A_∞-method in Lusternik-Schnirelmann category

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Abstract

To clarify the method behind [11], a generalisation of Berstein-Hilton Hopf invariants is defined as 'higher Hopf invariants'. They detect the higher homotopy associativity of Hopf spaces and are studied as obstructions not to increase the LS category by one by attaching a cone. Under a condition between dimension and LS category, a criterion for Ganea's conjecture on LS category is obtained using the generalised higher Hopf invariants, which yields the main result of [11] for all the cases except the case when $p = 2$. As an application, conditions in terms of homotopy invariants of the characteristic maps are given to determine the LS category of sphere-bundles-over-spheres. Consequently, a closed manifold M is found not to satisfy Ganea's conjecture on LS category and another closed manifold N is found to have the same LS category as its 'punctured submanifold' $N - \{P\}, P \in N$. But all examples obtained here support the conjecture in [11].

1 Introduction

In this paper, each space is assumed to have the homotopy type of a CW complex. The LS category of X is the least number m such that there is a covering of X by $m+1$ open subsets each of which is contractible in X, which is (by Whitehead [27]) the least number m such that the diagonal map $\Delta_{m+1} : X \to X^{m+1}$ can be compressed into the 'fat wedge' $T^{m+1}(X)$ or $X^{[m+1]}$. Hence $\operatorname{cat}\{*\} = 0$.

As is well-known, the LS category of a product space $\operatorname{cat} X \times S^n$ is either $\operatorname{cat} X$ or $\operatorname{cat} X + 1$. A problem was posed by Ganea in [6]: Can only the latter case occur for any X and $n \geq 1$?

The affirmative answer had been supposed to be true and came to be known as 'Ganea's conjecture' (see [8]) or 'the Ganea conjecture' (see [15]). A major advance in this subject was

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made by Jessup [16] and Hess [8] working in the rational category: the rational version of the conjecture is true for $n \geq 2$. Also by Singhof [20] and Rudyak [18], [19], the conjecture is true for a large class of manifolds.

From April to June in 1997, the author studied in Aberdeen University to provide a proof of Ganea's conjecture using higher Hopf invariants [12] on the projective spaces with the properties of Stasheff's fibrations (see Sugawara [23] and [24], Stasheff [22] and Mimura-Iwase [13]) associated with the A_∞-structure of a loop space. The author knows that Stanley was aiming to find out a counter example using the ordinary James-Hopf invariants, and also a related result on a sufficient condition to Ganea's conjecture was obtained by Vandembroucq [26] at about the same peirod.

The author's attempt was failed on showing the stable non-triviality of non-trivial higher Hopf invariants, but it turned to suggest a straight and right way to construct counter examples, and eventually the first counter example was found (see [10] or [11]). Soon after, the higher Hopf invariants given in [12] had begun to be studied by several authors, e.g. Stanley [21].

This paper is organised as follows: In Section 2, to clarify the method behind [11], a generalisation of the Berstein-Hilton Hopf invariants is defined with its related invariants to detect the higher homotopy associativity of a Hopf space. In Section 3, under a condition between dimension and LS category, some conjectures on LS category are verified by using fibrations associated with the A_∞-structure of a loop space. In Section 4, a result of Boardman-Steer is generalised to give a sufficient condition to determine LS category in terms of a generalised version of the Berstein-Hilton crude Hopf invariants. In Section 5, the relation between a homology decomposition and LS category of (product) spaces is shown, by extending a result of Curjel [4]. In Section 6, generalising the main result of [11] for all the cases except the case when $p = 2$, some more examples are obtained by the properties of the higher Hopf invariants given in Section 2. In Section 7, we give some conditions to determine the LS category of sphere-bundles-over-spheres. Using it, we construct, in Section 8, an orientable closed manifold N_p, for each prime $p \geq 5$, with the LS category same as its 'punctured submanifold' $N_p - \{P\}$, $P \in N_p$. Also another orientable closed manifold M is constructed as a counter example to Ganea's conjecture.

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2 Projective spaces and higher Hopf invariants

In this section, we introduce a generalised version of the Berstein-Hilton Hopf invariant (see [2]), a higher Hopf invariant for short, in terms of projective spaces associated with the A_∞-structure of a loop space, to detect the higher homotopy associativity, or the A_m-structure of a Hopf space (see Example 2.7 and Conjecture 2.8). We also show that a higher Hopf invariant gives the obstruction for increasing the LS category by one by attaching a cone, as in [11].

For a given space, its loop space is an A_∞-space with the given space as its A_∞-structure. More precisely, every space X has a filtration given by the projective spaces $P^m(\Omega X)$ of its loop space ΩX. There is a ladder of Stasheff's fibrations $E^{m+1}(\Omega X) \overset{p^{\Omega X}_m}{\to} P^m(\Omega X)$ with the fibre ΩX contractible in $E^{m+1}(\Omega X)$. The total space $E^{m+1}(\Omega X)$ has the homotopy type of the $m{+}1$-fold unreduced join of ΩX which is denoted by $\bar{E}^{m+1}(\Omega X)$ and the base space $P^{m+1}(\Omega X)$ has the homotopy type of the mapping cone of $p^{\Omega X}_m$, $m \geq 0$: the fibration is induced by the inclusion $e^X_m : P^m(\Omega X) \to P^\infty(\Omega X)$ from the *universal* fibration $E^\infty(\Omega X) \overset{p^{\Omega X}_\infty}{\to} P^\infty(\Omega X)$ with contractible total space. (see [22] for details.)

Theorem 2.1 (Ganea) *Let X be a connected CW complex. Then $\operatorname{cat} X \leq m$ if and only if the inclusion $e^X_m : P^m(\Omega X) \subset P^\infty(\Omega X) \simeq X$ has a right homotopy inverse (homotopy section).*

This result enables us to define local versions of cat, e.g, cat_p is defined in [11], for a prime p, as the least number m such that $e^X_m : P^m(\Omega X) \subset P^\infty(\Omega X) \simeq X$ has a homotopy section at p, e.g. $\operatorname{cat}_p S^1_{(p)} = 2$ while $\operatorname{cat} S^1 = 1$. We remark that in some of the literature, the composition functor $P^m{\circ}\Omega$ is abbreviated as G_m, the 'Ganea space' functor. We use the following fact.

Fact 2.2 ([11]) *Let X be a connected CW complex with $\operatorname{cat} X = m$. Then $\operatorname{cat} X{\times}S^n = m$ if and only if $(e^X_m{\times}1_{S^n})| : P^m(\Omega X){\times}\{*\} \cup P^{m-1}(\Omega X){\times}S^n \subset P^\infty(\Omega X){\times}S^n \simeq X{\times}S^n$ has a homotopy section.*

In [11], a conjecture was posed instead of Ganea's conjecture (Conjecture 1.4 in [11]):

Conjecture 2.3 ([11]) *For any space X, there exists an integer $n(X)$, $1 \leq n(X) \leq \infty$, such that* $\operatorname{cat} X \times S^n$ *is equal to* $\operatorname{cat} X + 1$, *if $n < n(X)$;* $\operatorname{cat} X$, *if $n \geq n(X)$.*

When V is the suspension of a co-H-space, say $V = \Sigma V_0$ with V_0 a co-H-space, we fix a canonical structure map $\sigma(V) = \Sigma\,\mathrm{ad}(1_V) : V \to \Sigma\Omega V$ for $\operatorname{cat} V \leq 1$, which makes V a homotopy commutative and homotopy associative co-H-space, where $\mathrm{ad}(1_V) : V_0 \to \Omega\Sigma V_0$ is the adjoint of the identity. In this section, we fix a non-contractible co-H-space V with right and left inversion (e.g, V is a suspension space), together with a structure map $\sigma(V) : V \to \Sigma\Omega V$ for $\operatorname{cat} V = 1$.

Definition 2.4 *Let X be a space with $\operatorname{cat} X \leq m$, $m \geq 1$. For a choice of the homotopy section $\sigma(X) : X \to P^m(\Omega X)$ $(m \geq 1)$, we define a higher Hopf invariant as*

$$H_m = H_m^{\sigma(X)} : [V, X] \to [V, E^{m+1}(\Omega X)],$$

which is a homomorphism when V is homotopy associative and homotopy commutative: For a map $f : V \to X$, the difference between $\sigma(X)\circ f$ and $P^m(\Omega f)\circ\sigma(V)$, in the algebraic loop $[V, P^m(\Omega X)]$, is given by a map $d_m^{\sigma(X)}(f) : V \to P^m(\Omega X)$ so that $\sigma(X)\circ f + d_m^{\sigma(X)}(f) \sim P^m(\Omega f)\circ\sigma(V)$. Since $d_m^{\sigma(X)}(f)$ vanishes in $[V, P^\infty(\Omega X)] \cong [V, X]$, it has a unique lift $H_m^{\sigma(X)}(f) : V \to E^{m+1}(\Omega X)$ to the total space of Stasheff's fibration

$$\Omega X \hookrightarrow E^{m+1}(\Omega X) \overset{p_m^{\Omega X}}{\to} P^m(\Omega X) \overset{e_m^X}{\to} X, \quad m \geq 1. \tag{2.1}$$

Remark 2.5 *1. When V is a homotopy associative co-H-space, $\sigma(V)$ is a co-H-map by Theorem 2.2 of Ganea [5]. Hence a simple calculation shows that $d_m^{\sigma(X)}$ and $H_m^{\sigma(X)}$ are homomorphisms, if V is a homotopy associative and homotopy commutative co-H-space.*

2. By Berstein-Dror [1], a co-associative co-H-space admits another co-associative co-H-structure which has right and left inversions.

3. When V is a Moore space of type (A, n), H_m can be regarded as the Berstein-Hilton Hopf invariant $H : \pi_n(X; A) \to \pi_n(X^{m+1}, T^{m+1}X; A)$ (see [2]), since $\pi_n(X^{m+1}, T^{m+1}X; A) = \pi_n(E^{m+1}(\Omega X); A)$ by Ganea (see the proof of Theorem 1.1 in [11]).

4. When $V = \Sigma V_0$ a suspension space, a map $f : V \to X$ factors through $\Sigma\Omega X$ as $f = e_1^X \circ \Sigma\,\mathrm{ad}(f)$, where e_1^X denotes the evaluation map and $\mathrm{ad}(f) = \Omega f \circ \mathrm{ad}(1_V) : V_0 \to \Omega X$ gives the adjoint of f. Then the higher Hopf invariant $H_m^{\sigma(V)}(e_1^X)$ of $e_1^X : \Sigma\Omega X \to X$ has *a kind of 'universality': $H_m^{\sigma(V)}(f) = H_m^{\sigma(\Sigma\Omega X)}(e_1^X)\circ\Sigma\,\mathrm{ad}(f)$. So we call $H_m^{\sigma(\Sigma\Omega X)}(e_1^X)$ the <u>universal Hopf invariant</u> for X and $m \geq \mathrm{cat}\, X$.*

5. *If X is a suspension space, say $X = \Sigma Y$, then we have $P^1(\Omega X) = \Sigma\Omega X = \Sigma J(Y) \simeq \Sigma Y \vee \Sigma\overset{2}{\wedge}Y \vee \Sigma\overset{3}{\wedge}Y \vee \ldots$. Let us recall that $H_1^{\sigma(X)}$ is uniquely determined by $d_1^{\sigma(X)}$, whose projection to $\Sigma\overset{j}{\wedge}Y$ gives exactly the j-th James Hopf invariant h_j, $j \geq 2$. Thus we may regard H_1 as the collection of all James Hopf invariants h_j, $j \geq 2$.*

The above definition of a higher Hopf invariant also allows us to define a generalisation of the Berstein-Hilton crude Hopf invariant as follows.

Definition 2.6

$$\bar{H}_m = \bar{H}_m^{\sigma(X)} : [V, X] \xrightarrow{H_m^{\sigma(X)}} [V, E^{m+1}(\Omega X)] \xrightarrow{\circ(\Sigma h_{m+1}^X{}_*)\circ\Sigma} [\Sigma V, \wedge^{m+1}\Sigma\Omega X] \xrightarrow{(\wedge^{m+1}e_1^X)_*} [\Sigma V, \wedge^{m+1}X],$$

where e_1^X is the evaluation map and $h_{m+1}^X : E^{m+1}(\Omega X) \to \bar{E}^{m+1}(\Omega X) \simeq \Omega X \wedge (\wedge^m \Sigma\Omega X)$ denotes the natural homotopy equivalence (see Stasheff [22]).

Example 2.7 *For an A_m-space G in the sense of Stasheff [22], the adjoint of the inclusion $\iota_{1,m}^G : \Sigma G \hookrightarrow P^m(G)$ is an A_m-map $\mathrm{ad}(\iota_{1,m}^G) : G \hookrightarrow \Omega P^m(G)$ whose A_m-structure map $P^m(\mathrm{ad}(\iota_{1,m}^G)) : P^m(G) \to P^m(\Omega P^m(G))$ is given by a splitting of $e_m^{P^m(G)} : P^m(\Omega P^m(G)) \to P^m(G)$ (see [13]). By putting $X = P^m(G)$, $V = \bar{E}^{m+1}(G)$ and $\sigma(X) = P^m(\mathrm{ad}(\iota_{1,m}^G))$, we have*

$$H_m = H_m^{\sigma(X)} : [\bar{E}^{m+1}(G), P^m(G)] \to [\bar{E}^{m+1}(G), \bar{E}^{m+1}(\Omega P^m(G))]. \tag{2.2}$$

The group $[\bar{E}^{m+1}(G), \bar{E}^{m+1}(\Omega P^m(G))]$ contains the image of $1 = 1_{\bar{E}^{m+1}(G)}$ under the homomorphism $\mathrm{ad}(\iota_{1,m}^G)_* : [\bar{E}^{m+1}(G), \bar{E}^{m+1}(G)] \to [\bar{E}^{m+1}(G), \bar{E}^{m+1}(\Omega P^m(G))]$. As is clearly seen, if G is an A_{m+1}-space, then there is a 'higher Hopf invariant one' element, i.e. there is a map $f : \bar{E}^{m+1}(G) \to P^m(G)$ such that $H_m(f)$ is the image of 1 as mentioned above. The converse is also verified when G is a sphere: Let $f : \bar{E}^{m+1}(G) \to P^m(G)$ be a 'higher Hopf invariant one' element. Then an easy calculation of the Serre spectral sequence shows that the fibre of f has the homotopy type of the sphere G. Thus G must be an A_{m+1}-space (see [22]).

This suggests the following conjecture.

Conjecture 2.8 *An A_m-space G has an A_{m+1}-structure extending the given A_m-structure if and only if there is a map $f : \bar{E}^{m+1}(G) \to P^m(G)$ with 'higher Hopf invariant one', where $P^m(G)$ is the G-projective m-space associated with the A_m-structure in the sense of Stasheff.*

We bring this higher Hopf invariant $H_m^{\sigma(X)}$ into stable homotopy theory:

Definition 2.9 *We define a stabilised higher Hopf invariant as*

$$\mathcal{H}_m = \mathcal{H}_m^{\sigma(X)} : [V, X] \stackrel{H_m^{\sigma(X)}}{\to} [V, E^{m+1}(\Omega X)] \stackrel{\Sigma_*^\infty}{\to} \{V, E^{m+1}(\Omega X)\}$$

and the stabilised crude higher Hopf invariant as

$$\bar{\mathcal{H}}_m = \bar{\mathcal{H}}_m^{\sigma(X)} : [V, X] \stackrel{\bar{H}_m^{\sigma(X)}}{\to} [\Sigma V, \wedge^{m+1} X] \stackrel{\Sigma_*^\infty}{\to} \{\Sigma V, \wedge^{m+1} X\}.$$

These definitions of higher Hopf invariants depend on the choice of the structure map $\sigma(X)$. So it might be useful to define the following set-valued functions.

Definition 2.10

$$\begin{aligned}
H_m^S(f) &= \{H_m^{\sigma(X)}(f) \mid \sigma(X) \text{ is a structure map for } \mathrm{cat}\, X = m\} \subset [V, E^{m+1}(\Omega X)], \\
\bar{H}_m^S(f) &= \{\bar{H}_m^{\sigma(X)}(f) \mid \sigma(X) \text{ is a structure map for } \mathrm{cat}\, X = m\} \subset [\Sigma V, \wedge^{m+1} X], \\
\mathcal{H}_m^S(f) &= \{\mathcal{H}_m^{\sigma(X)}(f) \mid \sigma(X) \text{ is a structure map for } \mathrm{cat}\, X = m\} \subset \{V, E^{m+1}(\Omega X)\} \\
\bar{\mathcal{H}}_m^S(f) &= \{\bar{\mathcal{H}}_m^{\sigma(X)}(f) \mid \sigma(X) \text{ is a structure map for } \mathrm{cat}\, X = m\} \subset \{\Sigma V, \wedge^{m+1} X\}.
\end{aligned}$$

We show the fundamental properties of higher Hopf invariants.

