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On a Gauss-Givental Representation of Quantum Toda Chain Wave Function

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Abstract

We propose group theory interpretation of the integral representation of the quantum open Toda chain wave function due to Givental. In particular we construct the representation of $U(\mathfrak{gl}(N))$ in terms of first order differential operators in Givental variables. The construction of this representation turns out to be closely connected with the integral representation based on the factorized Gauss decomposition. We also reveal the recursive structure of the Givental representation and provide the connection with the Baxter Q-operator formalism. Finally the generalization of the integral representation to the infinite and periodic quantum Toda wave functions is discussed.

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1 Introduction

In 1996 A. Givental proposed a new integral formula for the common eigenfunction of open Toda chain Hamiltonian operators [Gi] (see also [JK]). The first non trivial Hamiltonian operator of the N particle open Toda chain is given by

$$H=-\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{N-1}e^{x_i-x_{i+1}}\,. \tag{1.1}$$

The new integral formula for the common eigenfunction is given by the following multiple integral [Gi], [JK]:

$$\Psi_{\lambda_1,\ldots,\lambda_N}(T_{N,1},\ldots,T_{N,N})=\int_{\Gamma}e^{\frac{1}{\hbar}\mathcal{F}_N(T)}\prod_{k=1}^{N-1}\prod_{i=1}^{k}dT_{k,i}, \tag{1.2}$$

where $T_{N,i}:=x_i$, the function $\mathcal{F}_N(T)$ is given by

$$\mathcal{F}_N(T)=\imath\sum_{k=1}^{N}\lambda_k\Big(\sum_{i=1}^{k}T_{k,i}-\sum_{i=1}^{k-1}T_{k-1,i}\Big)-\sum_{k=1}^{N-1}\sum_{i=1}^{k}\Big(e^{T_{k+1,i}-T_{k,i}}+e^{T_{k,i}-T_{k+1,i+1}}\Big), \tag{1.3}$$

and the cycle Γ is a middle dimensional submanifold in the $N(N-1)/2$- dimensional complex torus with coordinates $\{\exp T_{k,i},\ i=1,\ldots,k;\ k=1,\ldots,N-1\}$ such that the integral converges. In particular one can choose $\Gamma=\mathbb{R}^{N(N-1)/2}$. This integral representation was motivated by the constructions of the Quantum Cohomology of the flag manifolds and its description in mirror dual Landau-Ginzburg model.

Basically two different regular approaches to the problem of the explicit construction of the integral representations of the open Toda chain wave function are known. The first goes back to B. Kostant (1978) and reduces the eigenvalue problem to the construction of a particular matrix element of an irreducible representation of the corresponding Lie algebras [Ko1], [STS]. Several integral representations of this matrix element are known [J], [Sch], [St], [GKMMMO]. This approach may be generalized to the periodic Toda chain using the representation theory of the affine Lie groups but has not yet lead to the explicit integral formulas for the wave functions in the periodic case.

Another approach is based on the Quantum Inverse Scattering Method (QISM) [F], [KS], applied to both open and periodic quantum Toda chains in [Gu], [Ga], [Sk1]. The explicit integral representations for the eigenfunctions in this framework were constructed in [KL1], [KL2], [KL3]. The explanation of this representation for the open Toda chain in terms of the representation theory of Lie groups was given in [GKL1] (see also [GKL2]) and was based on the generalization of the Gelfand-Zetlin construction [GZ], [GG] to the case of the infinite-dimensional representations of $U(\mathfrak{gl}(N))$ proposed in [GKL1].

Thus the representation theory provides a unifying framework for the constructions of the integral representations of the wave functions of open Toda chain. This naturally raises the question of the interpretation of the integral formula (1.2), (1.3) in terms of the representation theory.

In this note we demonstrate that the explicit integral representation (1.2), (1.3) for N particle open Toda chain naturally arises in representation theory approach. It is based on a particular parameterization of the upper/lower triangular parts of the Gauss decomposition of the group element of $GL(N,\mathbb{R})$. This parameterization is close to the factorization into the product of the elementary Jacobi matrices (see e.g. [BFZ]) but is slightly different. An interesting property of this parameterization is that it provides a simple construction of the principal series representation of the universal enveloping algebra $U(\mathfrak{gl}(N))$ acting in the space of functions on the *totally positive* unipotent upper-triangular matrices. The representation is given by the following explicit expressions for the images $E_{i,j}^{(N)}$, $i,j=1,\ldots,N$ of the generators of $\mathfrak{gl}(N)$ as the first order differential operators in variables $T_{k,i}$, $i=1,\ldots,k$; $k=1,\ldots,N-1$

$$E_{i,i}^{(N)}=\mu_i^{(N)}+\sum_{k=1}^{i-1}\frac{\partial}{\partial T_{N+k-i,k}}-\sum_{k=i}^{N-1}\frac{\partial}{\partial T_{k,i}}\,,$$

$$E_{i,i+1}^{(N)}=\sum_{n=1}^{i}\left(\sum_{k=n}^{i}e^{T_{N+k-i,k}-T_{N+k-i-1,k}}\right)\left(\frac{\partial}{\partial T_{N+n-i-1,n}}-\frac{\partial}{\partial T_{N+n-i-1,n-1}}\right), \tag{1.4}$$

$$E_{i+1,i}^{(N)}=\sum_{k=i}^{N-1}e^{T_{k,i}-T_{k+1,i+1}}\left(\mu_i^{(N)}-\mu_{i+1}^{(N)}+\sum_{s=i}^{k}\left(\frac{\partial}{\partial T_{s,i+1}}-\frac{\partial}{\partial T_{s,i}}\right)\right),$$

where $\mu_i^{(N)}=-\imath\hbar^{-1}\lambda_i-\rho_i^{(N)}$, $\lambda_i\in\mathbb{R}^N$, $\rho_i^{(N)}=\frac{1}{2}(N-2i+1)$, $i=1,\ldots N$ and we assume that $T_{N,i}=0$ and $\sum_i^j=0$ for $i>j$. The functions of the variables $T_{k,i}$ are identified with the functions on the subspace of the totally positive unipotent upper-triangular matrices. In the following we will use the term Gauss-Givental representation for this representation and the resulted representation of the open Toda wave function.

Let us stress that the Gauss-Givental representation is closely related with the recursive structure of the open Toda chains [Sk2]. In particular there exists the explicit integral recursion operator connecting the wave functions of the $N-1$ particle and the N particle Toda chains. The iterative application of this operator provides a simple independent derivation of the integral representation [Gi], [JK]. Note that the recursion operator is closely connected with the Baxter Q-operator for periodic Toda chain [Ga], [PG]. Thus from this point of view, the Gauss-Givental representation is a direct consequence of the generalized Baxter Q-operator formalism. This provides another relation between representation theory and QISM approaches to the solution of open Toda chain. Note that the similar iterative procedure also based on Baxter Q-operator was used in [GKL1] to get another integral representation in Gelfand-Zetlin parameterization. Finally let us remark that in [St] the iterative construction of the Whittaker function was proposed in the framework of the Iwasawa decomposition of the group element. It is easy to see that after simple manipulations this iterative procedure may be reduced to the one discussed in this paper.

The plan of the paper is as follows. In Section 2 we construct a representation of $U(\mathfrak{gl}(N))$ in terms of differential operators using a particular parameterization of the totally positive upper-triangular matrices. In Section 3 we derive the integral representation for the Toda chain wave function using this parameterization and show that integral representation obtained in this way coincides with the representation proposed by Givental. In Section 4 we rederive the integral representation using the iterative procedure and discuss the connection with Baxter Q-operator formalism. In Section 5, using the results of the previous section, we consider a generalization of the integral representations of wave function to infinite and periodic Toda chains. We conclude in Section 6 with the discussions of further directions of research. In the Appendices A and B the proofs of various statements from the main part of the text are given.

