R}^n)$ with $OF[\psi] \subset \Lambda$. To this end, we apply the transform U^* to functions of the form

$$f(x,p) = (\pi h)^{n/4} e^{\frac{i}{h}S} \varphi \delta_{(\Lambda,d\sigma)}, \tag{3.3}$$

[4] The last requirement can of course be weakened.

where S and φ are smooth functions on Λ, φ is compactly supported, S is real-valued, and $\delta_{(\Lambda,d\sigma)}$ is the delta function on Λ corresponding to a smooth measure $d\sigma$:

$$< \delta_{(\Lambda,d\sigma)}, \chi >= \int_\Lambda \chi\, d\sigma, \quad \chi \in C_0^\infty(\mathbf{R}^{2n}_{q,p}). \tag{3.4}$$

We have introduced the factor $e^{\frac{i}{h}S}$ in (3.3) for the following reason. Integration over Λ may cancel out the oscillations, and we shall choose S so as to exclude this possibility. On substituting (3.3) into (3.1) we obtain

$$\psi(x) \stackrel{\text{def}}{=} K_{(\Lambda,d\sigma)}\varphi = \left(\frac{1}{2\pi h}\right)^{n/2} \int_\Lambda e^{\frac{i}{h}\Phi(x,\alpha)}\varphi(\alpha)\, d\sigma(\alpha), \tag{3.5}$$

where

$$\Phi(x,\alpha) = S(\alpha) + (x - q(\alpha))p(\alpha) + \frac{i}{2}(x - q(\alpha))^2 \tag{3.6}$$

and $\alpha \mapsto (q(\alpha), p(\alpha))$ is the embedding $\Lambda \subset \mathbf{R}^{2n}_{q,p}$. Let $H(q,p)$ be a compactly supported symbol. Then

$$\begin{aligned} &H\left(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}\right)\psi(x) \\ = \;& \left(\frac{1}{2\pi h}\right)^{3n/2} \int_{\Lambda\times\mathbf{R}^{2n}_{q,p}} e^{\frac{i}{h}\{p(x-p)+\Phi(q,\alpha)\}} H(x,p)\varphi(\alpha) d\sigma(\alpha)\, dq\, dp. \end{aligned} \tag{3.7}$$

Obviously, the function (3.7) is $O(h^\infty)$ if the phase function

$$\psi(x,q,p,\alpha) = p(x-q) + (q - q(\alpha))p(\alpha) + S(\alpha) + \frac{i}{2}(q - q(\alpha))^2 \tag{3.8}$$

has no stationary points on the support of the integrand.

The stationary point equations read

$$\begin{cases} \dfrac{\partial\psi}{\partial p} = x - q = 0 \\ \dfrac{\partial\psi}{\partial q} = p(\alpha) - p + i(q - q(\alpha)) = 0 \\ \dfrac{\partial\psi}{\partial\alpha} = (q - q(\alpha))\dfrac{\partial p(\alpha)}{\partial\alpha} - p(\alpha)\dfrac{\partial q(\alpha)}{\partial\alpha} + \dfrac{\partial S(\alpha)}{\partial\alpha} + i(q(\alpha) - q)\dfrac{\partial q(\alpha)}{\partial\alpha}, \end{cases} \tag{3.9}$$

whence we obtain

$$\begin{cases} q(\alpha) = q = x, \ \ p = p(\alpha), \\ dS(\alpha) = p(\alpha)\, dx(\alpha) \end{cases} \tag{3.10}$$

From (3.7) we see that the validity of these equations for some point (x, p) is necessary and sufficient for this point to belong to $OF(\psi)$ (provided $\varphi(\alpha) \neq 0$).

If we require that $OF(\psi) = \Lambda$ (more precisely, $OF(\psi) = \operatorname{supp}\varphi$), we must require that

$$p\, dx = dS \tag{3.11}$$

on Λ, that is, Λ is a Lagrangian manifold. In this case, formula (3.5) defines the *Maslov canonical operator* [7] on Λ in the form[5] considered by Karasev [19], which itself is a paraphrase of the construction suggested by Cordoba and Fefferman [13] for Fourier integral operators.

