On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain

by

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I. Introduction.

Let Ω be a bounded regular and connected open set in $\mathbb{R}^N$ with $N \geq 3$. We are looking for a map u from Ω into $\mathbb{R}$ such that

$$(1) \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

We shall denote by $H_d(\Omega; \mathbb{Z}_2)$ the homology of dimension d of Ω with $\mathbb{Z}_2$-coefficients.

Our main result is the following

Theorem 1

If there exists a positive integer d such that

$H_d(\Omega; \mathbb{Z}_2) \neq 0$, then (1) has a solution.

Note that if $N = 3$ and Ω is not contractible then $H_1(\Omega; \mathbb{Z}_2)$ or $H_2(\Omega; \mathbb{Z}_2)$ is not trivial. Thus Theorem 1 implies:

Corollary 2.

If $N = 3$ and Ω is not contractible then (1) has a solution.

Remarks 3

a. Trudinger [24] has proved that any $H^1(\Omega)$-solution of (1) is in $L^\infty(\Omega)$ (and therefore in $C^\infty(\Omega)$) .

b. Pohozaev [15] has proved that if Ω is starshaped then (1) has no solution.

c. Kazdan-Warner [9] have pointed out that if Ω is an annulus then (1) has a solution.

d. It has been proved in [8] that if Ω has a "small hole" (see [8] for the precise statement) then (1) has a solution.

e. Corollary 2 has been announced in [4] with a sketch of a proof.

We start the proof of Theorem 1 by recalling some well known facts.

II. Well known facts

1) The Palais-Smale condition.

We first introduce some notations. Let, for u in $H_0^1(\Omega)$, $\| u \| = (\int |\nabla u|^2)^{1/2}$ where the integration is on Ω .

Let

$$\Sigma = \{u \in H_0^1(\Omega) \mid \| u \| = 1\}$$

$$\Sigma_+ = \{u \in \Sigma \mid u \geq 0\}$$

$$p = \frac{N+2}{N-2}$$

$$J(u) = \frac{1}{\int |u|^{p+1}} \quad \text{for } u \text{ in } \Sigma .$$

If u is a critical point of J in Σ_+, then $J(u)^{\frac{p-1}{2}} u$ is a solution of (1). Σ_+ is invariant by the flow associated to $-J'$. J does not satisfy the Palais-Smale condition on Σ_+ but the sequences which violate the Palais-Smale condition are known. In order to describe them, let us introduce some notations. Let, for a in $\mathbb{R}^N$ and λ in $(0,\infty)$, $\delta(a,\lambda)$ be the function from $\mathbb{R}^N$ into $(0,\infty)$ defined by

$$(\delta(a,\lambda))(x) = c_0 \left(\frac{\lambda}{1+\lambda^2 |x-a|^2}\right)^{\frac{N-2}{2}} , \tag{2}$$

where c_0 is such that $\int_{\mathbb{R}^N} |\nabla \delta(a,\lambda)|^2 = 1$ (c_0 is independent of a and λ).

For $\varepsilon > 0$ and n in $\mathbb{N}^*$ we denote by $V(n,\varepsilon)$ the set of functions u in Σ such that:

$\exists\ (a_1, a_2, \ldots, a_n) \in \Omega^n$, $\exists\ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0,\infty)^n$ such that

$$\left\| u - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} P\delta(a_i, \lambda_i) \right\| < \varepsilon \tag{3}$$

(4) $\lambda_i \, d(a_i,\partial\Omega) > \varepsilon^{-1}$ $\quad\forall i$

(5) $\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j \, |a_i-a_j|^2 > \varepsilon^{-1}$ $\quad\forall\ (i,j)$ with $i \neq j$,

where P is the projection on $H_0^1(\Omega)$ (i.e. $P\varphi = \varphi - h$ with $\Delta h = 0$ in Ω and $h = \varphi$ on $\partial\Omega$) and $d(a_i,\partial\Omega)$ is the distance from a_i to $\partial\Omega$. Let

$$S = \frac{1}{\int_{\mathbb{R}^N} \delta(a,\lambda)^{p+1}} .$$

S does not depend on a and λ . It is known, see [6], that

$$\operatorname*{Inf}_{u\in\Sigma} J(u) = S$$

and that this infimum is not achieved. Let $b_n = n^{\frac{p-1}{2}} S$.

We shall prove Theorem 1 by contradiction and so we shall assume throughout the whole paper that (1) has no solution.

<u>Proposition 4</u>

Let u_k be a sequence in Σ_+ such that $J'(u_k) \longrightarrow 0$ and $J(u_k)$ is bounded ; then there exists a positive integer n and a sequence (ε_k) with $\varepsilon_k > 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$ such that, for a subsequence of the u_k , $u_k \in V(n,\varepsilon_k)$.

Conversely, let n be a positive integer let (ε_k) be a sequence in $(0,\infty)$ with $\lim_{k\to\infty} \varepsilon_k = 0$ and let (u_k) be a sequence in Σ_+ such that $u_k \in V(n,\varepsilon_k)$ then $J'(u_k) \longrightarrow 0$ and $J(u_k) \longrightarrow b_n$.

The pioneers for this kind of conclusion are Sacks-Uhlenbeck [16] and Wente [27]; [16] deals with harmonic maps and [27] with H-systems. Improvements have been obtained by Meeks-Yau [13] and Siu-Yau [19] for harmonic maps and by [6] for H-systems. A similar description has been obtained, using Uhlenbeck [25] [26], by Taubes [21] [22] for the Yang-Mills and the Yang-Mills-Higgs equations (see also Donaldson [10] and Sedlacek [18]). Struwe [20] has obtained a result which is very close to Proposition 4. The conditions (3) and (4) appear for the first time in [6]. Lions [11] is also related to Proposition 4.

In order to prove Proposition 4 one can introduce the functional

$$E(u) = \frac{1}{2}\int|\nabla u|^2 - \frac{N-2}{2N}\int(u^+)^{\frac{2N}{N-2}} , \; u \in H_0^1(\Omega) .$$

Note that if $u_k \in \Sigma$ then $J'(u_k) \longrightarrow 0$ if and only if $E'(J(u_k)^{\frac{p-1}{2}} u_k) \longrightarrow 0$. To get Proposition 4 one can now follow [6] step by step with the functional E . Note that the proof of [6] is inspired by the method of concentration compactness due to Lions [11].

For c in $(0,\infty)$ let $J_+^c = \{u \in \Sigma_+ | \; J(u) \leq c\}$. It follows from Proposition 4 that if c_1, c_2 are two real numbers such that $b_n < c_1 \leq c_2 \leq b_{n+1}$ for some integer n , then $J_+^{c_1}$ is a strong deformation retract of $J_+^{c_2}$. In the following we set $W_n = J_+^{b_{n+1}}$.

Remarks 5.

a. Note that, if $J(u_k) \longrightarrow S$, then $J'(u_k) \longrightarrow 0$. Using this fact (or Lions [11] as in [8]) and Proposition 4 one can easily see that Ω is homeomorphic to a retract of W_1. This has been noticed and used in [8] (we also use it here - see (27)). It explains why the topology of Ω can play a role in the existence of a solution to (1). It has been conjectured in [8] that, if Ω is not contractible, then (1) has a solution. Corollary 2 solves this conjecture when $N = 3$ and Theorem 1 gives a partial answer when $N \geq 4$.

b. Bahri [2] [3] has studied the orbits in $V(n,\varepsilon)$ and has described the "critical points at infinity", i.e, the orbits of $-J'$ which stay in $V(n,\varepsilon)$. Their description involves Green's function and its regular part, which indicates that the geometry of the domain should be also important for the existence of a solution to (1), and leads to the formula for the topology of W_n/W_{n-1} given in [4]. Even if we do not need it, it has helped us to find the topological argument described in section III as one can see by looking at our sketch of proof in [4]. In that sketch we use the formula for the topology of W_n/W_{n-1}; it makes the topological argument more transparent. A similar method (i.e. to find the critical points at infinity and try to prove, in the absence of a solution that there is a topological contradiction) has been used in [1].

We continue section II with a classical deformation argument (see e.g. [14])

2. A classical deformation argument.

In this sub-section n is a positive integer which is fixed. Let θ and $\bar{\varepsilon}$ be two strictly positive real numbers; first θ will be fixed large enough and then $\bar{\varepsilon}$ is fixed small enough. Let μ be a function in $C^\infty([0,\infty[;\mathbb{R}^+)$ such that

$$(6)\qquad \begin{cases} \mu(0) = \bar{\varepsilon} \\ -\frac{2}{\theta} \leq \mu' \leq 0 \\ \mu(r) = 0 \qquad \text{for } r \text{ in } [\theta\bar{\varepsilon},+\infty[\; . \end{cases}$$

Let now $F : \Sigma_+ \longrightarrow \mathbb{R}$ be defined by

$$F(u) = J(u) - \mu(\| J'(u) \|^2) \quad \text{for} \quad J(u) \leq (n+\tfrac{1}{2})^{\frac{p-1}{2}} S$$

$$F(u) = J(u) \quad \text{elsewhere} \; .$$

F is C^1 (if $\theta\bar{\varepsilon}$ is small enough - use Proposition 4); Let

$$K(u) = \| J'(u) \|^2 \; .$$

An easy computation shows that there exists a constant M such that

$$(7)\quad |K'(v).J'(v)| \leq M \, \| J'(v) \|^2 \quad \forall\, v \in \Sigma^+ \text{ with } J(v) \leq b_{n+1} \; .$$

We now fix $\theta > 2M$. It follows from (6) and (7) that

(8) $F'(v) . J'(v) > 0 \qquad \forall\, v \in \Sigma^+$ with $J(v) \leq b_{n+1}$.

Let $$F_+^c = \{u \in \Sigma_+ \mid F(u) \leqq c\} .$$

We have (if $\bar{\varepsilon}$ is small enough, see below):

Proposition 6

The pair $(F_+^{b_{n-1}}, W_{n-1})$ is a strong deformation retract of the pair (W_n, W_{n-1}) .

Proof of Proposition 6

Let $f : [0,\infty[\times \Sigma \longrightarrow \Sigma$ be the solution of

$$(9) \qquad \begin{cases} \frac{\partial}{\partial t} f(t,u) = - J'(f(t,u)) \\ f(0,u) = u \end{cases}$$

(In [14], f is defined by $\frac{\partial f}{\partial t} = - F'(f)$, $f(0,u) = u$; it is possible to prove that even if F is not $C^{1,1}$ this equation has a unique solution and that Σ_+ is also stable by such an f - at least if θ^{-1} and $\theta\bar{\varepsilon}$ are small enough - ; but defining f by (9) we avoid these difficulties since J is C^2 and clearly, if f is defined by (9), $f([0,+\infty[\times \Sigma_+) \subset \Sigma_+$; this modification has been suggested to us by Jing)

Using Proposition 4 we have

$$\{t \geq 0 \mid F(f(t,u)) \leq b_{n-1}\} \neq \phi \qquad \forall\, u \in W_n .$$

Let, for u in W_n,

$$T(u) = \text{Min}\,\{t \geq 0 \mid F(f(t,u)) \leq b_{n-1}\} .$$

It follows from (8) and (9) that T is continuous. Moreover since $W_{n-1} \subset F_+^{b_{n-1}}$, we have

$$(10) \qquad T(u) = 0 \qquad \forall\, u \in W_{n-1} .$$

We now define $\beta : [0,1] \times W_n \longrightarrow W_n$ by

$$\beta(t,u) = f\left(\frac{t}{1-t}, u\right) \qquad \text{if} \quad \frac{t}{1-t} \leq T(u) \quad \text{and} \quad t \neq 1$$

$$\beta(t,u) = f\,(T(u), u) \qquad \text{if} \quad T(u) \leq \frac{t}{1-t}$$

$$\beta(t,u) = f\,(T(u), u) \qquad \text{if} \quad t = 1 .$$

Then β is continuous, $\beta(0,u) = u$ for any u in W_n, $f(1,u) \in F_+^{b_{n-1}}$ for any u in W_n and finally $\beta(t,u) = u$ for any u in $F_+^{b_{n-1}}$. It proves Proposition 6.

In Section III we conclude the proof of Theorem 1. In order not to interrupt the main thread of the topological argument we have placed many of the estimates needed in Appendices.

III. The topological argument

First let us remark that, with the notations of section II and n being fixed, we have, using Proposition 4,

$$(11) \quad \forall\, \varepsilon > 0\ \exists\, \varepsilon_1 > 0 \ \text{ such that } \ 0 < \bar{\varepsilon} < \varepsilon_1 \Rightarrow \overline{F_+^{b_{n-1}} \setminus W_{n-1}} \subset \widehat{\mathring{V}(n,\varepsilon)} .$$

Hence (for $\bar{\varepsilon}$ small enough, ε being given), if we denote by i the inclusion map

$$(F_+^{b_{n-1}} \cap V(n,\varepsilon),\ W_{n-1} \cap V(n,\varepsilon)) \longrightarrow (F_+^{b_{n-1}},\ W_{n-1}),$$ then

(12) $\qquad i_*$ is an isomorphism

We are now going to give a parametrization of $V(n,\varepsilon)$.

