REGULARITY AND INDEX THEORY FOR DIRAC-SCHRÖDINGER SYSTEMS WITH LIPSCHITZ COEFFICIENTS

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Dedicated to Robert Seeley on the occasion of his 75. birthday.

ABSTRACT. Dirac-Schrödinger systems play a central role when modeling Dirac bundles and Dirac-Schrödinger operators near the boundary, along ends or near other singularities of Riemannian manifolds. In this article we develop the Fredholm theory of Dirac-Schrödinger systems with Lipschitz coefficients.

INTRODUCTION

A *Dirac system* d consists of a bundle $\mathcal{H} \to \mathbb{R}_+$ of separable complex Hilbert spaces together with a differential operator, its *Dirac operator*

$$D = \gamma(\partial + A), \tag{0.1}$$

where $\gamma = (\gamma_t)_{t\in\mathbb{R}_+}$ is a family of unitary operators on the fibers H_t of $\mathcal{H}$ with $\gamma_t^{-1} = -\gamma_t$, $(A_t)_{t\in\mathbb{R}_+}$ is a family of self-adjoint operators on the fibers H_t with discrete spectrum and anti-commuting with γ, and ∂ is a metric connection on $\mathcal{H}$ derived from these data. The Dirac operator is symmetric on sections with compact support in $(0,\infty)$.

The notion of Dirac system is traditionally connected with the finite dimensional versions of (0.1) which derived from separating variables in Dirac's original equation describing the relativistic electron. A very influential discussion of an infinite dimensional case was carried out in the celebrated work of Atiyah, Patodi, and Singer [APS], where manifolds with cylindrical ends are considered. More generally, Dirac systems arise in the study of Dirac operators on Dirac bundles in the sense of Gromov-Lawson when studying boundary value problems or ends with special geometric features. This is the motivation underlying the investigation of Dirac systems we present here.

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In many situations encountered in geometry, the data of the relevant Dirac system do not depend smoothly on the parameter $t \in \mathbb{R}_+$. For example, if M is a complete, non-compact Riemannian manifold with finite volume and pinched negative sectional curvature, then the Busemann functions associated to the ends of the manifold are only C^2, so that the tangent and normal bundles of their level surfaces are only C^1. This is the situation studied in [BB1] and [BB2]. The natural setup seems to be Dirac systems with (locally) Lipschitz coefficients as we consider them here. The present work leads to generalizations of the results in [BB1] and [BB2]. We will discuss this in a future publication.

After [APS], where the so-called APS-projection is introduced, it became customary to state boundary conditions for Dirac systems in terms of orthogonal projections in $H = H_0$. The regularity theory of boundary conditions defined by orthogonal projections in H plays a central role in [BL2], see for example Theorem 4.3 in [BL2], an important predecessor of this article regarding the regularity theory of boundary conditions.

The first main contribution of the present work consists in a new way of looking at boundary value problems for Dirac systems. Let D_0 be the restriction of D to Lipschitz sections of $\mathcal{H}$ which vanish at $t = 0$. Then D_0 is symmetric and contained in $D_{\max} := D_0^*$, the *maximal extension* of D_0, with domain $\mathcal{D}_{\max}$. Denote by H^s, $s \in \mathbb{R}$, the domain of the operator $(I + |A_0|^2)^{s/2}$. For $I \subset \mathbb{R}$, denote by Q_I the spectral projection of A_0 associated to $I \cap \operatorname{spec} A_0$ and set $H^s_I := Q_I(H^s)$. We show that the space $\check{H} := \{\sigma(0) \,:\, \sigma \in \mathcal{D}_{\max}\}$ of admissible initial values is the hybrid Sobolev space

$$\check{H} = H^{1/2}_{(-\infty,0]} \oplus H^{-1/2}_{(0,\infty)}. \tag{0.2}$$

This leads us to say that a *boundary value problems* or a *boundary condition* for D is a closed subspace of $\check{H}$. By (0.2), the topology of the space $\check{H}$ is a mixture of the topologies of the spaces $H^{1/2}$ and $H^{-1/2}$ and is therefore not compatible with the topology of H or the Sobolev spaces H^s, which causes considerable technical problems when discussing boundary value problems given by projections.

Our first observation is that the closed extensions of D_0 are precisely the operators $D_{B,\max}$ with domain

$$\mathcal{D}_{B,\max} := \{\sigma \in \mathcal{D}_{\max} \,:\, \sigma(0) \in B\}, \tag{0.3}$$

given by boundary conditions $B \subset \check{H}$ as defined above. We show this in our discussion of constant coefficient Dirac systems (Proposition 1.50), but the same arguments also apply in the case of Dirac systems with Lipschitz coefficients, cf. Theorem B below. This characterization of closed extensions of D_0 is a first confirmation that our way of defining boundary value problems is the appropriate one.

The adjoint operator, $D^*_{B,\max}$, arises from the boundary form

$$
\begin{aligned}
(D_{\max}\sigma_1, \sigma_2)_{L^2(\mathcal{H})} - (\sigma_1, D_{\max}\sigma_2)_{L^2(\mathcal{H})} &= \langle \sigma_1(0), \gamma_0\sigma_2(0)\rangle_H \\
&=: \omega(\sigma_1(0), \sigma_2(0)),
\end{aligned}
\tag{0.4}
$$

a non-degenerate skew-Hermitian form on $\check{H}$. We show that

$$
D^*_{B,\max} = D_{B^a,\max},
\tag{0.5}
$$

where B^a denotes the annihilator of B with respect to ω.

With $H^1_{\mathrm{loc}}(d)$ the natural Sobolev space associated to d, we show an important regularity property of $\mathcal{D}_{\max}$,

$$
\mathcal{D}_{\max} \cap H^1_{\mathrm{loc}}(d) = \{\sigma \in \mathcal{D}_{\max} \,:\, \sigma(0) \in H^{1/2}\}.
\tag{0.6}
$$

Consequently we say that a boundary value problem B for D is *regular* if $B \subset H^{1/2}$. We say that a boundary value problem B is *elliptic* if B and its adjoint boundary value problem B^a are both regular. We prove next that elliptic boundary conditions coincide with the boundary conditions introduced in [BäB] (Proposition 1.65).

We say that a boundary condition B is *self-adjoint* if $B = B^a$. By definition, a self-adjoint boundary condition is elliptic if it is regular. In one of our main results on boundary value problems we characterize elliptic self-adjoint boundary conditions (Theorem 1.83 and Corollary 1.84). Part of this characterization is the following result.

Theorem A. *Let $H^\pm := \{x \in H \,:\, i\gamma x = \pm x\}$. Then $\check{H}$ contains an elliptic self-adjoint boundary condition if and only if the restriction A_0^+ of A_0 to H^+ is a Fredholm operator to H^- (in general unbounded) with index* $\operatorname{ind} A_0^+ = 0$.

Let $d = ((H_t), (A_t), (\gamma_t))$ be a Dirac system with Lipschitz coefficients, and denote by d^0 the Dirac system with constant coefficients (H_0, A_0, γ_0) and associated Dirac operator D^0. Our second main contribution to Dirac systems is the regularity theory for Dirac systems with Lipschitz coefficients. The first part of our work in this direction is concerned with the regularity theory of the maximal domain (Theorem 2.29):

Theorem B. *Let $\mathcal{D}_{\max}$ and $\mathcal{D}^0_{\max}$ be the domains of the maximal extension of D and D^0, respectively. If $\sigma \in L^2(\mathcal{H})$ has compact support, then $\sigma \in \mathcal{D}_{\max}$ if and only if $\sigma \in \mathcal{D}^0_{\max}$.*

This result underlies the asserted equalities in (0.2) and (0.6) above which we show for constant coefficients first and then extend to Lipschitz coefficients, by Theorem B.

For a satisfactory analysis of the index theory of Dirac systems it is necessary to consider extended solutions. This goes back to the work of Atiyah, Patodi, and Singer in [APS]. Here we rely on the approach of the third author and his related notion of non-parabolicity, compare [Ca1] and [Ca2]. The domain of the corresponding extended Dirac operator D_{ext} is denoted W, the operator and subdomain associated to a boundary condition B by $D_{B,\mathrm{ext}}$ and W_B, respectively.

In the second part of our work on the regularity theory of Dirac systems we study the space of Cauchy data of the spaces $\ker D_{\max}$ and $\ker D_{\mathrm{ext}}$. Before we formulate our results in this direction, some comments seem in order. Let M be a smooth compact manifold with boundary N and $E \to M$ be a smooth Hermitian vector bundle. Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic pseudo-differential operator of order one. In [Ca, Se], A. Calderón and R. Seeley studied the space of Cauchy data of $\ker D$. Let $\mathcal{C}^s$ be the space of such data which belong to the Sobolev space $H^{s+1/2}(M,E)$. By the Trace Theorem, $\mathcal{C}^s$ is a subspace of $H^s(N,E)$. Calderón and Seeley showed that there is a pseudo-differential projector P in $H^s(N,E)$ (of order 0) onto $\mathcal{C}^s$ and that the principal symbol of P is the projection onto the positive eigenspace of a certain operator derived from the symbol of D [1]. The projection P is obtained with a single layer potential and is not the orthogonal projection onto the L^2-closure of $\mathcal{C}^s$. However, B. Booß-Bavnbek and K. Wojciechowski remarked that the L^2-orthogonal projection has the same properties, see Lemma 12.8 in [BW]. Our result for Dirac systems with Lipschitz coefficients (and its adaptation to manifolds in Chapter 5) is a generalization of this result to a non-smooth setting (Theorems 3.6, 3.7, and 3.9); we emphasize that this generalization is achieved without any recourse to pseudo-differential techniques.

Theorem C. *Let d be a non-parabolic Dirac system with Lipschitz coefficients. Let $\check{\mathcal{C}}_{\max}$ and $\check{\mathcal{C}}_{\mathrm{ext}}$ be the Calderón spaces of Cauchy data $\sigma(0) \in \check{H}$ with $\sigma \in \ker D_{\max}$ and $\sigma \in \ker D_{\mathrm{ext}}$, respectively. Then*

$$\mathcal{C}^{1/2}_{\max} := \check{\mathcal{C}}_{\max} \cap H^{1/2} \quad \text{and} \quad \mathcal{C}^{1/2}_{\mathrm{ext}} := \check{\mathcal{C}}_{\mathrm{ext}} \cap H^{1/2}$$

are mutually adjoint elliptic boundary conditions.

Let $C_{\max}$ and C_{ext} be the orthogonal projections in H onto the closure of $\mathcal{C}_{\max} := \check{\mathcal{C}}_{\max} \cap H$ and onto $\mathcal{C}_{\mathrm{ext}} := \check{\mathcal{C}}_{\mathrm{ext}} \cap H$, respectively. Then $C_{\max}$ and C_{ext} restrict to H^s and extend to H^{-s}, $0 \le s \le 1/2$, and

$$C_{\max} - Q_{(0,\infty)} \quad \text{and} \quad C_{\mathrm{ext}} - Q_{(0,\infty)}$$

[1] Actually, Calderón and Seeley considered also elliptic operators of higher order and treat the L^p theory as well, see Theorem 2 in [Pa, p. 287] or Theorem 12.4 in [BW].

are compact in H^s *for all* $|s| \le 1/2$.

Recall Kato's notion of a Fredholm pair of closed subspaces in a Banach space [Ka, Section IV.4], compare Appendix A. Our main index formula is formulated in terms of such pairs (Theorem 3.12).

Theorem D. *Let* d *be a non-parabolic Dirac system with Lipschitz coefficients and* B *be an elliptic boundary condition. Then* $(\bar{B}, \mathcal{C}_{\mathrm{ext}})$ *is a Fredholm pair in* H *and*

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind}(\bar{B}, \mathcal{C}_{\mathrm{ext}}),$$

where $\bar{B}$ *denotes the closure of* B *in* H.

The boundary value problem considered by Atiyah, Patodi, and Singer corresponds to $B_{APS} := H^{1/2}_{(-\infty,0]}$. Another main index formula is of Agranovič-Dynin type and shows the fundamental character of the Atiyah-Patodi-Singer boundary condition (Theorem 3.14):

Theorem E. *Let* d *be a non-parabolic Dirac system with Lipschitz coefficients and* B *be an elliptic boundary condition. Then*

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind} D_{B_{APS},\mathrm{ext}} + \operatorname{ind}(\bar{B}, H_{(0,\infty)}).$$

The Cobordism Theorem for the chiral Dirac operator D^+ on the space of spinor fields of a closed spin manifold M of even dimension states that $\operatorname{ind} D^+ = 0$ if M is cobordant to a compact spin manifold, compare [BW, Corollary 21.6]. We prove a version of the Cobordism Theorem for Dirac systems with Lipschitz coefficients (Theorem 3.19). As in Theorem A above, let $H^\pm := \{x \in H : i\gamma x = \pm x\}$ and A_0^+ be the restriction of A_0 to H^+, a Fredholm operator to H^-.

Theorem F (Cobordism Theorem). *Let* d *be a Dirac system with Lipschitz coefficients. If the associated Dirac operator* D *is of Fredholm type in the sense that* d *is non-parabolic with* $W = \mathcal{D}_{\max}$, *then*

$$\operatorname{ind} A_0^+ = 0.$$

When cutting a manifold M into pieces M_1 and M_2 along a compact hypersurface $N = M_1 \cap M_2$, we may ask for the index of a Dirac operator D on sections of a Hermitian bundle E over M in terms of its restrictions to the pieces. The corresponding boundary condition along N, the so-called *transmission boundary condition*, requires that sections σ_1 and σ_2 of E over M_1 and M_2, respectively, coincide along N. In terms of Dirac systems, the decomposition of M and D corresponds to the direct sum of two Dirac systems which have compatible initial conditions at $t = 0$. Our first result concerning this type of boundary value problem is of Bojarski type (Theorem 3.23):

Theorem G. *Let d_1 and d_2 be non-parabolic Dirac systems with Lipschitz coefficients and Calderón spaces $\mathcal{C}_{1,\mathrm{ext}}$ and $\mathcal{C}_{2,\mathrm{ext}}$, respectively. Suppose that the initial conditions of d_1 and d_2 satisfy*

$$H := H_{1,0} = H_{2,0}, \quad A := A_{1,0} = -A_{2,0}, \quad \text{and} \quad \gamma_{1,0} = -\gamma_{2,0}.$$

Then $(\mathcal{C}_{1,\mathrm{ext}}, \mathcal{C}_{2,\mathrm{ext}})$ is a Fredholm pair in H.

Consider the Dirac operator D on $d = d_1 \oplus d_2$ with transmission boundary condition $B = \{(x,x) : x \in H^{1/2}\}$. Then B is an elliptic and self-adjoint boundary condition and

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind}(\mathcal{C}_{1,\mathrm{ext}}, \mathcal{C}_{2,\mathrm{ext}}).$$

Another convenient way of determining the index of a Dirac operator via decompositions is by decoupling the boundary conditions on the pieces M_1 and M_2. Our relevant result in this direction (Theorem 3.24) generalizes Theorem 4.3 of [BL1].

Theorem H. *Let d_1 and d_2 be non-parabolic Dirac systems with Lipschitz coefficients as in Theorem G. Then*

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind} D_{1,B_1,\mathrm{ext}} + \operatorname{ind} D_{2,B_2,\mathrm{ext}},$$

where B is the transmission boundary condition, B_1 is any elliptic boundary condition for d_1, and $B_2 = B_1^\perp \cap H^{1/2}$.

The above results are discussed and proved in Chapters 1–3 of the text. Many of our arguments and results here extend and simplify what is known from the literature. In Chapter 4, we discuss supersymmetric Dirac systems and derive the corresponding index formulas. In Chapter 5, we describe a geometric setup for non-smooth boundary value problems for differential operators of Dirac type and explain how our results extend to this situation. This will be important for our geometric applications in a forthcoming article, in which we will extend the results from [BB1, BB2]. We believe that it will also be useful in further work on boundary value problems and index theory of Dirac type operators. We derive our results not only for Dirac operators, but for the more general class of Dirac-Schrödinger operators, that is, operators of the form $D + V$, where D is a Dirac operator and V is a symmetric potential, see Definition 2.26.

In two appendices, we derive some results which are used in the main text and seem to be of independent interest, but are not closely connected with the program we are pursuing here.

In all our estimates, generic constants may change from line to line.

WB and JB would like to use this occasion to refer to the article [Kas], which already contains one of the main observations underlying

the proof of Theorem B of [BB2] and also similar applications. We would like to thank Tobias Ebel for pointing this out to us.