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2 Representation of $U(\mathfrak{gl}(N))$

In this section we construct a particular realization of the principal series representation of $U(\mathfrak{g}(N))$ acting in the space of functions on the subset of the totally positive elements of the nilpotent subgroup of $G=GL(N,\mathbb{R})$.

Let N_+ be a unipotent radical of the Borel subgroup B_+ of the group G. We fix the choice of this subgroups by considering the elements of B_+ as upper triangular matrices and the elements of N_+ as upper triangular matrices with unit diagonal. We define the subset $N_+^{(+)}\subset N_+$ of the totally positive unipotent upper-triangular matrices. The matrix $x\in N_+$ is called totally positive if all its not identically zero minors are positive real numbers (for details see e.g. [BFZ]). Consider the subspace of functions $M_{\boldsymbol{\mu}}\subset M=Fun(G)$ equivariant with respect to the left action of the opposite Borel subgroup of the lower-triangular matrices $B_-\subset G$: $f(bg)=\chi_{\boldsymbol{\mu}}(b)f(g)$, where $b\in B_-$, $\boldsymbol{\mu}=(\mu_1^{(N)},\ldots,\mu_N^{(N)})$ and $\chi_{\boldsymbol{\mu}}(b)=\prod\limits_{i=1}^{N}|b_{ii}|^{\mu_i^{(N)}}$ is a character of B_-.

The right action of the group on $M_{\boldsymbol{\mu}}$ defines a realization of the principal series representation of $GL(N,\mathbb{R})$ for $\mu_i^{(N)}=-\imath\hbar^{-1}\lambda_i-\rho_i^{(N)}$, where $\lambda_i\in\mathbb{R}$ and $\rho_i^{(N)}=\frac{1}{2}(N-2i+1)$, $i=1,\ldots N$ are the components of the vector $\boldsymbol{\rho}:=\frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}$ in the standard basis of $\mathbb{R}^N$. Let $M_{\boldsymbol{\mu}}^{(+)}$ by the set of functions on the totally positive unipotent upper-triangular matrices. We will consider the representation of the universal enveloping algebra $U(\mathfrak{gl}(N))$ in $M_{\boldsymbol{\mu}}^{(+)}$ induced by the above construction. Let $e_{i,j}$ stand for the elementary $N\times N$ matrix with the unit at the (i,j) place and zeros otherwise. Consider the set of the diagonal elements parameterized as follows $U_k=\sum\limits_{i=1}^{k}e^{T_{k,i}}e_{i,i}+\sum\limits_{i=k+1}^{N}e_{i,i}$. Define the following set of the upper-triangular matrices

$$\tilde{U}_k = \sum_{i=1}^{k} e^{T_{k,i}} e_{i,i} + \sum_{i=k+1}^{N} e_{i,i} + \sum_{i=1}^{k-1} e^{T_{k-1,i}} e_{i,i+1}. \tag{2.1}$$

Then any totally positive element $x \in N_+^{(+)}$ can be represented in the following form:

$$x = \tilde{U}_2 U_2^{-1} \tilde{U}_3 U_3^{-1} \cdots \tilde{U}_{N-1} U_{N-1}^{-1} \tilde{U}_N, \tag{2.2}$$

where we assume that $T_{N,i} = 0$. This can be easily verified using the connection with the parametrization of the upper triangular matrices in terms of the product of the elementary Jacobi matrices (see e.g. [BFZ]), namely, an arbitrary element $x \in N_+^{(+)}$ can be represented as follows:

$$x = \prod_{k=1}^{N-1} \left(1 + \sum_{i=1}^{k} y_{k,i} e_{i,i+1} \right), \tag{2.3}$$

where the variables $y_{k,i}$ are positive numbers. The comparison of (2.2), (2.3) leads to the following relations:

$$y_{k,i} = e^{T_{k,i} - T_{k+1,i+1}}. \tag{2.4}$$

Now let us use this parameterization to construct a representation of $U(\mathfrak{gl}(N))$ in $M_\mu^{(+)}$. The images, $E_{i,j}^{(N)}$, of the generators of $U(\mathfrak{gl}(N))$ in the representation induced from the one dimensional representation of the Borel subalgebra are defined as follows. Let $E_{i,j}^{(N)}$ be a first order differential operator in $T_{k,i}$ such that for any $f \in M_{\boldsymbol{\mu}}$ one has

$$E_{i,j}^{(N)} f(x) = \frac{d}{d\epsilon} f(x(1 + \epsilon e_{i,j}))|_{\epsilon=0}. \tag{2.5}$$

Proposition 2.1 *The following differential operators define a representation $\pi_{\boldsymbol{\mu}}$ of $\mathfrak{gl}(N)$ in $M_{\boldsymbol{\mu}}$:*

$$\begin{aligned}
E_{i,i}^{(N)} &= \mu_i^{(N)} + \sum_{k=1}^{i-1} \frac{\partial}{\partial T_{N+k-i,k}} - \sum_{k=i}^{N-1} \frac{\partial}{\partial T_{k,i}}, \\
E_{i,i+1}^{(N)} &= \sum_{n=1}^{i} \left(\sum_{k=n}^{i} e^{T_{N+k-i,k} - T_{N+k-i-1,k}} \right) \left(\frac{\partial}{\partial T_{N+n-i-1,n}} - \frac{\partial}{\partial T_{N+n-i-1,n-1}} \right), \\
E_{i+1,i}^{(N)} &= \sum_{k=i}^{N-1} e^{T_{k,i} - T_{k+1,i+1}} \left(\mu_i^{(N)} - \mu_{i+1}^{(N)} + \sum_{s=i}^{k} \left(\frac{\partial}{\partial T_{s,i+1}} - \frac{\partial}{\partial T_{s,i}} \right) \right),
\end{aligned} \tag{2.6}$$

where we assume that $T_{N,i} = 0$ and omit the derivatives over $T_{i,j}$, $i < j$.

The proof is given in the Appendix A.

3 Integral representation of the wave function

3.1 Matrix elements as open Toda wave functions

In this section we construct an integral representation of the wave function of the open Toda chain corresponding to the representation $\pi_{\boldsymbol{\mu}}$ of $U(\mathfrak{gl}(N))$ introduced in the previous section. The construction can be done using either Gauss or Iwasawa decompositions of the group. In the paper we will use the Gauss decomposition and the corresponding Whittaker model of representation.

We first recall some facts from [Ko2]. Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be two nilpotent subalgebras of $\mathfrak{gl}(N)$ generated, respectively, by positive and negative root generators. The homomorphisms (characters) $\chi_+ : n_+ \to \mathbb{C}$, $\chi_- : n_- \to \mathbb{C}$ are uniquely determined by their values on the simple root generators, and are called non-singular if the complex numbers $\chi_+(E^{(N)}_{i,i+1}) := \xi^{(i)}_R$ and $\chi_-(E^{(N)}_{i+1,i}) := \xi^{(N-i)}_L$ are non-zero for all $i = 1, \ldots, N-1$.