Let $\varphi : (0,\infty)^n \times \Omega^n \times (0,\infty)^n \longrightarrow \Sigma_+$ be defined by

$$\varphi(\alpha,x,\lambda) = \Big(\sum_{i=1}^{n} \alpha_i P\delta(x_i,\lambda_i)\Big) / \Big\| \sum_{i=1}^{n} \alpha_i P\delta(x_i,\lambda_i) \Big\|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x = (x_1, \ldots, x_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, and let B_ε be the set of (α,x,λ) in $(0,\infty)^n \times \Omega^n \times (0,\infty)^n$ such that

$$\lambda_i\, d(x_i,\partial\Omega) > \varepsilon^{-1} \qquad \forall i$$

$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j |x_i - x_j|^2 > \varepsilon^{-1} \qquad \forall i, \forall j \text{ with } i \neq j$$

$$\frac{1}{2\sqrt{n}} < \alpha_i < 2 \qquad \forall i .$$

Let:

$$e = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in (0,\infty)^n .$$

Hence

$$V(n,\varepsilon) = \{u \in \Sigma \mid \exists (x,\lambda) \text{ with } (e,x,\lambda) \in B_\varepsilon \text{ such that } \| u - \varphi(e,x,\lambda) \| \leq \varepsilon\} .$$

We have the following Proposition

Proposition 7.

$\forall$ n $\exists \varepsilon_0 > 0$ such that for any u in $V(n,\varepsilon_0)$ the problem

$$\text{Minimize} \quad \| u - \varphi(\alpha,x,\lambda) \| \quad \text{for} \quad (\alpha,x,\lambda) \quad \text{in} \quad B_{4\varepsilon_0}$$

has a unique solution (up to permutations) .

The proof of Proposition 7 is given in Appendix A.

For a function u in $V(n,\varepsilon_0)$ let (α,x,λ) be the unique solution (up to permutations) of the minimization problem in Proposition 7. Let $X : V(n,\varepsilon_0) \longrightarrow \Omega^n/_{\sigma_n}$ be the map defined by $X(u) = x$. Note that since one has uniqueness only up to permutations $X(u)$ is not in Ω^n but in $\Omega^n/_{\sigma_n}$ (as usual σ_n denotes the group of permutations of $\{1, \ldots ,n\}$) .

Let K be a compact set in Ω , and let

$$\Delta_{n-1} = \{(t_1, \ldots ,t_n) \mid t_i \in [0,1] \ \forall i \ \text{and} \ \sum_{i=1}^{n} t_i = 1\}$$

$$B_n(K) = \{\Sigma\, t_i \delta_{x_i} \mid (x_1, \ldots ,x_n) \in K^n, \ (t_1, \ldots ,t_n) \in \Delta_{n-1}\}$$

where δ_{x_i} is the (true) Dirac mass at the point x_i . We provide $B_n(K)$ with the weak topology of measures. $B_n(K)$, with its topology, can also be viewed as the quotient of $K^n \times \Delta_{n-1}$, with its usual topology, by some equivalence relation that we shall denote $\sim$. For example, when $n = 2$, $(x_1,x_1,t_1,t_2) \sim (x_1,x_1,t_1', t_2')$, $(x_1,x_2,t_1,t_2) \sim (x_2,x_1,t_2,t_1)$ and $(x_1,x_2,0,1) \sim (x_1',x_2,0,1)$.

Let $R : H_0^1(\Omega) \setminus \{0\} \longrightarrow \Sigma$

$$Ru = \frac{u}{\| u \|}$$

and let $g_n : K^n \times \Delta_{n-1} \longrightarrow \Sigma_+$ be defined by

$$(13) \quad g_n((x_1, \ldots, x_n), (\alpha_1, \ldots, \alpha_n)) = R(\sum_{i=1}^{n} \alpha_i P\delta(x_i, \lambda))$$

where λ is fixed in $(0,\infty)$ (λ will be taken large). Two elements of $K^n \times \Delta_{n-1}$ which are equivalent for $\sim$ have the same image by $\tilde{g}_n$; hence $\tilde{g}_n$ defines a map $g_n : B_n(K) \longrightarrow \Sigma_+$. It follows from corollary B.3 that if λ is large enough $g_n(B_n(K)) \subset W_n$. Moreover Proposition B.1 tells us that

Proposition 8

There exists a positive integer n_0 and λ_0 in $(0,\infty)$ such that if $\lambda \geq \lambda_0$, $g_{n_0}(B_{n_0}(K)) \subset W_{n_0 - 1}$.

Throughout this section we shall denote by $H_*(\ )$ (resp. $H^*(\ )$) the homology (resp. the cohomology) with $\mathbb{Z}_2$-coefficients. By convention $B_0(K)$ will be the empty set (note that W_0 is also the empty set) and we shall assume that K is a regular manifold (possibly with boundary). Let

$$S_n = \{x \in K^n \mid \exists\ i \in [1,n]\ \exists\ j \in [1,n] \text{ with } x_i \neq x_j \text{ and } i \neq j\}$$

and let T_n be an open neighborhood of S_n in K^n which is invariant by σ_n and such that (in order to construct such a T_n one can proceed as in Appendix C)

(14) $K_0^n = K^n \setminus T_n$ is a manifold (with boundary)

(15) S_n is a strong σ_n-equivariant deformation retract of $\bar{T}_n$,

(15) means that there exists a strong deformation retraction map of $\bar{T}_n$ to S_n which is σ_n-equivariant. Note that σ_n acts on $K^n \times \Delta_{n-1}$, $T_n \times \Delta_{n-1} \cup K^n \times \partial\Delta_{n-1}$ and $S_n \times \Delta_{n-1} \cup K^n \times \partial\Delta_{n-1}$ by

$$\tau((x_1, \ldots, x_n),(\alpha_1, \ldots, \alpha_n)) = ((x_{\tau(1)}, \ldots, x_{\tau(n)}),(\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)}))$$

for $\tau \in \sigma_n$;

we shall denote by $K^n \underset{\sigma_n}{\times} \Delta_{n-1}$, $T_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1}$ and $S_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1}$ the quotient spaces. Note that for any (x,α) in $K^n \times \Delta_{n-1}$ and any τ in σ_n we have $(x,\alpha) \sim \tau(x,\alpha)$ hence there exists a natural projection

$b_n : K^n \underset{\sigma_n}{\times} \Delta_{n-1} \longrightarrow B_n(K)$; b_n maps the pair $(K^n \underset{\sigma_n}{\times} \Delta_{n-1} , S_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1})$ into the pair $(B_n(K) , B_{n-1}(K))$ and so defines a map $b_{n*} : H_*(K^n \underset{\sigma_n}{\times} \Delta_{n-1} , S_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1})$ into $H_*(B_n(K) , B_{n-1}(K))$. Note that

(16) b_{n*} is an isomorphism .

Indeed b_n defines an homeomorphism between $K^n \underset{\sigma_n}{\times} \Delta_{n-1} \setminus (S_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1})$ and $B_n(V) \setminus B_{n-1}(V)$, and $S_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} K^n \times \partial\Delta_{n-1}$ is a strong deformation retract of one of its closed neighborhoods in $K^n \underset{\sigma_n}{\times} \Delta_{n-1}$.

The cap product $H^*(K^n \times_{\sigma_n} \Delta_{n-1}) \otimes H_*(K^n \times \Delta_{n-1} , S_n \times \Delta_{n-1} \cup_{\sigma_n} K^n \times \partial\Delta_{n-1})$ $\longrightarrow H_*(K^n \times \Delta_{n-1} , S_n \times \Delta_{n-1} \cup_{\sigma_n} K^n \times \partial\Delta_{n-1})$ provides $H_*(B_n(K) , B_{n-1}(K))$ with a structure of $H^*(\Omega^n/\sigma_n)$-module via the isomorphism b_{n*} and the homomorphism $a_n^* \; H^*(\Omega^n/\sigma_n) \longrightarrow H^*(K^n \times_{\sigma_n} \Delta_{n-1})$ defined from the map $a_n : K^n \times_{\sigma_n} \Delta_{n-1} \longrightarrow \Omega^n/\sigma_n$ $a_n(x,\alpha) = x$. We shall denote by . the product.

The map g_n defines a map: $(B_n(K) , B_{n-1}(K)) \longrightarrow (W_n , W_{n-1})$ and so a map $g_{n*} \; H_*(B_n(K) , B_{n-1}(K)) \longrightarrow H_*(W_n , W_{n-1})$. Our next proposition is

Proposition 9.

The homology $H_*(W_n , W_{n-1})$ has a natural structure of $H^*(\Omega^n/\sigma_n)$-module and g_{n*} is $H^*(\Omega^n/\sigma_n)$-linear.

Proof of Proposition 9.

The cap product $H^*(F_+^{b_{n-1}} \cap V(n,\varepsilon_0)) \otimes H_*(F_+^{b_{n-1}} \cap V(n,\varepsilon_0), W_{n-1} \cap V(n,\varepsilon_0))$ $\longrightarrow H_*(F_+^{b_{n-1}} \cap V(n,\varepsilon_0), W_{n-1} \cap V(n,\varepsilon_0))$ induces by Proposition 6 and (12) a structure of $H^*(F_+^{b_{n-1}} \cap V(n,\varepsilon_0))$-module on $H_*(W_n, W_{n-1})$. Moreover, using Proposition 7, we have defined a map $X : V(n,\varepsilon_0) \longrightarrow \Omega^n/\sigma_n$. Therefore $H_*(W_n, W_{n-1})$ is also, via the homomorphism $\left(X\big|_{V(n,\varepsilon_0) \,\cap\, F_+^{b_{n-1}}}\right)^*$, a $H^*(\Omega^n/\sigma_n)$-module . We shall denote by . the product.

We are now going to prove that g_{n*} is $H^*(\Omega^n/\sigma_n)$-linear

Let, for η in $(0,1]$,

$$\Delta_{n-1,\eta} = \left\{\left(\eta\left(\alpha_1 - \frac{1}{n}\right) + \frac{1}{n}, \ \dots \ ,\eta\left(\alpha_n - \frac{1}{n}\right) + \frac{1}{n}\right) \mid \alpha \in \Delta_{n-1}\right\}$$

$$\partial\Delta_{n-1,\eta} = \left\{\left(\eta\left(\alpha_1 - \frac{1}{n}\right) + \frac{1}{n}, \ \dots \ ,\eta\left(\alpha_n - \frac{1}{n}\right) + \frac{1}{n}\right) \mid \alpha \in \partial\Delta_{n-1}\right\}$$

$$\mathring{\Delta}_{n-1,\eta} = \Delta_{n-1,\eta} \setminus \partial\Delta_{n-1,\eta} \ .$$

Let $\bar{g}_n = g_n \circ b_n$, $d(T_n) = \underset{x\in T_n}{\mathrm{Max}} \ \underset{i\neq j}{\mathrm{Min}} \ |x_i - x_j|$;

it follows from the regularity of K that for any $d > 0$ there exists T_n satisfying (14) and (15) and such that $d(T_n) < d$. Note that if $(\alpha,x) \in \Delta_{n-1} \times K^n$ is such that, if $x_i \neq x_j$ $\forall i \neq j$, then

$$\lim_{\lambda\to+\infty} J\left(R \sum_{i=1}^{n} \alpha_i P\delta(x_i,\lambda)\right) = S \, \frac{\left(\sum_i \alpha_i^2\right)^{\frac{p+1}{2}}}{\left(\sum_i \alpha_i^{p+1}\right)} \ . \tag{17}$$

Therefore, using Lemma B.7 and Lemma B.4, one can choose η in $(0,1)$, d small enough and then λ large enough in such a way that

$$\bar{g}_n\left(K^n \times (\Delta_{n-1} \setminus \mathring{\Delta}_{n-1,\eta}) \underset{\sigma_n}{\cup} \bar{T}_n \times \Delta_{n-1}\right) \subset W_{n-1}$$

$$\bar{g}_n\left(K^n \underset{\sigma_n}{\times} \Delta_{n-1}\right) \subset F_+^{b_{n-1}}$$

$$\bar{g}_n\left(K_0^n \underset{\sigma_n}{\times} \Delta_{n-1,\eta}\right) \subset V(n,\varepsilon_0)$$

where we have chosen T_n satisfying (14) - (15) and such that $d(T_n) \leq d$. Hence the following diagram is commutative

$$\begin{tikzcd}
(K^n \times_{\sigma_n} \Delta_{n-1}, K^n \times (\Delta_{n-1} \setminus \mathring{\Delta}_{n-1,\eta}) \cup_{\sigma_n} \bar{T}_n \times \Delta_{n-1}) \arrow[r, "\bar{g}_n"] & (F_+^{b_{n-1}}, W_{n-1}) \\
(K_0^n \times_{\sigma_n} \Delta_{n-1}, K_0^n \times \partial \Delta_{n-1,\eta} \cup_{\sigma_n} \partial K_0^n \times \Delta_{n-1}) \arrow[r, "\bar{g}_n"] \arrow[u, "\bar{i}"'] \arrow[d] & (F_+^{b_{n-1}} \cap V(n,\varepsilon_0), W_{n-1} \cap V(n,\varepsilon_0)) \arrow[u, "i"'] \arrow[d] \\
K^n \times_{\sigma_n} \Delta_{n-1} \arrow[r, "a_n"] & \Omega^n / \sigma_n
\end{tikzcd} \tag{17}$$

where i and $\bar{i}$ are inclusion maps. Note that i_* and $\bar{i}_*$ are isomorphisms (see (12) and use (14)). Moreover if

$i_1 : (K^n \times_{\sigma_n} \Delta_{n-1}, S_n \times \Delta_{n-1} \cup_{\sigma_n} K^n \times \partial \Delta_{n-1}) \longrightarrow$

$(K^n \times_{\sigma_n} \Delta_{n-1}, K^n \times (\Delta_{n-1} \setminus \mathring{\Delta}_{n-1,\eta}) \cup_{\sigma_n} \bar{T}_n \times \Delta_{n-1})$ is the inclusion map, then it follows from (15) that i_{1*} is an isomorphism; hence Proposition 9 follows from the commutativity of (17).