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Contents

1. Dirac systems with constant coefficients

1.1. **Generalities.** Let H be a separable complex Hilbert space with Hermitian inner product $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle$ and norm $||\cdot||_H = ||\cdot||$. Let A be a self-adjoint operator in H with domain H_A such that, with respect to the graph norm $||\cdot||_A$, the embedding $H_A \to H$ is compact; equivalently, A is discrete in the sense that $\operatorname{spec} A$ consists only of isolated eigenvalues with finite multiplicity. The pair $e := (H, A)$ will be referred to as an *evolution system* since we will associate an evolution operator to it. To that end we note first that any local Lipschitz function $\sigma : \mathbb{R}_+ := [0, \infty) \to H$ is weakly differentiable with locally uniformly bounded weak derivative σ' a.e.; this is a well known fact, but for the sake of completeness we will give a proof below. Then we can introduce the space

$$\mathcal{L}_{\mathrm{loc}}(e) := \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}_+, H) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H_A) \tag{1.1}$$

and the operator

$$L = L(e) := \partial_t + A : \mathcal{L}_{\mathrm{loc}}(e) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H), \tag{1.2}$$

where $\partial_t\sigma = \sigma'$ denotes the derivative of σ with respect to t. We call L the *evolution operator* associated to the evolution system $e = (H, A)$.

1.3. Lemma. *If $f : \mathbb{R}_+ \to H$ is locally Lipschitz, then f is weakly differentiable almost everywhere with locally uniformly bounded derivative. More precisely, if $L_{[a,b]}(f)$ denotes the Lipschitz constant of f on $[a, b]$, then*

$$||f'(t)||_H \le L_{[a,b]}(f),$$

for almost all $t \in [a, b]$.

Proof. Since H is separable, there is a countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H. By Lebesgue's Theorem, there exists a measurable subset $\mathcal{R} \subset \mathbb{R}_+$ of full measure such that the functions $t \mapsto \langle f(t), e_n\rangle$ are differentiable in all points of $\mathcal{R}$ for all $n \in \mathbb{N}$. Hence the functions $t \mapsto \langle f(t), u\rangle$, where u is in the dense subspace of H generated by the chosen basis, are also differentiable in all points of $\mathcal{R}$. We have

$$|\langle h^{-1}(f(t+h) - f(t)), u\rangle| \le L_{[0,T]}(f)||u||,$$

for all $u \in H$ and $t, h \in \mathbb{R}_+$ with $t, t+h \in [0, T]$. It follows that $t \mapsto \langle f(t), u\rangle$ is differentiable in $\mathcal{R}$ for *all* $u \in H$ and thus that the function f has a weak derivative, $f'(t) \in H$, in each $t \in \mathcal{R}$ and with the asserted norm estimate. $\square$

We will also need the spaces

$$\mathcal{L}_c(e) := \{\sigma \in \mathcal{L}_{\mathrm{loc}}(e) \,:\, \operatorname{supp}\sigma \text{ compact}\}, \tag{1.4}$$

$$\mathcal{L}_{0,c}(e) := \{\sigma \in \mathcal{L}_c(e) \,:\, \sigma(0) = 0\}. \tag{1.5}$$

On $\mathcal{L}_c(e)$, we define the scalar product

$$(\sigma_1, \sigma_2) := \int_0^\infty \langle \sigma_1(t), \sigma_2(t)\rangle \, dt, \tag{1.6}$$

and we denote by $L^2(\mathbb{R}_+, H)$ the Hilbert space arising by completion.

The formal adjoint of L in $L^2(\mathbb{R}_+, H)$ is $-\partial_t + A$, hence L does not induce a symmetric operator on $\mathcal{L}_{0,c}(e)$. This defect can be cured if there is an operator $\gamma \in \mathcal{L}(H) \cap \mathcal{L}(H_A)$ which satisfies the following two relations:

$$-\gamma = \gamma^* = \gamma^{-1} \quad \text{on } H, \tag{1.7}$$

$$A\gamma + \gamma A = 0 \qquad \text{on } H_A. \tag{1.8}$$

Note that (1.8) implies that $\operatorname{spec} A$ is symmetric with respect to 0. Then the triple $d := (H, A, \gamma)$ is called a *Dirac system*. The associated *Dirac operator* is defined as

$$D = D(d) := \gamma(\partial_t + A) : \mathcal{L}_{\mathrm{loc}}(e) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H). \tag{1.9}$$

We find, for $\sigma_1, \sigma_2 \in \mathcal{L}_c(e)$,

$$\langle \gamma\sigma_1, \sigma_2\rangle' = \langle D\sigma_1, \sigma_2\rangle - \langle \sigma_1, D\sigma_2\rangle, \tag{1.10}$$

hence

$$(D\sigma_1, \sigma_2) - (\sigma_1, D\sigma_2) = \langle \sigma_1(0), \gamma\sigma_2(0)\rangle =: \omega(\sigma_1(0), \sigma_2(0)), \tag{1.11}$$

and therefore the restriction $D_{0,c}$ of D to $\mathcal{L}_{0,c}(e)$ is symmetric. The adjoint operator $D_{\max} := (D_{0,c})^*$ of $D_{0,c}$ is called the *maximal extension* of $D_{0,c}$; its domain is denoted by $\mathcal{D}_{\max}$. The closure $D_{\min}$ of $D_{0,c}$ is called the *minimal extension* of $D_{0,c}$, the domain of $D_{\min}$ is denoted by $\mathcal{D}_{\min}$. By definition,

$$D_{0,c} \subset D_{\min} = (D_{\max})^* \subset D_{\max}. \tag{1.12}$$

For later purposes it is useful to note that norm estimates for $L\sigma$ also hold for $D\sigma$,

$$||D\sigma(t)||_H = ||L\sigma(t)||_H \tag{1.13}$$

for all $\sigma \in \mathcal{L}_{\mathrm{loc}}(e)$ and $t \in \mathbb{R}_+$.

We denote by $H^1(e)$ the closure of $\mathcal{L}_c(e)$ under the norm

$$\|\sigma\|^2_{H^1(e)} := \|\sigma\|^2_{L^2(\mathbb{R}_+,H)} + \|\partial_t\sigma\|^2_{L^2(\mathbb{R}_+,H)} + \|A\sigma\|^2_{L^2(\mathbb{R}_+,H)}, \tag{1.14}$$

which is naturally associated to the data defining the evolution system[2]. We will also use the space

$$(1.15)\quad H^1_{\mathrm{loc}}(e) := \{\sigma : \mathbb{R}_+ \to H \text{ measurable} : \psi\sigma \in H^1(e) \text{ for all } \psi \in \mathrm{Lip}_c(\mathbb{R}_+)\}.$$

Note that the norm of $H^1(e)$ is stronger than the graph norm of D. In particular, we have a continuous extension

$$(1.16)\qquad D : H^1(e) \to L^2(\mathbb{R}_+, H).$$

Moreover, if the domain $\bar{\mathcal{D}}$ of some closed extension $\bar{D}$ of $D_{0,c}$ is contained in $H^1(e)$, then the $H^1(e)$-norm and the graph norm of $\bar{D}$ are equivalent on $\bar{\mathcal{D}}$, by the Closed Graph Theorem. This fact will be used repeatedly.

Spectral projections of A will play a specific role; we reserve the letter Q for these. For a subset $J \subset \mathbb{R}$, $Q_J = Q_J^*$ denotes the associated spectral projection of A in H. As shorthand, we use, for $\Lambda \in \mathbb{R}$,

$$(1.17)\qquad \begin{aligned} Q_{>\Lambda} &:= Q_{(\Lambda,\infty)}, \quad Q_{\geq\Lambda} := Q_{[\Lambda,\infty)}, \\ Q_{<\Lambda} &:= Q_{(-\infty,\Lambda)}, \quad Q_{\leq\Lambda} := Q_{(-\infty,\Lambda]}, \end{aligned}$$

We also use $Q_0 := Q_{\{0\}}$ and

$$(1.18)\qquad \begin{aligned} &Q_> := Q_{>0}, \quad Q_\geq := Q_{\geq 0}, \quad Q_< := Q_{<0}, \quad Q_\leq := Q_{\leq 0}, \\ &Q_{\neq} = Q_< + Q_> = I - Q_0. \end{aligned}$$

Since γ anticommutes with A, we have $\gamma^* Q_J \gamma = Q_{-J}$. In particular,

$$(1.19)\qquad \gamma^* Q_> \gamma = Q_<, \quad \gamma^* Q_\leq \gamma = Q_\geq, \quad \text{and} \quad \gamma^* Q_0 \gamma = Q_0.$$

Furthermore, we set $H_< := Q_<(H)$ and use a similar notation in the other cases and for the Sobolev spaces associated to A below.

1.2. **Sobolev spaces associated to A.** Let H and A be as above. For $s \geq 0$, let $H^s \subset H$ be the domain of $|A|^s$. Then $H = H^0$ and $H_A = H^1$. We set $H^\infty = \cap_{s\geq 0} H^s$, which is a dense subspace of H.

For $s \in \mathbb{R}$, we define a scalar product $\langle \cdot, \cdot \rangle_s$ on H^∞,

$$(1.20)\qquad \langle x, y \rangle_s := \langle (I + A^2)^{s/2} x, (I + A^2)^{s/2} y \rangle.$$

For $s \geq 0$, the norm $||\cdot||_s$ associated to $\langle \cdot, \cdot \rangle_s$ is equivalent to the graph norm of $|A|^s$ and H^s is equal to the completion of H^∞ with respect to $||\cdot||_s$. For $s < 0$, we define H^s to be the completion of H^∞ with respect to $||\cdot||_s$ and set $H^{-\infty} := \cup_{s\in\mathbb{R}} H^s$. The pairing

$$(1.21)\quad B_s : H^s \times H^{-s} \to \mathbb{C}, \quad B_s(x, y) := \langle (I + A^2)^{s/2} x, (I + A^2)^{-s/2} y \rangle,$$

[2]The notation $H_1(\mathbb{R}_+, A)$ is also common and was used e.g. in [BL2].

is perfect, that is, it identifies H^{-s} with the dual space of H^s. In particular, any $S \in \mathcal{L}(H^s)$ admits a dual operator $S' \in \mathcal{L}(H^{-s})$ with

$$B_s(Sx, y) = B_s(x, S'y). \tag{1.22}$$

This defines an algebra antimorphism $\mathcal{L}(H^s) \to \mathcal{L}(H^{-s})$. More generally, for $S \in \mathcal{L}(H^{s_1}, H^{s_2})$, we obtain a dual operator $S' \in \mathcal{L}(H^{-s_2}, H^{-s_1})$; in particular, if $s = s_1 = -s_2$, then $S, S' \in \mathcal{L}(H^s, H^{-s})$.

Since A is discrete, the embedding $i_{t,s} : H^t \hookrightarrow H^s$ is compact for $s < t$. For $0 \leq \theta \leq 1$ and $r = \theta s + (1-\theta)t$, H^r is (isomorphic to) the interpolation space $[H^s, H^t]_\theta$, see for example [Ta, Chapter 4.2]. If $S \in \mathcal{L}(H^s)$ satisfies $S(H^t) \subset H^t$, then $S : H^t \to H^t$ is continuous, by the Closed Graph Theorem. Moreover, $S(H^r) \subset H^r$ for any r as above, by interpolation. Note also that $(i_{t,s})' = i_{-s,-t}$.

We say that an operator $S \in \mathcal{L}(H)$ is *s-smooth*, for $s \geq 0$, if both S and S^* restrict to H^s; this implies that S, S^* restrict to H^t and extend (continuously) to H^{-t} for $0 \leq t \leq s$. In fact, the dual of the restriction of S and S^* to H^s extends S^* and S to H^{-s}, respectively.

An s-smooth operator S is said to be *$(-s, s)$-smoothing* if S maps H^{-s} into H^s; if S is $(-s, s)$-smoothing, then so is S^*. In the special case $s = 1/2$ we simply speak of *smoothing operators*. Note that $S \in \mathcal{L}(H)$ is smoothing if S extends to $H^{-1/2}$ with image in $H^{1/2}$.

We say that an operator $S \in \mathcal{L}(H)$ has *order* 0, if S and S^* restrict to H^s for any $s > 0$; that is, S is of order 0 if and only if S is s-smooth for all $s \geq 0$. The space of operators of order 0 is denoted $\mathrm{Op}^0(A)$. By definition, all spectral projections of A have order 0.

We are primarily interested in the cases $s = -1/2, 0, 1/2$ and $s = 1$. If $S \in \mathcal{L}(H)$ extends continuously to $H^{-1/2}$, then the extension is denoted by $\tilde{S}$,

$$\tilde{S} : H^{-1/2} \to H^{-1/2}; \tag{1.23}$$

if $S \in \mathcal{L}(H)$ restricts to $H^{1/2}$, then the restriction is denoted by $\hat{S}$,

$$\hat{S} : H^{1/2} \to H^{1/2}. \tag{1.24}$$

If there is no danger of confusion, we also write S instead of $\hat{S}$ or $\tilde{S}$.

If the adjoint operator S^* of $S \in \mathcal{L}(H)$ restricts to $H^{1/2}$, then S extends continuously to $H^{-1/2}$,

$$\tilde{S} = (\widehat{S^*})'. \tag{1.25}$$

In particular, if $S = S^*$ and $S(H^{1/2}) \subset H^{1/2}$, then $\tilde{S} = \hat{S}'$. If Q is a spectral projection of A, then $Q(H^s) \subset H^s$ for any $s \in \mathbb{R}$, by the definition of H^s. Since $Q^* = Q$, we have $\tilde{Q} = \hat{Q}'$ for any such Q.

The following lemma and corollary will only be used in the discussion of regular pairs of projections in Section 1.6.

1.26. LEMMA. *Let $S \in \mathcal{L}(H)$ be $1/2$-smooth. Then the following conditions are equivalent:*

(1) *Let $x \in H^{-1/2}$. If $\tilde{S}x \in H^{1/2}$ or $\widetilde{S^*}x \in H^{1/2}$, then $x \in H^{1/2}$.*
(2) $\ker \tilde{S} = \ker \hat{S}$, $\ker \widetilde{S^*} = \ker \widehat{S^*}$, *and there is a constant C with*

$$\left.\begin{array}{l} ||x||_{1/2} \le C(||\hat{S}x||_{1/2} + ||x||_{-1/2}) \\ ||x||_{1/2} \le C(||\widehat{S^*}x||_{1/2} + ||x||_{-1/2}) \end{array}\right. \quad \textit{for all } x \in H^{1/2}.$$

(3) $\hat{S}$ *and* $\widehat{S^*}$ *are Fredholm operators with* $\operatorname{ind} \hat{S} + \operatorname{ind} \widehat{S^*} = 0$.
(4) *There are a $1/2$-smooth operator U and smoothing operators K_r, K_l in $\mathcal{L}(H)$ such that*

$$\tilde{S}\tilde{U} = \widetilde{U^*}\widetilde{S^*} = I - \widetilde{K_l} \quad \textit{and} \quad \tilde{U}\tilde{S} = \widetilde{S^*}\widetilde{U^*} = I - \widetilde{K_r}.$$

Proof. (1) $\Rightarrow$ (2). The assertion on the kernels is an obvious consequence of (1). Consider next $\widehat{S^{(*)}}$ as an unbounded operator in $H^{-1/2}$ with domain and target space $H^{1/2}$. Then it follows from (1) that $\widehat{S^{(*)}}$ is closed. The projection $\pi_1 : H^{-1/2} \times H^{1/2} \to H^{-1/2}$ takes values in $H^{1/2}$ when restricted to the graph of $\widehat{S^{(*)}}$. Applying the Closed Graph Theorem to this map we derive the asserted inequalities in (2).
(2) $\Rightarrow$ (3). By Lemma A.11 in Appendix A, the a priori estimate in (2) implies that $\hat{S}$ and $\widehat{S^*}$ have finite-dimensional kernels and closed images in $H^{1/2}$. From the assumption on the kernels and duality we deduce that

$$\begin{aligned} \operatorname{codim} \hat{S} &= \dim (\operatorname{im} \hat{S})^0 = \dim \ker \widetilde{S^*} = \dim \ker \hat{S}^*, \\ \operatorname{codim} \widehat{S^*} &= \dim \, \ker \hat{S}, \end{aligned}$$

where the superscript 0 indicates the annihilator (or polar set) in $H^{-1/2}$. This establishes (3).
(3) $\Rightarrow$ (4). It is immediate from the assumptions that $\ker \tilde{S} = \ker \hat{S}$ and $\ker \widetilde{S^*} = \ker \widehat{S^*}$. Choose a basis $(e_j^{(*)}) \subset H^{1/2}$ of $\ker \widehat{S^{(*)}}$ which is orthonormal in H and set

$$K_{r(l)}x := \sum B_{-1/2}(x, e_j^{(*)})\, e_j^{(*)}, \quad x \in H^{-1/2}.$$

Then $K_{r(l)} \in \mathcal{L}(H^{-1/2}, H^{1/2})$ is a projection in $H^{-1/2}$ onto $\ker \widehat{S^{(*)}}$ and $\tilde{S} : \ker K_r \to \ker K_l$ is an isomorphism. It follows that there is a $1/2$-smooth operator $U \in \mathcal{L}(H)$ with

$$\tilde{S}\tilde{U} = I - \widetilde{K_l} \quad \text{and} \quad \tilde{U}\tilde{S} = I - \widetilde{K_r}.$$

Restricting to $H^{1/2}$ and computing the dual operators gives the remaining identities in (4).