Proposition 2.11 *Let V, X and f be as above. Then the following two statements hold.*

(1) Let V' be the suspension space of a co-H-space. If $g : V' \to V$ is a co-H-map (or equivalently, $P^1(\Omega g)\circ\sigma(V') \sim \sigma(V)\circ g$ in $P^1(\Omega V)$), then $H_m^{\sigma(X)}(f\circ g) \sim H_m^{\sigma(X)}(f)\circ g$.

(2) Let X' be a space of LS category $\leq m$ with a structure map $\sigma(X')$. If $h : X \to X'$ is m-primitive in the sense of Berstein-Hilton [2] (or equivalently, $\sigma(X')\circ h \sim P^m(\Omega h)\circ\sigma(X)$), then $H_m^{\sigma(X')}(h\circ f) \sim E^{m+1}(\Omega h)\circ H_m^{\sigma(X)}(f)$.

Corollary 2.11.1 *For any map $f : V \to X$, we have the following homotopy relation.*

$$H_m^{\sigma(X)}(f) \sim H_m^{\sigma(X)}(e_1^X)\circ\Sigma\Omega(f)\circ\sigma(V). \tag{2.3}$$

Remark 2.12 *The statement (2) in Proposition 2.11 is a generalisation of Proposition 3.2 in Berstein-Hilton [2].*

Proof of Proposition 2.11. Firstly we show (1): Let $d_m^{\sigma(X)}(f)$ be the difference between $\sigma(X)\circ f$ and $P^m(\Omega f)\circ\sigma(V)$, in the algebraic loop $[V, P^m(\Omega X)]$. Since g is a co-H-map, we obtain the following equation up to homotopy:

$$\sigma(X)\circ(f\circ g) + d_m^{\sigma(X)}(f)\circ g \sim \{(\sigma(X)\circ f) + d_m^{\sigma(X)}(f)\}\circ g \sim P^m(\Omega f)\circ\sigma(V)\circ g.$$

Again, since g is a co-H-map, $P^m(\Omega g)\circ\sigma(V') \sim \sigma(V)\circ g$, and hence we proceed as

$$\sigma(X)\circ(f\circ g) + d_m^{\sigma(X)}(f)\circ g \sim P^m(\Omega f)\circ P^m(\Omega g)\circ\sigma(V') = P^m(\Omega(f\circ g))\circ\sigma(V').$$

This implies the difference between $\sigma(X)\circ(f\circ g)$ and $P^m(\Omega(f\circ g))\circ\sigma(V')$ is given by $d_m^{\sigma(X)}(f\circ g)$ $= d_m^{\sigma(X)}(f)\circ g$, and hence $H_m^{\sigma(X)}(f\circ g) = H_m^{\sigma(X)}(f)\circ g$.

Secondly we show (2): Let $d_m^{\sigma(X)}(f)$ be the difference between $\sigma(X)\circ f$ and $P^m(\Omega f)\circ\sigma(V)$. Since h is an m-primitive map, it follows that $\sigma(X')\circ h \sim P^m(\Omega h)\circ\sigma(X)$. Hence we obtain the following equation up to homotopy:

$$\begin{aligned}\sigma(X')\circ(h\circ f) + P^m(\Omega h)\circ d_m^{\sigma(X)}(f) &\sim P^m(\Omega h)\circ\{\sigma(X)\circ f + d_m^{\sigma(X)}(f)\}\\ &\sim P^m(\Omega h)\circ P^m(\Omega f)\circ\sigma(V) = P^m(\Omega(h\circ f))\circ\sigma(V).\end{aligned}$$

Thus the difference between $\sigma(X')\circ(h\circ f)$ and $P^m(\Omega(h\circ f))\circ\sigma(V)$ is given by the map $d_m^{\sigma(X')}(h\circ f)$ $= P^m(\Omega h)\circ d_m^{\sigma(X)}(f)$, and hence $H_m^{\sigma(X')}(h\circ f) = P^m(\Omega h)\circ H_m^{\sigma(X)}(f)$. *QED.*

3 Higher Hopf invariant and LS category

In the remainder of this paper, we always assume that $V = \Sigma V_0$ a suspension space and we fix the structure map $\sigma(V) = \Sigma\,\mathrm{ad}(1_V)$, unless otherwise stated. We begin this section with the following results by James [14] and by Berstein and Hilton (see Proposition 2.5 in [2]).

Proposition 3.1 (James) *Let a CW complex X be $(d-1)$-connected, $d \geq 2$. Then the following inequality holds:* $\dim X \geq d\cdot\mathrm{cat}\,X$.

Proposition 3.2 (Berstein-Hilton) *Let a CW complex X be $(d-1)$-connected, $d \geq 2$. If* $\dim X \leq dm+d-2$*, then a structure map $\sigma(X) : X \to P^m(\Omega X)$ for* $\mathrm{cat}\,X \leq m$ *is determined uniquely up to homotopy. In particular, when X is simply connected and* $\dim X \leq 2\,\mathrm{cat}\,X$*, the structure map $\sigma(X) : X \to P^m(\Omega X)$ for* $\mathrm{cat}\,X \leq m$ *is determined uniquely up to homotopy.*

Proposition 3.1 implies that quite a few complexes satisfy the hypothesis of Proposition 3.2. Such spaces satisfy the uniqueness of higher Hopf invariants as follows:

Corollary 3.2.1 *Let CW complexes V and X be $(e-1)$-connected and $(d-1)$-connected resp., $d, e \geq 2$ with $\dim X \leq dm+d-2$, $\operatorname{cat} X \leq m$. Then the higher Hopf invariant $H_m = H_m^{\sigma(X)}$ is uniquely determined.*

Proof. By Proposition 3.2, the hypothesis on $\dim X$ and $\operatorname{cat} X$ implies that the structure map $\sigma(X)$ for $\operatorname{cat} X \leq m$ is uniquely determined. By the definition of the homomorphism $H_m = H_m^{\sigma(X)}$, H_m is uniquely determined. *QED.*

We describe the relationship between the higher Hopf invariant and the LS category:

Proposition 3.3 *For a given structure map $\sigma(X)$ for $\operatorname{cat} X = m$ and a given map $f : V \to X$, let W be the mapping cone of f. Then the following diagram without the dotted arrows commute up to homotopy.*

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "H_m(f)"'] & X \arrow[r, hook, "i"] \arrow[d, "\sigma(X)"] & W \arrow[r, "q"] \arrow[dd, dashed, "\sigma'(W)"] & \Sigma V \\
E^{m+1}(\Omega X) \arrow[r, dashed, "p_m^{\Omega X}"'] \arrow[d, "E^{m+1}(\Omega i)"'] & P^m(\Omega X) \arrow[d, "P^m(\Omega i)"] & & \\
E^{m+1}(\Omega W) \arrow[r, "p_m^{\Omega W}"'] & P^m(\Omega W) \arrow[r, hook, "\iota_m^{\Omega W}"'] & P^{m+1}(\Omega W) \arrow[r, "e_{m+1}^W"'] & W,
\end{tikzcd}$$

where $i : X \hookrightarrow W$ and $\iota_m^{\Omega W} : P^m(\Omega W) \to P^{m+1}(\Omega W)$ denote the inclusions.

Proof. Let us recall that $H_m(f)$ is given by the unique lift of the difference between $\sigma(X)\circ f$ and $P^m(\Omega f)\circ\sigma(V)$ in the group $[V, P^m(\Omega X)]$. Hence the composition $P^m(\Omega i)\circ p_m^{\Omega X}\circ H_m(f)$ gives the difference between $P^m(\Omega i)\circ\sigma(X)\circ f$ and $P^m(\Omega i)\circ P^m(\Omega f)\circ\sigma(V) = P^m(\Omega i\circ f)\circ\sigma(V) \sim 0$. Thus we obtain the commutativity of the diagram. *QED.*

Remark 3.4 *By the homotopy commutativity of the left rectangle of the diagram, there is a map $\sigma'(W) : W \to P^{m+1}(\Omega W)$ making the diagram commutative, since both top and bottom rows except for the map e_{m+1}^W are cofibration sequences.*

Now we discuss about the naturality of $\sigma'(W)$ which is determined as above. A suspension map $g = \Sigma g_0 : V \to V'$ between suspension spaces gives the equality

$$\sigma(V)\circ g = \Sigma\Omega g\circ\sigma(V').$$

For maps $f : V \to X$ and $f' : V' \to X$, we have their cofibres $i : X \hookrightarrow W = \{*\} \cup [0,1] \times V \cup_f X$ and $i' : X \hookrightarrow W' = \{*\} \cup [0,1] \times V' \cup_{f'} X$ with null-homotopies $F : 0 \sim i \circ f$ and $F' : 0 \sim i' \circ f'$:

$$F(t, v) = t \wedge v, \quad F(t', v') = t' \wedge v', \quad \text{for } v \in V,\ v' \in V' \text{ and } t, t' \in [0,1].$$

Lemma 3.5 *Let $g : V' \to V$ be a suspension map between suspension spaces. If $f' \sim f \circ g$ with a homotopy $L : [0,1] \times V' \to X$, there is a map $\tilde{L} : W' \to W$ extending $i : X \to W$. Moreover there is a homotopy $\sigma'(W) \circ \tilde{L} \sim P^{m+1}(\Omega \tilde{L}) \circ \sigma'(W')$ relative to X, where $\sigma'(W)$ and $\sigma'(W')$ above are given as $\sigma'(W)$ in Remark 3.4 from $\sigma(X)$ using $H_m(f)$ and $H_m(f')$, resp.*

Proof. Let $V = \Sigma V_0$ and $V' = \Sigma V_0'$ and let $g = \Sigma g_0$, $g_0 : V_0' \to V_0$. We define $\tilde{L}$ by the following equation.

$$\tilde{L}(x) = x, \quad \tilde{L}(t \wedge v') = \begin{cases} (2t) \wedge g(v') \in W, & t \leq \frac{1}{2}, \\ L(2 - 2t, v') \in X \subset W, & t \geq \frac{1}{2} \end{cases}$$

for $x \in X$ and $(t, v') \in [0,1] \times V'$. Let us consider the following diagram of homotopies:

$$\begin{tikzcd}[column sep=huge]
P^m(\Omega(\tilde{L}\circ i'))\circ\sigma(X)\circ f' \arrow[r, dashed, "P^m(\Omega i)\circ\sigma(X)\circ L"] \arrow[d, dashed] \arrow[dr, phantom, "(A)"] & P^m(\Omega i)\circ\sigma(X)\circ f\circ g \arrow[d, dashed] \\
P^m(\Omega(\tilde{L}\circ i'))\circ\sigma(X)\circ f' - 0 \arrow[r, dashed, "P^m(\Omega i)\circ\sigma(X)\circ L - c(0)"] \arrow[d, dashed, "\substack{c(P^m(\Omega(\tilde{L}\circ i'))\circ\sigma(X)\circ f') \\ -P^m(\Omega(\tilde{L}\circ F'))\circ\iota_{V'}\circ\sigma(V')}"'] \arrow[dr, phantom, "(B)"] & P^m(\Omega i)\circ\sigma(X)\circ f\circ g - 0 \arrow[d, dashed, "\substack{c(P^m(\Omega i))\circ\sigma(X)\circ f\circ g \\ -P^m(\Omega F)\circ\iota_{V'}\circ\sigma(V')}"] \\
\substack{P^m(\Omega(\tilde{L}\circ i'))\circ\sigma(X)\circ f' \\ -P^m(\Omega(\tilde{L}\circ i'\circ f'))\circ\iota_{V'}\circ\sigma(V')} \arrow[r, dashed, "\substack{P^m(\Omega i)\circ\sigma(X)\circ L \\ -P^m(\Omega(i\circ L))\circ\iota_{V'}\circ\sigma(V')}"] \arrow[d, equal] \arrow[dr, phantom, "(C)"] & \substack{P^m(\Omega i)\circ\sigma(X)\circ f\circ g \\ -P^m(\Omega(i\circ f))\circ\iota_V\circ\sigma(V)\circ g} \arrow[d, equal] \\
\substack{P^m(\Omega(\tilde{L}\circ i'))\circ(\sigma(X)\circ e_1^X \\ -\iota_X)\circ\Sigma\,\mathrm{ad}(f')} \arrow[r, dashed, "\substack{P^m(\Omega i)\circ(\sigma(X)\circ e_1^X \\ -\iota_X)\circ\Sigma\mathrm{ad}\,(L)}"] \arrow[d, dashed] \arrow[dr, phantom, "(D)"] & \substack{P^m(\Omega i)\circ(\sigma(X)\circ e_1^X \\ -\iota_X)\circ\Sigma\,\mathrm{ad}(f\circ g)} \arrow[d, dashed] \\
P^m(\Omega(\tilde{L}\circ i'))\circ p_m^{\Omega X}\circ H_m^{\sigma(X)}(e_1^X)\circ\Sigma\,\mathrm{ad}(f') \arrow[r, dashed] \arrow[d, equal] \arrow[dr, phantom, "(E)"] & P^m(\Omega i)\circ p_m^{\Omega X}\circ H_m^{\sigma(X)}(e_1^X)\circ\Sigma\,\mathrm{ad}(f\circ g) \arrow[d, equal] \\
P^m(\Omega(\tilde{L}\circ i'))\circ p_m^{\Omega X}\circ H_m^{\sigma(X)}(f') \arrow[r, "\substack{P^m(\Omega i)\circ p_m^{\Omega X}\circ \\ H_m^{\sigma(X)}(e_1^X)\circ\Sigma\mathrm{ad}\,(L)}"] & P^m(\Omega i)\circ p_m^{\Omega X}\circ H_m^{\sigma(X)}(f\circ g)
\end{tikzcd} \tag{3.1}$$

where $c(-)$ denotes a constant homotopy and $\iota_X : P^1(\Omega X) \hookrightarrow P^m(\Omega X)$, $\iota_V : P^1(\Omega V) \hookrightarrow P^m(\Omega V)$ and $\iota_{V'} : P^1(\Omega V') \hookrightarrow P^m(\Omega V')$ are inclusions. The homotopy commutativities of (C) and (E) are trivial.

To show the homotopy commutativity of (A), we define a map $\theta_A : [0,1]\times[0,1]\times V' \to W$.

$$\theta_A(s,t,u\wedge v_0') = \begin{cases} L(s, \left(\frac{2u}{2-t}\right) \wedge v_0') \in W, & u \leq \frac{2-t}{2} \\ * \in X \subset W, & u \geq \frac{2-t}{2} \end{cases} \qquad \text{for } s,t \in [0,1] \text{ and } u\wedge v_0' \in V'.$$

We have $\theta_A(s,0,u\wedge v_0') = L(s,u\wedge v_0')$, $\theta_A(s,1,u\wedge v_0') = (L-c(0))(s,u\wedge v_0')$ and that $\theta_A(0,t,u\wedge v_0')$ and $\theta_A(1,t,u\wedge v_0')$ give canonical homotopies from $f'(u\wedge v_0')$ and $f\circ g(u\wedge v_0') = f(u\wedge g_0(v_0')$ to $(f-0)(u\wedge v_0')$ and $(f\circ g - 0)(u\wedge v_0') = (f-0)(u\wedge g_0(v_0')$. Thus θ_A gives a homotopy

$$\begin{tikzcd} f' \arrow[r, dashed, "L"] \arrow[d, dashed] & f\circ g \arrow[d, dashed] \\ f'-0 \arrow[r, dashed, "L-c(0)"'] & f\circ g - 0. \end{tikzcd}$$

where the upper and lower rows are given by $t=0$ and $t=1$ and left and right columns are given by $s=0$ and $s=1$. By applying the composition with $P^m(\Omega i)\circ\sigma(X)$ from the left, we get the homotopy commutativity of (A).