Since $H_d(\Omega) \neq 0$ it follows from Thom [23] that there exists a d-dimensional compact connected C^∞- manifold without boundary V and a continuous map $h : V \longrightarrow \Omega$ such that if we denote by $[V]$ the class of orientation (mod. 2) of V then $h_*([V]) \neq 0$. Clearly there exists a compact C^∞ manifold with boundary K such that $h(V) \subset K \subset \Omega$. We define $B_n(V)$ as we have defined $B_n(K)$. We define also

$S'_n = \{x \in V^n \mid \exists\, i \in [1,n], \exists\, j \in [1,n]$ such that $x_i = x_j$ and $i \neq j\}$

$h_n : V^n \times_{\sigma_n} \Delta_{n-1} \longrightarrow K^n \times_{\sigma_n} \Delta_{n-1}$, $h_n(x,\alpha) = ((h(x_1), \dots, h(x_n)), \alpha)$

$g'_n : B_n(V) \longrightarrow W_n$, $g'_n(\sum_i \alpha_i \delta_{x_i}) = g_n(\sum_i \alpha_i \delta_{h(x_i)})$

$a'_n : V^n \times_{\sigma_n} \Delta_{n-1} \longrightarrow \Omega^n/\sigma_n$, $a'_n(x,\alpha) = (h(x_1), \dots, h(x_n))$

and finally,

$$b'_n : (V^n \times_{\sigma_n} \Delta_{n-1}, S'_n \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1}) \longrightarrow (B_n(V) , B_{n-1}(V))$$

is the natural projection. As above (see(16))

(18) b'_{n*} is an isomorphism

The cap product:

$$H^*(V^n \times_{\sigma_n} \Delta_{n-1}) \otimes H_*(V^n \times_{\sigma_n} \Delta_{n-1}, S'_n \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1}) \longrightarrow$$

$H_*(V^n \times_{\sigma_n} \Delta_{n-1}, S'_n \times \Delta_{n-1} \cup_{\sigma_n} K^n \times \partial\Delta_{n-1})$ provides $H_*(B_n(V), B_{n-1}(V))$ with a structure of $H^*(V^n \times_{\sigma_n} \Delta_{n-1})$-module via the isomorphism b'_{n*} . We shall denote $*$ this product. This product provides $H_*(B_n(V), B_{n-1}(V))$ with a structure of $H^*(\Omega^n/\sigma_n)$-module via the homomorphism a'^*_n ; we shall denote by . this new product. Note that g'_n maps the pair $(B_n(V), B_{n-1}(V))$ into the pair (W_n, W_{n-1})- we agree on $B_0(V) = \phi$. We have

(19) $g'_{n*} : H_*(B_n(V), B_{n-1}(V)) \longrightarrow H_*(W_n, W_{n-1})$ is $H_*(\Omega^n/\sigma_n)$-linear

Indeed we have the following commutative diagram

$$\begin{tikzcd}
\Omega^n/\sigma_n \arrow[rr, "\text{identity}"] & & \Omega^n/\sigma_n \\
(V^n \times_{\sigma_n} \Delta_{n-1}, S_n' \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1}) \arrow[u, "a_n'"] \arrow[d, "b_n'"'] \arrow[rr, "h_n"] & & (K^n \times_{\sigma_n} \Delta_{n-1}, S_n \times \Delta_{n-1} \cup_{\sigma_n} K^n \times \partial\Delta_{n-1}) \arrow[u, "a_n"] \arrow[d, "b_n"'] \\
(B_n(V), B_{n-1}(V)) \arrow[r, "g'_n"] & (W_n, W_{n-1}) & (B_n(K), B_{n-1}(K)) \arrow[l, "g_n"']
\end{tikzcd}$$

Hence (19) follows from Proposition 9 .

Let T_n' be an open neighborhood of S_n' in V^n σ_n-invariant and such that

(20) $V_0^n = V^n \setminus T_n'$ is a manifold with boundary

(21) S_n' is a strong σ_n-equivariant deformation retract of $\bar{T}_n'$

(see Appendix C for an example of T_n') .

Let $i_n' : (V_0^n \times_{\sigma_n} \Delta_{n-1}, \partial(V_0^n \times_{\sigma_n} \Delta_{n-1})) \longrightarrow (V^n \times_{\sigma_n} \Delta_{n-1}, \bar{T}_n' \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1})$,

$j_n' : (V^n \times_{\sigma_n} \Delta_{n-1}, S_n' \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1}) \longrightarrow (V^n \times_{\sigma_n} \Delta_{n-1}, \bar{T}_n' \times \Delta_{n-1} \cup_{\sigma_n} V^n \times \partial\Delta_{n-1})$ be inclusion maps.

It follows from (20) and (21) that i_{n*}' and j_{n*}' are isomorphism. Let k_n : $H_*(B_n(V), B_{n-1}(V)) \longrightarrow H_*(V_0^n \times_{\sigma_n} \Delta_{n-1}, \partial(V_0^n \times_{\sigma_n} \Delta_{n-1}))$ be defined by

(22) $$k_n = (i_{n*}')^{-1} \; j_{n*}' \; b_{n*}'^{-1} \; ;$$

k_n is an isomorphism.

Note that $V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}$ is a manifold with boundary; let $[V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}, \partial(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1})]$ be the (mod. 2) orientation class of this manifold and let:

$$[B_n(V), B_{n-1}(V)] = k_n^{-1}([V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}, \partial(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1})]) \in H_{nd+n-1}(B_n(V), B_{n-1}(V)) .$$

We are going to prove, by induction on n, that:

$$(23) \qquad g'_{n*}([B_n(V), B_{n-1}(V)]) \neq 0 \qquad \forall\, n \in \mathbb{N}\setminus\{0\}$$

which is in contradiction with Proposition 8. Let $\omega \in H^d(\Omega)$ be such that $< \omega, h_*([V]) > = 1$ and let $\omega_V = h^*(\omega)$. We denote by $\sigma_1 \times \sigma_{n-1}$ the subgroup of σ_n which contains the permutations of $\{1, \ldots, n\}$ which leaves invariant 1. The transfer - we will denote it by tr-defines (see e.g. Bredon [5]) a map from $H^*(\Omega^n/\sigma_1 \times \sigma_{n-1})$ into $H^*(\Omega^n/\sigma_n)$ and a map from $H^*(V^n \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1})$ into $H^*(V^n \underset{\sigma_n}{\times} \Delta_{n-1})$. Let $\pi : \Omega^n/\sigma_1 \times \sigma_{n-1} \longrightarrow \Omega$ be the projection on the first factor of Ω^n and let $p: V^n \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1} \longrightarrow V$ be also the projection on the first factor of $V^n \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1}$. Let us consider the following commutative diagram

$$(24) \qquad \begin{tikzcd}
H_*(B_n(V), B_{n-1}(V)) \arrow[r, "g'_{n*}"] \arrow[d, "\partial"'] & H_*(W_n, W_{n-1}) \arrow[d, "\partial"'] \\
H_{*-1}(B_{n-1}(V), B_{n-2}(V)) \arrow[r, "g'_{(n-1)*}"] & H_{*-1}(W_{n-1}, W_{n-2})
\end{tikzcd}$$

where ∂ are the usual connecting homomorphisms. In Appendix C we prove

(25) $$\partial((\mathrm{tr}\,p^*\omega_V)*[B_n(V),B_{n-1}(V)]) = [B_{n-1}(V),B_{n-2}(V)] \ .$$

Using (19), (24), (25) and the functoriality of the transfer (see [5]) we have

(26) $$\partial((\mathrm{tr}\,\pi^*\,\omega)\ .\ g'_{n*}[B_n(V),B_{n-1}(V)]) = g'_{(n-1)*}([B_{n-1}(V),B_{n-2}(V)]) \ .$$

Let e be the canonical generator of $H_0(V) = H_0(B_1(V),B_0(V))$. Using (19) again we have:

$$g'_{1*}(e) = g_{1*}\ (\omega\ .\ [V]) = \omega\ .\ g_{1*}\ ([V])$$

and, therefore, since $g'_{1*}\ (e) \neq 0$ and $[V] = [B_1(V),B_0(V)]$,

(27) $$g'_{1*}\ ([B_1(V),B_0(V)]) \neq 0 \ ;$$

(23) follows from (26) and (27) by induction on n .

<u>Comments 10</u>.

1. An important point in our proof is the "interaction" between the "particles" (i.e. the functions $P\delta(a,\lambda)$) . This interaction is computed in Appendix B (see in particular Proposition B.5) and it leads to Proposition 8. This interaction phenomena has been used by Siu-Yau [19]. It has been also computed by Taubes [22] for the Yang-Mills-Higgs equations on $\mathbb{R}^3$; it has allowed him to prove that for these equations the functional is a "good Morse function" (see [22] for the definition). This is also the

case for our equation but only in the set $\Sigma_+ \setminus J^c$ with c large (this c depends on Ω ; see [4]). Taubes has also computed in [21] the interaction between two particles for the Yang-Mills equations S^4 ; he has used it to prove the analogue of $J^s \cap \Sigma_+$ (which is not empty for these equations) is connected.

2. It follows from the universal-coefficients formula that $H_d(\Omega;\mathbb{Q}) \neq 0$ implies that $H_d(\Omega;\mathbb{Z}_2) \neq 0$. When d is odd and $H_d(\Omega;\mathbb{Q}) \neq 0$ one can prove the existence of a solution to (1) without using the transfer (see Appendix D).

3. One can find a different presentation of the topological argument in [3].

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Appendix A

In this Appendix we give a proof of Proposition 7. We recall that B_ε is the set of (α,x,λ) in $\mathbb{R}^n\times\Omega^n\times(0,\infty)^n$ such that

$$\lambda_i d(x_i,\partial\Omega) > \varepsilon^{-1} \qquad \forall i \tag{A.1}$$

$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\,|x_i - x_j|^2 > \varepsilon^{-1} \qquad \forall\ i\neq j \tag{A.2}$$

$$\frac{1}{2\sqrt{n}} < \alpha_i < 2 \qquad \forall i\ . \tag{A.3}$$

The symetric group σ_n acts on B_ε . We start with some Lemmas

Lemma A.1

Let (ε_k) be a sequence with $\varepsilon_k > 0$ and $\lim\limits_{k\to+\infty} \varepsilon_k = 0$ and let $\left(\alpha^k,x^k,\lambda^k\right) \in B_{\varepsilon_k}$, $\left(\tilde\alpha^k,\tilde x^k,\tilde\lambda^k\right) \in B_{\varepsilon_k}$ such that

$$\lim_{k\to+\infty} \left\| \varphi\left(\alpha^k,x^k,\lambda^k\right) - \varphi\left(\tilde\alpha^k,\tilde x^k,\tilde\lambda^k\right) \right\| = 0\ . \tag{A.4}$$

Then $\left(\text{modulo permutations on } \left(\tilde\alpha^k,\tilde x^k,\tilde\lambda^k\right)\right)$:

$$\lim_{k\to+\infty} \frac{\lambda_i^k}{\tilde\lambda_i^k} = 1 \qquad \forall\ i \in [1,n] \tag{A.5}$$

$$\lim_{k\to+\infty} \lambda_i^k\,\tilde\lambda_i^k\,|x_i^k - \tilde x_i^k|^2 = 0 \qquad \forall\ i \in [1,n] \tag{A.6}$$

$$\lim_{k\to+\infty} |\alpha_i^k - \tilde\alpha_i^k| = 0 \qquad \forall\ i \in [1,n] \tag{A.7}$$

Proof of Lemma A.1

Let $\bar{\delta}(a,\lambda) = P(\delta(a,\lambda))$. Note that

$$\text{(A.8)} \qquad \lim_{\lambda d(a,\partial\Omega)\to+\infty} \|\bar{\delta}(a,\lambda)\| = 1$$

and that

$$\text{(A.9)} \qquad \lim_{\substack{\lambda d(a,\partial\Omega)\to+\infty \\ \lambda' d(a',\partial\Omega)\to+\infty \\ \frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda}+\lambda\lambda' |a-a'|^2\to+\infty}} \int \nabla\bar{\delta}(a,\lambda)\ \nabla\bar{\delta}(a',\lambda') = 0$$

It follows from (A.8) and (A.9) that there exists c in $\mathbb{R}^+$ such that $\forall\ i \in [1,n]\ \forall k\ \exists\ j$ such that

$$\text{(A.10)} \qquad \frac{\lambda_i^k}{\tilde{\lambda}_j^k} + \frac{\lambda_j^k}{\lambda_i^k} + \lambda_i^k\ \tilde{\lambda}_j^k\ |x_i^k - \tilde{x}_j^k|^2 \leq c\ ,$$

and, clearly, if k is large enough, i and k being given there exists one and only one j which satisfies (A.10) (use the fact that $(\alpha^k, x^k, \lambda^k)$ and $(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k)$ are in B_{ε_k}) . Without loss of generality we may assume that $j = i$. In the following we shall denote by $o(1)$ various sequences which tends to 0 as k goes to ∞ and we shall omit the index k . Using (A.4) and (A.9) we have:

$$\forall\ i \in [1,n] \quad \int |\alpha_i\ \nabla\bar{\delta}_i(x_i,\lambda_i) - \tilde{\alpha}_i\ \nabla\bar{\delta}_i(\tilde{x}_i,\tilde{\lambda}_i)|^2 = o(1)$$

Hence using (A.8) we have (A.7) and also:

$$\forall\ i \in [1,n] \quad \int_{\mathbb{R}^N} |\nabla\delta_i(x_i,y_i) - \nabla\delta_i(\tilde{x}_i,\tilde{y}_i)|^2 = o(1)$$

Let $$\omega(x) = c_0\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}}$$

we have

$$\text{(A.11)} \qquad \int_{\mathbb{R}^N} \left|\nabla\omega - \nabla\delta\left(\lambda_i(x_i-\tilde{x}_i),\frac{\tilde{\lambda}_i}{\lambda_i}\right)\right|^2 = \int_{\mathbb{R}^N} |\nabla\delta_i(x_i,\lambda_i) - \nabla\delta_i(\tilde{x}_i,\tilde{\lambda}_i)|^2 = o(1)$$

and using (A.10) and (A.11) we deduce (A.6) and (A.7).