(4) $\Rightarrow$ (1). Consider $x \in H^{-1/2}$ with $y := \tilde{S}x \in H^{1/2}$. Then we obtain from (4)

$$x = \hat{U}y + K_r x \in H^{1/2},$$

since K_r is smoothing; a similar argument works for $\widetilde{S^*}$. $\square$

Since, by complex interpolation, both $\tilde{S}$ and $\tilde{U}$ restrict to H^s, for $|s| \leq 1/2$, we have the following consequence.

1.27. Corollary. *Under the conditions of Lemma 1.26, $\widetilde{S^{(*)}}$ restricts respectively extends to a Fredholm operator on each H^s, $|s| \leq 1/2$, with kernel and index independent of s.*

1.3. **The domain of the maximal extension.** In our approach, boundary conditions at 0 will play a prominent role; for that reason, the existence of restriction maps is of interest. We begin with the following regularity lemma; its third part reflects the usual trace properties of Sobolev spaces.

1.28. Lemma (Regularity I). *We have*

(1) $\mathcal{L}_{\mathrm{loc}}(e) \subset C^{0+1/2}(\mathbb{R}_+, H^{1/2})$.
(2) $H^1(e) \subset C(\mathbb{R}_+, H^{1/2})$.
(3) $\mathcal{R} : H^1(e) \to H^{1/2}$, $\mathcal{R}\sigma := \sigma(0)$, *is continuous.*

Proof. By the Cauchy-Schwarz inequality, we have, for any $x \in H_A$,

$$||x||^2_{H^{1/2}} \leq ||x||_{H_A}||x||_H.$$

Hence if $\sigma \in \mathcal{L}_{\mathrm{loc}}(e)$ with $||\sigma||_{H_A} \leq K$ and $||\sigma'||_H \leq L$ on $[0,T]$, then

$$\begin{aligned} ||\sigma(s) - \sigma(t)||^2_{H^{1/2}} &\leq ||\sigma(s) - \sigma(t)||_{H_A}||\sigma(s) - \sigma(t)||_H \\ &\leq 2KL|s-t|, \end{aligned}$$

for all $s, t \in [0,T]$. This shows the first claim.

As for the proof of the second and third claim, we choose an orthonormal basis, $(e_n)_{n\in\mathbb{N}}$, for H, consisting of eigenvectors of A, i.e. $Ae_n = a_n e_n$ for some $a_n \in \mathbb{R}$. For $\sigma \in \mathcal{L}_c(e)$ we set $\sigma_n(t) := \langle \sigma(t), e_n \rangle$. Then $\sigma_n \in \mathrm{Lip}_c(\mathbb{R}_+)$ and hence, by (B.3),

$$|a_n||\sigma_n(t) - \sigma_n(s)|^2 \leq 2||\sigma_n'||^2_{L^2([s,t])} + 2a_n^2||\sigma_n||^2_{L^2([s,t])}.$$

Therefore

$$\begin{aligned} (1.29)\quad & ||\sigma(t) - \sigma(s)||^2_{H^{1/2}} \\ &\leq C\big(||\sigma||^2_{L^2([s,t],H)} + ||\sigma'||^2_{L^2([s,t],H)} + ||A\sigma||^2_{L^2([s,t],H)}\big). \end{aligned}$$

Since σ has compact support,

$$\begin{aligned}(1.30)\quad & ||\sigma(s)||_{H^{1/2}} \\ & \leq C\big(||\sigma||^2_{L^2([s,\infty],H)} + ||\sigma'||^2_{L^2([s,\infty],H)} + ||A\sigma||^2_{L^2([s,\infty],H)}\big).\end{aligned}$$

In particular,

$$||\sigma(0)||_{H^{1/2}} \leq C||\sigma||_{H^1(e)}.$$

Since $H^1(e)$ is the closure of $\mathcal{L}_c(e)$ in the $H^1(e)$-norm, (1.29) and (1.30) hold for all $\sigma \in H^1(e)$. Claims (2) and (3) follow. $\square$

To get a satisfactory description of the domain $\mathcal{D}_{\max} \subset L^2(\mathbb{R}, H)$ of the maximal extension $D_{\max}$ of $D_{0,c}$, we employ the solution theory of the evolution operator L. For $\sigma \in L^2(\mathbb{R}_+, H)$ we set

$$(S_L\sigma)(t) := \int_0^t e^{(s-t)A_>}\sigma(s)ds - \int_t^\infty e^{(s-t)A_<}\sigma(s)ds, \tag{1.31}$$

where we have written $A_> := AQ_>$ and $A_< := AQ_<$. The *solution operator* S_L has been studied in [APS, Proposition (2.5)] via the corresponding ordinary differential equations in the eigenspaces of A. The result is that

$$S_L : L^2(\mathbb{R}_+, H_{\neq}) \to \{\sigma \in Q_{\neq}(H^1(e)) \,:\, \sigma(0) \in H^{1/2}_<\}$$

is continuous and bijective with inverse L. We conclude:

1.32. Lemma. *The solution operator $S_D := S_L\gamma^* : L^2(\mathbb{R}_+, H) \to H^1(e)$ of D is continuous with $(Q_>S_D\sigma)(0) = 0$ and*

$$DS_D\sigma = Q_{\neq}\sigma \tag{1.33}$$

for all $\sigma \in L^2(\mathbb{R}_+, H)$. Moreover,

$$S_DD\sigma = Q_{\neq}\sigma \tag{1.34}$$

for all $\sigma \in H^1(e)$ with $Q_>\sigma(0) = 0$. In particular, the map

$$\mathcal{R}S_D : L^2(\mathbb{R}_+, H) \to H^{1/2}_<, \quad \sigma \mapsto (S_D\sigma)(0), \tag{1.35}$$

is surjective. $\square$

We also use the *extension operator*

$$\mathcal{E}x(t) := e^{-t(|A|+Q_0)}x, \tag{1.36}$$

which is defined for $x \in H^{-\infty}$ and $t \geq 0$. We note that $\mathcal{E}x(t) \in H^\infty$ for all $t > 0$. The following assertions are readily verified by studying the respective ordinary differential equations in the eigenspaces of A.

1.37. Lemma. *For any $s \in \mathbb{R}$ and $x \in H^s$,*

(1) *$\mathcal{E}x \in C(\mathbb{R}_+, H^s)$ and $||(\mathcal{E}x)(t)||_s \leq ||x||_s$ for all $t \geq 0$.*
(2) *$\mathcal{E}x \in C^1(\mathbb{R}_+, H^{s-1})$ with $(\mathcal{E}x)' = -(|A| + Q_0)\mathcal{E}x$.*

(3) $C_s^{-1}||x||_{s-\frac{1}{2}} \le ||(|A|+Q_0)^s \mathcal{E}x||_{L^2(\mathbb{R}_+,H)} \le C_s||x||_{s-\frac{1}{2}}$. □

Since $||(|A|+Q_0)x||_{s-1} \le ||x||_s$, the second equation implies that, for any $x \in H_A = H^1$, the extension $\mathcal{E}x : \mathbb{R}_+ \to H$ is Lipschitz continuous with Lipschitz constant 1. In particular, $\mathcal{E}x \in \mathcal{L}_{\mathrm{loc}}(e)$ for any $x \in H_A$.

1.38. Proposition. *The map*

$$\mathcal{E}_> : H_>^{-1/2} \to \ker D_{\max}, \quad \mathcal{E}_> x := \mathcal{E}x,$$

is a continuous isomorphism. The restriction map $\mathcal{R}$ extends to a continuous map $\mathcal{R}$ on $\ker D_{\max}$ *with $\mathcal{R}\mathcal{E}_> x = x$.*

Proof. It follows from Lemma 1.37.2 that $\mathcal{E}_>$ maps $H_>^1$ to $\ker D_{\max}$. Lemma 1.37.3 implies that it extends to $H_>^{-1/2}$ as a continuous and injective map, where we recall that $\ker D_{\max} \subset L^2(\mathbb{R}_+,H)$ is closed.

To prove surjectivity, choose a unitary basis (e_n) of H of eigenvectors of A, $Ae_n = a_n e_n$. Let $\sigma \in \ker D_{\max}$ and set $\sigma_n(t) := \langle \sigma(t), e_n\rangle$. Then σ_n solves the ordinary differential equation $\sigma_n' + a_n\sigma_n = 0$ weakly, and hence $\sigma_n(t) = e^{-ta_n}x_n$, where $x_n = \sigma_n(0)$. Since σ is square integrable, $x_n = 0$ for $a_n \le 0$ and $x = \sum_{a_n>0} x_n e_n \in H_>^{-1/2}$. Hence $\sigma = \mathcal{E}_> x$.

The assertion about $\mathcal{R}$ is clear. □

We note that the Dirac operator D commutes with Q_0 and $Q_{\neq}$, hence

$$\mathcal{D}_{\max} = Q_{\neq}\mathcal{D}_{\max} \oplus Q_0\mathcal{D}_{\max}. \tag{1.39}$$

Moreover $Q_0\mathcal{D}_{\max} = H^1(\mathbb{R}_+, Q_0H)$, the standard Sobolev space.

1.40. Corollary (Representation Formula). *The map*

$$H_>^{-1/2} \oplus L^2(\mathbb{R}_+, H_{\neq}) \oplus H^1(\mathbb{R}_+, Q_0H) \to \mathcal{D}_{\max},$$
$$(x,\tau,\sigma_0) \mapsto \sigma = \mathcal{E}_> x + S_D\tau + \sigma_0,$$

is a continuous isomorphism with $D_{\max}\sigma = \tau + \gamma\sigma_0'$.

Proof. Clearly $\mathcal{E}_> x + S_D\tau + \sigma_0 \in \mathcal{D}_{\max}$ for all $x \in H_>^{-1/2}$, $\tau \in L^2(\mathbb{R}_+, H_{\neq})$, and $\sigma_0 \in H_1(\mathbb{R}_+, Q_0H)$. Vice versa, let $\sigma \in \mathcal{D}_{\max}$ and set $\tau = D_{\max}\sigma$ and $\sigma_0 = Q_0\sigma$. Then $\sigma - S_D\tau - \sigma_0 \in \ker D_{\max}$, by Lemma 1.32. Hence our map is a continuous isomorphism, by the continuity of S_D and Proposition 1.38. □

1.41. Proposition (Boundary Values). *Let*

$$\check{H} := H_>^{-1/2} \oplus Q_0H \oplus H_<^{1/2}.$$

Then $\mathcal{R}$ and $\mathcal{E}$ extend to respectively define continuous operators

$$\mathcal{R} : \mathcal{D}_{\max} \to \check{H} \quad \textit{and} \quad \mathcal{E} : \check{H} \to \mathcal{D}_{\max}$$

with $\mathcal{R}\mathcal{E} = I$ on $\check{H}$. In particular, $\mathcal{R}$ is surjective. □

Now we can derive the precise regularity properties of elements in $\mathcal{D}_{\max}$ which will make the special role of 0 even more apparent. For ease of notation, we set $\mathcal{R}\sigma =: \sigma(0)$.

1.42. LEMMA (Regularity II). *The maximal domain $\mathcal{D}_{\max}$ has the following properties:*

(1) *$\mathcal{L}_c(e)$ is dense in $\mathcal{D}_{\max}$.*
(2) *$H^1(e) = \{\sigma \in \mathcal{D}_{\max} : \sigma(0) \in H^{1/2}\} \subset \mathcal{D}_{\max}$.*
(3) *$\mathcal{D}_{\max} \subset C(\mathbb{R}_+, \check{H}) \cap C((0,\infty), H^{1/2})$.*
(4) *$\lim_{t\to\infty} \sigma(t) = 0$ in $H^{1/2}$ for any $\sigma \in \mathcal{D}_{\max}$.*
(5) *If $\phi \in \mathrm{Lip}(\mathbb{R}_+)$ is bounded and $\sigma \in \mathcal{D}_{\max}$, then $\phi\sigma \in \mathcal{D}_{\max}$ and $(\phi\sigma)(0) = \phi(0)\sigma(0)$.*

Proof. (1) By definition, $\mathcal{L}_c(e)$ is dense in $H^1(e)$. Hence it suffices to consider $\sigma \in \ker D_{\max}$, by Corollary 1.40. Write $\sigma = \mathcal{E}_> x$ with $x \in H_>^{-1/2}$. Choose a sequence (x_n) in $H_>^1$ with $x_n \to x$ in $H^{-1/2}$ and $\phi \in \mathrm{Lip}_c(\mathbb{R}_+)$ with $\phi = 1$ near 0. Set $\phi_n(t) := \phi(t/n)$, then by Lemma 1.37, $\phi_n \mathcal{E}_> x_n \in \mathcal{L}_c(e)$. It is easy to see that $\phi_n \mathcal{E}_> x_n \to \mathcal{E}_> x$ in $\mathcal{D}_{\max}$.

(2) Clearly $H^1(e) \subset \mathcal{D}_{\max}$. Since the image of S_D is contained in $H^1(e)$, the asserted characterization of $H^1(e)$ is immediate from Lemma 1.37.3 and Corollary 1.40.

(3) $\mathcal{D}_{\max} \subset C(\mathbb{R}_+, \check{H})$ is clear from Lemma 1.37.1. By Lemma 1.28.2, $H^1(e)$ is contained in $C(\mathbb{R}_+, H^{1/2})$, thus in $C(\mathbb{R}_+, \check{H})$. By Corollary 1.40, it is hence sufficient to consider the image of $\mathcal{E}_>$. Now $\mathcal{E}x(t) \in H^\infty$ and $\mathcal{E}x(t+t') = \mathcal{E}(\mathcal{E}x(t))(t')$ for all $x \in H^{-1/2}$ and $t, t' > 0$. Hence $\mathcal{E}_> x \in C((0,\infty), H^{1/2})$ for all $x \in H^{-1/2}$, by Lemma 1.37.1.

(4) Let $\sigma \in \mathcal{D}_{\max}$. It follows from (2) and (3) that σ shifted by $t > 0$, $\tau_t\sigma(t') := \sigma(t+t')$, is in $H^1(e)$. Hence by (1.30),

$$\begin{aligned} ||\sigma(t)||^2_{H^{1/2}} &= ||\tau_t\sigma(0)||^2_{H^{1/2}} \le C||\tau_t\sigma||^2_{H^1(e)} \\ &= C\int_t^\infty (||\sigma'||^2 + ||A\sigma||^2 + ||\sigma||^2). \end{aligned}$$

Hence $\sigma(t) \to 0$ in $H^{1/2}$ as $t \to \infty$.

(5) Let $\sigma \in \mathcal{D}_{\max}$ and $\phi \in \mathrm{Lip}(\mathbb{R}_+)$ be bounded. Choose a sequence (σ_n) in $\mathcal{L}_c(e)$ which converges to σ in $\mathcal{D}_{\max}$. Then $\phi\sigma_n \in \mathcal{L}_c(e)$ and $\phi\sigma_n \to \phi\sigma$ in $\mathcal{D}_{\max}$, hence the claim. □

Now we can extend (1.11) (cf. [BL2, Lemma 2.15]) to $\mathcal{D}_{\max}$. We only have to use Part 1 of Lemma 1.42 and to note that the skew-Hermitian form ω defined in (1.11) extends naturally to $(x,y) \in \check{H} \times \check{H}$ by

$$\omega(x,y) := B_{-1/2}(Q_{>\Lambda}x, \gamma Q_{<-\Lambda}y) + B_{1/2}(Q_{\le\Lambda}x, \gamma Q_{\ge-\Lambda}y), \tag{1.43}$$

where $\Lambda \in \mathbb{R}$ is arbitrary.

1.44. Corollary. *For $\sigma_1, \sigma_2 \in \mathcal{D}_{\max}$ we have*

$$(D_{\max}\sigma_1, \sigma_2) - (\sigma_1, D_{\max}\sigma_2) = \omega(\sigma_1(0), \sigma_2(0)). \quad \square$$

We note that ω is non-degenerate on $\check{H}$. For a linear subspace $B \subset \check{H}$, the annihilator of B with respect to ω is

$$B^a := \{y \in \check{H} \ : \ \omega(x, y) = 0 \text{ for all } x \in B\}; \tag{1.45}$$

$B^a \subset \check{H}$ is closed, and B^{aa} is the closure of B in $\check{H}$. The description of B^a is easy in the case where B is contained in $H^{1/2}$.

1.46. Lemma. *If $B \subset H^{1/2} \subset \check{H}$, then $B^a = (\gamma B^0) \cap \check{H}$, where*

$$B^0 = \{y \in H^{-1/2} \ : \ B_{1/2}(x, y) = 0 \textit{ for all } x \in B\}.$$

In particular, $B^a \cap H^{1/2} = \gamma(B^\perp \cap H^{1/2})$, where $B^\perp$ is the orthogonal complement of $B \subset H$ in H.