Let V be any $U(\mathfrak{gl}(N))$-module. A vector $\psi_R \in V$ is called a Whittaker vector with respect to the character χ_+ if

$$E^{(N)}_{i,i+1}\psi^{(N)}_R = \xi^{(i)}_R \psi^{(N)}_R\,, \qquad (i = 1, \ldots, N-1), \tag{3.1}$$

and an element $\psi_L \in V'$ is called a Whittaker vector with respect to the character χ_- if

$$E^{(N)}_{i+1,i}\psi^{(N)}_L = \xi^{(N-i)}_L \psi^{(N)}_L\,, \qquad (i = 1, \ldots, N-1). \tag{3.2}$$

A Whittaker vector is called cyclic in V if $U(\mathfrak{gl}(N))\psi = V$, and a $U(\mathfrak{gl}(N))$-module is a Whittaker module if it contains a cyclic Whittaker vector. The $U(\mathfrak{gl}(N))$-modules V and V' are called dual if there exists a non-degenerate pairing $\langle\,.\,,.\rangle : V' \times V \to \mathbb{C}$ such that $\langle v', Xv\rangle = -\langle Xv', v\rangle$ for all $v \in V$, $v' \in V'$ and $X \in \mathfrak{gl}(N)$. Let us assume that the action of the Cartan subalgebra is integrated to the action of the Cartan torus. An eigenfunction of the open Toda chain can be written in terms of the pairing as follows

$$\Psi^{(N)}_{\boldsymbol{\lambda}}(T_{N,1}, \ldots, T_{N,N}) = e^{-\sum T_{N,i}\rho^{(N)}_i}\langle \psi^{(N)}_L\,, \pi_{\boldsymbol{\mu}}(e^{-\sum T_{N,i}E^{(N)}_{ii}})\,\psi^{(N)}_R\rangle\,, \tag{3.3}$$

where $\mu^{(N)}_k = -\imath\hbar^{-1}\lambda_k - \rho^{(N)}_k$ and $\rho^{(N)}_k = \frac{1}{2}(N - 2k + 1)$, $k = 1, \ldots N$ are the components of the vector $\boldsymbol{\rho} := \frac{1}{2}\sum\limits_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}$ in the standard basis of $\mathbb{R}^N$. Let H_k be the radial projections of generators c_k of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(N))$, defined by

$$H_k\Psi^{(N)}_{\boldsymbol{\lambda}}(T_{N,1}, \ldots, T_{N,N}) = e^{-\sum T_{N,i}\rho^{(N)}_i}\langle \psi^{(N)}_L\,, \pi_{\boldsymbol{\mu}}(e^{-\sum T_{N,i}E^{(N)}_{ii}})\,c_k\,\psi^{(N)}_R\rangle. \tag{3.4}$$

One can show that H_k provide the complete set of commuting Hamiltonians of the open Toda chain model.

3.2 Matrix element in Gauss-Givental parameterization

We are going to write down the matrix element (3.3) explicitly in the Gauss-Givental representation introduced above. Let us start with construction of the Whittaker vectors $\psi^{(N)}_L$ and $\psi^{(N)}_R$.

Proposition 3.1 *The functions*

$$\psi_R^{(N)} = \exp\left\{\sum_{i=1}^{N-1}\xi_R^{(i)}\sum_{k=i}^{N-1}e^{T_{k,i}-T_{k+1,i+1}}\right\}, \tag{3.5}$$

and

$$\psi_L^{(N)} = \exp\left\{\sum_{k=1}^{N-1}\sum_{i=1}^{k}(\mu_k-\mu_{k+1})T_{k,i}+\sum_{i=1}^{N-1}\xi_L^{(i)}\sum_{k=1}^{N-i}e^{T_{k+i,k}-T_{k+i-1,k}}\right\}, \tag{3.6}$$

are solutions of the linear differential equations (3.1) and (3.2) (we assume that $T_{N,i}=0$).

Proof. The proof is based on the recursion properties of the Whittaker vectors $\psi_{L,R}^{(N)}$ and is given in the Appendix B.

Evidently, the Whittaker modules $V := U(\mathfrak{gl}(N))\psi_R^{(N)}$ and $V' := U(\mathfrak{gl}(N))\psi_L^{(N)}$ are spanned by the elements $\prod_{k=1}^{N-1}\prod_{i=1}^{k}e^{n_{k,i}T_{k,i}}\,\psi_{R,L}^{(N)}$, where $n_{k,i}\in\mathbb{Z}$.

Now we construct the pairing between V and V'. Define a measure of integration over $N_+^{(+)}$ as $\omega_N = \wedge_{k=1}^{N-1}\wedge_{i=1}^{k}e^{T_{k,i}}dT_{k,i}$, and for any $\phi\in V'$, $\psi\in V$ introduce the following pairing:

$$\langle\phi,\psi\rangle = \int\limits_{\Gamma}\omega_N\,\overline{\phi}(T)\psi(T)\,, \tag{3.7}$$

where the integration contour Γ is chosen in such a way that the integral (3.7) is convergent for $\phi=\psi_L^{(N)}$ and $\psi=\psi_R^{(N)}$. Then the convergence for arbitrary elements $\phi\in V'$ and $\psi\in V$ follows.

Lemma 3.1 *The pairing (3.7) is non-degenerate and satisfies the following condition*

$$\langle\phi,X\psi\rangle = -\langle X\phi,\psi\rangle\,, \tag{3.8}$$

for any $X\in\mathfrak{gl}(N)$.

Now we are ready to find the integral representation for the pairing (3.3). To get the explicit expression for the integrand one should make a choice. For example, the group element $e^{H^{(N)}}=\exp(-\sum_{i=1}^{N}T_{N,i}E_{i,i}^{(N)})$ can act on the right vector $\psi_R^{(N)}$ or on the left vector $\psi_L^{(N)}$. We use the most symmetric choice. Let us represent the Cartan group element in the following way

$$\begin{gathered} e^{H^{(N)}} = e^{H_L^{(N)}}e^{H_R^{(N)}} = \\ = \exp\left\{\sum_{i=1}^{N-1}T_{N,i}\sum_{k=1}^{N-1}\frac{\partial}{\partial T_{k,i}}\right\}\exp\left\{-\sum_{i=1}^{N}\mu_i^{(N)}T_{N,i}-\sum_{i=2}^{N}T_{N,i}\sum_{k=1}^{i-1}\frac{\partial}{\partial T_{N-i+k,k}}\right\}. \end{gathered} \tag{3.9}$$

Then we suppose that $H_L^{(N)}$ acts on the left vector and $H_R^{(N)}$ acts on the right vector in (3.3). Taking into account the results of the last lemma one obtains the following theorem.

Theorem 3.1 *The eigenfunction of the open Toda chain defined by (3.3), (3.7) has the following integral representation*

$$\Psi^{(N)}_{\boldsymbol{\lambda}}(T_{N,1},\ldots,T_{N,N}) = \int\limits_{\Gamma} \exp\left\{\frac{\imath}{\hbar}\sum_{k=1}^{N}\lambda_k\Big(\sum_{i=1}^{k}T_{k,i}-\sum_{i=1}^{k-1}T_{k-1,i}\Big)\right\}\times$$
$$\exp\left\{\sum_{i=1}^{N-1}\bar{\xi}_L^{(i)}\sum_{k=1}^{N-i}e^{T_{k+i,k}-T_{k+i-1,k}}+\sum_{i=1}^{N-1}\xi_R^{(i)}\sum_{k=i}^{N-1}e^{T_{k,i}-T_{k+1,i+1}}\right\}\prod_{k=1}^{N-1}\prod_{i=1}^{k}dT_{k,i}\,. \tag{3.10}$$

The integral representation (3.10) coincides with (1.2), (1.3), if we set $\xi_R^{(i)}=\xi_L^{(i)}=-\hbar^{-1}$. In this case one can choose the contour to be $\Gamma=\mathbb{R}^{N(N-1)/2}$.