Our next Lemma is

Lemma A.2

There exists $\varepsilon_0 > 0$ such that for any u in $V(n,\varepsilon)$ with $\varepsilon \leq \varepsilon_0$

$$\operatorname*{Inf}_{(\alpha,x,\lambda)\in B_{4\varepsilon}} \| u - \varphi(\alpha,x,\lambda) \|$$

is achieved in $B_{2\varepsilon}$ and is not achieved in $B_{4\varepsilon}\setminus B_{2\varepsilon}$.

Proof of Lemma A.2

Argue by contradiction and use Lemma A.1. Let us, for example, prove that the infimum can not be achieved in $B_{4\varepsilon_0}\setminus B_{2\varepsilon_0}$ if ε_0 is small enough. If it is not true, there exists a sequence (ε_k) with $\varepsilon_k > 0$ and $\varepsilon_k = o(1)$, there exists a sequence $((x^k,\lambda^k))$ such that (e,x^k,λ^k) is in B_{ε_k} with $e = \left(\frac{1}{\sqrt{n}},\ \dots\ ,\frac{1}{\sqrt{n}}\right) \in [0,1]^n$, there exists a sequence $((\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k))$ with $(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k) \in B_{4\varepsilon_k} \setminus B_{2\varepsilon_k}$ such that

$$\| \varphi(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k) - \varphi(e, x^k, y^k) \| = o(1) .$$

We now use Lemma A.2, we have (modulo permutations):

$$\text{(A.12)} \qquad \frac{\lambda_i^k}{\tilde{\lambda}_i^k} = o(1) + 1 \qquad \forall\ i \in [1,n]$$

$$\text{(A.13)} \qquad |x_i^k - \tilde{x}_i^k|^2\ \lambda_i^k \cdot \tilde{\lambda}_i^k = o(1) \qquad \forall\ i \in [1,n] ,$$

but one easily checks that (A.12), (A.13), $(e, x^k, \lambda^k) \in B_{\varepsilon_k}$ and $(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k) \in B_{4\varepsilon_k} \setminus B_{2\varepsilon_k}$ are not compatible for k large enough.

We are now going to prove Proposition 7. We argue by contradiction: if Proposition 7 is false then, by Lemma B.3, there exists a sequence (ε_k) with $\varepsilon_k > 0$ and $\varepsilon_k = o(1)$, there exists u^k in $V(n, \varepsilon_k)$, $(\alpha^k, x^k, \lambda^k)$ and $(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k)$ in $B_{2\varepsilon_k}$ such that

$$\text{(A.14)} \qquad (\alpha^k, x^k, \lambda^k) \neq (\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k)$$

and if $v^k = u^k - \varphi(\alpha^k, x^k, \lambda^k)$ $\tilde{v}^k = u^k - \varphi(\tilde{\alpha}^k, \tilde{x}^k, \tilde{\lambda}^k)$

$$\text{(A.15)} \qquad 0 = \int \nabla v^k \nabla \delta_i^k = \int \nabla v^k \nabla \frac{\partial \delta_i^k}{\partial \lambda_i^k} \qquad \forall i\ \forall k$$

(A.16) $$0 = \int \nabla v^k \nabla \frac{\partial \delta_i^k}{\partial x_i^k} \quad (\in \mathbb{R}^N) \qquad \forall i \ \forall k$$

(A.17) $$0 = \int \nabla \tilde{v}^k \ \nabla \tilde{\delta}_i^k = \int \nabla \tilde{v}^k \ \nabla \frac{\partial \tilde{\delta}_i^k}{\partial \lambda_i^k} \qquad \forall i \ \forall k$$

(A.18) $$0 = \int \nabla \tilde{v}^k \ \nabla \frac{\partial \delta_i^k}{\delta x_i^k} \quad (\in \mathbb{R}^N) \qquad \forall i \ \forall k$$

where $\delta_i^k = \delta(x_i^k, \lambda_i^k) \qquad \tilde{\delta}_i^k = \delta(\tilde{x}_i^k, \tilde{\lambda}_i^k)$.

As before we shall omit the index k . Using Lemma A.1 we have (modulo permutations)

$$\frac{\lambda_i}{\tilde{\lambda}_i} = 1 + o(1)$$

$$\lambda_i \tilde{\lambda}_i |x_i - \tilde{x}_i|^2 = o(1)$$

$$|\alpha_i - \tilde{\alpha}_i| = o(1) \ .$$

From (A.15) and (A.17) we get

(A.19) $$\sum_j \int (\alpha_j \nabla P \delta_j - \tilde{\alpha}_j \nabla P \tilde{\delta}_j) \nabla \delta_i = \int \nabla \tilde{v} (\nabla \delta_i - \nabla \tilde{\delta}_i) \ .$$

Let $a_i = \tilde{\lambda}_i (x_i - \tilde{x}_i)$, $\eta_i = \frac{\tilde{\lambda}_i}{\lambda_i} - 1$, $\mu_i = \alpha_i - \tilde{\alpha}_i$. Note that $|a_i| = o(1)$, $\eta_i = o(1)$, $\mu_i = o(1)$. In the following c will denote various constant which does not depend on k . It is easy to see that

(A.20) $$|\tilde{\delta}_j(y) - \delta_j(y)| \le c(|\eta_j| + |a_j|) \ \delta_j(y)$$

and since $-\Delta \delta_j \ge 0$ we have

$$(A.21) \qquad |(P\delta_j - P\tilde{\delta}_j)(y)| \leq c(|\eta_j| + |a_j|)\ \delta_j(y)$$

Note that

$$(A.22) \quad \int(\alpha_j\ \nabla P\delta_j - \tilde{\alpha}_j\ \nabla P\tilde{\delta}_j)\ \nabla\delta_i = (\alpha_j - \tilde{\alpha}_j)\ \int \nabla P\delta_j\ \nabla\delta_i + \tilde{\alpha}_j S \int \delta_i^p (P\delta_j - P\tilde{\delta}_j)$$

From (A.19), (A.21) and (A.22) we get (note that $\int|\nabla\tilde{v}|^2 = o(1)$):

$$(A.23) \quad \mu_i + \tilde{\alpha}_i S \int \delta_i^p\ P(\delta_i - \tilde{\delta}_i) = o(1)\ \left(\sum_j\ (|\eta_j| + |a_j| + |\mu_j|)\right) + o(1)\left(\int|\nabla\delta_i - \nabla\tilde{\delta}_i|^2\right)^{\frac{1}{2}} .$$

$$(A.24) \qquad \int \delta_i^p\ P(\delta_i - \tilde{\delta}_i) = \int \delta_i^p\ (\delta_i - \tilde{\delta}_i) - \int \delta_i^p (h_i - \tilde{h}_i) ,$$

but, using the maximum principle and (A.20), we have

$$(A.25) \qquad |h_i - \tilde{h}_i| \leq c(|\eta_i| + |a_i|)\left(\frac{1}{\lambda_i d(x_i, \partial\Omega)}\right)^{N-2} .$$

From (A.24), (A.25) and again (A.20) we get:

$$(A.26) \qquad \int \delta_i^p\ P(\delta_i - \tilde{\delta}_i) = \tau_i + o(1)(|a_i| + |\eta_i|)$$

with

$$\tau_i = \int_{\mathbb{R}^N} \delta_i^p (\delta_i - \tilde{\delta}_i) .$$

we have

$$\tau_i = \int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^{\frac{N+2}{2}} \left\{\left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} - \left(\frac{1+\eta_i}{1+|(1+\eta_i)y+a_i|^2}\right)^{\frac{N-2}{2}}\right\} dy ;$$

but

$$\left(\frac{1+\eta_i}{1+|(1+\eta_i)y+a_i|^2}\right)^{\frac{N-2}{2}} = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} \left\{1 + \frac{N-2}{2}\eta_i - (N-2)\eta_i \frac{|y|^2}{1+|y|^2} - (N-2)\frac{a_i \cdot y}{1+|y|^2}\right.$$

$$\left. + 0\ (|a_i|^2 + |\eta_i|^2)\right\} ,$$

where, as usual, $0(|a_i|^2 + |\eta_i|^2)$ denotes a sequence bounded by $c(|a_i|^2 + |\eta_i|^2)$.

Hence

$$\tau_i = -\frac{N-2}{2}\,\eta_i \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)} \left(1-\frac{2|y|^2}{1+|y|^2}\right) dy + O(|a_i|^2+|\eta_i|^2) \ .$$

But

$$\int_0^{+\infty} \frac{r^{N+1}}{(1+r^2)^{N+1}}\,dr = \frac{1}{2N}\int_0^{\infty}\left(-\frac{1}{(1+r^2)^N}\right)' r^N dr = \frac{1}{2}\int_0^{\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr$$

and therefore

$$\text{(A.27)} \qquad \tau_i = O\,(|a_i|^2 + |\eta_i|^2) \ .$$

From (A.23), (A.26) and (A.27) we deduce

$$\text{(A.28)} \qquad \mu_i = o(1)\,\Big(\sum_j (|\eta_j| + |a_j| + |\mu_j|)\Big) \qquad \forall i \ .$$

Using again (A.15) and (A.17) we have

$$\sum_j \int (\alpha_j \nabla P\delta_j - \widetilde{\alpha}_j \nabla P\widetilde{\delta}_j)\nabla \frac{\partial\delta_i}{\partial\lambda_i} = (\alpha_j-\widetilde{\alpha}_j)\int \nabla P\delta_j \nabla \frac{\partial\delta_i}{\partial\lambda_i} + \widetilde{\alpha}_j \int (\nabla P\delta_j - \nabla P\widetilde{\delta}_j)\nabla\frac{\partial\delta_i}{\partial\lambda_i}$$

and a similar computation as above leads to:

$$\text{(A.29)} \quad 0 = \frac{o(1)}{\lambda_i}\Big(\sum_{j\neq i}(|\eta_j|+|a_j|+|\mu_j|)\Big) + \widetilde{\alpha}_i S \int (P\delta_i - P\widetilde{\delta}_i)\,\frac{\partial\delta_i^p}{\partial\lambda_i} \ .$$

Proceeding still as above one gets

$$\text{(A.30)} \qquad \int (P\delta_i - P\widetilde{\delta}_i)\,\frac{\partial\delta_i^p}{\partial\lambda_i} = \tau_i' + \frac{o(1)}{\lambda_i}(|a_i| + |\eta_i|)$$

with

$$\tau_i' = \int_{\mathbb{R}^N} \frac{\partial\delta_i^p}{\partial\lambda_i}(\delta_i - \widetilde{\delta}_i)$$

$$\tau_i' = \frac{N+2}{2\lambda_i} \int_{\mathbb{R}^N} \left(\frac{1 - |y|^2}{(1+|y|^2)^{\frac{N+2}{2}}} \left(\left(\frac{1}{1+|y|^2} \right)^{\frac{N-2}{2}} - \left(\frac{1+\eta_i}{1+|(1+\eta_i)y+a_i|^2} \right)^{\frac{N-2}{2}} \right) dy$$

$$\text{(A.31)} \quad \tau_i' = -\eta_i \frac{(N-2)(N+2)}{4\lambda_i} \left(\int_{\mathbb{R}^N} \frac{(1-|y|^2)^2}{(1+|y|^2)^{\frac{N}{2}+2}} dy \right) + \frac{1}{\lambda_i^2} O(|a_i|^2+|\eta_i|^2)$$

It follows from (A.29), (A.30) and (A.31) that

$$\text{(A.32)} \qquad \eta_i = o(1) \left(\sum_j (|\eta_j| + |a_j| + |\mu_j|) \right) \qquad \forall i \ .$$

Finally we use (A.16) and (A.18) and get:

$$\sum_j \int (\alpha_j \nabla P\delta_j - \tilde{\alpha}_j \nabla P\tilde{\delta}_j) \nabla \frac{\partial \delta_i}{\partial x_i} = \int \nabla \tilde{v} \left(\nabla \frac{\partial \delta_i}{\partial x_i} - \nabla \frac{\partial \tilde{\delta}_i}{\partial \tilde{x}_i} \right)$$

and similar computations as above lead to

$$\text{(A.33)} \qquad a_i = o(1) \left(\sum_j (|\eta_j| + |a_j| + |\mu_j|) \right) .$$

From (A.28), (A.32) and (A.33) we deduce that, at least for k large enough,

$$\eta_i = 0 \ , \quad a_i = 0 \ , \quad \mu_i = 0 \qquad \forall \ i \in [1,n] \ ;$$

a contradiction with (A.14).