Proof. For $x, y \in \check{H}$ with $x \in H^{1/2}$, we have $\omega(x, y) = B_{1/2}(x, \gamma y)$. $\square$

1.4. **Boundary conditions and Fredholm properties.** With any linear subspace, $B \subset \check{H}$, we now associate various extensions of $D_{0,c}$. We define:

$$\begin{aligned} \mathcal{L}_{B,c}(e) &: = \{\sigma \in \mathcal{L}_c(e) \ : \ \sigma(0) \in B\}, \\ D_{B,c} &: = D|\mathcal{L}_{B,c}(e); \end{aligned} \tag{1.47}$$

$$\begin{aligned} \mathcal{D}_B &: = \{\sigma \in \mathcal{D}_{\max} \ : \ \sigma(0) \in B \cap H^{1/2}\} \\ &= \{\sigma \in H^1(e) \ : \ \sigma(0) \in B\}, \\ D_B &: = D|\mathcal{D}_B; \end{aligned} \tag{1.48}$$

$$\begin{aligned} \mathcal{D}_{B,\max} &: = \{\sigma \in \mathcal{D}_{\max} \ : \ \sigma(0) \in B\}, \\ D_{B,\max} &: = D_{\max}|\mathcal{D}_{B,\max}. \end{aligned} \tag{1.49}$$

Since the restriction map $\mathcal{R} : \mathcal{D}_{\max} \to \check{H}$ is continuous, $D_{B,\max}$ is a closed operator if B is a closed subspace of $\check{H}$. Vice versa, we have:

1.50. Proposition. *Let $\bar{D} \subset D_{\max}$ be a closed extension of $D_{0,c}$ and $\bar{\mathcal{D}}$ be the domain of $\bar{D}$. Then $\bar{D} = D_{B,\max}$, where $B = \{\sigma(0) \ : \ \sigma \in \bar{\mathcal{D}}\}$ is a closed subspace of $\check{H}$.*

Proof. Since $\bar{D}$ is a closed extension of $D_{0,c}$, the closure of $\mathcal{L}_{0,c}(e)$ in the $H^1(e)$-norm is contained in $\bar{\mathcal{D}}$,

$$H^1_0(e) := \{\sigma \in H^1(e) \ : \ \sigma(0) = 0\} \subset \bar{\mathcal{D}}.$$

Since the difference of any two elements from $\mathcal{D}_{\max}$ with the same value at 0 is in $H^1_0(e)$, by Lemma 1.42.2, we conclude that

$$\bar{\mathcal{D}} = \{\sigma \in \mathcal{D}_{\max} \, : \, \sigma(0) \in B\},$$

hence that $\bar{D} = D_{B,\max}$. Suppose now that (x_n) is a sequence in B such that $x_n \to x$ in $\check{H}$. Then, by what we just said, $(\mathcal{E}x_n)$ is a sequence in $\bar{\mathcal{D}}$ and $\mathcal{E}x_n \to \mathcal{E}x$ in $\mathcal{D}_{\max}$, by Proposition 1.41. Since $\bar{D}$ is a closed operator and $\mathcal{R}$ is continuous, we get that $x \in B$. □

1.51. Definition. A (linear) *boundary condition* for a Dirac system is a closed linear subspace $B \subset \check{H}$.

1.52. Remark. Since the seminal article [APS] of Atiyah, Patodi, and Singer, it is customary to state boundary conditions for Dirac systems in terms of projections in H. In our setup, the boundary condition introduced by Atiyah, Patodi, and Singer is given by the subspace $B_{APS} := \check{H}_{\leq}$ of $\check{H}$. We will discuss boundary conditions given by projections in Section 1.6. Our approach to the description of boundary conditions for Dirac systems, however, does not only seem to be more general but will also lead to a more satisfying analysis of the corresponding operators, as we are going to explain.

For any $\sigma \in \mathcal{L}_c(e)$, $\sigma(0) \in H_A = H^1$. Vice versa, for any $x \in H_A$ there is $\sigma \in \mathcal{L}_c(e)$ with $\sigma(0) = x$. Similarly, for any $x \in H^{1/2}$ there is $\sigma \in H^1(e)$ with $\sigma(0) = x$. Let $B \subset \check{H}$ be a boundary condition. We conclude, using (1.44), that the adjoint operators of the above operators are

$$(D_{B,c})^* = D_{B_1,\max} \quad \text{with } B_1 = (B \cap H_A)^a, \tag{1.53}$$

$$(D_B)^* = D_{B_2,\max} \quad \text{with } B_2 = (B \cap H^{1/2})^a, \tag{1.54}$$

$$(D_{B,\max})^* = D_{B^a,\max}. \tag{1.55}$$

Since the closure of a linear subspace of $\check{H}$ is the annihilator of its annihilator, the closures of the above operators are

$$D_{B,\min} = (D_{B,c})^{**} = D_{C_1,\max}, \tag{1.56}$$

$$(D_B)^{**} = D_{C_2,\max}, \tag{1.57}$$

$$(D_{B,\max})^{**} = D_{B,\max}. \tag{1.58}$$

where C_1 is the closure of $B \cap H_A$ in $\check{H}$ in (1.56) and C_2 is the closure of $B \cap H^{1/2}$ in $\check{H}$ in (1.57). In particular,

$$D_{B,\min} = D_{B,\max} \iff B \cap H_A \text{ is dense in } B. \tag{1.59}$$

1.60. Definition. We say that a boundary condition $B \subset \check{H}$ is *regular* if $D_{B,\max} = D_B$. We say that a boundary condition B is *elliptic* if B and B^a are regular.

By the representation formula 1.40, the boundary condition $B_{APS} = \check{H}_{\leq}$ of Atiyah, Patodi, and Singer is the most natural regular boundary condition. The following reformulations of regularity are immediate from the definition of regularity and the properties of the maximal domain.

1.61. Proposition. *A closed linear subspace B of $\check{H}$ is a regular boundary condition iff any of the following equivalent conditions is satisfied:*

1. $D_{B,\max} = D_B$.
2. $\mathcal{D}_{B,\max} \subset H^1(e)$.
3. $B \subset H^{1/2} \subset \check{H}$.

A closed linear subspace B of $H^{1/2}$ is a regular boundary condition iff one of the following two equivalent conditions is satisfied:

4. *The $H^{1/2}$ and $\check{H}$-norms are equivalent on B.*
5. *For some or, equivalently, any $\Lambda \in \mathbb{R}$, there is a constant C such that, for all $x \in B$,*
$$||Q_{>\Lambda}x||_{1/2} \leq C(||Q_{>\Lambda}x||_{-1/2} + ||Q_{\leq\Lambda}x||_{1/2}). \quad \Box$$

1.62. Lemma. *Let $B \subset \check{H}$ be a regular boundary condition and $\Lambda \in \mathbb{R}$. Then the map $Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda}$ is a left-Fredholm operator, that is, has finite-dimensional kernel and closed image. Moreover, $(H^{1/2}_{>\Lambda}, B)$ is a left-Fredholm pair in $H^{1/2}$ with*

$$\mathrm{null}(H^{1/2}_{>\Lambda}, B) = \dim\ker(Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda}) = \dim(H^{1/2}_{>\Lambda} \cap B),$$
$$\mathrm{def}(H^{1/2}_{>\Lambda}, B) = \dim\operatorname{coker}(Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda}) = \dim(\check{H}_{\geq-\Lambda} \cap B^a).$$

Proof. We use Hörmander's Criterion, see Lemma A.11. Suppose that (x_n) is a bounded sequence in B such that $Q_{\leq\Lambda}(x_n)$ converges in $H^{1/2}_{\leq\Lambda}$. Since the inclusion $H^{1/2} \to H^{-1/2}$ is compact and (x_n) is bounded in $H^{1/2}$, we may assume, by passing to a subsequence if necessary, that (x_n) converges in $H^{-1/2}$. But then $(Q_{>\Lambda}x_n)$ is a Cauchy sequence in $H^{1/2}$, by Proposition 1.61.5. It follows that $Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda}$ is a left-Fredholm operator and hence, by Proposition A.12, that $(H^{1/2}_{>\Lambda}, B)$ is a left-Fredholm pair. The formulas for the nullity and the first formula for the deficiency of the pair $(H^{1/2}_{>\Lambda}, B)$ are clear. As for the last equality, we have, using (A.6),

$$\begin{aligned}(H^{1/2}_{>\Lambda} + B)^0 &= (H^{1/2}_{>\Lambda})^0 \cap B^0 = H^{-1/2}_{\leq\Lambda} \cap B^0 \\ &= \{x \in H^{-1/2}_{\leq\Lambda} \,:\, B_{-1/2}(x,y) = 0 \text{ for all } y \in B\} \\ &= \gamma(\{x \in \check{H}_{\geq -\Lambda} \,:\, \omega(x,y) = 0 \text{ for all } y \in B\}) \\ &= \gamma(\check{H}_{\geq -\Lambda} \cap B^a). \end{aligned}$$

□

1.63. Proposition. *Let Λ be a real number, $\hat{U} \subset H^{1/2}_{\leq\Lambda}$ be a closed subspace, $F \subset H^{1/2}_{<-\Lambda}$ be a finite-dimensional subspace, $\hat{V} := F^0 \cap H^{1/2}_{<-\Lambda}$, and let $g : \hat{U} \to \hat{V}$ be a continuous linear map. Then*

$$B = \gamma F \oplus \{u + \gamma g u \,:\, u \in \hat{U}\}$$

is a regular boundary condition, and all regular boundary conditions arise in this way.

Proof. It is clear that any boundary condition B of the given form is regular. Conversely, let $B \subset H^{1/2}$ be a regular boundary condition. By Lemma 1.62,

$$\hat{U} := \operatorname{im}(Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda})$$

is a closed subspace of $H^{1/2}_{\leq\Lambda}$ and

$$F := \gamma(B \cap H^{1/2}_{>\Lambda}) = \gamma(\ker(Q_{\leq\Lambda} : B \to H^{1/2}_{\leq\Lambda}))$$

is a finite-dimensional subspace of $H^{1/2}_{<-\Lambda}$. It follows that $G = (\gamma F^\perp) \cap B$ is a complement of γF in B and that $Q_{\leq\Lambda} : G \to \hat{U}$ is an isomorphism. Hence there is a continuous linear map $g : \hat{U} \to H^{1/2}_{<-\Lambda}$ such that

$$G = \{u + \gamma g u \,:\, u \in \hat{U}\}.$$

Since $G \subset \gamma F^\perp$, g takes values in $\hat{V}$. □

1.64. Remark. In Proposition 1.63 above and Proposition 1.65 below, the roles of weak and strong inequalities can be interchanged.

1.65. Proposition. *Let Λ be a real number and let*

$$H_{\leq\Lambda} = E \oplus U \quad \text{and} \quad H_{<-\Lambda} = F \oplus V$$

be orthogonal decompositions, where $E, F \subset H^{1/2}_{<-\Lambda}$ are finite-dimensional subspaces, and $g : U \to V$ be a $1/2$-smooth linear map. Then

$$B = \gamma F \oplus \{u + \gamma g u \,:\, u \in U \cap H^{1/2}\}$$

is an elliptic boundary condition with

$$B^a = \gamma E \oplus \{v + \gamma g^* v \,:\, v \in V \cap H^{1/2}\}.$$

All elliptic boundary conditions arise in this way.

1.66. Remark. In previous work, but in a different context, the first author and Christian Bär considered boundary conditions of precisely this form. For details see the forthcoming article [BäB].

Proof of Proposition 1.65. With data as in Proposition 1.63, write

$$B = \gamma F \oplus \{u + \gamma \hat{g} u \,:\, u \in \hat{U}\},$$

where the map g there is decorated with a hat here. Since $F \subset H^{1/2}_{<-\Lambda}$ is of finite dimension,

$$F \oplus \tilde{V} = H^{-1/2}_{<-\Lambda}, \quad F \oplus V = H_{<-\Lambda}, \quad F \oplus \hat{V} = H^{1/2}_{<-\Lambda},$$

where

$$\tilde{V} = F^0 \cap H^{-1/2}_{<-\Lambda}, \quad V = F^0 \cap H_{<-\Lambda}, \quad \hat{V} = F^0 \cap H^{1/2}_{<-\Lambda}.$$

Let $x \in \gamma B^0 \subset H^{-1/2}$. Then there exist $f \in F$ and $v \in \tilde{V}$ with $Q_{<-\Lambda} x = f + v$. We compute $B_{-1/2}(x, f) = |f|^2$. Since $f \in \gamma B$, we conclude that $f = 0$ and hence that

$$Q_{<-\Lambda}(\gamma B^0) \subset \tilde{V}.$$

Conversely, let $v \in \tilde{V}$. Then $B_{-1/2}(v + \gamma w, f) = 0$ for all $w \in H^{-1/2}_{\le\Lambda}$ and $f \in F$, by the definition of $\tilde{V}$ and since $F \subset H^{1/2}_{<-\Lambda}$. With $u \in \hat{U}$, we compute

$$\begin{aligned} B_{-1/2}(v + \gamma w, \gamma u - \hat{g} u) &= B_{-1/2}(\gamma w, \gamma u) - B_{-1/2}(v, \hat{g} u) \\ &= B_{-1/2}(w, u) - B_{-1/2}(v, \hat{g} u) \\ &= B_{-1/2}(w, u) - B_{-1/2}(u', u) \end{aligned}$$

for some appropriate $u' \in H^{-1/2}_{\le\Lambda}$, by the duality $(H^{1/2}_{\le\Lambda})' = H^{-1/2}_{\le\Lambda}$. We conclude that $v + \gamma u' \in \gamma B^0$. In particular,

$$\tilde{V} = Q_{<-\Lambda}(\gamma B^0).$$

Since $\check{H} = H^{1/2}_{<-\Lambda} \oplus H^{-1/2}_{\ge\Lambda}$, we have $v + \gamma u' \in \check{H}$ if and only if $v \in H^{1/2}_{<-\Lambda}$.

We now use that B is elliptic. Then $B^a = (\gamma B^0) \cap \check{H}$ is regular and hence $(\gamma B^0) \cap \check{H} = (\gamma B^0) \cap H^{1/2}$. It follows that $v + \gamma u' \in \gamma B^0$ as above belongs to $H^{1/2}$ if and only if $v \in H^{1/2}$, and therefore

$$\hat{V} = Q_{<-\Lambda}(B^a).$$

By the symmetry of the roles of $B = (B^a)^a$ and B^a and switching the roles of weak and strong inequalities, see Remark 1.64, we get

$$\hat{U} = Q_{\le\Lambda}(B) = E^0 \cap H^{1/2}_{\le\Lambda},$$

where $E = \gamma(B^a \cap H^{1/2}_{\geq\Lambda})$. By Lemma 1.62, E is finite-dimensional. Hence the sesquilinear form $B_{-1/2}$ identifies $\tilde{U} = E^0 \cap H^{-1/2}_{\leq\Lambda}$ with the dual space of $\hat{U}$. In particular, in the above $v+u'$, we may take $u' = \hat{g}'v$, where $\hat{g}' : \tilde{V} \to \tilde{U}$ is the dual map of $\hat{g}$.

We now recall that $u' = \hat{g}'v$ is in $H^{1/2}$ if $v \in H^{1/2}$, by the regularity of B^a. By interpolation we get that $\hat{g}'$ is the extension of a 1/2-smooth linear map $g^* : V \to U$. By symmetry, g^* is the adjoint of a 1/2-smooth map $g : U \to V$ and $\hat{g}$ is the restriction of g to $\hat{U}$. □

1.67. Corollary. *Let $B \subset \check{H}$ be an elliptic boundary condition and $\Lambda \in \mathbb{R}$. Then $\gamma B^\perp$ is the closure of B^a in H and*

$$(1) \qquad \bar{B} \cap H_{\geq\Lambda} = B \cap H^{1/2}_{\geq\Lambda}, \quad B^\perp \cap H_{<\Lambda} = \gamma(B^a \cap H^{1/2}_{>-\Lambda}),$$

where $\bar{B}$ denotes the closure of B in H. Moreover, $(\bar{B}, H_{\geq\Lambda})$ is a Fredholm pair in H with index

$$(2) \qquad \begin{aligned} \operatorname{ind}(\bar{B}, H_{\geq\Lambda}) &= \dim(\bar{B} \cap H_{\geq\Lambda}) - \dim(B^\perp \cap H_{<\Lambda}) \\ &= \dim(B \cap H^{1/2}_{\geq\Lambda}) - \dim(B^a \cap H^{1/2}_{>-\Lambda}). \quad \square \end{aligned}$$

It is natural to ask whether the index formula in (1.67.2) gives the index of $D_{B,\max}$ for suitable Λ; this is in fact true for $\Lambda = 0$ if $\ker A = 0$.