2. Let us now study formula (3.2). In this case,

$$\tilde{f} = U[\psi], \tag{3.12}$$

and so an appropriate method is to start from the desired function ψ and try to see what $\tilde{f}$ must be.

We are primarily interested in the *semiclassical wave functions* of the form

$$\psi(x) = e^{i/h\, S(x)} \varphi(x) \tag{3.13}$$

or Fourier transforms of such functions.

Consider the wave packet transform of the function (3.13):

$$U[\psi](q,p) = 2^{-\frac{n}{2}} (\pi h)^{-\frac{3n}{4}} \int\limits_{\mathbf{R}^n_x} \exp \frac{i}{h} \left\{ S(x) + (q - x)p + \frac{i}{2}(q - x)^2 \right\} \varphi(x)\, dx. \tag{3.14}$$

The first obvious property of the function (3.14) is that it satisfies equations (1.16). Furthermore, we can obtain the asymptotic expansion of $U[\psi](q, p)$ in powers of h by using the version [8] of the stationary phase method with complex-valued phase function.

To this end, let us write out the equations of stationary points of the phase function

$$\Phi(x, q, p) = S(x) + (q - x)p + \frac{i}{2}(q - x)^2 \tag{3.15}$$

[5] For lack of space, our considerations are purely local and we do not even touch any issues pertaining to quantization conditions on Λ.

of the integral (3.14). They read

$$\frac{\partial \Phi}{\partial x} \equiv \frac{\partial S}{\partial x} - p + i(x-q) = 0. \tag{3.16}$$

We are interested in *real* stationary points, i.e., points at which the phase function (3.15) is real. Then we have

$$x = q, \;\; p = \frac{\partial S}{\partial x}(q). \tag{3.17}$$

Thus, the integral (3.14) has a real stationary point $x = x(q,p)$ if and only if

$$p = \frac{\partial S}{\partial x}(q), \tag{3.18}$$

that is, the point (p,q) lies on the Lagrangian manifold Λ_S generated by S. In this case,

$$x(q,p) = q. \tag{3.19}$$

This stationary point is nondegenerate. Indeed,

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 S}{\partial x^2} + iE \tag{3.20}$$

is a nondegenerate matrix since $\partial^2 S/\partial x^2$ is real symmetric[6]result:

$$U[\psi](q,p) = O(h^\infty) \tag{3.21}$$

outside a neighborhood of Λ_S, whereas in the vicinity of Λ_S for any $N > 0$ we have the asymptotic expansion

$$u[\psi](q,p) = h^{-n/4} e^{\frac{i}{h}\Phi(q,p)} \sum_{k=0}^{N=1} h^k a_k(q,p) + O(h^N), \tag{3.22}$$

where $a_k(q,p)$ are smooth functions independent of h and

$$\Phi(q,p) =^\infty \Phi(x(q,p),q,p) \tag{3.23}$$

is the almost *analytic continuation* of $\Phi(x,q,p)$ to the almost-solution $x(q,p)$ of equation (3.16) (see details in [8]). The phase function $\Phi(q,p)$ has the following properties:

$$\operatorname{Im}\Phi(q,p) \geq 0; \;\; \operatorname{Im}\Phi(q,p) \geq 0 \Leftrightarrow (q,p) \in \Lambda_S. \tag{3.24}$$

[6]Experienced reader will see that being appropriately modified, this arguments remains valid for a complex-valued phase function $S(x)$ with nonnegative imaginary part.

Now consider the semiclassical wave function of the form

$$\psi(x) = \left(\frac{1}{2\pi h}\right)^{n/2} \int e^{\frac{i}{h}(\xi x + \tilde{S}(\xi))} \varphi(\xi)\, d\xi. \tag{3.25}$$

Then

$$U[\psi](q,p) = 2^{-n}(\pi h)^{-\frac{5n}{4}} \int\limits_{\mathbf{R}^{2n}_{x,\xi}} \exp\frac{i}{h}\left[\tilde{S}(\xi) + \xi x + (q-x)p + \frac{i}{2}(q-x)^2\right] \varphi(\xi)\, d\xi\, dx. \tag{3.26}$$