Appendix B

In this section K is a fixed compact in Ω ; for $\alpha = (\alpha_1, \dots ,\alpha_n)$ in Δ_{n-1}, $x = (x_1, \dots ,x_n)$ in K^n , λ in $(0,\infty)$ one defines $\widetilde{\varphi}(\alpha,x,\lambda) = J(R(\sum_{i=1}^{n} P\delta_i(x_i,\lambda)))$.

Let $\psi(\alpha,x,\lambda) = J(\varphi(\alpha,x,\lambda))$ for (α,x,λ) in $\Delta_{n-1}\times K^n\times(0,\infty)^n$ and let $\widetilde{\psi}(\alpha,x,\lambda) = J(\widetilde{\varphi}(\alpha,x,\lambda))$ for (α,x,λ) in $\Delta_{n-1}\times K^n\times(0,\infty)$. In this appendix we are going to give some estimates on $\psi(\alpha,x,\lambda)$ and $\widetilde{\psi}(\alpha,x,\lambda)$. In particular we shall prove

Proposition B.1

There exist a positive integer n_0 and a positive real number λ_0 such that

$$\text{(B.1)} \qquad \lambda \geq \lambda_0 \Rightarrow \widetilde{\psi}(\alpha,x,\lambda) \leq n_0^{\frac{p-1}{2}} S \quad \forall\alpha \in \Delta_{n_0-1} , \forall x \in K^{n_0} .$$

For simplicity we write δ_i for $\delta(x_i,\lambda_i)$. We start with some Lemmas.

Lemma B.2

$$\text{(B.2)} \quad \psi(\alpha,x,\lambda) \leq S^{\frac{p+1}{2}} \left\{ \frac{\int(\sum_{i=1}^{n} \alpha_i\delta_i)^{p+1}}{\int(\sum_{i=1}^{n} \alpha_i P\delta_i)^{p+1}} \right\}^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} \int a_i \delta_i^{p+1} \right)^{\frac{p-1}{2}}$$

$$\forall x \in K^n , \quad \forall\alpha \in \Delta_{n-1} , \quad \forall\lambda \in (0,\infty)^n , \quad \forall\ n \geq 1 ,$$

where

$$a_i = \frac{\alpha_i\delta_i}{\sum_{j=1}^{n} \alpha_j\delta_j}$$

Proof of Lemma B.2

Let: $u = \sum_{i=1}^{n} \alpha_i \, P\delta_i$. We have

$$(\text{B.3}) \qquad J(Ru) = \frac{\left(\int |\nabla u|^2\right)^{\frac{p+1}{2}}}{\int u^{p+1}}$$

For simplicity we shall write $\sum\limits_i$ instead of $\sum\limits_{i=1}^{n}$.

We have

$$\int |\nabla u|^2 = S \int \Big(\sum_i \alpha_i \, P\delta_i \Big) \Big(\sum_i \alpha_i \delta_i^p \Big) ,$$

hence, by Hölder's inequality

$$(\text{B.4}) \quad \int |\nabla u|^2 \le S \left(\int \Big(\sum_i \alpha_i \, P\delta_i \Big)^{p+1} \right)^{\frac{1}{p+1}} \left(\int \Big(\sum_i \alpha_i \delta_i^p \Big)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} .$$

By the convexity of $x \longrightarrow |x|^{\frac{p+1}{p}}$

$$\Big(\sum_i a_i \delta_i^{p-1} \Big)^{\frac{p+1}{p}} \le \sum_i a_i \delta_i^{\frac{p^2-1}{p}} ,$$

and therefore:

$$\Big(\sum_i \alpha_i \delta_i^p \Big)^{\frac{p+1}{p}} \le \Big(\sum_i a_i \delta_i^{\frac{p^2-1}{p}} \Big) \Big(\sum_i \alpha_i \delta_i \Big)^{\frac{p+1}{p}} .$$

Using now Hölder's inequality

$$(\text{B.5}) \quad \int \Big(\sum_i \alpha_i \delta_i^p \Big)^{\frac{p+1}{p}} \le \left\{ \int \Big(\sum_i \alpha_i \delta_i \Big)^{p+1} \right\}^{\frac{1}{p}} \left\{ \int \Big(\sum_i a_i \delta_i^{\frac{p^2-1}{p}} \Big)^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} .$$

By the convexity of $x \longrightarrow |x|^{\frac{p}{p-1}}$ one has

$$\left(\sum_i a_i \delta_i^{\frac{p^2-1}{p}} \right)^{\frac{p}{p-1}} \leq \sum_i a_i \delta_i^{p+1} \ ,$$

and therefore, with (B.5) we have

$$\text{(B.6)} \qquad \int \left(\sum_i \alpha_i \delta_i^p \right)^{\frac{p+1}{p}} \leq \left\{ \int \left(\sum_i \alpha_i \delta_i \right)^{p+1} \right\}^{\frac{1}{p}} \left\{ \sum_i \int a_i \delta_i^{p+1} \right\}^{\frac{p-1}{p}}$$

(B.2) follows from (B.3), (B.4) and (B.6).

We are now going to deduce from Lemma B.2

Corollary B.3

$$\forall\ n > 0 \ ,\ \forall\ \varepsilon > 0 \ ,\ \exists\ \bar{\lambda} > 0 \ \text{ such that}$$

$$\lambda \in (\bar{\lambda}, \infty)^n \Rightarrow \psi(\alpha, x, \lambda) \leq (n+\varepsilon)^{\frac{p-1}{2}} S \ ,\ \forall \alpha \in \Delta_{n-1},\ \forall x \in K^n \ .$$

Proof of Corollary B.3

It follows from Lemma B.2 that for $(\alpha, x, \lambda) \in \Delta_{n-1} \times K^n \times (0,\infty)^n$:

$$\text{(B.7)} \qquad \psi(\alpha, x, \lambda) \leq n^{\frac{p-1}{2}} S \ \frac{\int (\sum_i \alpha_i \delta_i)^{p+1}}{\int (\sum_i \alpha_i P\delta_i)^{p+1}} \ .$$

By the maximum principle we have

$$\text{(B.8)} \qquad 0 \leq \delta_i - P\delta_i \leq \max_{\partial\Omega} \delta_i \leq \frac{c}{\lambda_i^{\frac{N-2}{2}}}$$

where c is a constant (we recall that K is fixed). Corollary B.3 follows from (B.7) and (B.8).

We now prove

Lemma B.4

For any integer n in $[2,\infty)$ there exists a strictly positive real number ε and λ_2 in $(0,\infty)$ such that for any x in K^n for any λ in $[\lambda_2,\infty)^n$ and for any α in Δ_{n-1} :

$$\text{(B.9)} \qquad \exists\, i \text{ with } \alpha_i \leq \varepsilon \Rightarrow \psi(\alpha,x,\lambda) \leq n^{\frac{p-1}{2}} S .$$

Proof of Lemma B.4

Let n be an integer in $[2,\infty)$. For x in K^n and α in Δ_{n-1} with $\alpha_1 \neq 1$ one defines $\tilde{\alpha}$ and $\tilde{x}$ by

$$\tilde{\alpha} = \frac{1}{\left(\sum\limits_{i\geq 2} \alpha_i\right)} (\alpha_2, \ldots ,\alpha_n) \in \Delta_{n-2}$$

$$\tilde{x} = (x_2, \ldots ,x_n) \in K^{n-1} .$$

Let η be in $(0,\infty)$; one easily sees that there exists ε in $(0,\infty)$ such that

$$\text{(B.10)} \quad \forall\alpha \in \Delta_{n-1}\ \forall x \in K^n\ \forall\lambda \in [1,\infty)^n, \alpha_1 \leq \varepsilon \Rightarrow \psi(\alpha,x,\lambda) \leq \eta+\psi(\tilde{\alpha},\tilde{x},\tilde{\lambda}) .$$

Lemma B.4 follows from Corollary B.3 and (B.10).

We are now going to give an expansion of $\widetilde{\Psi}(\alpha,x,\lambda)$ when $\lambda \operatorname*{Min}_{i \neq j} |x_i - x_j|$ is large. Let $H(x,y)$ be the regular part of the Green function, i.e.

$$\text{(B.11)} \quad \begin{cases} \Delta_y H(x,.) = 0 \\ H(x,y) = \dfrac{1}{|x-y|} \quad \text{if} \quad y \in \partial\Omega , \end{cases}$$

and let $G: \Omega \times \Omega \longrightarrow \mathbb{R}$ be the Green function:

$$G(x,y) = \frac{1}{|x-y|} - H(x,y) .$$

Let $d = d(x) = \operatorname*{Min}_{i \neq j} |x_i - x_j|$ and $\psi_1 : \Delta_{n-1} \times K^n \times (0,\infty) \longrightarrow \mathbb{R}$ be defined by

$$\psi_1(\alpha,x,\lambda) = S \frac{|\alpha|^{p+1}}{\|\alpha\|^{p+1}} \Big\{ 1 - \frac{c_1}{\lambda^{N-2}} \Big[\sum_{i=1}^{n} H(x_i,x_i) \Big(\frac{\alpha_i^2}{|\alpha|^2} - \frac{2\alpha_i^{p+1}}{\|\alpha\|^{p+1}} \Big) + \sum_{\substack{(i,j) \\ i \neq j}} \Big(2 \frac{\alpha_i^p \alpha_j}{\|\alpha\|^{p+1}} - \frac{\alpha_i \alpha_j}{|\alpha|^2} \Big) G(x_i,x_j) \Big] \Big\}$$

with

$$|\alpha| = \Big(\sum_i \alpha_i^2\Big)^{\frac{1}{2}}$$

$$\|\alpha\| = \Big(\sum_i \alpha_i^{p+1} \Big)^{\frac{1}{p+1}}$$

$$c_1 = \frac{p+1}{2} S c_0^{p+1} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} dy$$

(see (2) for the definition of c_0) .

We are going to prove

Proposition B.5

There exists a constant $c(n)$ which depends only n such that

$$| \psi_1(\alpha,x,\lambda) - \widetilde{\psi}(\alpha,x,\lambda) | \leq \frac{c(n)}{(\lambda d(x))^{N-1}}$$

for any α in Δ_{n-1}, any x in K^n with $d(x) > 0$ and any λ in $(1,\infty)$.

Proof of Proposition B.5

Let (α,x,λ) be in $\Delta_{n-1} \times K^n \times (1,\infty)$ and let $\bar{\delta}_i = P\delta_i$, $h_i = \delta_i - \bar{\delta}_i$ and $u = \sum_{i=1}^{n} \alpha_i \bar{\delta}_i$. We start with the estimate of $\int |\nabla u|^2$. We have

$$\text{(B.12)} \qquad \int \nabla\bar{\delta}_i \, \nabla\bar{\delta}_j = \int \nabla\bar{\delta}_i \, \nabla\delta_j = S \int \delta_j^p \, (\delta_i - h_i)$$

$$\text{(B.13)} \qquad \int \delta_i^{p+1} = \int_{\mathbb{R}^N} \delta_i^{p+1} - \int_{\mathbb{R}^N \setminus \Omega} \delta_i^{p+1} = S^{-1} - \int_{\mathbb{R}^N \setminus \Omega} \delta_i^{p+1}$$

Let $\ell = \text{dist}\,(K,\partial\Omega)$ and c be various constants which may depend on n but only on n (we recall that K is fixed) ; $0(a)$ will denote functions such that $|0(a)| \leq c|a|$ .Note that, using Corollary (B.3), we may assume that $\lambda d(x) \geq 1$.