To show the homotopy commutativity of (B), we define a map $\theta_B : [0,1]\times[0,1]\times V' \to W$.

$$\theta_B(s,t,v') = \begin{cases} \left(\frac{2t}{1+s}\right) \wedge g(v') \in W, & t \leq \frac{1+s}{2} \\ L(2+s-2t, v') \in X \subset W, & t \geq \frac{1+s}{2} \end{cases} \qquad \text{for } s,t \in [0,1] \text{ and } v' \in V'.$$

We have $\theta_B(s,0,v') = *$, $\theta_B(0,t,v') = \tilde{L}\circ F'(t,v')$, $\theta_B(s,1,v') = i\circ L(s,v')$ and $\theta_B(1,t,v') = F(t,g(v'))$. Thus θ_B gives a homotopy

$$\begin{tikzcd} 0 \arrow[r, dashed, "c(0)"] \arrow[d, dashed, "P^m(\Omega(\tilde{L}\circ F'))"'] & 0 \arrow[d, dashed, "P^m(\Omega F)"] \\ P^m(\Omega(\tilde{L}\circ i'\circ f')) \arrow[r, dashed, "P^m(\Omega(i\circ L))"'] & P^m(\Omega(i\circ f\circ g)) \end{tikzcd}$$

where the upper and lower rows are given by $t=0$ and $t=1$ and left and right columns are given by $s=0$ and $s=1$. By applying the composition with $\iota_{V'}\circ\sigma(V')$ from the right, we get the homotopy commutativity of (B).

To show the homotopy commutativity of (D), we fix a homotopy

$$\sigma(X)\circ e_1^X - \iota_X \sim p_m^{\Omega X}\circ H_m^{\sigma(X)}(e_1^X)$$

as a map $K : [0,1]{\times}\Sigma\Omega X \to P^m(\Omega X)$ with $K(0,-) = (\sigma(X){\circ}e_1^X - \iota_X)(-)$ and $K(1,-) = p_m^{\Omega X}{\circ}H_m^{\sigma(X)}(e_1^X)(-)$. Using it, we define a map $\theta_D : [0,1]{\times}[0,1]{\times}V' \to W$.

$$\theta_D(s,t,u{\wedge}v_0') = K(t, u{\wedge}\,\mathrm{ad}(L)(s,v_0')) \quad \text{for } s,t \in [0,1] \text{ and } u{\wedge}v_0' \in V'.$$

We have $\theta_D(s,0,u{\wedge}v_0') = (\sigma(X){\circ}e_1^X - \iota_X)(u{\wedge}\,\mathrm{ad}(L)(s,v_0'))$, $\theta_D(0,t,u{\wedge}v_0') = K(t,u{\wedge}\,\mathrm{ad}(f')(v_0'))$, $\theta_D(s,1,u{\wedge}v_0') = p_m^{\Omega X}{\circ}H_m^{\sigma(X)}(e_1^X)(u{\wedge}\,\mathrm{ad}(L)(s,v_0'))$ and $\theta_D(1,t,u{\wedge}v_0') = K(t,u{\wedge}\,\mathrm{ad}(f{\circ}g)(v_0'))$. Thus θ_D gives a homotopy

$$\begin{tikzcd}
(\sigma(X){\circ}e_1^X - \iota_X){\circ}\Sigma\,\mathrm{ad}(f') \arrow[r, dashed, "(\sigma(X){\circ}e_1^X - \iota_X){\circ}\Sigma\,\mathrm{ad}(L)"] \arrow[d, dashed] & (\sigma(X){\circ}e_1^X - \iota_X){\circ}\Sigma\,\mathrm{ad}(f{\circ}g) \arrow[d, dashed] \\
p_m^{\Omega X}{\circ}H_m^{\sigma(X)}(e_1^X){\circ}\Sigma\,\mathrm{ad}(f') \arrow[r, dashed, "P^m(\Omega i){\circ}\sigma(X){\circ}L - c(0)"'] & p_m^{\Omega X}{\circ}H_m^{\sigma(X)}(e_1^X){\circ}\Sigma\,\mathrm{ad}(f{\circ}g)
\end{tikzcd}$$

where the upper and lower rows are given by $t = 0$ and $t = 1$ and left and right columns are given by $s = 0$ and $s = 1$. By applying the composition with $P^m(\Omega i)$ from the left, we get the homotopy commutativity of (D).

The maps $\sigma'(W){\circ}\tilde{L}$ and $P^{m+1}(\Omega(\tilde{L})){\circ}\sigma'(W')$ are given by the homotopies in the diagram (3.1) indicated by the top and right arrows and left and bottom arrows respectively, and hence the homotopy commutativity of the homotopies in the diagram (3.1) implies that $\sigma'(W){\circ}\tilde{L} \sim P^{m+1}(\Omega\tilde{L}){\circ}\sigma'(W')$. *QED.*

Remark 3.6 *Since $e_{m+1}^W{\circ}\sigma'(W)$ and the identity 1_W coincide on X up to homotopy, the difference between them with respect to the co-action of ΣV is given by a map $\gamma : \Sigma V \to W$. Here, we know the fibration (2.1), induces the following split short exact sequence:*

$$0 \to [\Sigma V, E^{m+1}(\Omega W)] \overset{p_m^{\Omega W}}{\to}{}_* [\Sigma V, P^m(\Omega W)] \overset{e_m^W}{\to}{}_* [\Sigma V, W] \to 0. \tag{3.2}$$

Thus γ can be pulled back to a map $\gamma_0 : \Sigma V \to P^m(\Omega W) \subset P^{m+1}(\Omega W)$. By adding γ_0 to $\sigma'(W)$, we obtain a genuine compression $\sigma(W)$ of 1_W. We often call this $\sigma(W)$ a 'standard' structure map for $\mathrm{cat}\,W \leq m+1$ which gives a subset of $H_m^S(f)$ for $f \in [V', W]$ as follows:

$$H_m^{SS}(f) = \{H_m^{\sigma(W)}(f) \mid \sigma(W) \text{ is a 'standard' structure map for } \mathrm{cat}\,W \leq m+1\}$$

The cofibration sequence $V \overset{f}{\to} X \overset{i}{\to} W \overset{q}{\to} \Sigma V$ induces the following cofibration sequence:

$$V{*}S^{n-1} \overset{\hat{f}}{\to} W \cup X{\times}S^n \overset{\hat{i}}{\to} W{\times}S^n \overset{\hat{q}}{\to} \Sigma^{n+1}V;$$

here $\hat{f}$ is given by the relative Whitehead product $[\chi_f, 1_{S^{n-1}}]^r : V{*}S^n \to W \cup X{\times}S^n$, where $\chi_f : (C(V),V) \to (W,X)$ denotes the characteristic map given as a relative homeomorphism.

By the proof of Proposition 5.8 in [11], the following result is obtained using Remark 3.4.

Proposition 3.7 *For a given structure map $\sigma(X)$ for $\operatorname{cat} X = m$, the map $\hat{f}$ makes the following diagram without the dotted arrows commute up to homotopy.*

$$\begin{tikzcd}
V{*}S^{n-1} \arrow[r, "\hat{f}"] \arrow[d, "H_m(f){*}1_{S^{n-1}}"] & W \cup X{\times}S^n \arrow[r, hook, "\hat{i}"] \arrow[d, "\sigma'(W){\times}1_{S^n}|_{W\cup X\times S^n}"] & W{\times}S^n \arrow[r, "\hat{q}"] \arrow[dd, dashed, "\sigma(W\times S^n)"] & \Sigma^{n+1}V \\
E^{m+1}(\Omega X){*}S^{n-1} \arrow[r, dashed, "\hat{p}_m^{\Omega X}"'] \arrow[d, tail, "E^{m+1}(\Omega i){*}j_{n-1}"] & P^{m+1}(\Omega W) \cup P^m(\Omega X){\times}S^n \arrow[d, tail] & & \\
E^{m+1}(\Omega W){*}\Omega S^n \arrow[r, "\hat{p}_m^{\Omega W}"'] & P^{m+1}(\Omega W) \cup P^m(\Omega W){\times}S^n \arrow[r, hook] & P^{m+1}(\Omega W){\times}S^n \arrow[r, "e^W_{m+1}\times 1_{S^n}"'] & W{\times}S^n,
\end{tikzcd}$$

where $\hat{p}_m^{\Omega X} = [\chi_{p_m^{\Omega X}}, 1_{S^n}]^r$, $\hat{p}_m^{\Omega W} = [\chi_{p_m^{\Omega W}}, e_1^{S^n}]^r$ and j_{n-1} denotes the bottom cell inclusion $S^{n-1} \subset \Omega S^n$ and $\sigma'(W)$ is the extension of $\sigma(X)$ by Proposition 3.3 and Remark 3.4.

We then have the following result.

Theorem 3.8 *The following two statements hold for W with $\operatorname{cat} W \leq m+1$, $m = \operatorname{cat} X$.*

(1) $\operatorname{cat} W \leq \operatorname{cat} X$ if the set $H^S_m(f)$ contains a trivial element.

(2) $\operatorname{cat} W{\times}S^n \leq \operatorname{cat} X + 1$ if the set $\Sigma^n_* H^S_m(f)$ contains a trivial element.

Corollary 3.8.1 *Let W be the space constructed as in the above theorem with $\operatorname{cat} W = m+1$. Then W is a counter example to Ganea's conjecture if the set $\mathcal{H}^S_m(f)$ contains trivial element.*

Proof of Theorem 3.8. Firstly we show (1): If $E^{m+1}(\Omega i)\circ H^{\sigma(X)}_m(f) : V \to E^{m+1}(\Omega W)$ is trivial for some structure map $\sigma(X)$, then, by Proposition 3.3 and Remark 3.4, $\sigma(X)$ is extendible to W. By the argument given in Remark 3.6, we obtain a genuine compression $\sigma(W) : W \to P^m(\Omega W)$ of 1_W. Thus $\operatorname{cat} W \leq m$.

Secondly we show (2): If $(E^{m+1}(\Omega i){*}j_{n-1})\circ\Sigma^n H^{\sigma(X)}_m(f) : V{*}S^{n-1} \to E^{m+1}(\Omega W){*}\Omega S^n$ is trivial for some structure map $\sigma(X)$, then, by Proposition 3.7, the map $\sigma'(W){\times}1_{S^n}|_{W\cup X\times S^n} : W \cup X{\times}S^n \to P^{m+1}(\Omega W) \cup P^m(\Omega X){\times}S^n \subset P^{m+1}(\Omega W) \cup P^m(\Omega W){\times}S^n$ is extendible to $W{\times}S^n$. By the argument given in Remark 3.6 together with the fact that the natural map $[\Sigma^{n+1}V, Z{\times}B {\cup} C{\times}Y] \to [\Sigma^{n+1}V, Z{\times}Y] \cong [\Sigma^{n+1}V, Z]{\times}[\Sigma^{n+1}V, Y]$ is split surjective for any pointed pairs (Z, C) and (Y, B), we obtain a compression $\sigma'' : W{\times}S^n \to P^{m+1}(\Omega W) \cup P^m(\Omega W){\times}S^n$ of $\sigma'(W){\times}1_{S^n}$, as an extension of $\sigma'(W){\times}1_{S^n}|_{W\cup X\times S^n}$. Also by Remark 3.6, the difference between $e^W_{m+1}\circ\sigma'(W)$ and 1_W with respect to the co-action of ΣV can be pulled back to $\gamma_0 : \Sigma V \to \Sigma\Omega W \subset P^m(\Omega W)$. We define a map $\sigma(W\times S^n)$ by

$$\sigma(W\times S^n) : W\times S^n \overset{\mu\times 1_{S^n}}{\to} (W\vee\Sigma V)\times S^n \xrightarrow{\sigma''\cup(\gamma_0\times 1_{S^n})} P^{m+1}(\Omega W)\times\{*\}\cup P^m(\Omega W)\times S^n,$$

Since σ'' is homotopic to $\sigma'(W)\times 1_{S^n}$ in $P^{m+1}(\Omega W)\times S^n$ with the subspace $\{*\}\times S^n$ left fixed, $\sigma(W\times S^n)$ is homotopic to $(\sigma'(W)+\gamma)\times 1_{S^n}$ in $P^{m+1}(\Omega W)\times S^n$ which is a compression of $1_W\times 1_{S^n}$. Thus $\operatorname{cat} W\times S^n \le m+1$. *QED.*

Let V be a $(e-1)$-connected co-H-space and X a $(d-1)$-connected CW complex, $e\ge d\ge 2$.

Theorem 3.9 *Let $n\ge 1$ and X be of $\dim X \le d{\cdot}m+d-2$ and of $\operatorname{cat} X = m$. Then the following two statements hold for W and $H_m(f) = H^{\sigma(X)}_m(f)$ where $\sigma(X)$ is the unique structure map.*

(1) $\operatorname{cat} W = m+1$ *if* $E^{m+1}(\Omega i)\circ H_m(f)\ne 0$.

(2) $\operatorname{cat} W\times S^n = m+2$ *if* $(E^{m+1}(\Omega i)*j_{n-1})\circ\Sigma^n H_m(f)\ne 0$ *and* $n > \dim V - d(m+1)+2$.

(3) $\operatorname{cat} W\times S^n = m+2$ *if* $(E^{m+1}(\Omega i)*j_n)\circ\Sigma^{n+1}H_m(f)\ne 0$.

Remark 3.10 *If the condition $\dim V < d\cdot\operatorname{cat} X + e - 1$ is satisfied, then for dimensional reasons, we may remove the maps '$E^{m+1}(\Omega i)$', '$E^{m+1}(\Omega i)*j_{n-1}$' and '$E^{m+1}(\Omega i)*j_n$' from the statements in Theorem 3.9.*

Corollary 3.10.1 *Let W be the space constructed as the mapping cone of $f : V\to X$ from simply connected spaces V and X with $\dim X\le d\cdot\operatorname{cat} X+d-2$ and $\dim V\le d\cdot\operatorname{cat} X+e-2$, where V and X are $(e-1)$-connected and $(d-1)$-connected, resp. If $\operatorname{cat} W\times S^k = \operatorname{cat} W$ for some k, then $\operatorname{cat} W\times S^n = \operatorname{cat} W$ for all $n\ge k$. Thus Conjecture 2.3 is true for such W.*

Corollary 3.10.2 *Let W be the space constructed as the mapping cone of $f : V\to X$ from simply connected spaces V and X with $\dim X\le d\cdot\operatorname{cat} X+d-2$ and $\dim V\le d\cdot\operatorname{cat} X+e-2$, where V and X are $(e-1)$-connected and $(d-1)$-connected, resp. Then Ganea's conjecture for W is true if and only if the unique stabilised higher Hopf invariant $\mathcal{H}_m(f)$ is non-trivial.*

Proof of Theorem 3.9. Let $\dim X = k$. By the assumption, $\operatorname{cat} W\le m+1$.

Firstly we show (1): If $\operatorname{cat} W\le m$, then there exists a compression $\sigma : W\to P^m(\Omega W)$ of the identity 1_W. Since the pair $(P^\infty(\Omega W), P^m(\Omega W))$ is $(d(m+1)-1)$-connected, the inclusion map

$\iota_m : P^m(\Omega W) \to P^\infty(\Omega W)$ induces a bijection of homotopy sets $[Z, P^m(\Omega W)] \to [Z, P^\infty(\Omega W)]$ for any space Z of dimension up to $d(m+1)-2$. Then, for dimensional reasons, it follows that the restriction $\sigma_0 = \sigma|_X$ is unique up to homotopy in $P^m(\Omega W)$. Hence we may assume that σ_0 equals to $\sigma(X)$, the unique structure map for $\operatorname{cat} X = m$ by Proposition 3.2. Hence $P^m(\Omega i)\circ\sigma(X)\circ f \sim p^{\Omega W}_m\circ E^{m+1}(\Omega i)\circ H_m(f)$ is null-homotopic. Then by the short exact sequence in Remark 3.6, it follows that $E^{m+1}(\Omega i)\circ H_m(f)$ is null-homotopic. Thus the existence of the compression σ of 1_W implies the triviality of $E^{m+1}(\Omega i)\circ H_m(f) : V \to E^{m+1}(\Omega W)$.