The real stationary points of the phase function

$$\Phi(\xi, x, q, p) = \tilde{S}(\xi) + \xi x + (q-x)p + \frac{i}{2}(q-x)^2 \tag{3.27}$$

are given by the equations

$$\begin{cases} q = x, \\ \dfrac{\partial\Phi}{\partial\xi} = \dfrac{\partial\tilde{S}(\xi)}{\partial\xi} + x = 0, \\ \dfrac{\partial\Phi}{\partial x} = \xi - p + i(x-q) = 0, \end{cases} \tag{3.28}$$

whence it follows that

$$\xi = p,\ x = p,\ q = -\frac{\partial\tilde{S}}{\partial\xi}(p). \tag{3.29}$$

We see that the point (p,q) lies on the Lagrangian manifold

$$\Lambda_S = \left\{ (q,p) \,\middle|\, q + \frac{\partial\tilde{S}}{\partial\xi}(p) = 0 \right\}. \tag{3.30}$$

The stationary point (3.29) is nondegenerate. Indeed,

$$\begin{aligned} \det \mathrm{Hess}\Phi &= \det\begin{pmatrix} \frac{\partial^2\tilde{S}}{\partial\xi^2} & E \\ E & iE \end{pmatrix} = \det\begin{pmatrix} \frac{\partial^2\tilde{S}}{\partial\xi^2} + iE & 0 \\ E & iE \end{pmatrix} \\ &= i^n \det\left(\frac{\partial^2\tilde{S}}{\partial\xi^2} + iE\right) \neq 0. \end{aligned} \tag{3.31}$$

After some calculations, we see that $u[\psi]$ satisfies the same conditions (3.21)—(3.24).

3. Thus, we arrive at considering the following class of functions f to be used in the formula $\psi = U^{-1}[f]$.

Definition 5 Let $\Phi(q,p)$ be a smooth function on $\mathbf{R}^{2n}_{q,p}$,

$$\Phi(q,p) = \Phi_1(q,p) + i\Phi_2(q,p),$$

such that $\Phi_2(q,p) \geq 0$, and let Γ be the set of zeros of $\Phi_2(q,p)$. By $I(\Phi)$ we denote the class of functions $f(q,p,h)$, $(q,p) \in \mathbf{R}^{2n}$, $h \in (0,1]$, that satisfy the following conditions:

(a) $f \in H_\infty(\mathbf{R}^{2n}_{p,q}) \cap \mathcal{F}_2(\mathbf{R}^{2n}_{p,q})$;

(b) for any integer $N > 0$ one has the asymptotic expansion

$$f(q,p) = e^{\frac{i}{h}\Phi(x,p)} \sum_{k=0}^{N-1} h^k a_k(q,p) + O(h^N), \tag{3.32}$$

where $a_k(q,p)$, $k = 1,2,\ldots$, are smooth functions independent of h and rapidly decaying at infinity. Furthermore, we set

$$C_h^\infty(\Phi) = U^{-1}[I(\Phi)]. \tag{3.33}$$

Let us study the class $I(\Phi)$ in some detail.

Lemma 1 *Let $f \in I(\Phi)$, and let $a_k(q,p)$ be the corresponding functions occurring in* (3.32). *Then*

(a) osc-supp $f = WF(U^{-1}[f]) = \overline{\bigcup_{k=0}^{\infty} \operatorname{supp} a_k \bigcap \Gamma}$.

(b) *The functions $\Phi(x,p)$ and $a_k(q,p)$, $k = 0,1,2,\ldots$, satisfy the following system of equations in the interior of the support of a_0:*

$$i\frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial p} - ip = O(\Phi_2^\infty), \tag{3.34}$$

$$\frac{\partial a_k}{\partial q} - i\frac{\partial a_k}{\partial p} = O(\Phi_2^\infty). \tag{3.35}$$

Proof. (a) is obvious. To prove (b), note that substituting the asymptotic expansion (3.35) into (1.16), we obtain the equation

$$e^{\frac{i}{h}\Phi(x,p)} \left\{ \left[i\frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial p} - ip \right] a_0 + \sum_{k=1}^{N-1} h^k \left[\left(i\frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial p} - ip \right) a_k + \right.\right.$$
$$\left.\left. + i\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p} \right) a_{k-1} \right] + ih^N \left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p} \right) a_{N-1} \right\} = O(h^N). \tag{3.36}$$