3.3 Connection with another parametrization

It is useful to compare the integral representation obtained above with one in [GKMMMO], which is based on a more standard parameterization of the group element. Let us recall this construction. We parameterize the unipotent upper triangular matrices $\|x_{i,j}\|$ by its elements $x_{i,j}$, $1\le i<j\le N$. In the corresponding realization of the representation of $U(\mathfrak{gl}(N))$ the action of the generators of the algebra is given by the differential operators in $x_{i,j}$. The measure of integration entering the definition of the pairing between two vectors in this realization is given by

$$\omega'_N=\wedge_{i=1}^{N-1}\wedge_{j=i+1}^{N}dx_{i,j}.$$

Let us find the right Whittaker vector in this realization. Integrating the action of the algebra $\mathfrak{n}_+$ to the action of the group N_+ the equation for the right Whittaker vector can be written in the form

$$\pi_{\boldsymbol{\mu}}(z)\psi_R^{(N)}(x)=e^{\sum_{i=1}^{N-1}\xi_R^{(i)}z_{i,i+1}}\psi_R^{(N)}(x),$$

for any $z\in N_+$. Thus we should find the one-dimensional representation of the group of upper-triangular matrices. Note that the additive character of this group is easily constructed using the fact that the main diagonal behave additively under the multiplication. Exponentiating this character we get a one-dimensional representation:

$$\psi_R^{(N)}(x)=\prod_{i=1}^{N-1}e^{\xi_R^{(i)}x_{i,i+1}}. \tag{3.11}$$

Similarly, the left vector should satisfies the condition

$$\pi_{\boldsymbol{\mu}}(z^t)\psi_L^{(N)}(x)=e^{\sum_{i=1}^{N-1}\xi_L^{(i)}z_{i,i+1}}\psi_L^{(N)}(x),$$

for any $z\in N_+$. To construct such vector we use an inner automorphism which maps the unipotent upper-triangular matrices to the unipotent lower-triangular matrices. Consider the matrix $(w_0)_{i,j}=\delta_{i+j,N+1}$ acting as $z\to w_0^{-1}zw_0$. We have $w_0^{-1}N_+w_0=N_-$ where N_- is opposed to N_+. It is easy to check that we can define $\psi_L^{(N)}(x) = \psi_R^{(N)}(xw_0^{-1})$. An explicit calculation gives the following result

$$\psi_L^{(N)}(x) = \prod_{i=1}^{N-1} \left\{ \Delta_i(xw_0^{-1})^{\mu_i-\mu_{i+1}} e^{\xi_L^{(i)} \frac{\Delta_{i,i+1}(xw_0^{-1})}{\Delta_i(xw_0^{-1})}} \right\}, \tag{3.12}$$

where $\Delta_i(M)$ denotes the principle $i \times i$ minor of the matrix M and $\Delta_{i,i+1}(M)$ denotes the determinant obtained from $\Delta_i(M)$ by interchanging the i-th and $(i+1)$-th columns in M.

Consider the decomposition (2.3) of a totally positive unipotent upper-triangular matrix x where we assume that $y_{i,j}$ are defined through $T_{i,j}$ by the relation

$$y_{k,i} = e^{T_{k,i}-T_{k+1,i+1}-T_{N,i}+T_{N,i+1}}. \tag{3.13}$$

It is worth comparing the parameterizations (2.4) and (3.13). The difference between the two reflects the fact that in [GKMMMO] the group element entering (3.3) acts on the *right* vector in the integral representation. This difference may be compensated by the shift of the integration variables $T_{k,i} \to T_{k,i} - T_{N,i}$ thus reconciling (2.4) and (3.13).

Following [BFZ], one can express the minors through the variables $y_{k,i}$ as well as $T_{k,i}$ and establish the following

Proposition 3.2 *The following expressions for the minors of xw_0^{-1} in terms of the variables $y_{k,i}$ and $T_{k,i}$ hold:*

$$x_{i,i+1} = \sum_{k=i}^{N-1} y_{k,i} = \sum_{k=i}^{N-1} e^{T_{k,i}-T_{k+1,i+1}+T_{N,i+1}-T_{N,i}},$$

$$\Delta_i(xw_0^{-1}) = (-1)^{i-1} \prod_{k=i}^{N-1} \prod_{j=k-i+1}^{k} y_{k,j} = (-1)^{i-1} e^{\sum_{k=1}^{i} T_{i,k}-T_{N,k}}, \tag{3.14}$$

$$\frac{\Delta_{i,i+1}(xw_0^{-1})}{\Delta_i(xw_0^{-1})} = \sum_{k=i}^{N-1} \prod_{m=k}^{N-1} \frac{y_{m+1,m-i+1}}{y_{m,m-i+1}} = \sum_{k=1}^{N-i} e^{T_{k+i,k}-T_{k+i-1,k}},$$

where we assume $y_{N,k} = 1$.

Using the above relations it is easy to see that the integral representation given in [GKMMMO] reduces to the Gauss-Givental representation. The direct between the two integral representations gives also an explicit definition to the integration contour used in [GKMMMO].

4 The Baxter $Q-$operator induction over the rank

4.1 Induction over the rank and the Gauss-Givental construction

The easiest way to construct the Whittaker vectors in the Gauss-Givental parameterization uses the recursive structure of this parameterization. It is shown in Appendix B that using the conjugation by the operators

$$\Xi_L^{(n)} = e^{-\sum\limits_{i=1}^{n}\mu_i^{(n)}T_{n,i}} \exp\left\{\mu_n^{(n)}\left(\sum_{i=1}^{n}T_{n,i}-\sum_{i=1}^{n-1}T_{n-1,i}\right)+\sum_{i=1}^{n-1}\xi_L^{(n-i)}e^{T_{n,i}-T_{n-1,i}}\right\}e^{\sum\limits_{i=1}^{n-1}\mu_i^{(n)}T_{n-1,i}}, \tag{4.1}$$

and

$$\begin{aligned}\Xi_R^{(n)} &= \exp\left\{-\sum_{i=1}^{n-1}T_{n,i}\sum_{k=1}^{n-1}\frac{\partial}{\partial T_{k,i}}+\sum_{i=2}^{n}T_{n,i}\sum_{k=1}^{i-1}\frac{\partial}{\partial T_{k+(n-i),k}}\right\}\times \\ &\exp\left\{\sum_{i=1}^{n-1}\xi_R^{(i)}e^{T_{n-1,i}-T_{n,i+1}}\right\}\times \\ &\exp\left\{\sum_{i=1}^{n-2}T_{n-1,i}\sum_{k=1}^{n-2}\frac{\partial}{\partial T_{k,i}}-\sum_{i=2}^{n-1}T_{n-1,i}\sum_{k=1}^{i-1}\frac{\partial}{\partial T_{k+(n-1-i),k}}\right\},\end{aligned} \tag{4.2}$$

one can obtain the iterative representations for the Whittaker vectors in the form

$$\psi_R^{(N)} = e^{H_L^{(N)}}\Xi_R^{(N)}\cdots\Xi_R^{(2)}\cdot 1, \tag{4.3}$$

$$\psi_L^{(N)} = e^{H_L^{(N)}}\Xi_L^{(N)}\cdots\Xi_L^{(2)}\cdot 1. \tag{4.4}$$

Let us define

$$R = e^{-\sum_i T_{N,i}\rho_i^{(N)}}\cdot\prod_{i=1}^{N-1}e^{(N-i)(T_{N-1,i}-T_{N,i})}. \tag{4.5}$$