We have $$\int_{\mathbb{R}^N \setminus \Omega} \delta_i^{p+1} \leq c \int_{r \geq \ell} \left(\frac{\lambda}{1+\lambda^2 r^2}\right)^N r^{N-1} \, dr$$

$$\text{(B.14)} \qquad \int_{\mathbb{R}^N \setminus \Omega} \delta_i^{p+1} \leq \frac{c}{\lambda^N} .$$

On $\partial\Omega$

$$h_i(y) = c_0 \left(\frac{\lambda}{1+\lambda^2|y-x_i|^2}\right)^{\frac{N-2}{2}}$$

hence, on $\partial\Omega$

$$\left|h_i(y) - \frac{c_0\lambda^{\frac{2-N}{2}}}{|y-x_i|^{N-2}}\right| \leq \frac{c}{\lambda^{\frac{N+2}{2}}} .$$

Therefore, by the maximum principles,

$$\text{(B.15)} \qquad \left|h_i(y) - \frac{c_0}{\lambda^{\frac{N-2}{2}}} H(y,x_i)\right| \leq \frac{c}{\lambda^{\frac{N+2}{2}}} \quad \forall y \in \Omega .$$

We have $\displaystyle \int \delta_i^p \leq c \int_0^{+\infty} \left(\frac{\lambda}{1+\lambda^2 r^2}\right)^{\frac{N+2}{2}} r^{N-1}\, dr$,

hence

$$\text{(B.16)} \qquad \int \delta_i^p \leq \frac{c}{\lambda^{\frac{N-2}{2}}} .$$

$$\int \delta_i^p H(y,x_i)\,dy = \int_{B(x_i,\frac{\ell}{2})} \delta_i^p H(y,x_i)\,dy + \int_{\Omega\setminus B(x_i,\frac{\ell}{2})} \delta_i^p H(y,x_i)\,dy ,$$

where $B(x_i,\frac{\ell}{2}) = \{y \in \mathbb{R}^N \,|\, |x_i-y| < \frac{\ell}{2}\}$ (and $\delta_i=\delta_i(y)$) ; hence, using (B.16)

$$\int \delta_i^p H(y,x_i)\,dy = \int_{B(x_i,\frac{\ell}{2})} \delta_i^p H(y,x_i)\,dy + 0\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right) .$$

Note that $\Delta_y H(y,x_i) = 0$; therefore making an expansion of $H(y,x_i)$ near $y = x_i$ and, using the symmetries of δ_i^p we have

$$\int_{B(x_i,\frac{\ell}{2})} \delta_i^p \, H(y,x_i)\,dy = H(x_i,x_i) \int_{B(x_i,\frac{\ell}{2})} \delta_i^p + 0\left(\int_0^{\frac{\ell}{2}} \left(\frac{\lambda}{1+\lambda^2 r^2}\right)^{\frac{N+2}{2}} r^{N+3}\,dr\right);$$

hence:

$$\text{(B.17)} \qquad \int_{B(x_i,\frac{\ell}{2})} \delta_i^p \, H(y,x_i)\,dy = c_2 \, \frac{H(x_i,x_i)}{\lambda^{\frac{N-2}{2}}} + 0\left(\frac{1}{\lambda^{\frac{N+2}{2}}}\right)$$

with

$$\text{(B.18)} \qquad c_2 = c_0^p \int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^{\frac{N+2}{2}} dy \ .$$

From (B.15), (B.16) and (B.17) we get:

$$\text{(B.19)} \qquad \int \delta_i^p \, h_i = c_0 c_2 \, \frac{H(x_i,x_i)}{\lambda^{N-2}} + 0\left(\frac{1}{\lambda^N}\right)$$

and finally with (B.12), (B.13), (B.14) and (B.19) we have

$$\text{(B.20)} \qquad \int |\nabla \bar{\delta}_i|^2 = 1 - S\, c_0 c_2 \, \frac{H(x_i,x_i)}{\lambda^{N-2}} + 0\left(\frac{1}{\lambda^N}\right) .$$

Let now $i \neq j$

$$\text{(B.21)} \qquad \int \delta_j^p \, (\delta_i - h_i) = \int_{\mathbb{R}^N} \delta_j^p \delta_i - \int_{\mathbb{R}^N \setminus \Omega} \delta_i^p \delta_j - \int \delta_j^p h_i \ .$$

Similar computations to those which lead to (B.19) give:

$$\text{(B.22)} \qquad \int \delta_j^p h_i = c_0 c_2 \frac{H(x_i,x_j)}{\lambda^{N-2}} + 0\left(\frac{1}{\lambda^N}\right) .$$

We have

$$\int_{\mathbb{R}^N \setminus \Omega} \delta_j^p \delta_i \le \int_{\mathbb{R}^N \setminus \Omega} \left(\delta_i^{p+1} + \delta_j^{p+1}\right) ,$$

hence by (B.14)

$$\text{(B.23)} \qquad \int_{\mathbb{R}^N \setminus \Omega} \delta_j^p \delta_i \le \frac{c}{\lambda^N} .$$

Let $a_{ij} = x_i - x_j$ and $I = \int_{\mathbb{R}^N} \delta_j^p \delta_i$. We have

$$I = c_0^{p+1} \int \left(\frac{1}{1+|y|^2}\right)^{\frac{N+2}{2}} \left(\frac{1}{1+|y-\lambda a_{ij}|^2}\right)^{\frac{N-2}{2}} dy .$$

We have also

$$1+|y-\lambda a_{ij}|^2 = (1+\lambda^2 |a_{ij}|^2) \left\{ 1 + \frac{|y|^2 - 2\lambda y . a_{ij}}{1+\lambda^2|a_{ij}|^2} \right\} ,$$

hence

$$\text{(B.24)} \quad (1+|y-\lambda a_{ij}|^2)^{-\frac{N-2}{2}} = (1+\lambda^2|a_{ij}|^2)^{-\frac{N-2}{2}} \left\{ 1 + \frac{(N-2)\lambda a_{ij} . y}{1+\lambda^2 |a_{ij}|^2} + 0\left(\frac{|y|^2}{1+\lambda^2|a_{ij}|^2}\right)\right\}$$

$$\text{for} \quad |y| \le \frac{1}{4} \lambda |a_{ij}| .$$

Let $A(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N+2}{2}} \left(\frac{1}{1+|y-\lambda a_{ij}|^2}\right)^{\frac{N-2}{2}}$. We have

(B.25) $$\int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} A(y)\,dy = \frac{1}{(1+\lambda^2|a_{ij}|^2)^{\frac{N-2}{2}}}\left\{ \int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} \left(\frac{1}{1+|y|^2}\right)^{\frac{N+2}{2}} dy \right.$$
$$\left. + \frac{1}{(1+\lambda^2|a_{ij}|^2)}\, 0\left(\int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} \frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy\right)\right\},$$

(B.26) $$\int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} \frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy = 0\,(\log \lambda\, |a_{ij}|)\ ,$$

(B.27) $$\int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} \frac{dy}{(1+|y|^2)^{\frac{N+2}{2}}} = \frac{c_2}{c_0^p} + 0\left(\frac{1}{\lambda^2|a_{ij}|^2}\right).$$

From (B.25), (B.26), (B.27) we get

(B.28) $$\int_{|y|\leq \frac{\lambda|a_{ij}|}{4}} A(y)\,dy = \frac{c_2}{c_0^p}\ \frac{1}{\lambda^{N-2}|a_{ij}|^{N-2}} + 0\left(\frac{1}{\lambda^{N-1}|a_{ij}|^{N-1}}\right).$$

Let $B_1 = \left\{ y \in \mathbb{R}^N \mid\ |y-\lambda a_{ij}| \leq \frac{\lambda|a_{ij}|}{4} \right\}$ and

$B_2 = \left\{ y \in \mathbb{R}^N \mid\ |y| \leq \frac{\lambda|a_{ij}|}{4} \right\}$. We have

$$\int_{\mathbb{R}^N \setminus B_1 \cup B_2} A(y)\,dy \leq \frac{c}{\lambda^{N-2}|a_{ij}|^{N-2}} \int_{\lambda|a_{ij}|}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\, dr\ ,$$

$$\text{(B.29)} \qquad \int_{\mathbb{R}^N \setminus B_1 \cup B_2} A(y)\,dy = 0\left(\frac{1}{\lambda^N |a_{ij}|^N}\right),$$

$$\text{(B.30)} \qquad \int_{B_1} A(y)\,dy \leq \frac{c}{\lambda^{N+2}|a_{ij}|^{N+2}} \int_0^{\frac{\lambda|a_{ij}|}{4}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}\,dr$$

$$= 0\left(\frac{1}{\lambda^N |a_{ij}|^N}\right) .$$

From (B.28), (B.29), (B.30) it follows that

$$\text{(B.31)} \qquad \int_{\mathbb{R}^N} A(y)\,dy = \frac{c_2}{c_0^p} \frac{1}{\lambda^{N-2}|a_{ij}|^{N-2}} + 0\left(\frac{1}{\lambda^{N-1}|a_{ij}|^{N-1}}\right)$$

Finally from (B.21), (B.22), (B.23) and (B.31) we get, with $d = d(x)$,

$$\text{(B.32)} \qquad \int \nabla\bar{\delta}_i \nabla\bar{\delta}_j = S\, c_0 c_2\, G(x_i, x_j) \frac{1}{\lambda^{N-2}} + 0\left(\frac{1}{\lambda^{N-1} d^{N-1}}\right) .$$

Using now (B.32) and (B.20) we have

$$\int |\nabla u|^2 = |\alpha|^2 - \frac{2}{p+1} \frac{c_1}{\lambda^{N-2}} \left\{ \sum_i \alpha_i^2 \; H\,(x_i, x_i) - \sum_{\substack{(i,j)\\ i \neq j}} \alpha_i \alpha_j \; G(x_i, x_j) \right\}$$

$$+ 0\left(\frac{1}{\lambda^{N-1} d^{N-1}}\right)$$

and therefore

$$(B.33)\quad \left(\int|\nabla u|^2\right)^{\frac{p+1}{2}} = |\alpha|^{p+1}\left\{1-\frac{c_1}{|\alpha|^2\lambda^{N-2}}\left[\sum_i \alpha_i^2 H(x_i,x_i)-\sum_{\substack{i,j\\ i\neq j}}\alpha_i\alpha_j G(x_i,x_j)\right]\right\} .$$

$$+\ 0\left(\frac{1}{\lambda^{N-1}d^{N-1}}\right) .$$

We are now going to estimate $\int u^{p+1}$. Let

$B_i = \{\ y \mid |x_i-y| < \min\left(\frac{d}{2},\ell\right)\ \}$. We have

$$(B.34)\quad \int_{\Omega\setminus\cup B_i} u^{p+1} \leq c \int_{r>\lambda\min\left(\frac{d}{2},\ell\right)} \left(\frac{\lambda}{1+\lambda^2r^2}\right)^N r^{N-1}dr \leq 0\left(\frac{1}{\lambda^N d^N}\right) .$$

Let $d' = \min\left(\frac{d}{2},\ell\right)$. On B_i we have:

$$(B.35)\quad u^{p+1} = \alpha_i^{p+1}\delta_i^{p+1}+(p+1)\alpha_i^p\delta_i^p\left(\sum_{j\neq i}\alpha_j\bar{\delta}_j-\alpha_i h_i\right) + 0\left(\frac{\lambda^{N-2}}{(\lambda d)^{2(N-2)}}\,\delta_i^{p-1}\right)$$

$$\int_{B_i}\delta_i^{p-1} = \int_0^{d'}\left(\frac{\lambda}{1+\lambda^2r^2}\right)^2 r^{N-1}dr = \frac{1}{\lambda^{N-2}}\int_0^{\lambda d'}\frac{t^{N-1}}{(1+t^2)^2}\,dt$$

and then one easily sees that for any $N \geq 3$

$$(B.36)\quad \frac{1}{(\lambda d)^{2(N-2)}}\int_{B_i}\left(\frac{\lambda}{1+\lambda^2r^2}\right)^2 r^{N-1}\,dr = 0\left(\frac{1}{(\lambda d)^{N-1}}\right) .$$

Using (B.19), (B.22), (B.32), (B.34), (B.35) and (B.36) we have easily

$$(B.37)\quad \int_{B_i}u^{p+1} = \frac{\alpha_i^{p+1}}{S} - (p+1)\alpha_i^{p+1}c_0c_2\frac{H(x_i,x_i)}{\lambda^{N-2}} + \frac{(p+1)}{\lambda^{N-2}}c_0c_2\sum_{j\neq i}\alpha_i^p\alpha_j\,G(x_i,x_j)$$

$$+\ 0\left(\frac{1}{(\lambda d)^{N-1}}\right) .$$

Finally we have from (B.37) and (B.34):

$$(B.38)\quad \int u^{p+1} = \frac{\|\alpha\|^{p+1}}{S} - \frac{2c_1}{\lambda^{N-2}}\left(\sum_i \alpha_i^{p+1} H(x_i,x_i) - \sum_{\substack{i,j\\ i\neq j}} \alpha_i^p \alpha_j\, G(x_i,x_j)\right) + 0\left(\frac{1}{(\lambda d)^{N-1}}\right)$$

Proposition B.5 follows from (B.33) and (B.38).

Note that there exists $c' > 0$ and $\nu' > 0$ such that

$$H(y,y) \leq c' \qquad \forall y \in K$$

$$G(y_1,y_2) \geq \nu' \qquad \forall (y_1,y_2) \in K^2$$

Hence one easily gets from Proposition B.5

Corollary B.6

There exists two positive real numbers $\bar{c}$ and $\bar{\eta}$ such that for any positive integer n there exists a constant $c(n)$ such that for any λ in $[1,\infty)$ and any x in K^n with $d(x) \neq 0$

$$\operatorname*{Max}_{\alpha \in \Delta_{n-1}} \Psi(\alpha,x,\lambda) \leq n^{\frac{p-1}{2}}\left[S + \frac{2}{\lambda^{N-2}}(\bar{c} - n\bar{\eta})\right] + \frac{c(n)}{(\lambda d(x))^{N-1}} .$$

We are now going to prove

Lemma B.7

For any integer n in $[2,\infty)$ and any ε in $(0,\infty)$ there exists d_0 in $(0,\infty)$ and λ_3 in $[1,\infty)$ such that

$$(B.39)\quad \tilde{\psi}(\alpha,x,\lambda) \le n^{\frac{p-1}{2}} S \ \forall\alpha \in \Delta_{n-1}\cap [\varepsilon,1]^n,\ \forall\lambda \in [\lambda_3,+\infty[,$$

$$\forall x \in K^n \quad \text{with} \quad d(x) \le d_0 .$$

Proof of Lemma B.7

Clearly we may assume that

$$|x_1 - x_2| = d(x)$$

Note also that since $\displaystyle\lim_{\substack{|y_1-y_2|\to 0\\ (y_1,y_2)\in K^2}} G(y_1,y_2) = +\infty$, we have

from Proposition B.5:

$\exists\, d_1 > 0 \quad \exists\, C_1 > 0$ such that $\forall x \in K^n \quad \forall\alpha \in \Delta_{n-1} \quad \forall\lambda \in [1,\infty)$

$$(B.40)\quad \tilde{\psi}(\alpha,x,\lambda) \le n^{\frac{p-1}{2}} S \quad \text{if} \quad d(x) \le d_1 \quad \text{and} \quad \lambda|x_1-x_2| \ge C_1 .$$

Using (B.15) and (B.2) one sees that there exists $C_2 > 0$ such that

$$(B.41)\quad \tilde{\psi}(\alpha,x,\lambda) \le S^{\frac{p+1}{2}} \left(1+ \frac{C_2}{\lambda^{\frac{N-2}{2}}}\right) \left(\int_{\mathbb{R}^N} \frac{\delta_1}{\delta_1+\varepsilon\delta_2}\, \delta_2^{p+1} + \frac{n-1}{S}\right)^{\frac{p-1}{2}} ;$$

but there exists τ in $(0,\infty)$ such that

$$(B.42)\quad \int_{\mathbb{R}^N} \frac{\delta_1}{\delta_1+\varepsilon\delta_2}\, \delta_1^{p+1} \le \frac{1}{S}\,(1-\tau) \quad \text{if} \quad \lambda|x_1-x_2| \le C_1$$

(remark that by translation and dilation we may assume that $x_1 = 0$ and $\lambda = 1$)

Lemma B.7 follows from (B.40), (B.41) and (B.42) .