1.68. Proposition. *Let* $\ker A = 0$. *If $B \subset \check{H}$ is a regular boundary condition, then $D_B = D_{B,\max}$ is a left-Fredholm operator with*

$$(1) \qquad (\operatorname{im} D_B)^\perp = \ker D_{B^a,\max}.$$

If B is elliptic, then D_B is a Fredholm operator with

$$(2) \qquad \operatorname{ind} D_B = \dim \bar{B} \cap H_{\geq} - \dim B^\perp \cap H_{<}.$$

Proof. We again use Hörmander's Criterion from Lemma A.11. Since the kernel of A vanishes, we have the representation formula

$$\sigma = \mathcal{E} Q_{>}\sigma(0) + S_D D_{\max}\sigma,$$

characterizing elements $\sigma \in \mathcal{D}_{\max}$. Furthermore,

$$S_D : L^2(\mathbb{R}_+, H) \to \{\sigma \in H^1(e) \,:\, Q_{>}\sigma(0) = 0\}$$

is an isomorphism, by Lemma 1.32. Let (σ_n) be a bounded sequence in $\mathcal{D}_{B,\max}$ such that $D_{\max}\sigma_n$ converges in $L^2(\mathbb{R}_+, H)$. Then $(\sigma_n(0))$ is a bounded sequence in B and $(S_D D_{\max}\sigma_n)$ converges in $H^1(e)$. It follows that the sequence $(Q_{\leq}\sigma_n(0) = (S_D D_{\max}\sigma_n)(0))$ converges in $H^{1/2}_{\leq}$. By Lemma 1.62 and Hörmander's criterion again, $(\sigma_n(0))$ has a convergent subsequence in B. Hence $(\sigma_n = \mathcal{E} Q_{>}\sigma_n(0) + S_D D_{\max}\sigma_n)$ has a convergent subsequence in $\mathcal{D}_{B,\max}$. This shows that D_B is a left-Fredholm operator. Now $D_B^* = D_{B^a,\max}$, see (1.54), therefore $(\operatorname{im} D_B)^\perp = \ker D_{B^a,\max}$ as claimed. □

We note that the image of $D_{B,\max}$ is not closed if $\ker A \neq 0$ while the index formula in (1.67.2) holds in general. This suggests a possible extension of Proposition 1.68 which we achieve by conveniently enlarging the domain of $D_{\max}$. We recall that Q_0 and $Q_{\neq}$ commute with $D_{\max}$ and that $\mathcal{D}_{\max}$ splits perpendicularly with components $H^1(\mathbb{R}_+, Q_0 H)$ and $Q_{\neq}\mathcal{D}_{\max}$. As is well known, the source of trouble is the part

$$D_{\max} : H^1(\mathbb{R}_+, Q_0 H) \to L^2(\mathbb{R}_+, Q_0 H).$$

of $D_{\max}$. We restore Fredholm properties of D by enlarging $H^1(\mathbb{R}_+, Q_0 H)$. Our discussion is motivated by the work of the third author on non-parabolic Dirac operators, compare [Ca2] and Section 2.3 below.

By Corollary 1.40, we have equivalences of norms on $\mathcal{D}_{\max}$,

$$\begin{aligned} ||\sigma||^2_{D_{\max}} &\approx ||Q_> \sigma(0)||^2_{-1/2} + ||\tau||^2_{L^2(\mathbb{R}_+, H_{\neq})} + ||\sigma_0||^2_{H^1(\mathbb{R}_+, Q_0 H)} \\ &= ||Q_> \sigma(0)||^2_{-1/2} + ||D_{\max}\sigma||^2_{L^2(\mathbb{R}_+, H)} + ||\sigma_0||^2_{L^2(\mathbb{R}_+, Q_0 H)}, \qquad (1.69) \\ &\approx ||\sigma(0)||^2_{\check{H}} + ||D_{\max}\sigma||^2_{L^2(\mathbb{R}_+, H)} + ||\sigma_0||^2_{L^2(\mathbb{R}_+, Q_0 H)}, \qquad (1.70) \end{aligned}$$

where $\tau = D_{\max} Q_{\neq} \sigma$ and $\sigma_0 = Q_0 \sigma$ and where we note, for the last equivalence, that $\mathcal{R}$ is continuous on $\mathcal{D}_{\max}$. We now introduce a continuous seminorm $||\cdot||_W$ on $\mathcal{D}_{\max}$,

$$||\sigma||^2_W := ||\sigma(0)||^2_{\check{H}} + ||D_{\max}\sigma||^2_{L^2(\mathbb{R}_+, H)} \leq C \cdot ||\sigma||^2_{D_{\max}}. \tag{1.71}$$

Corollary 1.40 implies that $||\cdot||_W$ is actually a norm on $\mathcal{D}_{\max}$. Clearly, $||\cdot||_W$ and the graph norm of $D_{\max}$ are equivalent if $\ker A = 0$. On the other hand, if $\ker A \neq 0$, then $||\cdot||_W$ is strictly weaker than the graph norm of $D_{\max}$. However, one easily verifies that for any $T > 0$ there is a constant C_T such that

$$||\sigma||_{L^2([0,T], H)} \leq C_T ||\sigma||_W, \tag{1.72}$$

for all $\sigma \in \mathcal{D}_{\max}$.

We now let W be the closure of $\mathcal{D}_{\max}$ under the norm $||\cdot||_W$. By Lemma 1.42.1, $\mathcal{L}_c(e)$ is dense in W. By definition, $D_{\max}$ extends to a continuous operator

$$D_{\text{ext}} : W \to L^2(\mathbb{R}_+, H). \tag{1.73}$$

We observe now that

$$W = Q_{\neq} W \oplus Q_0 W = Q_{\neq}\mathcal{D}_{\max} \oplus Q_0 W. \tag{1.74}$$

The linear map $S_0 : Q_0H \oplus L^2(\mathbb{R}_+, Q_0H) \to Q_0W$ defined by

$$S_0(x,\tau)(t) := x + \gamma^* \int_0^t \tau(s)ds, \tag{1.75}$$

is an isomorphism with $D_{\mathrm{ext}} S_0(z,\tau) = \tau$. In particular,

$$Q_0W \subset H^1_{\mathrm{loc}}(\mathbb{R}_+, Q_0H). \tag{1.76}$$

With $\mathcal{R}(S_0(x,\tau)) := x$ we obtain a continuous extension

$$\mathcal{R} : W \to \check{H}, \quad \mathcal{R}\sigma =: \sigma(0), \tag{1.77}$$

of $\mathcal{R}$ to W. For a boundary condition $B \subset \check{H}$, we set

$$W_B := \{\sigma \in W \,:\, \sigma(0) \in B\} \quad \text{and} \quad D_{B,\mathrm{ext}} := D_{\mathrm{ext}}|W_B. \tag{1.78}$$

We see from the above that $L^2(\mathbb{R}_+, Q_0H) \subset \operatorname{im} D_{B,\mathrm{ext}}$, irrespective of the boundary condition B.

1.79. Theorem. *If B is regular, then $D_{B,\mathrm{ext}}$ is a left-Fredholm operator with* $(\operatorname{im} D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}$.

Proof. Use the representation $\mathcal{E}x + S_D\tau + S_0(y,\rho)$ of elements of W, where $x \in H^{-1/2}_>$, $\tau \in L^2(\mathbb{R}, Q_{\neq}H)$, $y \in Q_0H$, and $\rho \in L^2(\mathbb{R}, Q_0H)$, and adapt the argument from the proof of Proposition 1.68. □

For any boundary condition $B \subset \check{H}$,

$$\begin{aligned} \ker D_{B,\max} &= \mathcal{D}_{B,\max} \cap \ker D_{\max}, \\ \ker D_{B,\mathrm{ext}} &= W_B \cap \ker D_{\mathrm{ext}} = W_B \cap (\ker D_{\max} + Q_0H). \end{aligned} \tag{1.80}$$

In particular, we have isomorphisms

$$\begin{aligned} \mathcal{R} &: \ker D_{B,\max} \to B \cap \check{H}_>, \\ \mathcal{R} &: \ker D_{B,\mathrm{ext}} \to B \cap \check{H}_\geq. \end{aligned} \tag{1.81}$$

Recall that a boundary condition B is elliptic if B and B^a are regular. As above, we let $\bar{B}$ denote the closure of B in H.

1.82. Corollary and Definition. *If B is elliptic, then $D_{B,\mathrm{ext}}$ is a Fredholm operator with index*

$$\operatorname{ind} D_{B,\mathrm{ext}} = \dim(B \cap H_\geq) - \dim(B^\perp \cap H_<) = \operatorname{ind}(\bar{B}, H_\geq),$$

the extended index *of D_B, also denoted by* $\operatorname{ind}_{\mathrm{ext}} D_B$

Proof. Immediate from (1.81), Theorem 1.79, and Corollary 1.67. □

1.5. **Self-adjoint boundary conditions.** We say that a boundary condition $B \subset \check{H}$ is *self-adjoint* if $B = B^a$. By definition, a regular self-adjoint boundary condition is elliptic.

We say that (H_0, ω) is a *Hermitian symplectic vector space* if the ± 1-eigenspaces of the involution $i\gamma$ of H_0 have equal dimension. Then a subspace $L \subset H_0$ is *Lagrangian* if $L \perp \gamma L$ and $L \oplus \gamma L = H_0$.

1.83. Theorem. *Regular self-adjoint boundary conditions exist if and only if (H_0, ω) is a Hermitian symplectic vector space (where $H_0 = 0$ is not excluded). Then regular self-adjoint boundary conditions are given by the following data: a Lagrangian subspace $L \subset H_0$, an orthogonal decomposition $H_< = F \oplus V$, where $F \subset H_<^{1/2}$ is of finite dimension, and a $1/2$-smooth map $g : V \oplus L \to V \oplus L$ with $g^* = g$. The regular self-adjoint boundary condition B given by such data is*

$$B = \gamma F \oplus \{w + \gamma g w \,:\, w \in (V \oplus L) \cap H^{1/2}\}.$$

Write $H = H^+ \oplus H^-$, where $H^\pm$ is the ± 1 eigenspace of $i\gamma$. Since A anti-commutes with γ, A maps $H^\pm$ to $H^\mp$ so that the restriction of A to H^+ is a Fredholm operator (in general unbounded) to H^-. Since γ intertwines eigenspaces of A with opposite eigenvalues, it follows easily that (H_0, ω) is a Hermitian symplectic vector space if and only if the Fredholm operator A^+ has index 0.

1.84. Corollary. *With $H^\pm$ and A^+ as above, $\check{H}$ contains elliptic self-adjoint boundary conditions if and only if* $\operatorname{ind} A^+ = 0$. $\square$

Proof of Theorem 1.83. Any data as in the assertion give rise to a regular self-adjoint boundary condition. As for the existence, if $L \subset H_0$ is a Lagrangian subspace, then $L \oplus H_<$ is a regular self-adjoint boundary condition.

To prove the asserted characterization, we first observe that regular self-adjoint boundary conditions are elliptic, so that we can use the description of elliptic boundary conditions given in Proposition 1.65.

Let B be an elliptic boundary condition. By Proposition 1.65, there are orthogonal decompositions

$$H_\le = E \oplus U \quad \text{and} \quad H_< = F \oplus V,$$

where $E, F \subset H^{1/2}$ are of finite dimension, and a $1/2$-smooth linear map $b : U \to V$ such that

$$B = \gamma F \oplus \{u + \gamma b u \,:\, u \in U \cap H^{1/2}\},$$
$$B^a = \gamma E \oplus \{v + \gamma b^* v \,:\, v \in V \cap H^{1/2}\}.$$

From now on we assume that $B = B^a$. Then the H-closure $\bar{B} = \gamma B^\perp$, and hence any element in $\bar{B}$ can be written in any of the following two ways:

$$\gamma f + u + \gamma bu = \gamma f + u_< + u_0 + \gamma bu,$$

where $u_< = Q_< u$ and $u_0 = Q_0 u$, and

$$\gamma e + v + \gamma b^* v = \gamma e_< + \gamma e_0 + v + \gamma b^* v,$$

where $e_< = Q_< e$ and $e_0 = Q_0 e$. We are going to compare the $H_<$, H_0, and $H_>$ components of elements of $\bar{B}$ in the above two representation:

We observe first that $V = Q_<(U) = \{u_< \, : \, u \in U\}$. Since E and F are the orthogonal complements of U in $H_\le$ and V in $H_<$, it follows that

$$F = E \cap H_< \subset E.$$

Let $L := U \cap H_0$ and

$$B_L := \{u + \gamma bu \, : \, u \in L\} \subset \bar{B}.$$

Let $u \in L$. Then $u + \gamma bu \in \bar{B}$ and hence there exist $e \in E$ and $v \in V$ such that

$$u + \gamma bu = \gamma e + v + \gamma b^* v.$$

Clearly $v = 0$, hence $b^* v = 0$, and hence $u = \gamma e_0$ and $bu = e_<$. We get $\gamma u - bu = -e$ and hence

$$\gamma B_L = \{\gamma u - bu \, : \, u \in L\} \subset E.$$

Let $e \in E$. Then $\gamma e \in \bar{B}$ and hence there exist $f \in F$ and $u \in U$ such that

$$\gamma e = \gamma f + u_< + u_0 + \gamma bu.$$

We obtain $u_< = 0$, hence $\gamma e_0 = u_0 = u \in L$ and $e_< = f + bu$. Since $F \subset E$, we get

$$E = F \oplus \{\gamma u - bu \, : \, u \in L\} = F \oplus \gamma B_L.$$

Since U is the orthogonal complement of E in $H_\le$ and $Q_0(E) = \gamma L$, the orthogonal complement of γL in H_0 belongs to U, that is, to L, by the definition of L. We conclude that we have an orthogonal decomposition

$$H_0 = L \oplus \gamma L.$$

It follows that (H_0, ω) is a Hermitian symplectic vector space and that L is a Lagrangian subspace of (H_0, ω).

Since $\bar{B} = \gamma B^\perp$ and $E = F \oplus \gamma B_L$, we have orthogonal sums

$$\bar{B} = \gamma E \oplus \{v + \gamma b^* v \, : \, v \in V\} =: \gamma F \oplus B_L \oplus B_V.$$

Let $W := V \oplus L$. Then H decomposes orthogonally as

$$H = F \oplus W \oplus \gamma F \oplus \gamma W.$$

For a subspace $K \subset H$, let Q_K be the orthogonal projection in H onto K. Then $Q_W = Q_V + Q_L$.

Let $x \in B_L \oplus B_V$ and write $x = u + \gamma bu + v + \gamma b^* v$ with $u \in L$ and $v \in V$. Since $Q_F(b^*v) = Q_F(\gamma b^* v) = 0$, we have

$$\gamma Q_W b^* v = \gamma Q_{F\oplus W} b^* v = (I - Q_{F\oplus W})\gamma b^* v = (I - Q_W)\gamma b^* v.$$

Therefore

$$\begin{aligned} x &= u + \gamma bu + v + \gamma b^* v \\ &= u + Q_W \gamma b^* v + v + \gamma bu + (I - Q_W)\gamma b^* v \\ &= u + Q_W \gamma b^* v + v + \gamma(bu + Q_W b^* v). \end{aligned}$$

Since $\gamma b^* v \in H_{\geq}$, we have $Q_W \gamma b^* v = Q_L \gamma b^* v$. Hence

$$\begin{aligned} x &= (u + Q_L\gamma b^* v) + v + \gamma\big(b(u + Q_L \gamma b^* v) + (Q_W b^* - bQ_L\gamma b^*)v\big) \\ &= (Q_L + Q_V)x + \gamma g(Q_W x) = Q_W x + \gamma g(Q_W x), \end{aligned}$$

where $g : W \to W$ is the $1/2$-smooth linear map given by

$$gw = bQ_L w + (Q_W b^* - bQ_L\gamma b^*)Q_V w.$$

We conclude that

$$\bar{B} = \gamma F \oplus \{w + \gamma g w \,:\, w \in W\}.$$

Now

$$\gamma \bar{B} = B^\perp = F \oplus \{\gamma w - g^* w \,:\, w \in W\},$$

hence $g = g^*$. □

1.85. Example. Let $\beta : H \to H$ be $1/2$-smooth with

$$\beta^* = \beta^{-1} = \beta, \tag{1}$$

$$\gamma\beta + \beta\gamma = 0, \tag{2}$$

$$A\beta + \beta A = 0. \tag{3}$$

Then $B = \{x \in H^{1/2} \,:\, \beta x = x\}$ is a regular self-adjoint boundary condition.

For example, given a Dirac system $d = (H, A, \gamma)$, consider the Dirac system

$$\tilde{d} = (H \oplus H, (A, -A), (\gamma, -\gamma)).$$

Then $\beta : H \oplus H \to H \oplus H$, $\beta(x, y) = (y, x)$, satisfies (1)–(3). The corresponding boundary condition $B = \{(x, x) \,:\, x \in H^{1/2}\}$ is regular and self-adjoint. It arises as the transmission boundary condition when cutting a manifold along a hypersurface.