Then by direct calculation we can prove the following recursive relation for the Cartan elements

$$R\cdot\overline{\Xi_L^{(N)}}e^{-\sum T_{N,i}E_{i,i}^{(N)}}\Xi_R^{(N)} = e^{-\sum T_{N-1,i}\rho_i^{(N-1)}}Q_{\lambda_N}^{(N)}\cdot e^{-\sum T_{N-1,i}E_{i,i}^{(N-1)}}, \tag{4.6}$$

where

$$Q_{\lambda_N}^{(N)}(T_{N,1},\ldots,T_{N,N};T_{N-1,1},\ldots,T_{N-1,N-1}) = \exp\left\{\frac{\imath\lambda_N}{\hbar}\Big(\sum_{k=1}^{N}T_{N,k}-\sum_{k=1}^{N-1}T_{N-1,k}\Big)-\frac{1}{\hbar}\sum_{k=1}^{N-1}\Big(e^{T_{N,k}-T_{N-1,k}}+e^{T_{N-1,k}-T_{N,k+1}}\Big)\right\}. \tag{4.7}$$

Taking into account these recursive relations we obtain

Proposition 4.1

$$\Psi_{\lambda_1,\ldots,\lambda_N}^{(N)}(T_{N,1},\ldots,T_{N,N}) = \int\limits_{\mathbb{R}^{N-1}} Q_{\lambda_N}^{(N)}(T_{N,1},\ldots,T_{N-1,N-1})\Psi_{\lambda_1,\ldots,\lambda_{N-1}}^{(N-1)}(T_{N-1,1},\ldots,T_{N-1,N-1})\prod_{i=1}^{N-1}dT_{N-1,i}\,. \tag{4.8}$$

Proof. Direct calculation.

Thus, the integral operator $\mathcal{Q}^{(N)}_{\lambda_N}$ transforms the common eigenfunctions of $N-1$ particle open Toda chain Hamiltonians to the ones for N particle open Toda chain Hamiltonians. Therefore, one can rederive Givental's integral formula by applying iteratively integral operators $\mathcal{Q}^{(n)}_{\lambda_n}$, $1 < n \leq N$. In the next subsection we provide a more conceptual proof of the Proposition 4.1 and discuss the connection with Baxter's Q-operator formalism.

4.2 Recursive derivation and Baxter $Q-$operator

Surprisingly, the Gauss-Givental representation has a direct connection with the formalism of the recursion operator [KL1] and thus a close relation with the Baxter Q-operator and the QISM approach. Below we rederive this integral representation using the recursion procedure. Let us first recall the formalism of the Lax operators (see e.g. [Ga]). In this subsection we set $\xi^{(i)}_R = \xi^{(i)}_L = -\hbar^{-1}$ for simplicity.

Consider the following $N \times N$ Lax operators

$$L_N = \begin{pmatrix} p_1 & 1 & 0 & 0 & ... & 0 \\ e^{x_1-x_2} & p_2 & 1 & 0 & ... & 0 \\ 0 & ... & ... & ... & ... & \\ ... & ... & ... & ... & ... & \\ 0 & 0 & ... & e^{x_{N-2}-x_{N-1}} & p_{N-1} & 1 \\ 0 & 0 & ... & 0 & e^{x_{N-1}-x_N} & p_N \end{pmatrix}, \tag{4.9}$$

where $p_n = -\imath\hbar\frac{\partial}{\partial x_n}$. The generating function for the Hamiltonians $H^{(N)}_n$ of the N particle open Toda chain is defined as

$$\det(u - L_N) = \sum_{n=0}^{N}(-1)^n u^{N-n} H^{(N)}_n \,. \tag{4.10}$$

The determinant $A_N(u) \equiv \det(u-L_N)$ with non-commutative elements is defined iteratively

$$A_n(u) = (u - p_n)A_{n-1}(u) - e^{x_{n-1}-x_n}A_{n-2}(u), \tag{4.11}$$

for $n = 1, 2, \ldots$, where $A_{-1}(u) = 0$, $A_0(u) = 1$.

For any function $f \in \mathrm{Fun}\,\mathbb{R}^{N-1}$ not growing too fast at infinity, let

$$\begin{gathered}(\mathcal{Q}^{(N)}_u f)(x_1, \ldots, x_N) = \\ = \int\limits_{\mathbb{R}^{N-1}} Q^{(N)}_u(x_1, \ldots, x_N; y_1, \ldots, y_{N-1}) f(y_1, \ldots, y_{N-1}) dy_1 \ldots dy_{N-1}\,,\end{gathered} \tag{4.12}$$

where

$$Q_u^{(N)}(x_1,\dots,x_N;y_1,\dots,y_{N-1}) :=$$
$$= \exp\left\{\frac{\imath u}{\hbar}\Big(\sum_{i=1}^{N}x_i-\sum_{i=1}^{N-1}y_i\Big)-\frac{1}{\hbar}\sum_{i=1}^{N-1}\Big(e^{x_i-y_i}+e^{y_i-x_{i+1}}\Big)\right\}. \tag{4.13}$$

Lemma 4.1 *The following intertwining relation holds:*

$$A_N(u)\circ\mathcal{Q}_v^{(N)}=(u-v)\mathcal{Q}_v^{(N)}\circ A_{N-1}(u)\,, \tag{4.14}$$

where $A_{N-1}(u) := \sum\limits_{n=0}^{N-1}(-1)^n u^{N-n-1}H_n^{(N-1)}$ *is the generating function for the Hamiltonians of the* $N-1$ *particle open Toda chain acting as a differential operators in variables* $y_1,\dots,y_{N-1}$.

Proof. The proof is similar to the proof of Theorem 2 in [Gi] (see also [JK]).

■

Proposition 4.2 *Let* $\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}}(T_{N-1,1},\dots,T_{N-1,N-1})$ *be the eigenfunction of the* $N-1$ *particle open Toda chain. Then the function*

$$\Psi^{(N)}_{\lambda_1,\dots,\lambda_N}(T_{N,1},\dots,T_{N,N})=(\mathcal{Q}^{(N)}_{\lambda_N}\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}})(T_{N,1},\dots,T_{N,N}) \tag{4.15}$$

is the solution of the N *particle open Toda chain depending on coordinates* $T_{N,1},\dots,T_{N,N}$. *Here the integral kernel of the operator is defined by*

$$Q^{(N)}_{\lambda_N}(T_{N,1},\dots,T_{N,N};T_{N-1,1}\dots,T_{N-1,N-1})=$$
$$\exp\left\{\frac{\imath\lambda_N}{\hbar}\Big(\sum_{i=1}^{N}T_{N,i}-\sum_{i=1}^{N-1}T_{N-1,i}\Big)-\frac{1}{\hbar}\sum_{i=1}^{N-1}\Big(e^{T_{N,i}-T_{N-1,i}}+e^{T_{N-1,i}-T_{N,i+1}}\Big)\right\}. \tag{4.16}$$

Proof. Assume that the function $\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}}(T_{N-1,1},\dots,T_{N-1,N-1})$ satisfies the equation $A_{N-1}(u)\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}}=\prod_{k=1}^{N-1}(u-\lambda_k)\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}}$. Using (4.14) with $u=\lambda_N$, one obtains that the function (4.15) satisfies the equation $A_N(u)\Psi^{(N)}_{\lambda_1,\dots,\lambda_N}=\prod_{k=1}^{N}(u-\lambda_k)\Psi^{(N)}_{\lambda_1,\dots,\lambda_N}$. ■

Corollary 4.1 *The integral representation (1.2), (1.3) can be obtained by iteration of the recursive relation (4.15).*

Finally note that the recursive operator $\mathcal{Q}$ introduced above is closely connected with the Baxter Q-operator of the periodic Toda chain [PG]. The kernel of the operator $\mathcal{Q}$ is obtained form the kernel of the Baxter Q-operator by dropping out the terms corresponding to the "interaction" of the first and last nodes of the Toda chain. Moreover, the elementary blocks of the parameterization of the unipotent matrices introduce in Section 2 also have counterparts in the Baxter Q-operator formalism (see e.g. eq.(54) in [Sk2]).