Secondly we show (2): Let $\sigma(W)$ be a structure map for $\operatorname{cat} W \leq m+1$ obtained from $\sigma(X)$ by Proposition 3.3, Remarks 3.4 and 3.6, and let $\hat{j}_{m,n} : P^{m+1}(\Omega W) \cup P^m(\Omega W)\times S^n \subset P^\infty(\Omega W) \cup P^m(\Omega W)\times S^n$ and $\hat{i}_{m,n} : P^\infty(\Omega W) \cup P^m(\Omega W)\times S^n \subset P^\infty(\Omega W)\times S^n$ be inclusions. If $\operatorname{cat} W\times S^n \leq m+1$, then by Fact 2.2, there exists a compression $\sigma : W\times S^n \to P^{m+1}(\Omega W) \cup P^m(\Omega W)\times S^n$ of the identity $1_{W\times S^n}$. Hence we have

$$e^W_\infty\times 1_{S^n}|_{P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n}\circ\hat{j}_{m,n}\circ\sigma|_{W\cup X\times S^n} \sim 1_{W\times S^n} \quad \text{and}$$
$$e^W_\infty\times 1_{S^n}|_{P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n}\circ\hat{j}_{m,n}\circ(\sigma(W)\times 1_{S^n}|_{W\cup X\times S^n}) \sim 1_{W\times S^n}.$$

Here we know the pair $(P^\infty(\Omega W)\times S^n, P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n)$ is $(dm+n+d-1)$-connected and there is a co-action $\mu'_W : W\cup X\times S^n \to W\cup X\times S^n \vee \Sigma V$ associated with the cofibration sequence $V \to X\times S^n \to W \cup X\times S^n$. Since the subspace $X\times S^n$ is of dimension $k+n \leq dm+d+n-2$, the restrictions $\hat{j}_{m,n}\circ\sigma|_{X\times S^n}$ and $\hat{j}_{m,n}\circ\sigma(W)\times 1_{S^n}|_{X\times S^n} = \sigma(X)\times 1_{S^n}$ are homotopic and the difference between $\hat{j}_{m,n}\circ\sigma$ and $\sigma(W)\times 1_{S^n}|_{W\cup X\times S^n}$ with respect to the co-action μ'_W is given by a map into $E^{m+1}(\Omega W)*\Omega S^n$ the fibre of $e^W_\infty\times 1_{S^n}|_{P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n}$:

$$\sigma|_{W\cup X\times S^n} \sim \sigma(W)|_{W\cup X\times S^n} + \gamma, \qquad \gamma = \hat{j}_{m,n}\circ\hat{p}^{\Omega W}_m\circ\gamma_0, \quad \gamma_0 : \Sigma V \to E^{m+1}(\Omega W)*\Omega S^n.$$

If we assume that $n > \dim V - d(m+1) + 2$, then we also obtain that $\hat{j}_{m,n}\circ\sigma|_{W\cup X\times S^n} \sim \hat{j}_{m,n}\circ\sigma(W)\times 1_{S^n}|_{W\cup X\times S^n}$ for dimensional reasons. In this case, we have

$$\hat{j}_{m,n}\circ\hat{p}^{\Omega W}_m\circ(E^{m+1}(\Omega i)*j_{n-1})\circ\Sigma^n H_m(f) \sim \hat{j}_{m,n}\circ(\sigma(W)\times 1_{S^n}|_{W\cup X\times S^n})\circ\hat{f} \sim \hat{j}_{m,n}\circ\sigma|_{W\cup X\times S^n}\circ\hat{f}$$

which is trivial, since $\sigma|_{W\cup X\times S^n}$ is extendible on $W\times S^n$. Thus $(E^{m+1}(\Omega i)*j_{n-1})\circ\Sigma^n H_m(f)$ is trivial provided that $n > \dim V - d(m+1) + 2$.

Thirdly we show (3): In this case, the difference γ might not be trivial. Since the composition $\mathrm{pr}_1\circ\hat{j}_{m,n}\circ\sigma : W\times S^n \overset{\hat{j}_{m,n}\circ\sigma}{\to} P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n \overset{\mathrm{pr}_1}{\to} P^\infty(\Omega W)$ factors through $P^\infty(\Omega W)\times S^n$ as $\mathrm{pr}_1\circ\hat{j}_{m,n}\circ\sigma = \mathrm{pr}_1\circ\hat{i}_{m,n}\circ\hat{j}_{m,n}\circ\sigma$, we have $e^W_\infty\circ\mathrm{pr}_1\circ\hat{j}_{m,n}\circ\sigma = \mathrm{pr}_1\circ(e^W_\infty\times 1_{S^n})\circ\hat{i}_{m,n}\circ\hat{j}_{m,n}\circ\sigma \sim \mathrm{pr}_1$, which yields the following commutative diagram up to homotopy.

$$\begin{tikzcd}
W \arrow[r, "(e_\infty^W)^{-1}"] & P^\infty(\Omega W) \\
W\times S^n \arrow[u, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_1"'] \arrow[r, "\hat{j}_{m,n}\circ\sigma"] & P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_1"] \\
W \arrow[r, "(e_\infty^W)^{-1}"] & P^\infty(\Omega W).
\end{tikzcd}$$

Taking push-outs of both right and left columns of the above diagram, we have a map

$$\sigma_1 : W\times S^{n+1} \to P^\infty(\Omega W)\cup P^m(\Omega W)\times S^{n+1} \quad \text{with} \quad \sigma_1\circ(1_W\times E_n) = E_{m,n}\circ\hat{j}_{m,n}\circ\sigma,$$

where E_n denotes the inclusion $S^n = S^n\times\{0\} \hookrightarrow S^{n+1} = S^n\times[-1,1]/\sim$, $(x,-1)\sim(x,1)\sim(*,t)$ for $x\in S^n$ and $t\in[-1,1]$ and $E_{m,n} = 1_{P^\infty(\Omega W)}\times E|_{P^\infty(\Omega W)\cup P^m(\Omega W)\times S^n}$. By the definition of σ_1, we have $\sigma_1|_{X\times S^{n+1}} \sim \hat{j}_{m,n+1}\circ(\sigma(X)\times 1_{S^{n+1}})$ and the difference between $\sigma_1|_{W\cup X\times S^{n+1}}$ and $\hat{j}_{m,n+1}\circ(\sigma(W)\times 1_{S^{n+1}})|_{W\cup X\times S^{n+1}}$ are given by $E_{m,n}\circ\gamma$. Since $P^\infty(\Omega W)\cup P^m(\Omega W)\times(S^n\times 0)$ is compressible into $P^\infty(\Omega W)$, so is $E_{m,n}\circ\gamma$. Thus $E_{m,n}\circ\gamma \sim \mathrm{in}_1\circ\mathrm{pr}_1\circ E_{m,n}\circ\gamma = \mathrm{in}_1\circ\mathrm{pr}_1\circ\gamma \sim \mathrm{in}_1\circ\mathrm{pr}_1\circ\hat{i}_{m,n}\circ\hat{j}_{m,n}\circ\gamma \sim 0$, where in_1 or pr_1 denotes an appropriate inclusion or projection. Thus by the same argument as in (2), we obtain $(E^{m+1}(\Omega i)*j_n)\circ\Sigma^{n+1}H_m(f) \sim 0$. *QED.*

4 Higher Hopf invariants and the reduced diagonal

We state here another property of the higher Hopf invariants, which is a generalisation of Theorem 5.14 of Boardman-Steer [3].

Theorem 4.1 *Let* $V \overset{f}{\to} X \overset{i}{\to} W \overset{q}{\to} \Sigma V$ *be a cofibration sequence as above with* $\mathrm{cat}\, X = m$. *Then the reduced diagonal* $\hat{\Delta}_{m+1} : W \to W^{m+1} \to \wedge^{m+1}W$ *is given by the composition* $\wedge^{m+1}e_1^W\circ\Sigma h_{m+1}^W\circ\Sigma(E^{m+1}(\Omega i)\circ H_m(f))\circ q$.

Corollary 4.1.1 *Let* V, X, f *and* W *be as above with* $\mathrm{cat}\, X = m$ *and* $\mathrm{cat}\, W = m+1$. *Then the n-fold suspension of the reduced diagonal* $\hat{\Delta}_{m+1}\wedge 1_{S^n} : W\times S^n \to W^{m+1}\times S^n \to \wedge^{m+1}W\wedge S^n$ *is given by the composition* $\Sigma^n(\wedge^{m+1}e_1^W\circ\Sigma h_{m+1}^W\circ\Sigma(E^{m+1}(\Omega i)\circ H_m(f)))\circ\hat{q}$, *where* $\hat{q}$ *denotes the collapsing map* $W\times S^n \to W\times S^n/(W\cup X\times S^n) = (W/X)\wedge S^n = \Sigma^{n+1}V$.

For any space Z, Ganea showed that there is a commutative ladder of fibrations up to homotopy (see [5] or the proof of Theorem 1.1 of [11]):

$$\begin{tikzcd}
E^{m+1}(\Omega Z) \arrow[r, "p_m^Z"] \arrow[d, "h_{m+1}^Z"] & P^m(\Omega Z) \arrow[r, "e_m^Z"] \arrow[d, "\Delta'_{m+1}"] & Z \arrow[d, "\Delta_{m+1}"] \\
\bar{E}^{m+1}(\Omega Z) \arrow[r, "q_{m+1}^Z"] & Z^{[m+1]} \arrow[r, hook] & Z^{m+1},
\end{tikzcd} \tag{4.1}$$

where the fibration $q^Z_{m+1} : \bar{E}^{m+1}(\Omega Z) \to Z^{[m+1]}$ is given by $q^Z_{m+1}(\Sigma t_i \ell_i) = \{\ell_i(t_i/t_M)\}_{0\le i\le m}$, $t_M = \max(t_0, ..., t_m)$ and Δ'_{m+1} denotes a map which makes the right hand square of (4.1) a homotopy pull-back diagram.

Using the precise description of the fibration q^Z_{m+1}, we obtain the following proposition.

Proposition 4.2 *There exists a map $e_{m+1} : Z^{[m+1]} \cup \hat{C}(\bar{E}^{m+1}(\Omega Z)) \to Z^{m+1}$ which makes the following diagram commute up to homotopy, where $\hat{C}$ denotes the functor taking unreduced cones with base point $0\wedge *$:*

$$\begin{tikzcd}
\bar{E}^{m+1}(\Omega Z) \arrow[r, "q^Z_{m+1}"] \arrow[d, equal] & Z^{[m+1]} \arrow[r, hook] \arrow[d, equal] & Z^{[m+1]} \cup \hat{C}(\bar{E}^{m+1}(\Omega Z)) \arrow[r] \arrow[d, "e_{m+1}"] & \wedge^{m+1}\Sigma\Omega Z \arrow[d, "\wedge^{m+1} e^Z_1"] \\
\bar{E}^{m+1}(\Omega Z) \arrow[r, "q^Z_{m+1}"] & Z^{[m+1]} \arrow[r, hook] & Z^{m+1} \arrow[r] & \wedge^{m+1} Z,
\end{tikzcd}$$

where the upper row is a cofibration sequence.

Proof. We define $e_{m+1} : Z^{[m+1]} \cup \hat{C}(\bar{E}^{m+1}(\Omega Z)) \to Z^{m+1}$ by

$$e_{m+1}(t, \Sigma t_i \ell_i) = (\ell_1(t\cdot t_1/t_M), \cdots, \ell_m(t\cdot t_m/t_M)),$$

where $\sum_{0\le i\le m} t_i = 1$ and $t_M = \max(t_0, ..., t_m) > \frac{1}{m+1}$. By collapsing $Z^{[m+1]}$, we obtain a map $\hat{e}_{m+1} : \hat{\Sigma}\bar{E}^{m+1}(\Omega Z) \to \wedge^{m+1} Z$ from e_{m+1}, where $\hat{\Sigma}$ denotes the functor taking unreduced suspensions with base point $0\wedge *$. $\hat{e}_{m+1}$ is given by $\hat{e}_{m+1}(t \wedge \Sigma t_i \ell_i) = \ell_0(tt_0/t_M)\wedge ...\wedge \ell_m(tt_m/t_M)$.

Let us recall the homotopy equivalence $s_{m+1} : \hat{\Sigma}\bar{E}^{m+1}(\Omega Z) \to \wedge^{m+1}\hat{\Sigma}\Omega Z$ given by

$$s_{m+1}(t, \Sigma t_i \ell_i) = (u_i \wedge \ell_i), \quad u_i = t_i \frac{t}{t_M}.$$

Then we have the following relation of maps: $\wedge^{m+1} e^Z_1 \circ h \circ s_{m+1} = \hat{e}_{m+1}$, where $h : \wedge^{m+1}\hat{\Sigma}\Omega Z \to \wedge^{m+1}\Sigma\Omega Z$ is the canonical homotopy equivalence. Thus the proposition follows. *QED.*

Proof of Theorem 4.1. By Remarks 3.4 and 3.6 and Propositions 3.3 and 4.2, we have the following commutative diagram up to homotopy.

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "E^{m+1}(\Omega i)\circ H_m(f)"'] & X \arrow[r, hook, "i"] \arrow[d, "\sigma(X)"] & W \arrow[r, "q"] \arrow[d, "\sigma(W)"] & \Sigma V \arrow[d, "\Sigma(E^{m+1}(\Omega i)\circ H_m(f))"] \\
E^{m+1}(\Omega W) \arrow[r] \arrow[d, "h^W_{m+1}"'] & P^m(\Omega W) \arrow[r, hook] \arrow[d] & P^{m+1}(\Omega W) \arrow[r] \arrow[d, "\Delta_{m+1}\circ e^W_{m+1}"] & \Sigma E^{m+1}(\Omega W) \arrow[d, "\wedge^{m+1} e^W_1 \circ \Sigma h^W_{m+1}"] \\
\bar{E}^{m+1}(\Omega W) \arrow[r, "q^W_m"'] & W^{[m+1]} \arrow[r, hook] & W^{m+1} \arrow[r, "\wedge"'] & \wedge^{m+1} W.
\end{tikzcd}$$

QED.

Remark 4.3 *For any structure map $\sigma(X)$, by naturality, we have*

$$\begin{aligned}\hat{\Delta}_{m+1} &= (\wedge^{m+1}e_1^W)\circ\Sigma h_{m+1}^W\circ\Sigma(E^{m+1}(\Omega i)\circ H_m^{\sigma(X)}(f))\circ q\\ &= (\wedge^{m+1}e_1^W)\circ\Sigma h_{m+1}^W\circ\Sigma E^{m+1}(\Omega i)\circ\Sigma H_m^{\sigma(X)}(f)\circ q\\ &= (\wedge^{m+1}e_1^X)\circ(\wedge^{m+1}\Sigma\Omega i)\circ\Sigma h_{m+1}^X\circ\Sigma(H_m^{\sigma(X)}(f))\circ q\\ &= (\wedge^{m+1}i)\circ(\wedge^{m+1}e_1^X)\circ\Sigma h_{m+1}^X\circ\Sigma(H_m^{\sigma(X)}(f))\circ q = (\wedge^{m+1}i)\circ\bar{H}_m^{\sigma(X)}(f)\circ q,\end{aligned}$$

where the composition $\bar{H}_m^{\sigma(X)}(f) = (\wedge^{m+1}e_1^X)\circ\Sigma h_{m+1}^X\circ\Sigma(H_m^{\sigma(X)}(f))$ is the generalised version of the Berstein-Hilton crude Hopf invariant (see Definition 2.6). Thus we have

$$(\wedge^{m+1}i)_*\circ q^*\bar{H}_m^S(f) = \{\hat{\Delta}_{m+1}\}.$$

5 Homology decomposition and product spaces

In this section, we always assume that X is simply connected finite complex. Then there is a homology decomposition $\{X_t, f_t : S_t(X) \to X_t\}_{t\geq 1}$ of X, where $S_t(X)$ is the Moore space of type $(H_{t+1}(X), t)$ for $t \geq 1$. By modifying the arguments given in Curjel [4], we obtain the following result.