By [2], Lemma 4.1 (page 470 of the English translation), Eq. (3.36) implies that

$$\left(i\frac{\partial \Phi}{\partial q}+\frac{\partial \Phi}{\partial p}-ip\right)a_0=O(\Phi_2^N), \tag{3.37}$$

$$\left(i\frac{\partial \Phi}{\partial q}+\frac{\partial \Phi}{\partial p}-ip\right)a_k+i\left(\frac{\partial}{\partial q}-i\frac{\partial}{\partial p}\right)a_{k-1}=O(\Phi_2^{N-k}), \tag{3.38}$$

Since N is arbitrary, the assertion of the lemma follows.

At this stage, it might seem that our construction provides functions $\psi=U^{-1}[f]$ with arbitrary closed oscillation fronts Γ. But this is not the case, as shown by the following remarkable theorem.

Theorem 12 *Let $f\in I(\Phi)$ have the asymptotic expansion* (3.35), *and let $(q_0,p_0)\in\Gamma$. Suppose, furthermore, that $a_0(q_0,p_0)\neq 0$ and Γ is a submanifold in a neighborhood of (q_0,p_0). Then Γ is* isotropic *in a neighborhood of (q_0,p_0), that is,*

$$dp\wedge dq|_{\Gamma}=0. \tag{3.39}$$

Proof. Since Φ_2 is nonnegative everywhere and $\Phi_2|_\Gamma=0$, we have

$$\frac{\partial \Phi_2}{\partial q}=0,\ \ \frac{\partial \Phi_2}{\partial p}=0 \tag{3.40}$$

on Γ. Since $a_0(q_0,p_0)\neq 0$, Eq. (3.34) is valid on Γ in a neighborhood of (q_0,p_0). Let us separate the real and the imaginary parts in Eq. (3.34):

$$\frac{\partial \Phi_1}{\partial p}-\frac{\partial \Phi_2}{\partial q}=O(\Phi_2^\infty),\ \ \frac{\partial \Phi_1}{\partial q}+\frac{\partial \Phi_2}{\partial p}-p=O(\Phi_2^\infty). \tag{3.41}$$

In view of (3.40), (3.41), on Γ we have

$$\frac{\partial \Phi_1}{\partial p}=0,\ \ \frac{\partial \Phi_1}{\partial q}=p. \tag{3.42}$$

Differentiating (3.42) yields the following equations on the *tangent space of* Γ:

$$\frac{\partial^2 \Phi_1}{\partial p\,\partial q}dq+\frac{\partial^2 \Phi_1}{\partial p\,\partial p}=0,\ \ dp=\frac{\partial^2 \Phi_1}{\partial q\,\partial p}dp+\frac{\partial^2 \Phi_1}{\partial q\,\partial q}dq. \tag{3.43}$$

Let us multiply the second equation by dq:

$$\begin{aligned} dp\wedge dq &= \left(\frac{\partial^2 \Phi_1}{\partial q\,\partial p}dp\right)\wedge dq+\left(\frac{\partial^2 \Phi_1}{\partial q\,\partial q}dq\right)\wedge dq \\ &= dp\wedge\left(\frac{\partial^2 \Phi_1}{\partial p\,\partial q}dq\right)+\left(\frac{\partial^2 \Phi_1}{\partial q\,\partial q}dq\right)\wedge dq \\ &= -dp\wedge\left(\frac{\partial^2 \Phi_1}{\partial p\,\partial p}dp\right)+\left(\frac{\partial^2 \Phi_1}{\partial q\,\partial q}dq\right)\wedge dq, \end{aligned} \tag{3.44}$$

where we have used the first equation in (3.43). Since the matrices

$$\frac{\partial^2 \Phi_1}{\partial p\, \partial p} \text{ and } \frac{\partial^2 \Phi_1}{\partial q\, \partial q}$$

are symmetric, we obtain $dp \wedge dq = 0$, as desired.