1.6. **Regular pairs of projections.** Let P and Q be 1/2-smooth projections in H. We say that the ordered pair (P,Q) is *regular* if

$$x \in H^{-1/2},\ \tilde{P}x = 0,\ \tilde{Q}x \in H^{1/2} \Longrightarrow x \in H^{1/2}. \tag{1.86}$$

Roughly speaking, this means that Q is close to $I-P$; compare Proposition 1.93 below.

1.87. Lemma. *Let (P,Q) be a pair of 1/2-smooth projections in H. Then (P,Q) is regular if and only if*

$$x \in H^{-1/2},\ \tilde{P}x \in H^{1/2},\ \tilde{Q}x \in H^{1/2} \Longrightarrow x \in H^{1/2}.$$

Proof. Assume that (P,Q) is regular. Consider $x \in H^{-1/2}$ with $\tilde{P}x$ and $\tilde{Q}x$ in $H^{1/2}$. Set $y := (I-\tilde{P})x \in H^{-1/2}$. Then $\tilde{P}y = 0$ and

$$\tilde{Q}y = \tilde{Q}x - \tilde{Q}\tilde{P}x = \tilde{Q}x - \hat{Q}\tilde{P}x \in H^{1/2}.$$

By regularity, $y \in H^{1/2}$ and hence $x = \tilde{P}x + y \in H^{1/2}$. □

1.88. Corollary (Symmetry and Stability).

(1) *The regularity relation on pairs of* 1/2*-smooth projections is symmetric.*
(2) *The regularity relation is stable under smoothing perturbations, i.e. if P_1, P_2, Q_1, Q_2 are* 1/2*-smooth projections in H with $P_1 - P_2$ and $Q_1 - Q_2$ smoothing, then (P_1,Q_1) is regular if and only if (P_2,Q_2) is regular.* □

We need stronger regularity conditions: The pair (P,Q) is called *strongly regular* if both (P,Q) and $(I-P, I-Q)$ are regular.

1.89. Theorem. *Let P and Q be 1/2-smooth projections in H. Then the following conditions are equivalent.*

(1) *The pair (P,Q) is strongly regular.*
(2) *The operator*

$$T = T(P,Q) := P - Q = P(I-Q) - (I-P)Q$$

satisfies half of the condition 1.26.1, i.e.,

$$x \in H^{-1/2},\ \tilde{T}x \in H^{1/2} \Longrightarrow x \in H^{1/2}.$$

Proof. Assume that the pair (P,Q) is strongly regular. Let $x \in H^{-1/2}$ with $\tilde{T}x \in H^{1/2}$. Then $(I-\tilde{P})\tilde{P}x = 0$ and $(I-\tilde{Q})\tilde{P}x = (I-\hat{Q})\tilde{T}x$ is in $H^{1/2}$. Hence $\tilde{P}x \in H^{1/2}$, by the regularity of $(I-P, I-Q)$. A similar argument shows that $\tilde{Q}x \in H^{1/2}$, Hence $x \in H^{1/2}$, by the regularity of (P,Q). The other direction is obvious. □

In order to link strong regularity to Fredholm properties of suitable operators, as in [BL2], we have to require regularity of the adjoint projections, too.

1.90. Theorem. *Let P and Q be $1/2$-smooth projections in H. Then the following conditions are equivalent:*

(1) *The pairs (P,Q) and (P^*,Q^*) are strongly regular.*
(2) *With $T = T(P,Q) = P - Q$ as before, the operators $\hat{T}$ and $\widehat{T^*}$ are Fredholm in $H^{1/2}$ with $\operatorname{ind}\hat{T} + \operatorname{ind}\widehat{T^*} = 0$.*

If any of these conditions holds then both $\tilde{T}$ and $\widetilde{T^*}$ restrict to Fredholm operators in each H^s, $|s| \le 1/2$, with kernels independent of s.

Proof. From Theorem 1.89 we know that the strong regularity of the pairs (P,Q) and (P^*,Q^*) is equivalent to the condition 1.26.1 for T and T^*. By Lemma 1.26, this condition is equivalent to Condition (2) of the theorem. The Fredholm property of the restrictions and the constancy of their kernels follows from Corollary 1.27. □

1.91. Remarks. 2) $(I - P^*, I - Q^*)$ is (strongly) regular if and only if (P_γ, Q_γ) is (strongly) regular.
1) If P and Q are orthogonal, that is, $P = P^*$ and $Q = Q^*$, then (P,Q) is strongly regular if and only if (P,Q) and (P_γ, Q_γ) are regular.

1.92. Corollary. *For any pair P,Q of orthogonal $1/2$-smooth projections in H, the following conditions are equivalent.*

(1) *The pairs (P,Q) and (P_γ, Q_γ) are regular.*
(2) *$\hat{T}$ is a Fredholm operator, necessarily of index 0, in $H^{1/2}$.* □

With any projection Q in H, we associate the involution $J(Q) := I - 2Q$.

1.93. Proposition. *If there is a representation $P = I - Q + R_1 + R_2$ in $\mathcal{L}(H^{1/2})$, where R_2 and R_2^* are compact in $H^{1/2}$ and*

$$\|J(Q)R_1\|_{H^{1/2}}, \|R_1^* J(Q^*)\|_{H^{1/2}} < 1,$$

then (P,Q) and (P^*,Q^*) are strongly regular.

Proof. We show that Condition 2 of Theorem 1.90 holds. We have

$$T = J(Q) + R_1 + R_2 = J(Q)(I + J(Q)R_1) + R_2$$

and, similarly,

$$T^* = (I + R_1^* J(Q^*))J(Q^*) + R_2^*.$$

The bound on the norms now implies that both $\hat{T}$ and $\widehat{T^*}$ are Fredholm operators in $H^{1/2}$ of index 0, hence the assertion. □

In [BL2, Theorem 1.3] a criterion for regularity is given which uses only properties of P and Q in H, without referring to other Sobolev spaces, at the expense of introducing more conditions on P and Q. This result is a special case of our analysis as we will show now.

1.94. LEMMA. *Let S be a $1/2$-smooth Fredholm operator in H and denote by $K_{r(l)}$ the orthogonal projections onto* $\ker S$ *and* $\ker S^*$*, respectively. Then the following conditions are equivalent:*

(1) *S admits a $1/2$-smooth parametrix $U \in \mathcal{L}(H)$ such that*
$$US = I - K_r \quad \textit{and} \quad SU = I - K_l.$$
(2) *S and S^* restrict respectively extend to Fredholm operators in each H^s, $|s| \le 1/2$, with index independent of s.*

Proof. (1) $\Rightarrow$ (2). If U restricts to $H^{1/2}$ then $K_r = I - US$ and $K_l = I - SU$ as well. Since both projections have finite rank and since $H^{1/2}$ is dense in H, it follows that both projections are actually smoothing. Now (2) follows from Lemma 1.26 and Corollary 1.27.
(2) $\Rightarrow$ (1). This follows from the explicit construction of $K_{r(l)}$ in the proof of Lemma 1.26.4. □

This lemma gives a useful criterion for linking the regularity of a $1/2$-smooth projection P to Fredholm properties of $T = P - Q_>$ in H, provided that we can control the mapping properties of parametrices. To construct a parametrix U satisfying Condition (1) of Lemma 1.94, we start with the polar decomposition $T = V|T|$ of T, where
$$|T| = (T^*T)^{1/2}, \quad V^*V = I - K_r, \quad VV^* = I - K_l.$$
Now 0 is an isolated point in $\operatorname{spec}(T^*T)$ if T is a Fredholm operator, hence $\{\operatorname{Re} z > 0\} \cap \operatorname{spec}(T^*T)$ is a compact subset of $(0, \infty)$. The function $f = f(z) = 1/\sqrt{z}$ is holomorphic in $\{\operatorname{Re} z > 0\}$. Thus we can define the operator $|T|^{-1} := f(T^*T)$ by the Dunford-Taylor integral of f along a simple closed curve in $\{\operatorname{Re} z > 0\}$ surrounding $\operatorname{spec} T^*T \setminus \{0\}$ (cf. [Yo, p.225]). Then we have $|T||T|^{-1} = I - K_r$, which implies that
$$U := |T|^{-1}V^*$$
satisfies $UT = I - K_r$ and $TU = I - K_l$. Now it is apparent that this parametrix construction leads to a $1/2$-smooth parametrix for all Fredholm operators inside an operator algebra, $\mathcal{A} \subset \mathcal{L}(H)$, if $\mathcal{A}$ has the following properties:

(1) $\mathcal{A}$ is a *-algebra with identity,
(2) $\mathcal{A}$ admits holomorphic functional calculus, i.e., is closed under forming Dunford-Taylor integrals,
(3) $\mathcal{A}$ is contained in the space of $1/2$-smooth operators.

We combine these facts in the following result which generalizes Theorem 1.3 in [BL2].

1.95. THEOREM. *Let P and Q be $1/2$-smooth projections in H and assume that P and Q are contained in some operator algebra $\mathcal{A} \subset \mathcal{L}(H)$ which satisfies the above properties. Then the following conditions are equivalent:*

(1) *The pairs (P, Q) and (P^*, Q^*) are strongly regular.*
(2) *The operator $T := P - Q$ is Fredholm in H.*

The conditions imposed on the algebra $\mathcal{A}$ are not unnatural; e.g., they are satisfied for the algebra of pseudodifferential operators of order zero on a compact manifold.

We now come back to Dirac systems and study the more traditional boundary conditions defined by projections in H. Let P be a $1/2$-smooth projection in H. Then P induces a continuous projection in $\check{H}$ iff $Q_{\le}\tilde{P}Q_{>}$ is smoothing. In any case,

$$B_P := \ker \tilde{P} \cap \check{H} \tag{1.96}$$

is a closed subspace of $\check{H}$, that is, a boundary condition in the sense of Definition 1.51. Furthermore, $\ker \hat{P}$ is a closed subspace of $\hat{H}$. In their work, Atiyah, Patodi, and Singer consider the boundary condition given by $P_{APS} := Q_{>}$, see (2.3) in [APS].

1.97. REMARK. Let P be a $1/2$-smooth projection in H that induces a continuous projection $\check{P}$ in $\check{H}$. Since $H^{1/2}$ is dense in $\check{H}$ and $\check{P}(H^{1/2}) \subset H^{1/2}$, $B_P \cap H^{1/2}$ is dense in $B_P = \ker \check{P}$. Hence B_P is equal to the closure of $B_P \cap H^{1/2}$ in $\check{H}$.

Suppose there is an $x \in \operatorname{im} \check{P} \setminus H^{1/2}$ and set $B = \ker \check{P} \oplus \mathbb{R}x$, a closed subspace of $\check{H}$. If $z = y + \alpha x \in B$ is in $H^{1/2}$, then also $Pz = \alpha x$, hence $\alpha = 0$. It follows that $H^{1/2}$ is not dense in B. By what we just said, B is a boundary condition that is not realizable as the boundary condition B_R of a $1/2$-smooth projection R that induces a continuous projection in $\check{H}$.

The Dirac operators and domains corresponding to the boundary condition B_P posed by a $1/2$-smooth projection P in H will be denoted as above, except that we substitute the subscript P for B_P.

1.98. DEFINITION. We say that a projection $P : H \to H$ is *regular* if it is $1/2$-smooth and B_P is a regular boundary condition.

1.99. PROPOSITION. *For a $1/2$-smooth projection P in H, the following are equivalent:*

(1) *P is regular.*
(2) *$B_P = \ker \hat{P}$.*
(3) *For some or, equivalently, any $\Lambda \in \mathbb{R}$, we have*
$$x \in H^{-1/2}, \tilde{P}x = 0, Q_{\le\Lambda}x \in H^{1/2} \Longrightarrow x \in H^{1/2}.$$

Proof. The condition in (2) expresses that $B_P \subset H^{1/2}$, hence that B_P is a regular boundary condition, by Proposition 1.61.3. Since $\check{H}$ is equal to the direct sum $H^{1/2}_{\le\Lambda} \oplus H^{-1/2}_{>\Lambda}$, the condition in (3) is just another way of saying that $B_P \subset H^{1/2}$. □

Part 3 of the preceding result is the regularity criterion introduced in condition (4.6c) of [BL2].

We note that for a regular projection P in H with corresponding boundary condition $B_P = \ker \hat{P}$, the adjoint boundary condition is given by

$$(B_P)^a = \ker \tilde{P}_\gamma \cap \check{H} \quad \text{with} \quad P_\gamma := \gamma^*(I - P^*)\gamma. \tag{1.100}$$

We say that P is *elliptic* if P and P_γ are regular. Then

$$(B_P)^a = \ker \hat{P}_\gamma = \gamma \operatorname{im} \widehat{P^*}. \tag{1.101}$$

1.102. Corollary. *If P is an elliptic orthogonal projection in H, then $D_{P,\mathrm{ext}}$ is a Fredholm operator with extended index*
$$\operatorname{ind} D_{P,\mathrm{ext}} = \dim(\ker P \cap H_{\ge}) - \dim(\operatorname{im} P \cap H_{<}). \quad \square$$

2. Dirac-Schrödinger systems

2.1. **Dirac systems with Lipschitz coefficients.** In this section, we construct and describe a model for the geometric operators we are interested in; this model will be introduced axiomatically.

Let H be a separable complex Hilbert space. For $t \in \mathbb{R}_+$, let $\langle \cdot, \cdot \rangle_t$ be a family of scalar products with norm $|| \cdot ||_t$ compatible with the Hilbert space structure of H.

I. Axiom. For all $T \in \mathbb{R}_+$, there is a constant C_T such that

$$|\langle u, v\rangle_r - \langle u, v\rangle_s| \leq C_T ||u||_t ||v||_t |r - s|$$

for all $u, v \in H$ and $r, s, t \in [0, T]$.

It would be equivalent to require the estimate for $t = 0$ only instead of requiring it for arbitrary $t \in [0, T]$.

In the following we will write $\langle \sigma, \tau \rangle$ for the function $t \mapsto \langle \sigma(t), \tau(t)\rangle_t$, and similarly for related expressions.

Our data define a Lipschitz Hilbert bundle $\mathcal{H}$ over $\mathbb{R}_+$ with fibers $H_t = (H, \langle \cdot, \cdot \rangle_t)$, $t \in \mathbb{R}_+$. Any bundle $\mathcal{H} = (H_t)_{t \in \mathbb{R}_+}$ of Hilbert spaces which is (locally) Lipschitz over $\mathbb{R}_+$ is isometric to such a model bundle.

For $t \in \mathbb{R}_+$, define a positive definite operator $G_t \in \mathcal{L}(H)$ by

$$\langle G_t u, v\rangle_0 = \langle u, v\rangle_t, \quad u, v \in H. \tag{2.1}$$

The operators G_t and G_t^{-1} are locally Lipschitz functions of t in $\mathcal{L}(H)$. An easy application of Lemma 1.3 gives the following result.

2.2. Lemma. *The operator function G is weakly differentiable almost everywhere in $\mathbb{R}_+$ with symmetric derivative $G'_t \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H))$.*

More generally, if H_1 and H_2 are separable Hilbert spaces, then any function in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H_1, H_2))$ is weakly differentiable almost everywhere, and the norm of the derivative is locally uniformly bounded.

Now we set

$$\Gamma := \frac{1}{2} G_t^{-1} G'_t \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H)). \tag{2.3}$$

If ∂_t denotes the derivative with respect to t, $\partial_t \sigma = \sigma'$, then

$$\partial := \big(\partial_t + \Gamma\big) : \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H) \tag{2.4}$$

is a continuous metric connection, where *metric* means that

$$\langle \sigma_1, \sigma_2\rangle' = \langle \partial\sigma_1, \sigma_2\rangle + \langle \sigma_1, \partial\sigma_2\rangle, \tag{2.5}$$

for all $\sigma_1, \sigma_2 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, H)$.

2.6. Remark. Any other continuous metric connection

$$\tilde{\partial} : \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H)$$

is of the form $\tilde{\partial} = \partial + \tilde{\Gamma}$, where $\tilde{\Gamma} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H))$ takes values in the space of skew-Hermitian operators.

II. Axiom. There is a family $\mathcal{A}$ of self-adjoint operators A_t on H_t, $t \in \mathbb{R}_+$, with common domain H_A and graph norm $||\cdot||_{A_t}$ such that

(1) with respect to the graph norm $||\cdot||_{A_0}$ on H_A, the embedding $H_A \to H$ is compact;
(2) for all $T \in \mathbb{R}_+$, there is a constant C_T such that

$$|\langle A_r u, v\rangle_r - \langle A_s u, v\rangle_s| \le C_T ||u||_{A_t} ||v||_t |r - s|$$

for all $u \in H_A, v \in H$, and $r, s, t \in [0, T]$.

As above in Axiom I, it would be equivalent to require the estimate for $t = 0$ only instead of requiring it for arbitrary $t \in [0, T]$.

2.7. Remark. It would be tempting to use the metric connection ∂ to identify $\mathcal{H}$ with $\mathbb{R}_+ \times H_0$. But this parallel transport may not preserve H_A if Γ does not, and this happens indeed in important examples.