Theorem 5.1 *The homology decomposition $\{X_t, f_t : S_t(X) \to X_t\}_{t\geq 1}$ of X satisfies $0 = \operatorname{cat} X_1 \leq \operatorname{cat} X_2 \leq \ldots \leq \operatorname{cat} X_{t-1} \leq \operatorname{cat} X_t \leq \ldots \leq \operatorname{cat} X$. If, in addition, $\operatorname{cat} X_{k-1} = \operatorname{cat} X_k = m$, then we can choose structure maps $\sigma_{k-1}(X) : X_{k-1} \to P^m(\Omega X_{k-1})$ and $\sigma_k(X) : X_k \to P^m(\Omega X_k)$ for $\operatorname{cat} X_{k-1} = m$ and $\operatorname{cat} X_k = m$ to be compatible with each other, i.e. $\sigma_k(X)|_{X_k} \simeq \sigma_{k-1}(X)$ in $P^m(\Omega X_k)$.*

Proof. To prove the former part of the theorem, it is sufficient to show $\operatorname{cat} X_k \leq \operatorname{cat} X_{k+1}$ for $k \geq 1$. So we may assume that $X = X_{k+1}$ and $\operatorname{cat} X = m$. If $m = 0$, then Theorem 5.1 is clearly true, and hence we may assume that $m \geq 1$. Then there is a homotopy section $\sigma(X) : X \to P^m(\Omega X)$. By induction on k, we show the existence of a compression $\sigma_k(X) : X_k \to P^m(\Omega X_k)$ of $\sigma(X)|_{X_k} : X_k \to P^m(\Omega X)$. In the case $k = 1$, we have $X_1 = \{*\}$, and hence the existence of $\sigma_1(X)$ is clear. In the general case $k > 1$, by the induction hypothesis, we have a compression $\sigma_{k-1}(X) : X_{k-1} \to P^m(\Omega X_{k-1})$ of $\sigma(X)\circ i_{k-1} : X_{k-1} \to P^m(\Omega X)$, where $i_t : X_t \hookrightarrow X$ denotes the canonical inclusion, $t \geq 1$; in addition, $e_m^{X_{k-1}}\circ\sigma_{k-1}(X) \sim 1_{X_{k-1}}$. Let us consider the following commutative diagram:

$$
\begin{tikzcd}
 & & & E^{m+1}(\Omega X_k) \arrow[rr, hook, "E^{m+1}(\Omega i_k)"] \arrow[dd, "p_m^{\Omega X_k}"] & & E^{m+1}(\Omega X) \arrow[dd, "p_m^{\Omega X}"] \\
X_{k-1} \arrow[rr, hook, "i_{k-1,k}"] \arrow[dr, "\sigma_{k-1}(X)"'] & & X_k \arrow[rr, hook, "i_k"] \arrow[dr, dashed] & & X \arrow[dr, "\sigma(X)"] & \\
 & P^m(\Omega X_{k-1}) \arrow[rr, hook, "P^m(\Omega i_{k-1,k})"'] \arrow[d, "e_m^{X_{k-1}}"'] & & P^m(\Omega X_k) \arrow[rr, hook, "P^m(\Omega i_k)"'] \arrow[d, "e_m^{X_k}"'] & & P^m(\Omega X) \arrow[d, "e_m^X"] \\
 & X_{k-1} \arrow[rr, hook, "i_{k-1,k}"] & & X_k \arrow[rr, hook, "i_k"] & & X,
\end{tikzcd}
\tag{5.1}
$$

where $i_{k-1,k} : X_{k-1} \hookrightarrow X_k$ denotes the canonical inclusion. Also the k'-invariant f_{k-1} induces the following cofibre sequence:

$$S_{k-1} \overset{f_{k-1}}{\to} X_{k-1} \overset{i_{k-1,k}}{\hookrightarrow} X_k \overset{g_k}{\to} \Sigma S_{k-1},$$

The obstruction to extend $\sigma_{k-1}(X)$ to X_k is given by a map $\gamma = P^m(\Omega i_{k-1,k})\circ\sigma_{k-1}(X)\circ f_{k-1} : S_{k-1} \to P^m(\Omega X_k)$. But the commutativity of the diagram (5.1) implies that $p_m^{\Omega X_k}\circ\gamma = i_{k-1,k}\circ e_m^{X_{k-1}}\circ\sigma_{k-1}(X)\circ f_{k-1} \sim i_{k-1,k}\circ f_{k-1} \sim 0$. Hence γ has a unique lift $\hat{\gamma} : S_{k-1} \to E^{m+1}(\Omega X_k)$, which vanishes in $E^{m+1}(\Omega X)$, since $P^m(\Omega i_k)\circ\gamma = P^m(\Omega i_{k-1})\circ\sigma_{k-1}(X)\circ f_{k-1} \sim \sigma(X)\circ i_{k-1}\circ f_{k-1} \sim 0$. Since X and X_k are simply connected and (X, X_k) is k-connected, $(E^{m+1}(\Omega X), E^{m+1}(\Omega X_k))$ is $(k+2m-1)$-connected, $m \geq 1$. Hence γ vanishes since the dimension of S_{k-1} is at most $k \leq k+2m-2$. Thus there is a map $\sigma'_k : X_k \to P^m(\Omega X_k)$ such that $\sigma'_k\circ i_{k-1,k} \sim P^m(\Omega i_{k-1,k})\circ\sigma_{k-1}(X)$.

The difference between $e_m^{X_k}\circ\sigma'_k$ and the identity 1_{X_k} with respect to the co-action of ΣS_{k-1} is given by a map $\delta : \Sigma S_{k-1} \to X_k$. By (3.2), with $V = S_{k-1}$, $X = X_{k-1}$ and $W = X_k$, we have a surjection $[\Sigma S_{k-1}, \Sigma\Omega X_k] \to [\Sigma S_{k-1}, X_k]$ and hence δ can be pulled back to $\delta_0 : \Sigma S_{k-1} \to \Sigma\Omega X_k \subset P^m(\Omega X_k)$. By adding δ_0 to σ'_k, we have a compression σ_k of 1_{X_k} which is an extension of $\sigma_{k-1}(X)$. Thus $\operatorname{cat} X_k \leq m$.

The difference between $P^m(\Omega i_k)\circ\sigma_k$ and $\sigma(X)\circ i_k$ with respect to the co-action of ΣS_{k-1} is given by a map $\varepsilon : \Sigma S_{k-1} \to P^m(\Omega X)$ which vanishes in $P^\infty(\Omega X) \simeq X$. Thus ε can be lifted to $\varepsilon_0 : \Sigma S_{k-1} \to E^{m+1}(\Omega X)$. Since $(E^{m+1}(\Omega X), E^{m+1}(\Omega X_k))$ is $(k+2m-1)$-connected and the dimension of ΣS_{k-1} is at most $k+1 \leq k+2m-1$, ε_0 can be compressed into $E^{m+1}(\Omega X_k)$; $\varepsilon_0 : \Sigma S_{k-1} \overset{\varepsilon'_0}{\to} E^{m+1}(\Omega X_k) \subset E^{m+1}(\Omega X)$. Again by adding $p_m^{\Omega X_k}\circ\varepsilon'_0$ to σ_k, we obtain new structure map $\sigma_k(X) : X_k \to P^m(\Omega X_k)$ for $\operatorname{cat} X_k \leq m$ as a compression of $\sigma(X)\circ i_k$, which is also an extension of $\sigma_{k-1}(X)$. Thus we obtain the compression $\sigma_k(X)$ of $\sigma(X)\circ i_k$ for all k. The latter part of the theorem is clear by the definition of $\sigma_k(X)$. *QED.*

Now we apply the results in Section 3 for homology decompositions. Then we can show Theorems 3.8 (1) and 3.9 (1) in a slightly stronger form. Let $\operatorname{cat} X_k = m$ for some $k \geq 1$, and let $f_k(X) : S_k(X) \to X_k$ be the k'-invariant of the k-th stage. We show that the obstruction for X_{k+1} to satisfy $\operatorname{cat} X_{k+1} = m$ is the set of higher Hopf invariants $H^S_m(f_k(X))$.

Theorem 5.2 *$\operatorname{cat} X_{k+1} = m$ if and only if $E^{m+1}(\Omega i_{k,k+1})_* H^S_m(f_k(X))$ contains 0. Moreover, if (i) $m \geq 2$ or (ii) X is 2-connected or more generally $\operatorname{Ext}(H_{k+1}(X), H_2(X) \otimes H_{k+1}(X)) = 0$, then $\operatorname{cat} X_{k+1} = m$ if and only if $H^S_m(f_k(X))$ contains 0.*

Proof. If $\operatorname{cat} X_{k+1} = m$, then by Theorem 5.1, there exists a structure map $\sigma_k(X)$ for $\operatorname{cat} X_k = m$ such that $P^m(\Omega i_{k,k+1}) \circ \sigma_k(X)$ is extendible to X_{k+1}. For this particular choice of $\sigma_k(X)$, we obtain, by Proposition 3.3, the following diagram except for the dotted arrows commutative up to homotopy.

$$\begin{tikzcd}
 & S_k(X) \arrow[rr, "f_k(X)"] \arrow[d, "H^{\sigma_k(X)}_m(f_k(X))"'] & & X_k \arrow[r, hook, "i_{k,k+1}"] \arrow[d, "\sigma_k(X)"] & X_{k+1} \arrow[dd, dashed, "\sigma_{k+1}"] \\
\Omega X_k \arrow[r, hook] \arrow[d, tail, "\Omega i_{k,k+1}"'] & E^{m+1}(\Omega X_k) \arrow[rr, dashed, "p^{\Omega X_k}_m"] \arrow[d, tail, "E^{m+1}(\Omega i_{k,k+1})"'] & & P^m(\Omega X_k) \arrow[d, tail, "P^m(\Omega i_{k,k+1})"] & \\
\Omega X_{k+1} \arrow[r, hook] & E^{m+1}(\Omega X_{k+1}) \arrow[rr, "p^{\Omega X_{k+1}}_m"'] & & P^m(\Omega X_{k+1}) \arrow[r, hook, "\iota^{\Omega X_{k+1}}_m"'] & P^{m+1}(\Omega X_{k+1}).
\end{tikzcd}$$

Thus the extendibility of $P^m(\Omega i_{k,k+1}) \circ \sigma_k(X)$ implies that $p^{\Omega X_{k+1}}_m \circ E^{m+1}(\Omega i_{k,k+1}) \circ H^{\sigma_k(X)}_m(f_k(X)) \sim P^m(\Omega i_{k,k+1}) \circ \sigma_k(X) \circ f_k(X)$ is trivial. Since $p^{\Omega X_{k+1}}_m$ induces a split monomorphism of homotopy groups with any coefficient groups, it follows that $E^{m+1}(\Omega i_{k,k+1}) \circ H^{\sigma_k(X)}_m(f_k(X))$ is trivial. We know the pair $(E^{m+1}(\Omega X_{k+1}), E^{m+1}(\Omega X_k))$ is $(k+2m-1)$-connected, and hence, by Universal Coefficient Theorem for homotopy groups, we obtain

$$\begin{aligned}
&\pi_k(E^{m+1}(\Omega X_{k+1}), E^{m+1}(\Omega X_k); H_{k+1}(X)) = 0 \quad \text{and} \\
&\pi_{k+1}(E^{m+1}(\Omega X_{k+1}), E^{m+1}(\Omega X_k); H_{k+1}(X)) \\
&\qquad = \operatorname{Ext}(H_{k+1}(X), H_{k+2}(E^{m+1}(\Omega X_{k+1}), E^{m+1}(\Omega X_k)))
\end{aligned}$$

is trivial unless $m = 1$. When $m = 1$, for dimensional reasons, we have

$$\begin{aligned}
H_{k+2}(E^2(\Omega X_{k+1}), E^2(\Omega X_k)) &\cong H_{k+2}(S_1(X) * S_k(X) \vee S_k(X) * S_1(X)) \\
&\cong H_2(X) \otimes H_{k+1}(X) \oplus H_{k+1}(X) \otimes H_2(X),
\end{aligned}$$

and hence

$$\begin{aligned}&\pi_{k+1}(E^{m+1}(\Omega X_{k+1}), E^{m+1}(\Omega X_k); H_{k+1}(X))\\&\qquad\cong \operatorname{Ext}(H_{k+1}(X), H_2(X)\otimes H_{k+1}(X))\oplus \operatorname{Ext}(H_{k+1}(X), H_{k+1}(X)\otimes H_2(X))\end{aligned}$$

is trivial if the condition (ii) is satisfied. Hence $E^{m+1}(\Omega i_{k,k+1})_* : [S_k(X), E^{m+1}(\Omega X_k)] = \pi_k(E^{m+1}(\Omega X_k)) \to [S_k(X), E^{m+1}(\Omega X_{k+1})] = \pi_k(E^{m+1}(\Omega X_{k+1}))$ is an isomorphism. Thus we obtain that the set $H^S_m(f_k(X))$ contains $0 = H^{\sigma_k(X)}_m(f_k(X))$. The converse is an immediate consequence of Theorem 3.8 (1). *QED.*

Let $\operatorname{cat} X_{k+1} = m+1$, in other words, the set $H^S_m(f_k(X))$ does not contain 0. Then the following theorem is an immediate consequence of Theorem 3.8 (2).

Theorem 5.3 *If the set* $\Sigma^n_* H^S_m(f_k(X))$ *contains* 0*, then* $\operatorname{cat} X_{k+1}\times S^n = \operatorname{cat} X_{k+1} = m+1$.

Corollary 5.3.1 *If the set* $\mathcal{H}^S_m(f_k(X))$ *contains* 0*, then* X_{k+1} *is a counter example to Ganea's conjecture.*

Also let $n \geq 1$, $\operatorname{cat} Y \geq n+1$. Then, by Theorem 5.1, one can find an integer $h \geq 1$ such that $\operatorname{cat} Y_{h+1} = n+1$ and $\operatorname{cat} Y_h = n$. Let $f_h(Y) : S_h(Y) \to Y_h$ be the k'-invariant of the h-th stage. We know, by Theorem 5.2, that the obstruction for Y_{h+1} to satisfy $\operatorname{cat} Y_{h+1} \leq n$ is the set of Hopf invariants $H^S_h(f_h(Y))$. We define another set

Definition 5.4

$$H^S_{m,n}(f_k(X), f_h(Y)) = \{g_X * g_Y \mid g_X \in H^S_m(f_k(X)) \text{ and } g_Y \in H^S_n(f_h(Y))\}$$

Then we have the following theorem.

Theorem 5.5 *If the set* $H^S_{k,h}(X,Y)$ *contains* 0*, then* $\operatorname{cat} X_{k+1}\times Y_{h+1} < m+n+2 = \operatorname{cat} X_{k+1} + \operatorname{cat} Y_{h+1}$.