5 On a generalization to infinite and periodic Toda chain

The Gauss-Givental representation of the open Toda chain wave function seems to imply a natural generalization to the case of the infinite and periodic Toda chain wave function. The details of this generalizations will be discussed elsewhere. In this section we discuss the formal expressions arising in this case. For simplicity we will consider below only the case of zero eigenvalues $\lambda_i = 0$ of the Hamiltonians.

Let us start with the infinite Toda chain. The quantum Hamiltonians of the infinite Toda chain are obtained by taking the formal limit $N \to \infty$ of the finite open Toda case (see for example [Sk1]). Thus the first nontrivial Hamiltonian is given by

$$H = -\frac{\hbar^2}{2} \sum_{i=-\infty}^{\infty} \frac{\partial^2}{\partial x_i^2} + \sum_{i=-\infty}^{\infty} e^{x_i - x_{i+1}}. \tag{5.1}$$

It is natural to guess that the generalization of the integral representation for the common eigenfunctions of the commuting set of Hamiltonians for the infinite Toda chain is given by the appropriately defined limit $N \to \infty$ of the Gauss-Givental integral representation

$$\Psi^{(\infty)}(T_{0,i}) = \int_{\Gamma} e^{\frac{1}{\hbar}\mathcal{F}^{(\infty)}(T)} \prod_{k=1}^{\infty} \prod_{i=-\infty}^{\infty} dT_{k,i}\,, \tag{5.2}$$

where the function $\mathcal{F}^{(\infty)}(T)$ is given by

$$\mathcal{F}^{(\infty)}(T) = -\sum_{k=1}^{\infty} \left(\sum_{i=-\infty}^{\infty} e^{T_{k-1,i} - T_{k,i}} + \sum_{i=-\infty}^{\infty} e^{T_{k,i} - T_{k-1,i+1}} \right), \tag{5.3}$$

and the cycle Γ is a semiinfinite-dimensional submanifold in the infinite-dimensional complex torus with coordinates $\{e^{T_{k,j}} |\ k \in \mathbb{N}, i \in \mathbb{Z}\}$. Note that we relabel the variables $T_{k,i}$ to take the limit $N \to \infty$ properly.

Similarly one defines the formal integral representation for the periodic Toda chain. It is well known that the Hamiltonians of the N particle periodic Toda chain can be obtained from the Hamiltonians of the infinite Toda chain by imposing the periodicity condition $x_{i+N} = x_i$. Thus the first non-trivial Hamiltonian is given by

$$H = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} e^{x_i - x_{i+1}}, \tag{5.4}$$

and $x_{N+1} = x_1$. It is natural to guess that similar factorization works on the level of the integral representation. Thus we get formally

$$\Psi^{(per)}(T_{0,i}) = \int_{\Gamma} e^{\frac{1}{\hbar}\mathcal{F}^{(per)}(T)} \prod_{k=1}^{\infty} \prod_{i=1}^{N} dT_{k,i}\ , \tag{5.5}$$

where the function $\mathcal{F}_{(per)}(T)$ is given by

$$\mathcal{F}^{(per)}(T) = -\sum_{k=1}^{\infty}\left(\sum_{i=1}^{N} e^{T_{k-1,i}-T_{k,i}} + \sum_{i=1}^{N} e^{T_{k,i}-T_{k-1,i+1}}\right), \tag{5.6}$$

where we assume $T_{k,N+i} = T_{k,i}$ and the cycle Γ is a semi-infinite dimensional submanifold in the infinite-dimensional complex torus with coordinates $\{e^{T_{k,j}}|\ k \in \mathbb{N}, i = 1, \ldots, N\}$. Although these integral representations are rather meaningless, they seem reveal the important properties of the solution. For example the infinite number of integrations in the periodic case is a manifestation of the integration over the "positive part" of the loop group $LGL(N, \mathbb{R})$ which is a formal analog of the unipotent upper triangular matrices in the affine case.

Let us note that the formal expressions (5.2) and (5.5) have an interesting interpretation from the point of view of the recursive relations discussed in the previous section. Consider the formal limit $N \to \infty$ of the integral kernel of the recursion operator in the case of the infinite Toda chain

$$Q^{(\infty)}(T_{k,*}, T_{k-1,*}) = \exp\left\{-\frac{1}{\hbar}\sum_{i=-\infty}^{\infty}\left(e^{T_{k-1,i}-T_{k,i}} + e^{T_{k,i}-T_{k-1,i+1}}\right)\right\}. \tag{5.7}$$

It is easy to see that the integral operator defined by this kernel function commutes with the Hamiltonians and is a particular example of the Bäcklund transformation. This kind of transformations for infinite Toda chain defined by the integral kernel (5.7) was studied previously in detail [T]. Similarly in the case of the periodic Toda chain we have that

$$\begin{gathered} Q^{(per)}(T_{k,1}, \ldots, T_{k,N}, T_{k-1,1} \ldots, T_{k-1,N}) = \\ \exp\left\{-\frac{1}{\hbar}\sum_{i=1}^{N}\left(e^{T_{k-1,i}-T_{k,i}} + e^{T_{k,i}-T_{k-1,i+1}}\right)\right\}, \end{gathered} \tag{5.8}$$

where we assume $T_{k,i+N} = T_{k,i}$ and $T_{k-1,i+N} = T_{k-1,i}$. As in the case of the infinite chain the corresponding integral transformation provides an example of a Bäcklund transformation for the periodic chain. Exactly these transformations were considered previously in [Ga], [PG], [Sk2]. Taking into account the explicit expressions (5.7), (5.8) for the kernels the integral formulas (5.2) and (5.5) can be also interpreted as follows. Let us apply K times the integral operator with the kernel (5.7) to the constant function. It is easy to see that when $K \to \infty$ the resulting expression tends to the one given by the integral formula (5.2). This is consistent with the fact that the action of $Q^{(\infty)}$ transforms an eigenfunction of the infinite Toda chain Hamiltonians into another eigenfunction. Completely similar picture holds for the periodic case.

Finally let us mention that the solution for the common eigenfunction of the Hamiltonians of the periodic Toda chain has already appeared elsewhere [KL2], [KL3]. It is given by the finite-dimensional integral with rather complex integrand. Hopefully it could be obtained after an infinite number of the integrations from an appropriately defined formal integral representation (5.5).

6 Conclusions

In this paper we have proposed an interpretation of the solution of the open Toda chain given in [Gi] (see also [JK]) in terms of representation theory. This leads to a particular realization of the principal series representation of $U(\mathfrak{gl}(N))$ in terms of the differential operators acting on the space of functions defined on the subspace of the totally positive unipotent upper-triangular matrices. The resulting representation is very close to the representation constructed (rather implicitly) in [BFZ] which is based on the standard factorization of the unipotent upper-triangular matrices into the product of the elementary Jacobi matrices. Note that the representation of [BFZ] can be generalized to the case of an arbitrary semisimple Lie algebra. Explicitly the parameterization is given in terms of the matrix elements of the fundamental representations (see e.g [Li]). This implies that the Gauss-Givental representation discussed in this paper may be also rather straightforwardly generalized to the case of an arbitrary semisimple Lie algebra. Let us remark that the concept of the total positivity was generalized to the case of an arbitrary semisimple group by Lusztig (see [Lu] for details) in connection with the study of canonical bases. It is interesting that the same concept seems to play an important role in the choice of the proper coordinates for the construction of the Gauss-Givental representation proposed in this paper.