We can now prove Proposition B.1. We first use Corollary B.6 and choose n_0 such that

(B.43) $$\bar{c} - n_0\bar{\eta} < 0$$

We now use Lemma B.4 and then, Lemma B.7: there exists $\varepsilon > 0$, $d_3 > 0$ and $\lambda_4 > 0$ such that:

(B.44) $$\forall x \in K^{n_0}\ \forall \alpha \in \Delta_{n_0-1} \setminus [\varepsilon,1]^{n_0}\ \forall \lambda \in [\lambda_4,+\infty),\tilde{\psi}(\alpha,x,\lambda) \le n_0^{\frac{p-1}{2}}\ S$$

and

(B.45) $$\forall x \in K^{n_0}\ \forall \alpha \in \Delta_{n_0-1} \cap [\varepsilon,1]^{n_0}\ \forall \lambda \in [\lambda_4,+\infty)\ ,$$
$$d(x) \le d_3 \Rightarrow \tilde{\psi}(\alpha,x,\lambda) \le n_0^{\frac{p-1}{2}}\ S\ .$$

We use Corollary B.6 once more and (B.43), there exists λ_5 such that

(B.46) $$\forall x \in K^{n_0}\quad \forall \alpha \in \Delta_{n_0}\quad \forall \lambda \in [\lambda_5,+\infty)$$
$$d(x) \ge d_3 \Rightarrow \tilde{\psi}(\alpha,x,\lambda) \le n_0^{\frac{p-1}{2}}\ S\ .$$

Let now $\lambda_0 = \mathrm{Max}(\lambda_5,\lambda_4)$, using (B.44), (B.45) and (B.46) we have

(B.47) $$\forall x \in K^{n_0}\quad \forall \alpha \in \Delta_{n_0}\quad \forall \lambda \in [\lambda_0,+\infty)\quad \tilde{\psi}(\alpha,x,\lambda) \le n_0^{\frac{p-1}{2}}\ S\ ,$$

hence Proposition B.1.

Comments

1. The regular part H of the Green's function appears in the expression of $\widetilde{\psi}(\alpha,x,\lambda)$; originally it came out of expansions along the gradient flow (see [2] - [3] for further precisions). The role of the regular part of the Green function in connection with the critical Sobolev exponents has been pointed out for the first time by McLeod [12] for a Dirichlet problem and by Schoen [17] in the framework of the Yamabe conjecture. (But the computations in [2] - [3] where made independently of [12] and [17]).

2. More generally one finds the following expansion of ψ :

$$\Bigg|\psi(\alpha,x,\lambda) - S\frac{|\alpha|^{p+1}}{\|\alpha\|^{p+1}}\Bigg\{1-c_1\sum_i \frac{H(x_i,x_i)}{\lambda_i^{N-2}}\left(\frac{\alpha_i^2}{|\alpha|^2}-\frac{2\alpha_i^{p+1}}{\|\alpha\|^{p+1}}\right)$$

$$+\sum_{\substack{(i,j)\\ i\neq j}}\left(\frac{\alpha_i^{p+1}\alpha_j}{\|\alpha\|^{p+1}}-\frac{\alpha_i\alpha_j}{|\alpha|^2}\right)\varepsilon_{ij}\Bigg\}\Bigg| \leq c(n,K)\left(\sum_i\frac{1}{\lambda_i^{N-1}}+\sum_{\substack{(i,j)\\ i\neq j}}\varepsilon_{ij}^{\frac{N-1}{N-2}}\right)$$

for $x \in K^n$ and with $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i\lambda_j}{G^2(x_i,x_j)}\right)^{-\frac{N-2}{2}}$.

Appendix C

This Appendix is due to J. Lannes. We use here the notations of section III and we prove

Proposition C.1

$$\partial((\mathrm{tr}\ p^* \omega_V) * [B_n(V), B_{n-1}(V)]) = [B_{n-1}(V), B_{n-2}(V)] \quad . \tag{C.1}$$

Proof of Proposition C.1

For simplicity we shall write B_n instead of $B_n(V)$. Let ξ be a fixed point in V , and let CB_{n-1} the subset of B_n defined by

$$CB_{n-1} = \{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} \in B_n \setminus B_{n-1} \mid \exists\ i \in [1,n] \text{ such that } x_i = \xi \} \cup B_{n-1} \ .$$

CB_{n-1} is contractible in itself and therefore $H_*(CB_{n-1}, B_{n-1}) \simeq \tilde{H}_{*-1}(B_{n-1})$. Let τ be the natural injection of CB_{n-1} into B_n ; τ maps the pair (CB_{n-1}, B_{n-1}) into the pair (B_n, B_{n-1}) and the following diagram is commutative (γ and ∂' are the usual derivations)

$$\begin{tikzcd}
H_*(CB_{n-1}, B_{n-1}) \arrow[r, "\simeq", "\gamma"'] \arrow[d, "\tau_*"'] & \tilde{H}_{*-1}(B_{n-1}) \arrow[d, "\text{identity}"] \\
H_*(B_n, B_{n-1}) \arrow[r, "\partial'"] & \tilde{H}_{*-1}(B_{n-1})
\end{tikzcd} \quad . \tag{C.2}$$

Let $p_0 : V^n_0 \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1} \longrightarrow V$ be the projection on the first factor.

and $\nu : V^n/\sigma_1 \times \sigma_{n-1} \longrightarrow V$ be also the projection on the first factor. We choose an open neighborhood T'_n of S'_n in V^n, σ_n - invariant satisfying (20)-(21) and such that

(C.3) $\quad \operatorname{Ker} \; (d\nu)(x) \neq T_x(\partial V_0^n) \qquad \forall x \in \partial V_0^n$ with $x_1 = \xi$

(C.4) $\quad S_n^\xi$ is a strong σ_n-equivariant deformation retract of T_n^ξ,

where, in (C.3), $(d\nu)(x)$ denotes the differential of ν at x and $T_x(\partial V_0^n)$ the tangent space of ∂V_0^n at x and where in (C.4)

$$S_n^\xi = \{x \in S'_n \mid \exists \; i \in [1,n] \text{ with } x_i = \xi\}$$

$$T_n^\xi = \{x \in T'_n \mid \exists \; i \in [1,n] \text{ with } x_i = \xi\}$$

We give at the end of this Appendix an example of such a T'_n.

Note that it follows from (C.3) that $p_0^{-1}(\xi)$ is a manifold (with boundary). In Section III we have defined an isomorphism k_n between $H_*(B_n, B_{n-1})$ and $H_*(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}, \partial(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}))$ -see (22). In a similar way we are going to define an isomorphism k_n^ξ between $H_*(CB_{n_1}B_{n-1})$ and $H_*(p_0^{-1}(\xi), \partial(p_0^{-1}(\xi)))$. Let

$$V_\xi^n = \{x \in V^n \mid \exists \; i \in [1,n] \text{ with } x_i = \xi\}$$

and let $\quad b_n^\xi : (V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, S_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \longrightarrow (CB_{n-1}, B_{n-1})$

be the natural projection. As in Section III (see(16)) one easily proves

(C.5) b_n^ξ is an isomorphism.

Let now $j_n^\xi : (V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, S_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \longrightarrow$ $(V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, T_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1})$; it follows from (C.4) that

(C.6) j_{n*}^ξ is an isomorphism.

Let now $i_n^\xi : (p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))) \longrightarrow (V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, V_\xi^n \times \partial\Delta_{n-1} \underset{\sigma_n}{\cup} \overline{T_n^\xi} \times \Delta_{n-1})$ be the restriction of the projection: $V_0^n \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1} \longrightarrow V^n \underset{\sigma_n}{\times} \Delta_{n-1}$; i_n^ξ defines an homeomorphism between $p_0^{-1}(\xi) \setminus \partial(p_0^{-1}(\xi))$ and $V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1} \setminus (V_\xi^n \times \partial\Delta_{n-1} \underset{\sigma_n}{\cup} \overline{T_n^\xi} \times \Delta_{n-1})$; moreover $\partial(p_0^{-1}(\xi))$ is a strong deformation retract of one of its closed neighborhoods in $p_0^{-1}(\xi)$; therefore

(C.7) i_{n*}^ξ is an isomorphism.

We define $$\ell_n = \left(i_{n*}^\xi\right)^{-1} j_{n*}^\xi \left(b_{n*}^\xi\right)^{-1} .$$

We next remark that the following diagram is commutative

(C.8)
$$\begin{tikzcd}
(\dot{B}_n, \partial\dot{B}_n) \arrow[d, "q"'] & (p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))) \arrow[l, "s=q\circ t"'] \arrow[dl, "t"] \\
(V_0^n \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1}, \partial(V_0 \underset{\sigma_1 \times \sigma_{n-1}}{\times} \Delta_{n-1})) \arrow[d, "p_0"'] & \\
V &
\end{tikzcd}$$

where $\dot{B}_n = V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}$, q is the natural projection and t is the inclusion map. We have

$$k_n \circ \tau_* = s_* \circ k_n^\xi \,. \tag{C.9}$$

Indeed (C.9) is a consequence of the commutativity of the following diagrams

$$\begin{tikzcd}
(V^n \underset{\sigma_n}{\times} \Delta_{n-1}, S'_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} V^n \times \partial\Delta_{n-1}) \arrow[r, "b'_n"] & (B_n, B_{n-1}) \\
(V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, S_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \arrow[r, "b_n^\xi"] \arrow[u] & (CB_{n-1}, B_{n-1}) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
(V^n \underset{\sigma_n}{\times} \Delta_{n-1}, S'_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} V^n \times \partial\Delta_{n-1}) \arrow[r, "j'_n"] & (V^n \underset{\sigma_n}{\times} \Delta_{n-1}, T'_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} V^n \times \partial\Delta_{n-1}) \\
(V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, S_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \arrow[r, "j_n^\xi"] \arrow[u] & (V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, T_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1}, \partial(V_0^n \underset{\sigma_n}{\times} \Delta_{n-1})) \arrow[r, "i'_n"] & (V^n \underset{\sigma_n}{\times} \Delta_{n-1}, T'_n \times \Delta_{n-1} \underset{\sigma_n}{\cup} V^n \times \partial\Delta_{n-1}) \\
(p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))) \arrow[r, "i_n^\xi"] \arrow[u] & (V_\xi^n \underset{\sigma_n}{\times} \Delta_{n-1}, T_n^\xi \times \Delta_{n-1} \underset{\sigma_n}{\cup} V_\xi^n \times \partial\Delta_{n-1}) \arrow[u]
\end{tikzcd}$$

where the maps which are not labeled are inclusion maps.

Since B_{n-2} is contractible in B_{n-1}, the map $\theta : \tilde{H}_*(B_{n-1}) \to H_*(B_{n-1}, B_{n-2})$ of the reduced homology sequence of (B_{n-1}, B_{n-2}) is one to one; moreover (see (22)) $H_{(n-1)d+n-2}(B_{n-1}, B_{n-2}) = \mathbb{Z}_2$; hence

$$\theta\gamma(k_n^\xi)^{-1} \;([p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))]) = [B_{n-1}, B_{n-2}] \tag{C.10}$$

where $[p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))]$ is the class of orientation (modulo $\mathbb{Z}_2$) of the manifold with boundary $p_0^{-1}(\xi)$.

We denote by $\cap$ cap products. We are going to prove that

$$(C.11) \qquad s_*([p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))]) = (\mathrm{tr}_0\ p_0^*\ \omega_V) \cap [\dot{B}_n, \partial\dot{B}_n]$$

where tr_0 is the transfer map: $H^*(V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1}) \to H^*(V_0^n \times_{\sigma_n} \Delta_{n-1})$.