Proof. The proof is obtained by a similar argument given in the proof of Theorem 3.8 (2) using the following diagram instead of the diagram in Proposition 3.7:

$$\begin{tikzcd}
S_{k+h+1}(X_{k+1}\times Y_{h+1}) \arrow[r, "{[f_k(X),f_h(Y)]^r}"] \arrow[d, "H^{\sigma_k(X)}_{m+1}(f_k(X))*H^{\sigma_h(Y)}_{n+1}(f_h(Y))"] & X_{k+1}\times Y_h\cup X_k\times Y_{h+1} \arrow[r, hook] \arrow[d, "(\sigma_{k+1}(X)\times\sigma_{h+1}(Y))|"] & X_{k+1}\times Y_{h+1} \arrow[dd, dashed] \\
E^{m+1}(\Omega X_k)*E^{n+1}(\Omega Y_h) \arrow[r, dashed, "{[p^{\Omega X_k}_m, p^{\Omega Y_h}_n]^r}"] \arrow[d, tail, "E^{m+1}(\Omega i_{k,k+1})*E^{n+1}(\Omega i_{h,h+1})"] & \begin{array}{c}P^{m+1}(\Omega X_{k+1})\times P^n(\Omega Y_h)\\ \cup P^m(\Omega X_k)\times P^{n+1}(\Omega Y_{h+1})\end{array} \arrow[d, tail] & \\
E^{m+1}(\Omega X_{k+1})*E^{n+1}(\Omega Y_{h+1}) \arrow[r, "{[p^{\Omega X_{k+1}}_m, p^{\Omega Y_{h+1}}_n]^r}"] & \begin{array}{c}P^{m+1}(\Omega X_{k+1})\times P^n(\Omega Y_{h+1})\\ \cup P^m(\Omega X_{k+1})\times P^{n+1}(\Omega Y_{h+1})\end{array} \arrow[r, hook] & P^{m+1}(\Omega X_{k+1})\times P^{n+1}(\Omega Y_{h+1}),
\end{tikzcd}$$

The details are left to the reader. *QED.*

6 Higher Hopf invariants for some examples

In this section, we compute the higher Hopf invariants for well-known examples, which yields a generalisation of the main result of [11]: We denote by $\mathbb{C} \approx \mathbb{R}^2$ the field of complex numbers, by $\mathbb{H} \approx \mathbb{R}^4$ the algebra of quaternion numbers and by $\mathbb{O} \approx \mathbb{R}^8$ the Cayley algebra:

Example 6.1 *We know that* $\operatorname{cat} \mathbb{C}P^m = m$ *and* $\dim \mathbb{C}P^m = 2m$. *Hence* $\dim \mathbb{C}P^m = 2m \leq 2m + 2 - 2$. *Thus* $\mathbb{C}P^m$ *has a unique structure for* $\operatorname{cat} \mathbb{C}P^m = m$. *The higher Hopf invariant* $H_m : [V, \mathbb{C}P^m] \to [V, \Omega\mathbb{C}P^m * \cdots * \Omega\mathbb{C}P^m]$ *gives a (unique) homomorphism*

$$H_m : \pi_{2m+1}(\mathbb{C}P^m) \to \pi_{2m+1}(\Omega\mathbb{C}P^m * \cdots * \Omega\mathbb{C}P^m) \cong \mathbb{Z}$$

with the canonical projection $p_m^{S^1} : S^{2m+1} \to \mathbb{C}P^m$ *a 'higher Hopf invariant one' element.*

Example 6.2 *We know that* $\operatorname{cat} \mathbb{H}P^m = m$ *and* $\dim \mathbb{H}P^m = 4m$. *Hence* $\dim \mathbb{H}P^m = 4m \leq 4m + 4 - 2$. *Thus* $\mathbb{H}P^m$ *has a unique structure for* $\operatorname{cat} \mathbb{H}P^m = m$. *The higher Hopf invariant* $H_m : [V, \mathbb{H}P^m] \to [V, \Omega\mathbb{H}P^m * \cdots * \Omega\mathbb{H}P^m]$ *gives a (unique) homomorphism*

$$H_m : \pi_{4m+3}(\mathbb{H}P^m) \to \pi_{4m+3}(\Omega\mathbb{H}P^m * \cdots * \Omega\mathbb{H}P^m) \cong \mathbb{Z}$$

with the canonical projection $p_m^{S^3} : S^{4m+3} \to \mathbb{H}P^m$ *a 'higher Hopf invariant one' element.*

Example 6.3 *We know that* $\operatorname{cat} \mathbb{O}P^2 = 2$ *and* $\dim \mathbb{O}P^2 = 16$. *Hence* $\dim \mathbb{O}P^2 = 16 \leq 8 \times 2 + 8 - 2$. *Thus* $\mathbb{O}P^2$ *has a unique structure for* $\operatorname{cat} \mathbb{O}P^2 = 2$. *The higher Hopf invariant* $H_2 : [V, \mathbb{O}P^2] \to [V, \Omega\mathbb{O}P^2 * \Omega\mathbb{O}P^2 * \Omega\mathbb{O}P^2]$ *gives a (unique) homomorphism*

$$H_2 : \pi_{23}(\mathbb{O}P^2) \to \pi_{23}(\Omega\mathbb{O}P^2 * \Omega\mathbb{O}P^2 * \Omega\mathbb{O}P^2) \cong \mathbb{Z}.$$

But there are no elements of 'higher Hopf invariant one': The existence of such a higher Hopf invariant one element implies that the Hopf space S^7 *is homotopy associative. As is well-known, the p-local Hopf space* $S^7_{(p)}$ *is homotopy associative for* $p \geq 5$ *(in view of [24], [22] and [13], it actually is an* A_{p-1}*-space). However, by using primary cohomology operations, one can easily see that any Hopf structure of* $S^7_{(3)}$ *is not homotopy associative. Hopf space theorists were, however, much more interested in the case* $p = 2$. *And it was known by Goncalves [7], using higher order cohomology operations, and by Hubbuck [9], using K-theory Adams operations, that any Hopf structure of* $S^7_{(2)}$ *is not homotopy associative (but the result itself had already been known by James). Hence, the image of the higher Hopf invariant homomorphism is in* $6\mathbb{Z} \subset \mathbb{Z}$.

Example 6.4 *For $m,n \geq 1$ and $p \geq m+2$, let $f_{m,p} = p_m^{S^1} \circ g_{m,p} : S^{2mp+2(p-1)-1} \to \mathbb{C}P^m$, where $g_{m,p} : S^{2mp+2(p-1)-1} \to S^{2m+1}$ denotes the generator of $\pi_{2mp+2(p-1)-1}(S^{2m+1}) \cong \mathbb{Z}/p\mathbb{Z}$ and $p_m^{S^1} : S^{2m+1} \to \mathbb{C}P^m$ denotes the projection which gives a 'higher Hopf invariant one' element as in Example 6.1. For the dimensional reasons, the map $g_{m,p}$ is a co-H-map, and hence $H_m(f_{m,p}) = g_{m,p} \neq 0$, and hence $\Sigma^n H_m(f_{m,p})$ is trivial if and only if $n \geq 2$, by Theorem 13.4 in Toda [25].*

Let $Q_{m,p}$ be the mapping cone of $f_{m,p}$: $Q_{m,p} = \mathbb{C}P^m \cup_{f_{m,p}} e^{2mp+2(p-1)}$. Then, by Theorems 3.8 and 3.9 and Remark 3.10, it follows that $\operatorname{cat} Q_{m,p} = m+1$ and $m+1 \leq \operatorname{cat}(Q_{m,p} \times S^1) \leq m+2$ but $\operatorname{cat}(Q_{m,p} \times S^n) = m+1$ for $n \geq 2$ by Theorem 5.3.

Remark 6.5 *(1) Every example in this section supports Conjecture 2.3.*

(2) The space $Q_{m,p}$ in Example 6.4 is a generalisation of Q_p in [11] except Q_2. Actually, Ganea's conjecture for $Q_{m,p}$ is true if we consider cat_q, for $q \neq p$, instead of cat or cat_p.

7 LS category of sphere bundles over spheres

Let $r \geq 1$, $t \geq 1$ and E be a fibre bundle over S^{t+1} with fibre S^r. Then E has a CW decomposition $S^r \cup_\alpha e^{t+1} \cup_\psi e^{t+r+1}$ with $\alpha : S^t \to S^r$ and $\psi : S^{t+r} \to Q$, $Q = S^r \cup_\alpha e^{t+1}$. We identify $H_1^S(\alpha)$ with its unique element, say $H_1(\alpha)$, since a sphere S^k has the unique structure map for $\operatorname{cat} S^k = 1$.

Fact 7.1 *Let $\alpha = \pm 1_{S^r}$ the identity. Then clearly $\operatorname{cat} Q = 0$ and $\operatorname{cat} E = 1$. In addition, $\operatorname{cat} Q \times S^n = 1$ and $\operatorname{cat} E \times S^n = 2$ for $n \geq 1$.*

The following fact is an immediate consequence of Berstein-Hilton [2] and a cup length consideration.

Fact 7.2 *Let $\alpha \neq \pm 1_{S^r}$. Hence $1 \leq \operatorname{cat} Q \leq 2$. Then $\operatorname{cat} Q = 2$ if and only if $H_1(\alpha) \neq 0$. In particular if $H_1(\alpha) = 0$, we can easily obtain that $\operatorname{cat} Q = 1$ and $\operatorname{cat} E = 2$. In this case, it also follows that $\operatorname{cat} Q \times S^n = 2$ and $\operatorname{cat} E \times S^n = 3$ for $n \geq 1$.*

By Theorem 3.9 and Remark 3.10, we can extend the main result of [11].

Theorem 7.3 *Let $H_1(\alpha) \neq 0$. Hence $\operatorname{cat} Q = 2$. Then for $n \geq 1$, $\operatorname{cat} Q \times S^n = 3$ if $\Sigma^n H_1(\alpha) \neq 0$ with $n \geq t - 2r + 2$ or $\Sigma^{n+1} H_1(\alpha) \neq 0$.*

We give a partial answer to Ganea's Problem 4 (see [6]) for the total space of sphere-bundles-over-spheres. To show this, we need the following lemma.

Lemma 7.4 *The collapsing map $q : E \to E/Q = S^{t+r+1}$ induces a map with trivial kernel*

$$(\Sigma^n q)^* : [S^{n+r+t+1}, S^{n+kr}] \to [\Sigma^n E, S^{n+kr}] \quad \textit{for all } k \geq 3 \textit{ and } n \geq 0.$$

Proof. The cofibration sequences $S^{r+t} \xrightarrow{\psi} Q \xrightarrow{j} E$ and $S^t \xrightarrow{\alpha} S^r \xrightarrow{i} Q$ together with the bundle projection $p : E \to S^{t+1}$ induce the following commutative diagram:

$$\begin{tikzcd}
 & {[S^{n+t+2}, S^{n+kr}]} \arrow[dl, "\Sigma^{n+1}p^*"'] \arrow[d, "(\Sigma^{n+1}p|_{\Sigma^{n+1}Q})^*"] & & \\
{[\Sigma^{n+1}E, S^{n+kr}]} \arrow[r, "\Sigma^{n+1}j^*"'] & {[\Sigma^{n+1}Q, S^{n+kr}]} \arrow[r, "\Sigma^{n+1}\psi^*"'] \arrow[d, "\Sigma^{n+1}i^*"] & {[S^{n+r+t+1}, S^{n+kr}]} \arrow[r, "(\Sigma^n q)^*"'] & {[\Sigma^n E, S^{n+kr}]} \\
 & {[S^{n+r+1}, S^{n+kr}],} & &
\end{tikzcd}$$

where the column and row are exact sequences. Since $k \geq 3$, we have $n + r + 1 < n + kr$ and $\pi_{n+r+1}(S^{n+kr}) = 0$. Hence $(\Sigma^{n+1}p|_{\Sigma^{n+1}Q})^*$ is surjective, and so is $(\Sigma^{n+1}j)^*$. Thus $(\Sigma^{n+1}\psi)^*$ is trivial and the map $(\Sigma^n q)^*$ has trivial kernel for $k \geq 3$ and $n \geq 0$. *QED.*

Theorem 7.5 *Let $H_1(\alpha) \neq 0$. Hence $2 \leq \operatorname{cat} E \leq 3$. Then $\operatorname{cat} E = 3$ if $\Sigma^{r+1}h_2(\alpha) \neq 0$. Also $\operatorname{cat} E = 2$ if $H_2^S(\psi)$ or its subset $H_2^{SS}(\psi)$ (see Remark 3.6 for its definition) contains 0.*

Remark 7.6 *In the latter case of $\operatorname{cat} E = 2$, it is known that $\operatorname{cat} E \times S^n = 3$ for $n \geq 1$ by using cup-length argument on the cohomology ring (see Singhof [20]).*

Proof of Theorem 7.5. Let $q' : E \to E/S^r$, $q'' : Q \to Q/S^r$ be respectively the collapsing maps. The reduced diagonal map $\hat{\Delta}_2 : E \to E \wedge E$ factors as $E \xrightarrow{q'} S^{t+1} \cup e^{r+t+1} \xrightarrow{\hat{\Delta}_2} (S^r \cup e^{t+1}) \wedge (S^r \cup e^{t+1}) \subset E \wedge E$, which is an extension of the map $Q \xrightarrow{q} S^{t+1} \xrightarrow{\Sigma h_2(\alpha)} S^r \wedge S^r \subset E \wedge E$ by Theorem 5.14 of Boardman-Steer [3]. In this case, the generator in $H^{r+t+1}(E;\mathbb{Z})$ is a cup product of generators in $H^r(E;\mathbb{Z})$ and $H^{t+1}(E;\mathbb{Z})$. Then it follows that the mapping degrees of $\hat{\Delta}_2$ on $e^r \wedge e^{t+1}$ and $e^{t+1} \wedge e^r$ are 1. Hence the reduced diagonal map $\hat{\Delta}_3 = (\hat{\Delta}_2 \wedge 1_E) \circ \hat{\Delta}_2 : E \to E \wedge E \wedge E$ factors as $E \xrightarrow{q} S^{r+t+1} \xrightarrow{\simeq} S^{t+1} \wedge S^r \xrightarrow{\Sigma^{r+1}h_2(\alpha)} S^r \wedge S^r \wedge S^r \subset E \wedge E \wedge E$. Hence $\hat{\Delta}_3$ is the composition of $\Sigma^{r+1}h_2(\alpha) \circ q$ with a suitable inclusion, which does not depend on the choice of $\sigma(Q)$. By

Lemma 7.4, $q^* : [S^{r+t+1}, S^{3r}] \to [E, S^{3r}]$ has trivial kernel, and hence the non-triviality of $\Sigma^{r+1}h_2(\alpha)$ implies the non-triviality of $\Sigma^{r+1}h_2(\alpha)\circ q$. Then, for dimensional reasons, it follows that $\hat{\Delta}_3$ is also non-trivial. Then by Theorem 4.1, it follows that $H^S_2(\psi)$ does not contain 0, and hence we see that $\mathrm{cat}\, E = 3$ by Theorem 5.2. The latter part is clear by Theorem 5.2. *QED.*

We next study the LS category of $E\times S^n$. To do this, we need the following lemma.

Lemma 7.7 *The map $\hat{q} : E\times S^n \to S^{n+r+t+1}$ induces a map with trivial kernel*

$$\hat{q}^* : [S^{n+r+t+1}, S^{n+kr}] \to [E\times S^n, S^{n+kr}] \quad \textit{for all } k \geq 3 \textit{ and } n \geq 1.$$

Proof. Let us recall that the space $E \vee S^n$ is a retractile subspace (see Zabrodsky [28]) of both $E\times S^n$ and $E \cup Q\times S^n$. The cofibration sequences $S^{n+r+t} \overset{\psi'}{\to} E \cup Q\times S^n \overset{j'}{\hookrightarrow} E\times S^n$, $E \vee S^n \overset{i_0}{\hookrightarrow} E\times S^n \overset{q_0}{\to} E\wedge S^n$ and $E \vee S^n \overset{i'_0}{\hookrightarrow} E \cup Q\times S^n \overset{q'_0}{\to} Q\wedge S^n$ induce the following commutative diagram:

$$\begin{tikzcd}
{[\Sigma(E\wedge S^n), S^{n+kr}]} \arrow[r, "(\Sigma^{n+1}q_0)^*"] \arrow[d, "(\Sigma^{n+1}j)^*"'] & {[\Sigma(E\times S^n), S^{n+kr}]} \arrow[r, "(\Sigma i_0)^*"] \arrow[d, "(\Sigma j')^*"] & {[\Sigma E\vee\Sigma S^n, S^{n+kr}]} \arrow[d, equal] \\
{[\Sigma(Q\wedge S^n), S^{n+kr}]} \arrow[r, "(\Sigma^{n+1}q'_0)^*"] \arrow[d, "(\Sigma^{n+1}\psi)^*"'] & {[\Sigma(E \cup Q\times S^n), S^{n+kr}]} \arrow[r, "(\Sigma i'_0)^*"] \arrow[d, "(\Sigma\psi')^*"] & {[\Sigma E\vee\Sigma S^n, S^{n+kr}]} \\
{[S^{n+r+t+1}, S^{n+kr}]} \arrow[r, equal] & {[S^{n+r+t+1}, S^{n+kr}]} \arrow[d, "\hat{q}^*"] & \\
 & {[E\times S^n, S^{n+kr}],} &
\end{tikzcd}$$

where the columns are exact sequences and the rows are split short exact sequences with natural splittings (see Zabrodsky [28]). By the proof of Lemma 7.4, $(\Sigma^{n+1}j)^*$ is surjective, and hence so is $(\Sigma j')^*$. Thus $(\Sigma\psi')^*$ is trivial, and hence the map $\hat{q}^*$ has trivial kernel for $k \geq 3$ and $n \geq 0$. *QED.*

Theorem 7.8 *Let $\Sigma^{r+1}h_2(\alpha) \neq 0$. Hence $\mathrm{cat}\, E = 3$. Then for $n \geq 1$, $\mathrm{cat}\, E\times S^n = 4$ if $\Sigma^{n+r+1}h_2(\alpha) \neq 0$. Also for $n \geq 1$, $\mathrm{cat}\, E\times S^n = 3$ if $H^S_3(\psi')$, $\Sigma^n_* H^S_2(\psi)$ or $\Sigma^n_* H^{SS}_2(\psi)$ contains 0.*

Proof. By the proof of Theorem 7.5, the n-fold suspension of the reduced diagonal map $\hat{\Delta}_3\wedge 1_{S^n} : E\wedge S^n \to E\wedge E\wedge E\wedge S^n$ is the composition of $\Sigma^n \bar{H}^\sigma_2(\psi)\circ\hat{q} = \Sigma^{n+r+1}h_2(\alpha)\circ\hat{q}$ with a suitable inclusion, which does not depend on the choice of $\sigma = \sigma(E \cup Q \times S^n)$. By Lemma 7.7, $\hat{q}^* : [S^{n+r+t+1}, S^{n+3r}] \to [E \times S^n, S^{n+3r}]$ has trivial kernel, and hence the non-triviality of $\Sigma^{n+r+1} h_2(\alpha)$ implies the non-triviality of $\Sigma^{n+r+1} h_2(\alpha) \circ \hat{q}$. Then, for dimensional reasons, it follows that $\hat{\Delta}_3 \wedge 1_{S^n}$ is also non-trivial, and hence the four-fold reduced diagonal $\hat{\Delta}'_4$ of $E \times S^n$ is non-trivial. Thus by Theorem 4.1, $H^S_3(\psi')$ does not contain 0, and hence we see that $\operatorname{cat} E \times S^n = 4$ by Theorem 5.2. The latter part is clear by Theorem 5.3. *QED.*

8 Manifold examples

The Hopf fibration $\sigma_4 : S^7 \to S^4$ is given as a principal $Sp(1)$-bundle. Taking orbits of the action of $U(1) \subset Sp(1)$ on S^7, we obtain a fibre bundle $\mathbb{C}P^3 \to S^4$ with fibre $Sp(1)/U(1) \approx S^2$; the structure group $Sp(1)$ acts on the fibre S^2 via a map, say $\mu_0 : S^3 \times S^2 \to S^2$. Here the CW decomposition of $\mathbb{C}P^3$ is known as $\mathbb{C}P^3 = \mathbb{C}P^2 \cup_{p_2^{S^1}} e^6 = S^2 \cup_\eta e^4 \cup_{p_2^{S^1}} e^6$. Hence the attaching map $p_2^{S^1}$ of the top cell of $\mathbb{C}P^3$ is given by the composition

$$S^3 * S^1 \xrightarrow{[C(1_{S^3}), \Sigma 1_{S^1}]^r} C(S^3) \cup S^3 \times S^2 \xrightarrow{\hat{\chi}_4} \mathbb{C}P^2,$$

where $\hat{\chi}_4$ denotes the map defined by $\hat{\mu}|_{S^3 \times S^2} = \mu_0$ and $\hat{\mu}|_{C(S^3)} = \chi_4$ the characteristic map of the top cell of $\mathbb{C}P^2$.