The connection between the Baxter Q-operator and the recursion operator discussed in Section 4.2 provides a simple construction of the Givental type integral representations for the common eigenfunctions of Hamiltonians for a wide class of the models (e.g. XXX and XXY open chains). These applications will be discussed elsewhere [GKLO2]. Also note that the integral representations of the wave function of the periodic Toda chain corresponding to the affine algebra $\hat{\mathfrak{gl}}_N$ provides an interesting description of the correlators in the Landau-Ginzburg topological model. Mirror symmetry predicts that on the other hand one gets the description of the Gromov-Witten invariants of the affine flag manifold.

Finally one should say that there is a generalization of the results of this paper to the case of the quantum groups and the centrally extended loop algebras. We are going to discuss various generalization of the Gauss-Givental representation constructed in this paper elsewhere.

A Appendix A

In this appendix we give the proof of the Proposition 2.1. We will organize our calculations in two steps. First we use the recursive structure of (2.3) and obtain recursive formulas for generators $E^{(N)}_{i+1,i}$, $E^{(N)}_{i,i}$ and $E^{(N)}_{i,i+1}$ in terms of $y_{k,i}$. Then, changing variables and rewriting the recursive relations in variables $T_{k,i}$, we solve them to obtain the final form of the generators.

A.1 The case of $E^{(N)}_{i,i+1}$

In order to find an explicit form of the generators $E^{(N)}_{i,i+1}$ in the variables $y_{i,k}$ we use the following commutation relations between elementary unipotent matrices:

$$\left(1+\sum_{i=1}^{N-1} y_{N-1,i}\cdot e_{i,i+1}\right)(1+\varepsilon\cdot e_{j,j+1}) = \\ \left(1+\frac{y_{N-1,j-1}}{y_{N-1,j}}\varepsilon\cdot e_{j-1,j}\right)\left(1+\sum_{i=1}^{N-1} y'_{N-1,i}\cdot e_{i,i+1}\right) mod(\varepsilon^2), \tag{A.1}$$

where

$$y'_{N-1,j-1} = y_{N-1,j-1} - \frac{y_{N-1,j-1}}{y_{N-1,j}}\varepsilon, \\ y'_{N-1,j} = y_{N-1,j}+\varepsilon,\ \ y'_{N-1,i} = y_{N-1,i},\ i\neq j-1,j. \tag{A.2}$$

Using the decomposition (2.3), definition (2.5) and the relation (A.1) it is simple to check that, in variables $y_{i,j}$, the generators $E^{(N)}_{i,i+1}$ obey the following recursion relation:

$$E^{(N)}_{i,i+1} f(x) = \\ \left(\frac{y_{N-1,i-1}}{y_{N-1,i}}E^{(N-1)}_{i-1,i} - y^{-1}_{N-1,i}\left(y_{N-1,i-1}\frac{\partial}{\partial y_{N-1,i-1}} + y_{N-1,i}\frac{\partial}{\partial y_{N-1,i}}\right)\right) f(x). \tag{A.3}$$

Resolving the recursion procedure we obtain:

$$E^{(N)}_{i,i+1} = \sum_{k=0}^{i-1}\prod_{s=0}^{k}\frac{y_{N-s,i-s}}{y_{N-s,i+1-s}}\frac{\partial}{\partial y_{N-1-k,i-k}} - \prod_{s=0}^{k}\frac{y_{N-(s+1),i-(s+1)}}{y_{N-(s+1),i-s}}\frac{\partial}{\partial y_{N-1-k,i-(k+1)}} = \\ \sum_{k=1}^{i}\left(\sum_{s=k}^{i} e^{T_{N+s-i,s}-T_{N+s-i-1,s}}\right)\times\left(\frac{\partial}{\partial T_{N+k-i-1,k}} - \frac{\partial}{\partial T_{N+k-i-1,k-1}}\right). \tag{A.4}$$

Here we have used:

$$y_{p,n}\frac{\partial}{\partial y_{p,n}} = \sum_{k=1}^{n}\frac{\partial}{\partial T_{p-k+1,n-k+1}}, \qquad n=1,\ldots,p. \tag{A.5}$$

A.2 The case of $E^{(N)}_{i,i}$

For the Cartan generators we get the same type of recursion procedure in the variables $y_{k,i}$:

$$E^{(N)}_{i,i} f(x) = \left(E^{(N-1)}_{i,i} + y_{N-1,i-1}\frac{\partial}{\partial y_{N-1,i-1}} - y_{N-1,i}\frac{\partial}{\partial y_{N-1,i}}\right) f(x), \quad i\neq N, \tag{A.6}$$

$$E^{(N)}_{N,N} f(x) = (\mu^{(N)}_N + y_{N-1,N-1}\frac{\partial}{\partial y_{N-1,N-1}}) f(x), \tag{A.7}$$

using the following relation:

$$\left(1+\sum_{i=1}^{N-1} y_{N-1,i}\cdot e_{i,i+1}\right)(1+\varepsilon\cdot e_{j,j})=(1+\varepsilon\cdot e_{j,j})\left(1+\sum_{i=1}^{N-1} y'_{N-1,i}\cdot e_{i,i+1}\right), \tag{A.8}$$

where

$$\begin{gathered} y'_{N-1,j-1}=y_{N-1,j-1}(1+\varepsilon), \qquad y'_{N-1,j}=y_{N-1,j}(1+\varepsilon)^{-1}, \\ \widetilde{y}_{N-1,i}=y_{N-1,i},\ i\neq j,j-1. \end{gathered} \tag{A.9}$$

Resolving recursion relations one can find

$$E^{(N)}_{i,i}=\mu^{(N)}_i+\sum_{k=1}^{i-1}\frac{\partial}{\partial T_{N+k-i,k}}-\sum_{k=i}^{N-1}\frac{\partial}{\partial T_{k,i}}. \tag{A.10}$$

A.3 The case of $E^{(N)}_{i+1,i}$

In this case the same line of reasoning leads to the following recursion expression:

$$\begin{gathered} E^{(N)}_{i+1,i}f(x)=\left(E^{(N-1)}_{i+1,i}+y_{N-1,i}\left(E^{(N-1)}_{i,i}-E^{(N-1)}_{i+1,i+1}\right)-\right. \\ \left. y_{N-1,i}\left(y_{N-1,i}\frac{\partial}{\partial y_{N-1,i}}+y_{N-1,i+1}\frac{\partial}{\partial y_{N-1,i+1}}\right)\right)f(x). \end{gathered} \tag{A.11}$$

Resolving the recursion procedure and using (A.10) one obtains:

$$\begin{gathered} E^{(N)}_{i+1,i}=\sum_{k=1}^{N-1}\left[(\mu^{(N)}_i-\mu^{(N)}_{i+1})y_{k,i}-y_{k,i}(y_{k,i}\frac{\partial}{\partial y_{k,i}}-y_{k,i+1}\frac{\partial}{\partial y_{k,i+1}})+\right. \\ \left. y_{k,i}\sum_{s=1}^{k-1}(y_{s,i-1}\frac{\partial}{\partial y_{s,i-1}}-2y_{s,i}\frac{\partial}{\partial y_{s,i}}+y_{s,i+1}\frac{\partial}{\partial y_{s,i+1}})\right]= \\ \sum_{k=1}^{N-1}e^{(T_{k,i}-T_{k+1,i+1})}\left(\mu^{(N)}_i-\mu^{(N)}_{i+1}+\sum_{s=1}^{k}\left(\frac{\partial}{\partial T_{s,i+1}}-\frac{\partial}{\partial T_{s,i}}\right)\right). \end{gathered} \tag{A.12}$$

■

Notice that the generators $E^{(N)}_{i,i}$, $E^{(N)}_{i,i+1}$, $E^{(N)}_{i+1,i}$ written as differential operators in terms of $y_{k,i}$ has a natural recursive structure. However the variables $T_{k,i}$ are more appropriate when the problem of solving the equations on Whittaker vectors arises.