A pair $e := (\mathcal{H}, \mathcal{A})$ satisfying Axioms I and II will be called an *evolution system*. To any evolution system e we can naturally associate a family of constant coefficient system e^t, $t \in \mathbb{R}_+$, defined by

$$e^t := (H_t, A_t). \tag{2.8}$$

For any evolution system e, we introduce the Hilbert space $L^2(\mathcal{H})$ as completion of the space $\mathcal{L}_c(e^0)$ under the norm

$$||\sigma||^2_{L^2(\mathcal{H})} := \int_0^\infty ||\sigma||_t^2 dt. \tag{2.9}$$

Then we can form the linear operator

$$L := \partial + A : \mathcal{L}_c(e^0) \to L^2(\mathcal{H}), \tag{2.10}$$

which we call the *evolution operator* associated to e. Note that the domain of L only depends on the constant coefficient system e^0.

The evolution operator L introduced above is not symmetric on the dense subspace $\mathcal{L}_{0,c}(e^0)$ of $L^2(\mathcal{H})$. A modification as in the case of constant coefficients leads to a symmetric operator.

III. Axiom. There is a section

$$\gamma \in \mathrm{Lip}_{\mathrm{loc}}\big(\mathbb{R}_+, \mathcal{L}(H)\big) \cap L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+, \mathcal{L}(H_A)\big),$$

such that the following relations hold:

(1) $$-\gamma_t = \gamma_t^* = \gamma_t^{-1} \quad \text{on } H_t,$$

(2) $$A_t\gamma_t + \gamma_t A_t = 0 \quad \text{on } H_A,$$

(3) $$[\partial, \gamma] = 0 \quad \text{on } \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, H).$$

Note that $\gamma\mathcal{L}_c(e^0) \subset \mathcal{L}_c(e^0)$, by assumption.

A triple $d := (\mathcal{H}, \mathcal{A}, \gamma)$ as above satisfying Axioms I–III, is called a *Dirac system.* Now we are ready to introduce our first model operator, the *Dirac operator*

$$D := \gamma(\partial + A) : \mathcal{L}_{\mathrm{loc}}(e^0) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H), \tag{2.11}$$

associated to the Dirac system $(\mathcal{H}, \mathcal{A}, \gamma)$.

For later purposes it is important to note that, pointwise,

$$||D\sigma|| = ||L\sigma|| \tag{2.12}$$

for all $\sigma \in \mathcal{L}_{\mathrm{loc}}(e^0)$, so that estimates for the usual norms of $L\sigma$ also hold for $D\sigma$.

The restriction $D_{0,c}$ of D to the domain $\mathcal{D}_{0,c} := \mathcal{L}_{0,c}(e^0)$ is symmetric; we denote by $D_{\min}$, with domain $\mathcal{D}_{\min}$, the closure of $D_{0,c}$ in $L^2(\mathcal{H})$, and by $D_{\max}$, with domain $\mathcal{D}_{\max}$, the adjoint operator. In order to define self-adjoint extensions of $D_{\min}$, we will introduce boundary conditions as in Chapter 1. Again, this approach is based on integration by parts and the boundary form ω: (1.10) and (1.11) translate literally in view of the following computation, valid for all $\sigma_1, \sigma_2 \in \mathcal{L}_c(e^0)$,

$$\langle\gamma\sigma_1, \sigma_2\rangle' = \langle\gamma\partial\sigma_1, \sigma_2\rangle + \langle\gamma\sigma_1, \partial\sigma_2\rangle = \langle D\sigma_1, \sigma_2\rangle - \langle\sigma_1, D\sigma_2\rangle, \tag{2.13}$$

which is an easy consequence of our axioms; therefore, we also get

$$(D\sigma_1, \sigma_2) - (\sigma_1, D\sigma_2) = \omega(\sigma_1(0), \sigma_2(0)). \tag{2.14}$$

In particular, we have $\mathcal{L}_c(e^0) \subset \mathcal{D}_{\max}$.

2.2. **Comparison with constant coefficients.** Let d be a Dirac system with Lipschitz coefficients. Our strategy in dealing with d aims at some kind of comparison with constant Dirac systems, where we have substantial control over the solution theory. Any such attempt meets with two difficulties, firstly that we lack any a priori control on the domain of the maximal operator $D_{\max} = (D_{0,c})^*$ and secondly, that the domain of the adjoint operator to A_t in H_0 varies with t.

For any $t \geq 0$, we introduce the Dirac system

$$d^t = (H_t, A_t, \gamma_t) \tag{2.15}$$

with constant coefficients and the Dirac system d^{ct} with coefficients

$$
\begin{aligned}
&H_s^{ct} = (H, \langle .,. \rangle_s), \quad A_s^{ct} = A_s, \quad \gamma_s^{ct} = \gamma_s \quad \text{for } s \le t, \\
&H_s^{ct} = (H, \langle .,. \rangle_t), \quad A_s^{ct} = A_t, \quad \gamma_s^{ct} = \gamma_t \quad \text{for } s \ge t.
\end{aligned} \tag{2.16}
$$

Objects associated to d^t and d^{ct} will be decorated with a superscript t and ct, respectively. We think of d^{ct} as a kind of interpolation between $d^0 = d^{c0}$ and d.

2.17. THEOREM. *The Dirac systems d^{ct} compare with d^0 as follows:*

(1) *For all $t \ge 0$, we have $\mathcal{D}_{\min}^{ct} = \mathcal{D}_{\min}^0$ and $\mathcal{D}_{\max}^{ct} = \mathcal{D}_{\max}^0$.*
(2) *For all $T \ge 0$, there is a constant C_T such that*

$$
C_T^{-1} || \cdot ||_{D_{\max}^0} \le || \cdot ||_{D_{\max}^{ct}} \le C_T || \cdot ||_{D_{\max}^0}
$$

for all $t \in [0, T]$.

The proof of Theorem 2.17 will be given below. In preparation, we will study the operators $G^{ct} D^{ct}$, which are symmetric in $L^2(\mathcal{H}^0)$ with domain $\mathcal{L}_c(e^0)$.

We start with some estimates. Axioms I and III imply that, for any $t \ge 0$, there is a constant C_T such that, for all $r, s \in [0, T]$,

$$
||G_s \gamma_s \gamma_r^{-1} - G_r||_0 \le C_T |r - s|, \tag{2.18}
$$

$$
||G_r \gamma_r \Gamma_r||_0 \le C_T, \tag{2.19}
$$

$$
||G_r \gamma_r||_0 \le C_T. \tag{2.20}
$$

We will also need estimates on the operators A_t. From Axiom II we get, for $0 \le s, t \le T$ and $x \in H_A$,

$$
\begin{aligned}
||A_s x||_s^2 &\le C_T ||x||_{A_t} ||A_s x||_t + \langle A_t x, A_s x \rangle_t \\
&\le C_T ||x||_{A_t} ||A_s x||_t + ||A_t x||_t ||A_s x||_t \\
&\le C_T ||x||_{A_t} ||A_s x||_s,
\end{aligned}
$$

where the constant C_T may change from line to line. Therefore

$$
|| \cdot ||_{A_s} \le C_T || \cdot ||_{A_t} \tag{2.21}
$$

for all $s, t \in [0, T]$. In other words, the graph norms $||\cdot||_{A_t}$ are locally uniformly equivalent. For all $r, s, t \in [0, T]$ and $x \in H_A$, we also have

$$\begin{aligned}
||A_r x - A_s x||_t^2 &\le C_T ||A_r x - A_s x||_r^2 \\
&= C_T \cdot \langle A_r x - A_s x, A_r x - A_s x\rangle_r \\
&\qquad + C_T \cdot \langle A_s x - A_s x, A_r x - A_s x\rangle_s \\
&\le C_T |r-s| ||x||_{A_t} ||A_r x - A_s x||_t \\
&\qquad + C_T \big| \langle A_s x, A_r x - A_s x\rangle_r - \langle A_s x, A_r x - A_s x\rangle_s \big| \\
&\le C_T |r-s| (||x||_{A_t} + ||A_s x||_t) ||A_r x - A_s x||_t \\
&\le C_T |r-s| \cdot ||x||_{A_t} ||A_r x - A_s x||_t,
\end{aligned}$$

where we use Axiom I and (2.21) in the last two inequalities. Therefore

$$||A_r x - A_s x||_t \le C_T |r-s| \cdot ||x||_{A_t} \tag{2.22}$$

for all $0 \le r, s, t \le T$ and $x \in H_A$.

The main estimate we need is of Kato-Rellich type:

2.23. Lemma. *Given $T \ge 0$, there is a constant C_T such that, for all $r \le s$ in $[0, T]$ and $\sigma \in \mathcal{L}_c(e^0)$,*

$$\begin{aligned}
||G^{cr} D^{cr} \sigma - G^{cs} D^{cs} \sigma||_{L^2(\mathcal{H}^0)} \\
\le C_T ||\sigma||_{L^2(\mathcal{H}^0)} + C_T |r-s| \cdot ||G^{cr} D^{cr} \sigma||_{L^2(\mathcal{H}^0)}.
\end{aligned}$$

Proof. We start by comparing the coefficients of the two operators $G^{cr} D^{cr}$ and $G^{cs} D^{cs}$. On $[0, r]$, they coincide. At $t \in (r, s]$, we have

$$\begin{aligned}
(G^{cs} D^{cs})|_t &= G_t \gamma_t (\partial_t + \Gamma_t + A_t) \\
&= G_t \gamma_t \gamma_r^{-1} D^{cr} + G_t \gamma_t \Gamma_t + G_t \gamma_t (A_t - A_r).
\end{aligned}$$

At $t \in [s, \infty)$, we have

$$(G^{cs} D^{cs})|_t = G_s \gamma_s (\partial_t + A_s) = G_s \gamma_s \gamma_r^{-1} D^{cr} + G_s \gamma_s (A_s - A_r).$$

Let $\sigma \in \mathcal{L}_c(e^0)$. Then $G^{cr} D^{cr} \sigma$ and $G^{cs} D^{cs} \sigma$ coincide on $[0, r]$. Using (2.18), (2.19), and (2.20), we get

$$\begin{aligned}
||G^{cr} D^{cr} \sigma - G^{cs} D^{cs} \sigma||_{L^2(\mathcal{H}^0)} &\le C_T |r-s| \cdot ||D^{cr} \sigma||_{L^2(\mathcal{H}^0)} \\
&\quad + C_T ||\sigma||_{L^2(\mathcal{H}^0)} + C_T ||(A^{cr} - A^{cs}) \sigma||_{L^2(\mathcal{H}^0)}.
\end{aligned} \tag{2.24}$$

By Axiom I,

$$||D^{cr} \sigma||_{L^2(\mathcal{H}^0)} \le C_T ||G^{cr} D^{cr} \sigma||_{L^2(\mathcal{H}^0)},$$

hence the first two terms on the right in (2.24) are under control as desired. It remains to get a good upper bound for $||(A^{cr} - A^{cs}) \sigma||_{L^2(\mathcal{H}^0)}$. By (2.21) and (2.22),

$$||(A^{cr} - A^{cs}) \sigma||_{L^2(\mathcal{H}^0)} \le C_T |r-s| \cdot ||\sigma||_{L^2(\mathcal{H}^0)} + C_T ||\varphi A_r \sigma||_{L^2(\mathbb{R}_+, H_r)},$$

where $\varphi(t) = \inf(t-r, s-r)$ for $t \geq r$ and $\varphi(t) = 0$ for $t \leq r$. It remains to estimate the second term on the right of this inequality. We compute

$$||(\varphi\sigma)' + \varphi A_r\sigma||_r^2 = ||(\varphi\sigma)'||_r^2 + ||\varphi A_r\sigma||_r^2 + \langle\varphi A_r\sigma, \varphi\sigma\rangle_r'.$$

Now $\varphi\sigma \in \mathcal{L}_c(e^0)$ vanishes at 0, hence

$$||D^r(\varphi\sigma)||^2_{L^2(\mathbb{R}_+,H_r)} = ||(\varphi\sigma)'||^2_{L^2(\mathbb{R}_+,H_r)} + ||\varphi A_r\sigma||^2_{L^2(\mathbb{R}_+,H_r)}.$$

Since $D^r(\varphi\sigma) = \varphi'\gamma^r\sigma + \varphi D^r\sigma$, we conclude

$$\begin{aligned} ||\varphi A_r\sigma||_{L^2(\mathbb{R}_+,H_r)} &\leq C_T \cdot ||\sigma||_{L^2(\mathcal{H}^0)} + ||D^r(\varphi\sigma)||_{L^2(\mathbb{R}_+,H_r)} \\ &\leq C_T \cdot ||\sigma||_{L^2(\mathcal{H}^0)} + |s-r|||D^{cr}\sigma||_{L^2(\mathbb{R}_+,H_r)} \\ &\leq C_T \cdot ||\sigma||_{L^2(\mathcal{H}^0)} + C_T|s-r| \cdot ||G^{cr}D^{cr}\sigma||_{L^2(\mathcal{H}_0)}. \quad \square \end{aligned}$$

Proof of Theorem 2.17. We note first that the Hilbert spaces $L^2(\mathcal{H}^{ct})$ and $L^2(\mathcal{H}^0)$ coincide as vector spaces of (equivalence classes of) maps. The operators D^{ct} and $G^{ct}D^{ct}$ have the same minimal and maximal domains. Hence we may as well consider the family of operators $G^{ct}D^{ct}$ on $L^2(\mathcal{H}^0)$. We introduce operators

$$S^t = \begin{pmatrix} 0 & G^{ct}D^{ct} \\ G^{ct}D^{ct}_{0,c} & 0 \end{pmatrix} \quad \text{and} \quad T^t = \begin{pmatrix} 0 & G^{ct}D^{ct}_{\max} \\ G^{ct}D^{ct}_{\min} & 0 \end{pmatrix} \tag{2.25}$$

in $L^2(\mathcal{H}^0) \oplus L^2(\mathcal{H}^0)$ with domain $\mathcal{L}_{0,c}(e^0) \oplus \mathcal{L}_c(e^0)$ and $\mathcal{D}^{ct}_{\min} \oplus \mathcal{D}^{ct}_{\max}$, respectively. We note that S^t is symmetric and that T^t is self-adjoint with $S^t \subset T^t$.

Fix $T \geq 0$ and assume that, for some $r \in [0,T]$, the closure of S^r is equal to T^r with domain $\mathcal{D}^0_{\min} \oplus \mathcal{D}^0_{\max}$. By the results of the first section, this holds for $r = 0$. By the Kato-Rellich Theorem, see Theorem V.4.4 in [Ka] and Lemma 2.23, we get that the closure of S^s is self-adjoint with domain $\mathcal{D}^0_{\min} \oplus \mathcal{D}^0_{\max}$ for all $s \geq r$ in $[0,T]$ with $(s-r)C < 1/2$, where $C = C_T$ is the constant from Lemma 2.23. Since $S^s \subset T^s$ and T^s is self-adjoint, we conclude that the closure of S^s is equal to T^s for all such s. By the connectedness of $[0,T]$, we get that the closure of S^r is equal to T^r with domain $\mathcal{D}^0_{\min} \oplus \mathcal{D}^0_{\max}$ for all $r \in [0,T]$. This proves the first assertion.

As for the proof of the second assertion, we note that Lemma 2.23 implies that $D^{cr}_{\max}$ and $D^{cs}_{\max}$ have equivalent graph norms on their common domain $\mathcal{D}^0_{\max}$ as soon as $|r-s|C < 1$. Again by the connectedness of $[0,T]$, the graph norm of $D^{ct}_{\max}$ is equivalent to the one of $D^0_{\max}$. Hence there is a constant as claimed. $\square$

For applications it is useful to pass to a somewhat more general class of systems and operators.

2.26. Definition. A *Dirac-Schrödinger system* is a pair (d, V) consisting of a Dirac system d with Lipschitz coefficients and a *potential* $V \in L^\infty_{\mathrm{loc}}(\mathcal{L}(\mathcal{H}))$ with $V = V^*$. The associated *Dirac-Schrödinger operator* is given by

$$D := D^d + V : \mathcal{L}_{\mathrm{loc}}(e^0) \to L^\infty_{\mathrm{loc}}(\mathcal{H}),$$

where D^d denotes the Dirac operator of d.

2.27. Remark. It is not really necessary to assume that the potential is Hermitian, $V = V^*$. However, assuming $V = V^*$ keeps the notation a bit simpler. For most purposes, passing to the Dirac-Schrödinger system with operator

$$\begin{pmatrix} 0 & D^d + V^* \\ D^d + V & 0 \end{pmatrix}$$

reduces the general case to the case where V is Hermitian.