Definition 8.1 *For any map $\beta : S^q \to S^3$, we may assume that the suspension $\Sigma\beta : S^{q+1} \to S^4$ is a C^∞-map by suitably deforming it up to homotopy, since S^{q+1} and S^4 are closed C^∞-manifolds. We define $E(\beta)$ to be the total space of the $Sp(1)$-bundle $E(\beta) \to S^{q+1}$ induced by the C^∞-map $\Sigma\beta$ from the $Sp(1)$-bundle $\mathbb{C}P^3 \to S^4$. Hence $E(\beta)$ is an orientable, closed C^∞-manifold with CW decomposition $E(\beta) = S^2 \cup_{\eta\circ\beta} e^{q+1} \cup_{\psi(\beta)} e^{q+3}$.*

For a map $\beta : S^q \to S^3$ and a suspension map $\gamma : S^{q'} \to S^q$ with $3 \le q \le q'$, we denote $\beta' = \beta \circ \gamma$, and then we have $E(\beta') = S^2 \cup_{\eta\circ\beta'} e^{q'+1} \cup_{\psi(\beta')} e^{q'+3}$. By putting $Q(\beta) = S^2 \cup_{\eta\circ\beta} e^{q+1} \subset E(\beta)$ and $Q(\beta') = S^2 \cup_{\eta\circ\beta'} e^{q'+1} \subset E(\beta')$, we have the following commutative ladder of cofibre sequences:

$$\begin{tikzcd}
S^{q'} \arrow[r, "\eta\circ\beta'"] \arrow[d, "\gamma"] & S^2 \arrow[r, hook] \arrow[d, equal] & Q(\beta') \arrow[r] \arrow[d, "\hat{\gamma}_0"] & S^{q'+1} \arrow[r] \arrow[d, "\Sigma\gamma"] & S^3, \arrow[d, equal] \\
S^q \arrow[r, "\eta\circ\beta"] & S^2 \arrow[r, hook] & Q(\beta) \arrow[r] & S^{q+1} \arrow[r] & S^3,
\end{tikzcd} \tag{8.1}$$

where $\hat{\gamma}_0 = \hat{\gamma}|_{Q(\beta')}$ and $\hat{\gamma} : E(\beta') \to E(\beta)$ is the bundle map induced from $\Sigma\gamma : S^{q'+1} \to S^{q+1}$. To compare the higher Hopf invariant of $\psi(\beta)$ with that of $\psi(\beta')$, we show the following proposition.

Proposition 8.2 *With respect to the 'standard' structure maps (see Remark 3.6), $\hat{\gamma}_0$ is 'primitive' in the sense of Berstein and Hilton.*

Remark 8.3 *$\mathbb{C}P^2$ has a unique structure map $\sigma_2(\mathbb{C}P^2)$ for $\mathrm{cat}\,\mathbb{C}P^2 \leq 2$.*

Proof of Proposition 8.2. By Proposition 3.3 and Remark 3.4, there are maps $\sigma_2'(Q(\beta)) : Q(\beta) \to P^2(\Omega Q(\beta))$ and $\sigma_2'(Q(\beta')) : Q(\beta') \to P^2(\Omega Q(\beta'))$. Since γ is a suspension map, the 'naturality' of Lemma 3.5 imply a homotopy (relative to S^2):

$$P^2(\Omega\hat{\gamma}_0)\circ\sigma_2'(Q(\beta')) \sim \sigma_2'(Q(\beta))\circ\hat{\gamma}_0. \tag{8.2}$$

The difference between $e_2^{Q(\beta)}\circ\sigma_2'(Q(\beta))$ and $1_{Q(\beta)}$ with respect to the co-action of S^{q+1} is given by a map $\delta : S^{q+1} \to Q(\beta)$ which can be pulled back to a map $\delta_0 : S^{q+1} \to \Sigma\Omega Q(\beta)$. Also the difference between $e_2^{Q(\beta')}\circ\sigma_2'(Q(\beta'))$ and $1_{Q(\beta')}$ with respect to the co-action of $S^{q'+1}$ is given by a map $\delta' : S^{q'+1} \to Q(\beta')$ which can be pulled back to a map $\delta_0' : S^{q'+1} \to \Sigma\Omega Q(\beta')$. Let $\sigma_2(Q(\beta)) = \sigma_2'(Q(\beta)) + \iota_1^{\Omega Q(\beta)}\circ\delta_0$ and $\sigma_2(Q(\beta')) = \sigma_2'(Q(\beta')) + \iota_1^{\Omega Q(\beta')}\circ\delta_0'$, where the addition is induced from the co-actions of $S^{q+1} = Q(\beta)/S^2$ on $Q(\beta)$ and of $S^{q'+1} = Q(\beta')/S^2$ on $Q(\beta')$, respectively. Then $\sigma_2(Q(\beta))$ and $\sigma_2(Q(\beta'))$ are genuine compressions of $1_{Q(\beta)}$ and $1_{Q(\beta')}$.

Using the homotopy (8.2), we obtain a homotopy (relative to S^2)

$$\hat{\gamma}_0\circ e_2^{Q(\beta')}\circ\sigma_2'(Q(\beta')) \sim e_2^{Q(\beta)}\circ P^2(\Omega\hat{\gamma}_0)\circ\sigma_2'(Q(\beta')) \sim e_2^{Q(\beta)}\circ\sigma_2'(Q(\beta))\circ\hat{\gamma}_0,$$

and hence a homotopy (relative to S^2)

$$\begin{aligned} e_2^{Q(\beta)}\circ\sigma_2'(Q(\beta))\circ\hat{\gamma}_0 + \delta\circ\Sigma\gamma &\sim (e_2^{Q(\beta)}\circ\sigma_2'(Q(\beta)) + \delta)\circ\hat{\gamma}_0 \sim \hat{\gamma}_0 \sim \hat{\gamma}_0\circ e_2^{Q(\beta')}\circ\sigma_2(Q(\beta')) \\ &\sim \hat{\gamma}_0\circ e_2^{Q(\beta')}\circ\sigma_2'(Q(\beta')) + \hat{\gamma}_0\circ\delta' \sim e_2^{Q(\beta)}\circ\sigma_2'(Q(\beta))\circ\hat{\gamma}_0 + \hat{\gamma}_0\circ\delta'. \end{aligned}$$

Thus $\delta\circ\Sigma\gamma$ and $\hat{\gamma}_0\circ\delta'$ are homotopic in $\pi_{q'+1}(Q(\beta'))$. Hence the difference of $\delta_0\circ\Sigma\gamma$ and $\Sigma\Omega\hat{\gamma}_0\circ\delta_0'$ is given by a map ϵ_0 in the homotopy group $\pi_{q'+1}(\Sigma\Omega Q(\beta))$ which vanishes in $\pi_{q'+1}(Q(\beta'))$. Thus ϵ_0 can be lifted to a map $\hat{\epsilon}_0 \in \pi_{q'+1}(E^2(\Omega Q(\beta)))$ by the arguments given in Remark 3.6. This implies that $\iota_1^{\Omega Q(\beta)}\circ\epsilon_0$ is trivial in $\pi_{q'+1}(P^2(\Omega Q(\beta)))$. This yields the following homotopy relative to S^2:

$$\begin{aligned} P^2(\Omega\hat{\gamma}_0)\circ\sigma_2(Q(\beta')) &\sim P^2(\Omega\hat{\gamma}_0)\circ\sigma_2'(Q(\beta')) + P^2(\Omega\hat{\gamma}_0)\circ\iota_1^{\Omega Q(\beta')}\circ\delta_0' \\ &\sim P^2(\Omega\hat{\gamma'}_0)\circ\sigma_2'(Q(\beta')) + \iota_1^{\Omega Q(\beta)}\circ\delta_0\circ\Sigma\gamma + \iota_1^{\Omega Q(\beta)}\circ\epsilon_0 \\ &\sim \sigma_2'(Q(\beta))\circ\hat{\gamma}_0 + \delta\circ\Sigma\gamma \sim \sigma_2(Q(\beta))\circ\hat{\gamma}_0. \end{aligned}$$

This completes the proof of Proposition 8.2. *QED.*

The attaching map $\psi(\beta)$ of the top cell of $E(\beta)$ is given by the composition

$$S^q{*}S^1 \xrightarrow{[C(1_{S^q}),\Sigma 1_{S^1}]^r} C(S^q)\cup S^q\times S^2 \xrightarrow{\hat{\chi}_\beta} Q(\beta),$$

where $\hat{\chi}_\beta$ denotes the map defined by $\hat{\mu}_\beta|_{S^q\times S^2} = \mu_0\circ(\beta\times 1_{S^2})$ and $\hat{\mu}_\beta|_{C(S^q)} = \chi_{q+1}$ the characteristic map of the top cell of $Q(\beta)$. Then a direct calculation shows that the following diagram is strictly commutative:

$$\begin{tikzcd}
S^{q'}{*}S^1 \arrow[r, "{[C(1_{S^{q'}}),\Sigma 1_{S^1}]^r}"] \arrow[d, "\gamma{*}1_{S^1}"] & C(S^{q'})\cup S^{q'}\times S^2 \arrow[r, "\hat{\chi}_{\beta'}"] \arrow[d, "\bar{\gamma}_0"] & Q(\beta') \arrow[d, "\hat{\gamma}_0"] \\
S^q{*}S^1 \arrow[r, "{[C(1_{S^q}),\Sigma 1_{S^1}]^r}"] & C(S^q)\cup S^q\times S^2 \arrow[r, "\hat{\chi}_\beta"] & Q(\beta)
\end{tikzcd} \tag{8.3}$$

where $\bar{\gamma}_0$ is given by $\bar{\gamma}_0|_{C(S^{q'})} = C(\gamma)$ and $\bar{\gamma}_0|_{S^{q'}\times S^2} = \gamma\times 1_{S^2}$. Thus we have that $\hat{\gamma}_0\circ\psi(\beta\circ\gamma) \sim \psi(\beta)\circ(\gamma{*}1_{S^1})$. By Proposition 2.11 (2) and Proposition 8.2, we have the following theorem.

Theorem 8.4 *For a map $\beta : S^q \to S^3$ and a suspension map $\gamma : S^{q'} \to S^q$ with $3 \leq q \leq q'$, we have that $E^3(\Omega\hat{\gamma}_0)_* H_2^{SS}(\psi(\beta\circ\gamma)) = (\gamma{*}1_{S^1})^* H_2^{SS}(\psi(\beta)) = \pm(\Sigma^2\gamma)^* H_2^{SS}(\psi(\beta))$*

Corollary 8.4.1 *(1) If $\beta\circ\gamma = 0$, in $\pi_{q'+1}(S^3)$, then $(\Sigma^2\gamma)^* H_2^{SS}(\psi(\beta)) = \{0\}$.*

(2) If $\beta : S^q \to S^3$ is of finite order ℓ with $q \geq 3$, then each element of $H_2^{SS}(\psi(\beta))$ is also of finite order which divides ℓ.

Fact 7.2, Theorems 7.3, 7.5 and 7.8 imply the following lemma.

Lemma 8.5 *Let p be an odd prime, β be the co-H-map $\alpha_1(3) : S^{2p} \to S^3$ and γ be the suspension map $\alpha_2(2p) = \Sigma^{2p-3}\alpha_2(3) : S^{6p-5} \to S^{2p}$ for the prime p. Then $\Sigma_* H_2^S(\psi(\alpha_1(3)\circ\alpha_2(2p)))$ contains $\pm\alpha_1(6)\circ\alpha_2(2p+3)$.*

Remark 8.6 *The composition $\alpha_1(5)\circ\alpha_2(2p+2)$ is trivial for all odd primes p except the prime 3. At the prime 3, $\alpha_1(5)\circ\alpha_2(8) = -3\beta_1(5) \neq 0$ and $\alpha_1(7)\circ\alpha_2(10) = 0$ by Lemma 13.8 and Theorem 13.9 in [25].*

Proof. Firstly we summarise here some well-known results on odd primary components of stable and unstable homotopy groups of spheres.

By Theorem 6.2 of Oka [17], we know the following fact:

Fact 8.7 *Let $t = (p-1)$ where p is an odd prime. For $k < 2(p+3)t - 4$, the p-component of stable homotopy group of sphere spectrum ${}_p\pi_k^S$ is trivial unless $k = 2rt - 1$ $(1 \le r \le p+2)$, $2pt - 2$ and $2(p+1)t - 3$. In addition, all the non-trivial groups are given as follows:*

$$
\begin{aligned}
{}_p\pi_{2rt-1}^S &\cong \mathbb{Z}/p\mathbb{Z} \quad \textit{generated by } \alpha_r,\ r \neq p, \\
{}_p\pi_{2pt-1}^S &\cong \mathbb{Z}/p^2\mathbb{Z} \quad \textit{generated by } \alpha'_p, \\
{}_p\pi_{2pt-2}^S &\cong \mathbb{Z}/p\mathbb{Z} \quad \textit{generated by } \beta_1, \\
{}_p\pi_{2(p+1)t-3}^S &\cong \mathbb{Z}/p\mathbb{Z} \quad \textit{generated by } \alpha_1 \circ \beta_1,
\end{aligned}
$$

where α_r is defined inductively using Toda brackets: $\alpha_r = \langle \alpha_{r-1}, p\iota, \alpha_1 \rangle$. Note that $\alpha_p = p\alpha'_p$. At $p = 3$, we remark that β_1 is given by a Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$.

On the other hand by (13.5) in [25], we know the following fact:

Fact 8.8 $\pi_{2m-1+k}(S^{2m-1})_{(p)} \cong {}_p\pi_k^S$ *if* $\frac{k+3}{2t} \le m$.

Since $\frac{2pt-2+3}{2t} < p+1$ and $2(p+1)-1 = 2t+3$, we have $\pi_{2(p+1)t+1}(S^{2t+3})_{(p)} \cong {}_p\pi_{2pt-2}^S \cong \mathbb{Z}/p\mathbb{Z} \ni \beta_1$. Hence there is a generator $\beta_1(2t+3)$ of $\pi_{2(p+1)t+1}(S^{2t+3})_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$ corresponding to β_1.