B Appendix B

In this Appendix we give the proof of the Proposition 3.1. To derive the explicit expressions for the Whittaker vectors $\psi_{L,R}$ we use the iteration over the rank of the algebra $\mathfrak{gl}(N)$.

Consider the twisted generators of the algebra $\mathfrak{gl}(N)$

$$\widehat{E}^{(N)}_{i,i} = \mu^{(N)}_i + \sum_{k=1}^{i-1} \frac{\partial}{\partial T_{N+k-i,k}} - \sum_{k=i}^{N-1} \frac{\partial}{\partial T_{k,i}}, \tag{B.1}$$

$$\widehat{E}^{(N)}_{i,i+1} = \sum_{n=1}^{i} \left(\sum_{k=n}^{i} e^{T_{N+k-i,k} - T_{N+k-i-1,k}} \right) \left(\frac{\partial}{\partial T_{n+N-i-1,n}} - \frac{\partial}{\partial T_{n+N-i-1,n-1}} \right), \tag{B.2}$$

$$\widehat{E}^{(N)}_{i+1,i} = \sum_{k=1}^{N-1} e^{(T_{k,i}-T_{N,i})-(T_{k+1,i+1}-T_{N,i+1})} \left(\mu^{(N)}_i - \mu^{(N)}_{i+1} + \sum_{s=1}^{k} \left(\frac{\partial}{\partial T_{s,i+1}} - \frac{\partial}{\partial T_{s,i}} \right) \right), \tag{B.3}$$

that are related with the generators (2.1) as follows

$$\widehat{E}^{(N)}_{i,j} = e^{-H^{(N)}_L} E^{(N)}_{i,j} e^{H^{(N)}_L}. \tag{B.4}$$

To reveal the recursive structure it is useful to modify the generators $\widehat{E}^{(N)}_{i,i+1}$ and $\widehat{E}^{(N)}_{i+1,i}$ separately

$$\widetilde{E}^{(N)}_{i,i+1} := (\Xi^{(N)}_R)^{-1} \widehat{E}^{(N)}_{i,i+1} (\Xi^{(N)}_R) \tag{B.5}$$

$$\widetilde{\widetilde{E}}^{(N)}_{i+1,i} := (\Xi^{(N)}_L)^{-1} \widehat{E}^{(N)}_{i+1,i} (\Xi^{(N)}_L) \tag{B.6}$$

where the operators $\Xi^{(N)}_{L,R}$ are defined as

$$\Xi^{(n)}_L = e^{-\sum\limits_{i=1}^{n} \mu^{(n)}_i T_{n,i}} \exp\left\{ \mu^{(n)}_n \left(\sum_{i=1}^{n} T_{n,i} - \sum_{i=1}^{n-1} T_{n-1,i} \right) + \sum_{i=1}^{n-1} \xi^{(n-i)}_L e^{T_{n,i}-T_{n-1,i}} \right\} e^{\sum\limits_{i=1}^{n-1} \mu^{(n)}_i T_{n-1,i}}, \tag{B.7}$$

and

$$\Xi^{(n)}_R = \exp\left\{ -\sum_{i=1}^{n-1} T_{n,i} \sum_{k=1}^{n-1} \frac{\partial}{\partial T_{k,i}} + \sum_{i=2}^{n} T_{n,i} \sum_{k=1}^{i-1} \frac{\partial}{\partial T_{k+(n-i),k}} \right\} \times \exp\left\{ \sum_{i=1}^{n-1} \xi^{(i)}_R e^{T_{n-1,i}-T_{n,i+1}} \right\} \times \exp\left\{ \sum_{i=1}^{n-2} T_{n-1,i} \sum_{k=1}^{n-2} \frac{\partial}{\partial T_{k,i}} - \sum_{i=2}^{n-1} T_{n-1,i} \sum_{k=1}^{i-1} \frac{\partial}{\partial T_{k+(n-1-i),k}} \right\}. \tag{B.8}$$

Here $n = 1, \ldots, N$ and we also set $\Xi^{(1)}_{L,R} = 1$. We have the following relations

$$e^{T_{N-1,i}-T_{N,i+1}} \cdot \left(\widetilde{E}^{(N)}_{i,i+1} - \xi^{(i)}_R \right) = e^{T_{N-1,i-1}-T_{N,i}} \cdot \left(\widehat{E}^{(N-1)}_{i-1,i} - \xi^{(i-1)}_R \right) + \left(\sum_{k=i}^{N-1} \frac{\partial}{\partial T_{k,i}} - \sum_{k=i-1}^{N-1} \frac{\partial}{\partial T_{k,i-1}} \right) \tag{B.9}$$

and

$$
\begin{aligned}
e^{T_{N,i}-T_{N-1,i}} \cdot \left(\widetilde{\widetilde{E}}^{(N)}_{i+1,i} - \xi^{(i)}_L \right) = \\
e^{T_{N,i+1}-T_{N-1,i+1}} \cdot \left(\widehat{E}^{(N-1)}_{i+1,i} - \xi^{(i-1)}_L \right) - \left(\sum_{k=i}^{N-1} \frac{\partial}{\partial T_{k,i}} - \sum_{k=i+1}^{N-1} \frac{\partial}{\partial T_{k,i+1}} \right).
\end{aligned} \tag{B.10}
$$

Lemma B.1 *The equations*

$$
\widehat{E}^{(N)}_{i,i+1} \widehat{\psi}^{(N)}_R = \xi^{(i)}_R \widehat{\psi}^{(N)}_R \,, \qquad (i = 1, \ldots, N-1), \tag{B.11}
$$

$$
\widehat{E}^{(N)}_{i+1,i} \widehat{\psi}^{(N)}_L = \xi^{(N-i)}_L \widehat{\psi}^{(N)}_L \,, \qquad (i = 1, \ldots, N-1). \tag{B.12}
$$

admit the following solution

$$
\widehat{\psi}^{(N)}_R = \Xi^{(N)}_R \widehat{\psi}^{(N-1)}_R = \Xi^{(N)}_R \cdots \Xi^{(2)}_R \cdot 1, \tag{B.13}
$$

$$
\widehat{\psi}^{(N)}_L = \Xi^{(N)}_L \widehat{\psi}^{(N-1)}_L = \Xi^{(N)}_L \cdots \Xi^{(2)}_L \cdot 1. \tag{B.14}
$$

Proof. The statement follows from the recursion representation (B.9) and (B.10) for the generators. ■

Now the Whittaker vectors (3.5) and (3.6) are obtained as follows

$$
\widehat{\psi}^{(N)}_{L,R} = e^{-H^{(N)}_L} \psi^{(N)}_{L,R}. \tag{B.15}
$$

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