If $\dim \Gamma = n$, then Γ is Lagrangian and the elements of $C_h^\infty(\Phi)$ correspond to the canonical operator on Γ. If, however, $\dim \Gamma < n$, then elements of C_h^∞ correspond to the canonical operator on the isotropic manifold Γ with Lagrangian complex germ. In fact, the interpretation of elements of $C_h^\infty(\Phi)$ as the functions represented by the canonical operator corresponding to a general complex germ (e.g., see [20] and references therein) remains valid in the case of general set Γ. However, here we do not touch this subject any more; the corresponding study will be carried out elsewhere.

4 Quantization of symplectic transforms

In this section we shall show that the quantization of some symplectic transform

$$g : T^*\mathbf{R}^n \to T^*\mathbf{R}^n \tag{4.1}$$

is essentially the conjugation with the help of the U-transform of canonical change of variables (4.1). More exactly, the following affirmation is valid.

Theorem 13 *The operator*

$$T_g = U^* e^{\frac{i}{h}S(q,p)} g^* U,$$

or, in another form,

$$f(x) \mapsto U^* \left\{ \left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{h}S(q,p)} U\, f[g(q,p)] \right\} (x), \tag{4.2}$$

where the function $S(q,p)$ is determined by the relation

$$dS = p\, dq - g^*(\xi\, dy),$$

is the Fourier integral operator $T(g,1)$ [21] with symbol 1 corresponding to the symplectic transform (4.1).

Remark 6 Similarly, the operator

$$f \mapsto U^{-1}\left\{\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{h}S(q,p)}\varphi(q,p)U\, f[g(q,p)]\right\}(x)$$

coincides with the Fourier integral operator $T(g,\varphi)$.

Proof of Theorem 13. Let the functions

$$y = y(q,p),\ \ \xi = \xi(q,p)$$

determine the symplectic transform (4.1). We write down operator (4.2) in the integral form using the definitions of the transforms U and U^*:

$$U^*\left[\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{h}S(x,p)}U\, f[g(y,q)]\right](x) = \frac{i^{n/2}}{(2\pi h)^{3n/2}}\int G_{(x',p')}(x)e^{\frac{i}{h}S(x',p')}$$

$$\left\{\int \overline{G_{(y,q)}(y')}f(y')\,dy'\right\}\Bigg|_{y=y(x',p'),\,q=q(x',p')} dx'dp'.$$

Using formula (1.9), one can rewrite the latter formula in the form

$$\begin{aligned} T(g,1)f &= \frac{(-i)^{n/2}}{(2\pi h)^{3n/2}}\int \exp\left\{\frac{i}{h}\left[S(x',p') + p'(x-x') + \frac{i}{2}(x-x')^2\right.\right. \\ &- \left.\left. q(x',p')(y'-y(x',p')) + \frac{i}{2}(y'-y(x',p'))^2\right]\right\} f(y')dy'dx'dp' \\ &= \left(-\frac{i}{2\pi h}\right)^{n/2}\int K(x,y')f(y')dy', \end{aligned}$$

where the kernel $K(x,y')$ is given by

$$\begin{aligned} K(x,y') &= \left(\frac{1}{2\pi h}\right)^{n}\int \exp\left\{\frac{i}{h}\left[S(x',p') + p'(x-x') + \frac{i}{2}(x-x')^2\right.\right. \\ &- \left.\left. q(x',p')(y'-y(x',p')) + \frac{i}{2}(y'-y(x',p'))^2\right]\right\} dx'dp'. \end{aligned}$$

The latter expression exactly coincides with the expression for the canonically represented function

$$K(x,y') = K_{(\Lambda_g,d\sigma)}(1)$$

on the Lagrangian manifold $\Lambda_g = \operatorname{graph} g$ with the measure $d\sigma = (dp \wedge dx)^{\wedge n}$, written in the coordinates (x', p') of the manifold Λ_g. This follows from the fact that the nonsingular action S on the Lagrangian manifold Λ_g is determined by the formula

$$S = \int (p\,dq - \xi\,dy)|_{\Lambda_g} = \int p\,dq - g^*(\xi\,dy).$$

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