In $\pi_{2(p+2)t+3}(S^6)$, we know the following fact:

Fact 8.9 $\alpha_2(6) \circ \beta_1(4t+5) = 0$ *in* $\pi_{2(p+2)t+3}(S^6)_{(p)}$*, which is obtained by a similar argument given in Page 184 of [25] using (13.8) and Propositions 1.4 and 1.3 in [25]:*

$$
\begin{aligned}
\alpha_2(6) \circ \beta_1(4t+5) &\in 2\{\alpha_1(6), \alpha_1(2t+5), p\iota_{4t+4}\}_1 \circ \beta_1(4t+5) \\
&= 2\alpha_1(6) \circ \Sigma\{\alpha_1(2t+4), p\iota_{4t+3}, \beta_1(4t+3)\} \\
&\subset 2\alpha_1(6) \circ \{\alpha_1(2t+5), p\iota_{4t+4}, \beta_1(4t+4)\}_1,
\end{aligned}
$$

where $\{\alpha_1(2t+5), p\iota_{4t+4}, \beta_1(4t+4)\}_1$ is a subset of $\pi_{2(p+2)t+3}(S^{2t+5})_{(p)} \cong {}_p\pi_{2(p+1)t-2}^S = 0$, since $\frac{2(p+1)t-2+3}{2t} < p+2$ and $2(p+2) - 1 = 2t+5$.

In $\pi_{6t+1}(S^3)$, we also know the following fact:

Fact 8.10 $\alpha_1(3) \circ \alpha_2(2t+2) = 2\alpha_2(3) \circ \alpha_1(4t+2)$*, which is obtained by a similar argument given in Page 184 of [25] using (3.9) (i) and Propositions 1.4 and 13.6 in [25]:*

$$
\begin{aligned}
\alpha_2(3) \circ \alpha_1(4t+2) &\in \{\alpha_1(3), \Sigma(p\iota_{2t+1}), \Sigma\alpha_1(2t+1)\}_1 \circ \Sigma^2\alpha_1(4t) \\
&= \alpha_1(3) \circ \Sigma\{p\iota_{2t+1}, \alpha_1(2t+1), \alpha_1(4t)\},
\end{aligned}
$$

where the unstable Toda bracket $\{p\iota_{2t+1}, \alpha_1(2t+1), \alpha_1(4t)\} \subset \pi_{6t}(S^{2t+1};p) \cong {}_p\pi^S_{4t-1} \cong \mathbb{Z}/p\mathbb{Z}\{\alpha_2\}$ *corresponds to the stable Toda bracket* $\langle p\iota, \alpha_1, \alpha_1\rangle = \frac{1}{2}\alpha_2$*, since* $\frac{4t-1+3}{2t} < 3 \le t+1$ *and* $2(t+1)-1 = 2t+1$*. Thus* $\{p\iota_{2t+1}, \alpha_1(2t+1), \alpha_1(4t)\}$ *determines* $\frac{1}{2}\alpha_2(2t+1)$*, and hence* $\alpha_1(3)\circ\alpha_2(2t+2) = 2\alpha_2(3)\circ\alpha_1(4t+2)$ *in* $\pi_{6t+1}(S^3;p)$*.*

Next we apply these facts on higher Hopf invariants.

By Fact 8.9 we have $\alpha_2(2t+2)\circ\beta_1(6t+1) = 0$, since $t \ge 2$. Then by Theorem 8.4, it implies

$$\begin{aligned}&(\Sigma^2\beta_1(6t+1))^* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2t+2)))\\ &\quad = \pm E^3(\Omega\beta_1\widehat{(6t+1)}_0)_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2t+2)\circ\beta_1(6t+1))) = \{0\}.\end{aligned} \tag{8.4}$$

By Fact 8.10, we have

$$\begin{aligned}&E^3(\Omega\alpha_1\widehat{(4t+2)}_0)_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2t+1)))\\ &\quad = E^3(\Omega\alpha_1\widehat{(4t+2)}_0)_* H_2^{SS}(\psi(2\alpha_2(3)\circ\alpha_1(4t+2)))\\ &\quad = 2e(\Sigma^2\alpha_1(4t+2))^* H_2^{SS}(\psi(\alpha_2(3))) = 2e\alpha_1(4t+4)^* H_2^{SS}(\psi(\alpha_2(3))),\end{aligned}$$

where $e = \pm 1$. Here we see that $E^3(\Omega Q(\alpha_2(3))) \simeq \Omega Q(\alpha_2(3)) * \Omega Q(\alpha_2(3)) * \Omega Q(\alpha_2(3))$ has the homotopy type of a wedge sum of spheres up to dimension $4t+5 (> 4t+2)$. Since the suspension of Whitehead products are trivial, each element of $\Sigma H_2^{SS}(\psi(\alpha_2(3)))$ has the form of $a\alpha_2(6) + b\Sigma\iota\circ\alpha_1(2t+6)$ for dimensional considerations by Corollary 8.4.1 (2), where a, b are integers and $\iota : S^{2t+5} \hookrightarrow E^3(\Omega Q(\alpha_2(3)))$ denotes an appropriate map. Since $\alpha_1(5)\circ\alpha_1(2t+4) = 0$, we have $\alpha_1(2t+6)\circ\alpha_1(4t+5) = 0$, and hence each element of $2\alpha_1(4t+5)^*\Sigma_* H_2^{SS}(\psi(\alpha_2(3)))$ has the form of $2a\alpha_2(6)\circ\alpha_1(4t+5) = a\alpha_1(6)\circ\alpha_2(2t+5)$. On the other hand, we know

$$\begin{aligned}&2h_2(\eta_2\circ\alpha_2(3))\circ\alpha_1(4t+2) = 2h_2(\eta_2\circ\alpha_2(3)\circ\alpha_1(4t+2))\\ &\quad = h_2(\eta_2\circ\alpha_1(3)\circ\alpha_2(2t+2)) = \alpha_1(3)\circ\alpha_2(2t+2) = 2\alpha_2(3)\circ\alpha_1(4t+2),\end{aligned}$$

and hence $\Sigma^3 h_2(\eta_2\circ\alpha_2(3))\circ\alpha_1(4t+2) = \alpha_2(6)\circ\alpha_1(4t+5)$. By the proof of Theorem 7.5, $\Sigma^3 h_2(\eta_2\circ\alpha_2(3))\circ\alpha_1(4t+2)$ corresponds to $\alpha_1(4t+5)^*\Sigma_* H_2^{SS}(\psi(\alpha_2(3)))$ up to sign. Thus we have $a = \pm 1$ and $2\alpha_1(4t+5)^*\circ\Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{a\alpha_1(6)\circ\alpha_2(2t+5)\}$, and hence we have

$$\begin{aligned}&\Sigma E^3(\Omega\alpha_1\widehat{(4t+2)}_0)_*\circ\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_1(2t+1)))\\ &\quad = 2e\alpha_1(4t+5)^*\circ\Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{e'\alpha_1(6)\circ\alpha_2(2t+5)\}, \quad e' = \pm 1.\end{aligned} \tag{8.5}$$

By (13.10) in [25], we know that $h_p(\alpha_2(3)) = x\alpha_1(2t+3)$ for some $x \ne 0 \in \mathbb{Z}/p\mathbb{Z}$ and there are no other non-trivial James Hopf invariants $h_j(\alpha_2(3)), 1 < j \ne p$ for dimensional reasons.

Thus the attaching map of the $4t+3$ cell in $\Sigma\Omega Q(\alpha_2(3)) - \Sigma\Omega(S^2)$ corresponding to that in $Q(\alpha_2(3)) = S^2 \cup_{\eta_2\circ\alpha_2(3)} e^{4t+3}$ is given by

$$\begin{aligned}\Sigma\mathrm{ad}(\eta\circ\alpha_2(3)) &= \Sigma\Omega\eta\circ\Sigma\mathrm{ad}(\alpha_2(3)) = \Sigma\Omega\eta\circ(\Sigma\mathrm{ad}(\alpha_2(3)) - \alpha_2(3) + \alpha_2(3)) \\ &= \Sigma\Omega\eta\circ(\Sigma\mathrm{ad}(\alpha_2(3)) - \alpha_2(3)) + \Sigma\Omega\eta\circ\alpha_2(3) = x\Sigma\Omega\eta\circ\alpha_1(2t+3) + \Sigma\Omega\eta\circ\alpha_2(3).\end{aligned}$$

whose suspensions give the attaching maps of cells in

$$\Omega Q(\alpha_2(3)) * \Omega Q(\alpha_2(3)) * \Omega Q(\alpha_2(3)) - \Omega(S^2) * \Omega(S^2) * \Omega(S^2)$$

up to dimension $8t+6 (> 6t+4)$. Hence by (8.5), there is an integer y such that each element of $\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2t+2)))$ has the form

$$\begin{aligned}&e'\alpha_1(6)\circ\alpha_2(2t+5) + y\Sigma^{2t+1}\Sigma\mathrm{ad}(\eta\circ\alpha_2(3)) \\ &\quad = e'\alpha_1(6)\circ\alpha_2(2t+5) + y\{x\Sigma^{2t+2}\Omega\eta\circ\alpha_1(4t+5) + \Sigma^{2t+2}\Omega\eta\circ\alpha_2(2t+5)\}.\end{aligned}$$

If y is non-zero, then each element of $\beta_1(6t+4)^*\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2t+2)))$ has the form

$$\begin{aligned}&e'\alpha_1(6)\circ\alpha_2(2t+5)\circ\beta_1(6t+4) + y\{x\Sigma^{2t+2}\Omega\eta\circ\alpha_1(4t+5)\circ + \Sigma^{2t+2}\Omega\eta\circ\alpha_2(2t+5)\}\circ\beta_1(6t+4) \\ &\quad = yx\Sigma^{2t+2}\Omega\eta\circ\alpha_1(4t+5)\circ\beta_1(6t+4) \neq 0\end{aligned}$$

which contradicts to (8.4). Thus we obtain $\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(4t+2))) = \{e'\alpha_1(6)\circ\alpha_2(2t+5)\}$, $e' = \pm 1$. *QED.*

Using this, we show the following theorem.

Theorem 8.11 *There is a series of simply connected closed C^∞-manifolds N_p indexed by odd primes $p \geq 5$ with* $\mathrm{cat}\, N_p = \mathrm{cat}\,(N_p - \{P\})$, *where P is a point in N_p.*

Remark 8.12 *The manifold N_p does not have the property in Theorem 8.11 if we consider* cat_q, *for any prime $q \neq p$, instead of* cat *or* cat_p.

Proof of Theorem 8.11. We fix the prime $p \geq 5$ and let $L_p = E(\alpha_1(3)\circ\alpha_2(2p))$ for the prime p (see Theorem 13.4 in [25]). Then L_p is a C^∞-manifold with a CW decomposition $S^2 \cup_\alpha e^{6p-4} \cup_{\psi(\beta)} e^{6p-2}$, where $\alpha = \eta\circ\beta$ and $\beta = \alpha_1(3)\circ\alpha_2(2p)$. As is well-known (see [25] or [11]), $\alpha_1(3)$ is a co-H-map and $\alpha_2(2p)$ is a suspension map, and hence we have $h_2(\eta\circ\alpha_1(3)\circ\alpha_2(2p)) = \alpha_1(3)\circ\alpha_2(2p)$. Also by Proposition 13.6 and (13.7) in [25] and by the fact that S^3 is an H-space, we know that

$$\alpha_1(3)\circ\alpha_2(2p) \neq 0, \quad \Sigma^1(\alpha_1(3)\circ\alpha_2(2p)) \neq 0 \quad \text{but} \quad \Sigma^2(\alpha_1(3)\circ\alpha_2(2p)) \in \pi_{6p-3}(S^5)_{(p)} = 0,$$

which implies that $h_2(\eta\alpha_1(3)\circ\alpha_2(2p)) \neq 0$ and $\Sigma_* H^S_2(\psi(\alpha_1(3)\circ\alpha_2(2p))) \ni 0$ by Lemma 8.5. Hence by Fact 7.2 and Theorem 7.5, we have $\mathrm{cat}\,(S^2 \cup_\alpha e^{6p-4}) = 2$, $\mathrm{cat}\,(S^2 \cup_\alpha e^{6p-4}) \cup_{\psi(\beta)} e^{6p-2} = 2$, $2 \leq \mathrm{cat}\,L_p \leq 3$. If $\mathrm{cat}\,L_p = 2$, then we put $N_p = L_p$ which satisfies $\mathrm{cat}\,(N_p - \{P\}) = \mathrm{cat}\,(S^2 \cup_\alpha e^{6p-4}) \cup_{\psi(\beta)} e^{6p-2} = 2 = \mathrm{cat}\,L_p$. Otherwise we have $\mathrm{cat}\,L_p = 3$ and then by Theorem 7.8, we also have $\mathrm{cat}\,L_p \times S^n = \mathrm{cat}\,L_p = 3$, $n \geq 1$, and then we put $N_p = L_p \times S^2$ which satisfies $\mathrm{cat}\,(N_p - \{P\}) = \mathrm{cat}(L_p \times \{*\} \cup (S^2 \cup_\alpha e^{6p-4}) \times S^2) = 3 = \mathrm{cat}\,N_p$. Thus, in each case, N_p satisfies the required property. *QED.*

Theorem 8.13 *There is a simply connected closed manifold M with* $\mathrm{cat}\,M = 3$, $\mathrm{cat}\,M \times S^n = 3$ *for any $n \geq 2$ while we know only* $3 \leq \mathrm{cat}\,M \times S^1 \leq 4$ *for $n = 1$.*

Corollary 8.13.1 *There is a connected orientable closed manifold N with* $\mathrm{cat}\,N = \mathrm{cat}\,N \times S^n$ *for any $n \geq 1$.*

Remark 8.14 *Ganea's conjecture for the manifold M is true if we consider* cat_p*, for any prime $p \neq 3$, instead of* cat *or* cat_3*.*

Proof of Theorem 8.13. Let $M = E(\alpha_1(3)\circ\alpha_2(6))$ for the prime $p = 3$ (see [25]). Then M is a C^∞-manifold with a CW decomposition $S^2 \cup_\alpha e^{14} \cup_{\psi(\beta)} e^{16}$, where $\alpha = \eta\circ\beta$ and $\beta = \alpha_1(3)\circ\alpha_2(6)$. Also by Theorem 13.4 in [25] and by the fact that S^5 is an H-space at $p = 3$, we know that

$$\alpha_1(3)\circ\alpha_2(6) \neq 0, \quad \Sigma^3(\alpha_1(3)\circ\alpha_2(6)) \neq 0 \in \pi_{16}(S^6)_{(3)} \quad \text{but} \quad \Sigma^4(\alpha_1(3)\circ\alpha_2(6)) \in \pi_{17}(S^7)_{(3)} = 0,$$

which implies that $\Sigma^3_* h_2(\psi(\eta\alpha_1(3)\circ\alpha_2(2p))) \neq 0$ and $\Sigma^2_* H^S_2(\psi(\alpha_1(3)\circ\alpha_2(2p))) \ni 0$ by Lemma 8.5. Hence by Theorems 7.5 and 7.8, we have that $\mathrm{cat}\,M = 3$, $3 \leq \mathrm{cat}\,M \times S^1 \leq 4$ and $\mathrm{cat}\,M \times S^n = 3$ for $n \geq 2$. *QED.*

Proof of Corollary 8.13.1. Let $M = E(\alpha_1(3)\circ\alpha_2(6))$ for the prime $p = 3$ as in the proof of Theorem 8.13. Then M is a C^∞-manifold with $\mathrm{cat}\,M = 3$, $3 \leq \mathrm{cat}\,M \times S^1 \leq 4$ and $\mathrm{cat}\,M \times S^n = 3$ for $n \geq 2$. If $\mathrm{cat}\,M \times S^1 = 3$, we put $N = M$ which satisfies the required property. Otherwise, we put $N = M \times S^1$ with $\psi'(\alpha_1(3)\circ\alpha_2(6))$ as the attaching map of the top cell, where we know the set $\Sigma_* H^S_3(\psi'(\alpha_1(3)\circ\alpha_2(6)))$ include the set $\Sigma^2_* H^S_2(\psi(\alpha_1(3)\circ\alpha_2(6)))$ which contains $\Sigma^4\alpha_1(3)\circ\alpha_2(6) = 0$. Then Theorem 5.3 implies that N satisfies the required properties. *QED.*

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