Note that (C.1) follows (C.2),(C.9),(C.10),(C.11) and the functioriality of the transfer (see [5]). Since $q\colon V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1} \to V_0^n \times_{\sigma_n} \Delta_{n-1}$ is a covering between two manifolds, tr_0 is the Gysin's homomorphism; hence for any u in $H^*(V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1})$ we have

$$(\mathrm{tr}_0 u) \cap [\dot{B}_n, \partial\dot{B}_n] = q_*(u \cap [V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1}, \partial(V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1})]) \ .$$

In particular

$$(\mathrm{tr}_0 p_0^* \omega_V) \cap [\dot{B}_n, \partial\dot{B}_n] = q_*(p_0^* \omega_V \cap [V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1}, \partial(V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1})]) \ ;$$

but

$$p_0^* \omega_V \cap [V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1}, \partial(V_0^n \times_{\sigma_1 \times \sigma_{n-1}} \Delta_{n-1})] = t_*([p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))])$$

hence

$$(\mathrm{tr}\ p_0^* \omega_V) \cap [\dot{B}_n, \partial\dot{B}_n] = q_* t_*([p_0^{-1}(\xi), \partial(p_0^{-1}(\xi))])$$

which gives (C.11).

Finally we give an example of an open neighborhood T'_n of S'_n in V^n, σ_n-invariant satisfying (20), (21), (C.3) and (C.4). We provide V with a C^∞ Riemannian metric and denote by $d(x_1,x_2)$ the geodesic distance between two points x_1 and x_2 of V. Let $A: V^2 \to \mathbb{R}$ be a C^∞ map such that

$$A(x_1,x_2) = d^2(x_1,x_2) \quad \text{in a neighborhood of } S'_2$$

$$A(x_1,x_2) > 0 \quad \text{if} \quad (x_1,x_2) \in V^2 \setminus S'_2$$

Let ε be in $(0,\infty)$ and let

$$T'_n = \{x \in V^n | \prod_{i \neq j} A(x_i,x_j) < \varepsilon\}$$

T'_n is open, σ_n-invariant and contains S'_n. Moreover one easily verifies that, if ε is small enough, T'_n satisfies (20) (21) (C.3) and (C.4).

Appendix D

In this Appendix we give a proof, which does not need the transfer, of the existence of a solution to (1) when there exists some odd integer d such that $H_d(\Omega;\mathbb{Q}) \neq 0$. We shall consider here only rational homology and cohomology; we shall write $H_*(\ )$, $H^*(\ )$ instead of $H_*(\ ;\mathbb{Q})$, $H^*(\ ;\mathbb{Q})$.

Let K be a compact in Ω ; we have defined in (13) a map $g_n\ K^n \times \Delta_{n-1} \to \Sigma_+$ which depends on some parameter λ in $(0,\infty)$. If λ is large enough g_n maps the pair $(K^n \times \Delta_{n-1}, K^n \times \partial\Delta_{n-1})$ into the pair (W_n, W_{n-1}) and it is clear that $g_{n*} : H_*(K^n \times \Delta_{n-1}, K^n \times \partial\Delta_{n-1}) \longrightarrow H_*(W_n, W_{n-1})$ is independent of the choice of λ provided that λ is large enough. On the other hand the homology of $(\Omega^n \times \Delta_{n-1}, \Omega^n \times \partial\Delta_{n-1})$ is the direct limit of the homology of $(K^n \times \Delta_{n-1}, K^n \times \partial\Delta_{n-1})$ where K are compact sets in Ω ; hence one can define a natural map $\ell_n : H_*(\Omega^n \times \Delta_{n-1}, \Omega^n \times \partial\Delta_{n-1}) \longrightarrow H_*(W_n, W_{n-1})$.

We have

$$H_*(\Omega^n \times \Delta_{n-1}, \Omega^n \times \partial\Delta_{n-1}) = H_*(\Omega^n) \otimes H_*(\Delta_{n-1}, \partial\Delta_{n-1}) \ .$$

Let e_{n-1} be the canonical generator of $H_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1})$. Let $D : H_*(\Omega^n \times \Delta_{n-1}, \Omega^n \times \partial\Delta_{n-1}) \longrightarrow H_{*-1}(\Omega^{n-1} \times \Delta_{n-2}, \Omega^n \times \partial\Delta_{n-2})$ be defined by:

$$D(f \times e_{n-1}) = (-1)^{|f|}\left(\sum_{i=1}^{n} (-1)^{i-1} (p_i)_* f\right) \times e_{n-2}$$

where $p_i : \Omega^n \to \Omega^{n-1}$ is defined by $p_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and where $|f|$ is the degree of f.

Our first Lemma is

<u>Lemma</u> D.1

The following diagram is commutative

$$\begin{tikzcd}
H_*(\Omega^n\times\Delta_{n-1},\Omega^n \times \partial\Delta_{n-1}) \arrow[r, "\ell_n"] \arrow[d, "D"] & H_* (W_n,W_{n-1}) \arrow[d, "\partial"] \\
H_{*-1}(\Omega^{n-1}\times\Delta_{n-2},\Omega^{n-1}\times\partial\Delta_{n-2}) \arrow[r, "\ell_{n-1}"] & H_{*-1}(W_{n-1},W_{n-2})
\end{tikzcd}$$
.

<u>Proof of Lemma</u> D.1

Lemma D.1 is a consequence of the commutativity of the diagram:

$$\begin{tikzcd}
K^n \times \Delta_{n-2} \arrow[r, "Id\times f_i"] \arrow[d, "p_i\times Id"] & K^n \times \Delta_{n-1} \arrow[r, "g_n"] & W_n \\
K^{n-1} \times \Delta_{n-2} \arrow[rr, "g_{n-1}"] & & W_{n-1} \arrow[u]
\end{tikzcd}$$

where $f_i(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.

The cap product $H^*(\Omega^n\times\Delta_{n-1}) \otimes H_*(\Omega^n\times\Delta_{n-1}, \Omega^n\times\partial\Delta_{n-1}) \longrightarrow H_*(\Omega^n\times\Delta_{n-1},\Omega^n\times\partial\Delta_{n-1})$ provides $H_*(\Omega^n\times\Delta_{n-1},\Omega^n\times\partial\Delta_{n-1})$ with a structure of $H^*(\Omega^n)$-module and hence a structure of $H^*(\Omega^n/\sigma_n)$-module via the homomorphism $\pi_*: H^*(\Omega^n/\sigma_n) \longrightarrow H^*(\Omega^n)$ where π

is the projection $\Omega^n \longrightarrow \Omega^n/\sigma_n$. We denote by . the product.

We have seen in Proposition 9 that $H_*(W_n,W_{n-1})$ has also a structure of $H^*(\Omega^n/\sigma_n)$ module. Our next Lemma is

Lemma D.2

The map ℓ_n is $H^*(\Omega^n/\sigma_n)$ -linear.

Proof of Lemma D.2

We give a direct proof of (one could also use Proposition 9). Let K be a compact in Ω and let $\tilde{g}_n : K^n \times \Delta_{n-1} \longrightarrow \Sigma_+$

be defined by

$$\tilde{g}_n(x,\alpha) = R(\sum_{i=1}^{n} \alpha_i \, P\delta(x_i,\lambda^i)) .$$

Note that

$$\text{(D.1)} \qquad \lim_{\lambda\to+\infty} J(\tilde{g}_n(x,\alpha)) = S \frac{\left(\sum_i \alpha_i^2\right)^{\frac{p+1}{2}}}{\sum_i \alpha_i^{p+1}} \qquad \forall (x,\alpha) \in K^n \times \Delta_{n-1}$$

and hence (if λ is large enough) $\tilde{g}_n$ maps the pair $(K^n \times \Delta_{n-1}, K^n \times \partial\Delta_{n-1})$ into the pair (W_n,W_{n-1}) . We prove first that (if λ is large enough)

$$\text{(D.2)} \qquad \tilde{g}_{n*} = g_{n*} .$$

Let $h_n : [0,1] \times K^n \times \Delta_{n-1} \longrightarrow \Sigma_+$ be defined by

$$h_n(t,x,\alpha) = R(\sum_{i=1}^{n} \alpha_i \, P\delta(x_i, t\lambda + (1-t)\lambda^i)) ;$$

h_n is continuous and we have

(D.3) $$h_n(0,x,\alpha) = \tilde{g}_n(x,\alpha) \qquad \forall (x,\alpha) \in K^n \times \Delta_{n-1}$$

(D.4) $$h_n(1,x,\alpha) = g_n(x,\alpha) \qquad \forall (x,\alpha) \in K^n \times \Delta_{n-1}$$

Moreover, using Corollary (B.3), we have (if λ is large enough)

(D.5) $\forall t \in [0,1]$ $h_n(t,\cdot,\cdot)$ maps the pair $(K^n \times \Delta_{n-1}, K^n \times \partial\Delta_{n-1})$ into the pair (W_n, W_{n-1}).

The equality (D.2) follows from (D.3), (D.4) and (D.5).

We next remark (see in particular (D.1)) that there exists η_0 in $(0,\infty)$ and λ_0 in $(0,\infty)$ such that (where Δ_{n-1,η_0} is defined in the proof of Proposition 9)

(D.6) $$\lambda \geq \lambda_0 \Rightarrow \tilde{g}_n(K^n \times \Delta_{n-1,\eta_0}) \subset F_+^{b_{n-1}} \cap V(n,\varepsilon_0) .$$

It is also clear from (D.1) that (η_0 being now fixed) for λ large enough

(D.7) $$\tilde{g}_n\ (K^n \times (\Delta_{n-1} \setminus \mathring{\Delta}_{n-1,\eta_0})) \subset W_{n-1} .$$

Let $b(x,\alpha) = x$ for $(x,\alpha) \in K^n \times \Delta_{n-1}$. Clearly on $K^n \times (\Delta_{n-1} \setminus \mathring{\Delta}_{n-1,\eta})$:

(D.8) $$X \circ \pi = q \circ b$$

It follows from (D.6), (D.7) and (D.8) that the diagram

$$\begin{tikzcd}
(K^n\times\Delta_{n-1,n_0}, K^n\times(\Delta_{n-1}\setminus\mathring{\Delta}_{n-1,n_0})) \arrow[r, "\tilde{g}_n"] & (F_+^{b_{n-1}}, W_{n-1}) \\
(K^n\times\Delta_{n-1,n_0}, K^n\times\partial\Delta_{n-1,n_0}) \arrow[u] \arrow[r, "\tilde{g}_n"] \arrow[d, "b"] & (F_+^{b_{n-1}} \cap V(n,\varepsilon_0), W_{n-1} \cap V(n,\varepsilon_0)) \arrow[u] \arrow[d, "X"'] \\
K^n \arrow[r, "\pi"] & \Omega^n/\sigma_n
\end{tikzcd}$$

is commutative. Lemma D.2 is a consequence of this commutativity and (D.2).

Let now z be in $H_d(\Omega)$ and u be in $H^d(\Omega)$ such that $< u,z > = 1$. We are going to prove by induction on n that if d is odd then

(D.9) $$\ell_n(z^n\times e_{n-1}) \neq 0$$

where $z^n = z\times \dots \times z \in H_{nd}(\Omega^n)$ which is in contradiction with Proposition 8. First note that

(D.10) $$\ell_1(z^1\times e_0) \neq 0 \ .$$

Indeed let v be the canonical generator of $H_0(\Omega)$; we have $\ell_1(v) \neq 0$; by Lemma D.2 $\ell_1(u.(z\times e_0)) = u.\ell_1(z\times e_0)$ and $u.(z\times e_0) = v$ hence (D.10) .

Since the cohomology we consider is with rational coefficients the map $\pi^*: H^*(\Omega^n/\sigma_n) \longrightarrow H^*(\Omega^n)$ induces an isomorphism between $H^*(\Omega^n/\sigma_n)$ and the elements of $H^*(\Omega^n)$ which are invariant by σ_n (see e.g. [5]) . In particular there exists a class, that we shall

denote $\tilde{u}$, such that $\pi^*(\tilde{u}) = w$ with
$w = (u\times 1\times \dots \times 1) + (1\times u\times \dots \times 1) + \dots + (1\times \dots \times 1\times u) \ (\in H^d(\Omega^n))$,
where 1 denotes the unit element of $H^0(\Omega)$.

We are going to prove that

$$\text{(D.11)}\quad \partial(\tilde{u}.\ell_n(z^n\times e_{n-1})) = (-1)^{d-1}\Big(\sum_{i=1}^{n}(-1)^{(n-i)d+i}\Big)\ell_{n-1}(z^{n-1}\times e_{n-1})$$

which gives, when d is odd,

$$\text{(D.12)}\qquad \partial(\tilde{u}.\ell_n(z^n\times e_{n-1})) = (-1)^n\, n\ell_{n-1}(z^n\times e_{n-1}) ,$$

and then (D.9) follows from (D.10) and (D.12)

(Note that, if d is even, (D.11) gives $\partial(\tilde{u}.\ell_n(z^n\times e_{n-1})) = 0$ when n is even).

In order to prove (D.11) we remark that, in $H_*(\Omega^n)$,

$$w \cap z^n = \sum_{i=1}^{n} (-1)^{(n-i)d}\, z^{i-1}\times v\times z^{n-i}$$

and therefore, if we denote by $\underline{1}$ the unit element of $H^0(\Delta_{n-1})$, we have, in $H_*(\Omega^n\times\Delta_{n-1},\Omega^n\times\partial\Delta_{n-1})$:

$$\text{(D.13)}\quad (w\times\underline{1}) \cap z^n\times e_{n-1} = (-1)^{(n-1)d}\Big(\sum_{i=1}^{n}(-1)^{(n-i)d} z^{i-1}\times v\times z^{n-i}\Big)\times e_{n-1},$$

and (D.11) follows from (D.13) Lemma D.1 and Lemma D.2.

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