In what follows, D is the Dirac-Schrödinger operator associated to a Dirac-Schrödinger system (d, V). From (2.14) we get

$$(D\sigma_1, \sigma_2) - (\sigma_1, D\sigma_2) = \omega(\sigma_1(0), \sigma_2(0)), \tag{2.28}$$

where $\sigma_1, \sigma_2 \in \mathcal{L}_c(e^0)$. Therefore the restriction $D_{0,c}$ of D to the domain $\mathcal{D}_{0,c}$ is symmetric. We denote by $D_{\min}$, with domain $\mathcal{D}_{\min}$, the closure of $D_{0,c}$ in $L^2(\mathcal{H})$ and by $D_{\max} := (D_{0,c})^*$, with domain $\mathcal{D}_{\max}$, the adjoint operator of $D_{0,c}$ in $L^2(\mathcal{H})$.

We let D^0 be the Dirac operator associated to the constant coefficient Dirac system d^0 and $\mathcal{D}^0_{\max}$ be its domain. The following result is crucial.

2.29. Theorem. *If $\sigma \in L^2(\mathcal{H})$ has compact support, then $\sigma \in \mathcal{D}_{\max}$ if and only if $\sigma \in \mathcal{D}^0_{\max}$.*

Proof. Suppose that $\sigma \in L^2(\mathcal{H})$ has compact support in $[0, R]$. Since $V \in L^\infty_{\mathrm{loc}}(\mathcal{L}(\mathcal{H}))$, V is uniformly bounded on $[0, R]$, and hence we may assume that $V = 0$. Choose $T > R$. For any $t \in (R, T)$, the coefficients of D and D^{ct} coincide on $[0, R] \subset [0, t]$, compare (2.16). Hence $\sigma \in \mathcal{D}_{\max}$ if and only if $\sigma \in \mathcal{D}^{ct}_{\max}$, and from Theorem 2.17, $\mathcal{D}^{ct}_{\max} = \mathcal{D}^0_{\max}$. ∎

2.30. Proposition (Regularity). *The maximal domain $\mathcal{D}_{\max}$ satisfies:*

1. *$\mathcal{L}_c(e^0)$ is dense in $\mathcal{D}_{\max}$.*
2. *$\sigma \in \mathcal{D}_{\max}$ is in $H^1_{\mathrm{loc}}(e^0)$ if and only if $\sigma(0) \in H^{1/2}$.*
3. *$\mathcal{D}_{\max} \subset C(\mathbb{R}_+, \check{H}) \cap C((0, \infty), H^{1/2})$.*
4. *The restriction map on $\mathcal{L}_c(e^0)$ extends to a continuous surjective map $\mathcal{R} : \mathcal{D}_{\max} \to \check{H}$ and $\mathcal{D}_{\min} = \mathcal{R}^{-1}(0)$.*

(5) *For $\sigma_1, \sigma_2 \in \mathcal{D}_{\max}$, we have*

$$(D_{\max}\sigma_1, \sigma_2)_{L^2(\mathcal{H})} - (\sigma_1, D_{\max}\sigma_2)_{L^2(\mathcal{H})} = \omega\big(\sigma_1(0), \sigma_2(0)\big).$$

Proof. The first assertion follows from Lemma 1.42.1 and Theorem 2.29. As for the proof of the second and third assertion, multiply $\sigma \in \mathcal{D}_{\max}$ by a Lipschitz cutoff function χ which is equal to 1 on some interval $[0, R]$ and equal to 0 after $2R$. Then $\chi\sigma$ is in $\mathcal{D}^0_{\max}$, by Theorem 2.29, and $\chi\sigma$ has the asserted regularity properties, by Lemma 1.42. By Theorem 2.29, multiplication by χ defines a continuous operator from $\mathcal{D}_{\max}$ to $\mathcal{D}^0_{\max}$, hence the fourth assertion is immediate from Proposition 1.41. By (1) it is enough to check the last assertion for $\sigma_1, \sigma_2 \in \mathcal{L}_c(e^0)$. This case was already observed in (2.28). □

2.3. **Boundary conditions and Fredholm properties.** We now turn to the description of closed extensions of D, following closely the outline given in Section 1.4; most proofs carry over easily via the link given by Theorem 2.29. In what follows, we fix a Dirac-Schrödinger system (d, V) and define the Sobolev spaces H^s and $\check{H}$ with respect to A_0 as in Section 1.2.

As before, a boundary condition is a closed linear subspace $B \subset \check{H}$. Associated to a boundary condition B, we consider extensions of $D_{0,c}$ as in Section 1.4:

$$\begin{aligned} \mathcal{L}_{B,c} &:= \{\sigma \in \mathcal{L}_c(e^0) \, : \, \sigma(0) \in B\}, \\ D_{B,c} &:= D|\mathcal{L}_{B,c}; \end{aligned} \tag{2.31}$$

$$\begin{aligned} \mathcal{D}_B &:= \{\sigma \in \mathcal{D}_{\max} \cap H^1_{\mathrm{loc}}(e^0) \, : \, \sigma(0) \in B\}, \\ D_B &:= D_{\max}|\mathcal{D}_B; \end{aligned} \tag{2.32}$$

$$\begin{aligned} \mathcal{D}_{B,\max} &:= \{\sigma \in \mathcal{D}_{\max} \, : \, \sigma(0) \in B\}, \\ D_{B,\max} &:= D_{\max}|\mathcal{D}_{B,\max}. \end{aligned} \tag{2.33}$$

As before, since the restriction map $\mathcal{R} : \mathcal{D}_{\max} \to \check{H}$ is continuous and B is closed in $\check{H}$, $D_{B,\max}$ is a closed operator. Moreover, any closed extension of $D_{0,c}$ with domain contained in $\mathcal{D}_{\max}$ is of this form.

2.34. Remark. The same formulas for the adjoint operators and the closures as in (1.53)–(1.58) continue to hold and for the same reasons. We do not repeat them here.

As before, we say that a boundary condition $B \subset \check{H}$ is *regular* if $D_{B,\max} = D_B$. It is immediate from Proposition 2.30.2. that

(1) in the case of constant coefficients with potential $V = 0$, the present definition coincides with the one in Section 1.4;

(2) a boundary condition B is regular relative to (d, V) if and only if it is regular relative to d^0.

As in Section 1.4, we say that a boundary condition B is *elliptic* if B and B^a are regular.

In the case of constant coefficients with potential $V = 0$, D_B is not a Fredholm operator whenever $\ker A_0 \neq 0$, even if B is elliptic. However, we may look for an analogue of the space W which worked so nicely in the constant coefficient case. From the continuity of $\mathcal{R}$, established in Theorem 2.30.4 we get that there is a constant C such that

$$||\sigma||_W^2 := ||\sigma(0)||_{\check{H}}^2 + ||D_{\max}\sigma||_{L^2(\mathcal{H})}^2 \leq C||\sigma||_{\mathcal{D}_{\max}}^2. \tag{2.35}$$

for all $\sigma \in \mathcal{D}_{\max}$. The converse of (2.35) is not available in general, as we know, but a localized version may hold. This requires the inequality (1.72) which we now introduce as an additional axiom.

IV. Axiom. For each $T > 0$ there is a constant C_T such that

$$||\sigma||_{L^2([0,T],\mathcal{H})} \leq C_T||\sigma||_W \quad \text{for all } \sigma \in \mathcal{L}_c(e^0).$$

Following G. Carron (cf. the introduction to [Ca2]) we will call a Dirac-Schrödinger system (d, V) satisfying Axiom IV *non-parabolic* (at infinity). We say that a Dirac-Schrödinger system (d, V) is *of Fredholm type*, if there is a constant C such that

$$||\sigma||_{L^2(\mathcal{H})} \leq C||\sigma||_W \quad \text{for all } \sigma \in \mathcal{L}_c(e^0). \tag{2.36}$$

If (d, V) is non-parabolic, then (d, V) is of Fredholm type if and only if, for some $\psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ which is equal to 1 near $t = 0$,

$$||(1-\psi)\sigma||_{L^2(\mathcal{H})} \leq C_\psi||\sigma||_W \quad \text{for all } \sigma \in \mathcal{L}_c(e^0). \tag{2.37}$$

In the geometric setting considered by Carron, it is enough to work with smooth sections supported near infinity, hence the space $\check{H}$ does not enter his discussion. However, the two formulations of non-parabolicity here and there are equivalent in the following sense.

2.38. Lemma. *The inequality of Axiom IV holds for all $\sigma \in \mathcal{L}_c(e^0)$ if it holds for all $\sigma \in \mathcal{L}_{0,c}(e^0)$.*

Proof. Choose $\psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ with $\psi(0) = 1$. Let D^0 be the Dirac operator and $\mathcal{E}^0$ be the extension operator for d^0, see (1.36). Let $\sigma \in \mathcal{L}_c(e^0)$ and set

$$\sigma_0 := \psi\mathcal{E}^0\sigma(0) \quad \text{and} \quad \sigma_1 := \sigma - \sigma_0.$$

Since $\sigma(0) \in H_A$, we have $\sigma_0 \in \mathcal{L}_c(e^0)$; hence $\sigma_1 \in \mathcal{L}_{0,c}(e^0)$. Now we can estimate, using the assumption, Lemma 1.37, and Theorem 2.29,

$$\begin{aligned}
||\sigma||_{L^2([0,T],H)} &\leq ||\sigma_1||_{L^2([0,T],H)} + ||\sigma_0||_{L^2([0,T],H)} \\
&\leq C_{T,\psi}(||D_{\max}\sigma_1||_{L^2(\mathcal{H})} + ||\sigma(0)||_{-1/2}) \\
&\leq C_{T,\psi}(||D_{\max}\sigma||_{L^2(\mathcal{H})} + ||D_{\max}\psi\mathcal{E}^0\sigma(0)||_{L^2(\mathcal{H})} + ||\sigma(0)||_{-1/2}) \\
&\leq C_{T,\psi}(||D_{\max}\sigma||_{L^2(\mathcal{H})} + ||D^0_{\max}\psi\mathcal{E}^0\sigma(0)||_{L^2(\mathcal{H})} + ||\sigma(0)||_{-1/2}) \\
&= C_{T,\psi}(||D_{\max}\sigma||_{L^2(\mathcal{H})} \\
&\qquad\qquad + ||(A_0 - |A_0| - Q_0)\psi\mathcal{E}^0\sigma(0)||_{L^2(\mathcal{H})} + ||\sigma(0)||_{-1/2}) \\
&\leq C_{T,\psi}(||D_{\max}\sigma||_{L^2(\mathcal{H})} + ||\sigma(0)||_{\check{H}}),
\end{aligned}$$

where we allow the constant $C_{T,\psi}$ to change from line to line. □

As a first implication of non-parabolicity we note that the seminorm $||\cdot||_W$, as defined in (2.35), is actually a norm on $\mathcal{D}_{\max}$. Thus we can introduce again the space W as the completion of $\mathcal{D}_{\max}$ under this norm. Since $\mathcal{L}_c(e^0)$ is dense in $\mathcal{D}_{\max}$ with respect to the graph norm of $D_{\max}$, $\mathcal{L}_c(e^0)$ is dense in W with respect to the W-norm.

2.39. Lemma. *If (d, V) is a non-parabolic Dirac-Schrödinger system, then we have:*

(1) *The restriction map $\mathcal{R}$ and $D_{\max}$ extend to continuous maps $\mathcal{R}_{\text{ext}}$ and D_{ext} on W, respectively; $\mathcal{R}_{\text{ext}}$ induces an isometry from $\ker D_{\text{ext}}$ into $\check{H}$.*
(2) *If $\psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ and $\sigma \in W$, then $\psi\sigma \in \mathcal{D}_{\max} \subset W$. Moreover, there is a constant C_ψ such that*

$$||\psi\sigma||_{D_{\max}} \leq C_\psi ||\sigma||_W.$$

In particular, W can be viewed as a space of locally integrable functions and $W \cap L^2(\mathcal{H}) = \mathcal{D}_{\max}$.
(3) *$W = \mathcal{D}_{\max}$ if and only if (d, V) is a Dirac-Schrödinger system of Fredholm type; that is, there is a constant C such that*

$$||\sigma||_{L^2(\mathcal{H})} \leq C||\sigma||_W \quad \textit{for all } \sigma \in \mathcal{L}_c(e^0).$$

Proof. (1) and (3) are immediate from the definition of W. As for (2), we note that, by non-parabolicity, there is a constant C_ψ such that

$$||\psi\sigma||_{\mathcal{D}_{\max}} \leq C_\psi ||\sigma||_W$$

for all $\sigma \in \mathcal{L}_c(e^0)$, hence for all $\sigma \in W$ by the density of $\mathcal{L}_c(e^0)$.

Let $\sigma \in W \cap L^2(\mathcal{H})$ and $\tau \in \mathcal{L}_c(e^0)$. Choose $\psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ with $\psi\tau = \tau$. Then, by the first part of (2) and the choice of ψ,

$$\begin{aligned}(D_{\mathrm{ext}}\sigma, \tau)_{L^2(\mathcal{H})} &= (D_{\mathrm{ext}}(\psi\sigma), \tau)_{L^2(\mathcal{H})} = (D_{\max}(\psi\sigma), \tau)_{L^2(\mathcal{H})} \\ &= (\psi\sigma, D\tau)_{L^2(\mathcal{H})} = (\sigma, D\tau)_{L^2(\mathcal{H})},\end{aligned} \tag{2.40}$$

and hence $\sigma \in \mathcal{D}_{\max}$. The converse inclusion is clear. □

2.41. Lemma. *Let U be a bounded subset of W. Then U is precompact if and only if $D_{\mathrm{ext}}(U) \subset L^2(\mathcal{H})$ and $Q_{\geq}\mathcal{R}(U) \subset \check{H}$ are both precompact.*

Proof. If U is precompact, then also its image under the continuous maps D_{ext} and $Q_{\geq}\mathcal{R}$.

Vice versa, assume that $D_{\mathrm{ext}}(U) \subset L^2(\mathcal{H})$ and $Q_{\geq}\mathcal{R}(U) \subset \check{H}$ are both precompact. By the definition of W, it suffices to show that $\mathcal{R}(U)$ is precompact in $\check{H}$.

Let D^0 be the Dirac operator associated to d^0. Let $\varphi, \psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ such that $\varphi\psi = \psi$. The operator $S_{D^0}\varphi$ is the norm limit of the Hilbert-Schmidt operators $S_{D^0}\varphi Q_{[-n,n]}$ on $L^2(\mathbb{R}_+, H_0)$, hence $S_{D^0}\varphi$ is a compact operator. On the other hand, $\psi U \subset \mathcal{D}^0_{\max}$ and $D^0(\psi U)$ is bounded in $L^2(\mathbb{R}_+, H_0)$, see Theorem 2.29 and Lemma 2.39.2. It follows that $S_{D^0}\varphi(D^0(\psi U))$ is precompact in $L^2(\mathbb{R}_+, H_0)$. By Corollary 1.40,

$$\psi U \subset \psi(0)\mathcal{E}^0 Q_{>}\mathcal{R}(U) + S_{D^0}\varphi(D^0(\psi U)) + Q_0(\psi U),$$

hence ψU is precompact in $L^2(\mathbb{R}_+, H_0)$.

Now choose φ, ψ as above with ψ smooth and equal to 1 in a neighborhood of 0. We have

$$D_{\mathrm{ext}}(\psi U) \subset \gamma\psi' U + \psi D_{\mathrm{ext}}(U).$$

Since ψ' is in $\mathrm{Lip}_c(\mathbb{R}_+)$ with $\varphi\psi' = \psi'$, $\psi' U$ is precompact in $L^2(\mathbb{R}_+, H_0)$, by the first part of the proof. By assumption, $\psi D_{\mathrm{ext}}(U)$ is precompact in $L^2(\mathbb{R}_+, H_0)$. Hence ψU and $D_{\mathrm{ext}}(\psi U)$ are precompact in $L^2(\mathbb{R}_+, H_0)$, hence ψU is precompact in $\mathcal{D}_{\max}$. We conclude that $\mathcal{R}(U) = \mathcal{R}(\psi U)$ is precompact in $\check{H}$, and hence that U is precompact in W. □

For a boundary condition $B \subset \check{H}$, set

$$W_B := \{\sigma \in W \,:\, \sigma(0) \in B\} \quad \text{and} \quad D_{B,\mathrm{ext}} := D_{\mathrm{ext}}|W_B. \tag{2.42}$$

2.43. Theorem and Definition. *Assume that (d, V) is non-parabolic and that B is regular. Then $D_{B,\mathrm{ext}} : W_B \to L^2(\mathcal{H})$ is a left-Fredholm operator with $(\operatorname{im} D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}$ and index*

$$\operatorname{ind} D_{B,\mathrm{ext}} = \dim\ker D_{B,\mathrm{ext}} - \dim\ker D_{B^a,\max},$$

called the extended index *of D_B, also denoted* $\operatorname{ind}_{\mathrm{ext}} D_B$.

Proof. Let (σ_n) be a bounded sequence in W_B such that the sequence $(D_{\mathrm{ext}}\sigma_n)$ converges in $L^2(\mathcal{H})$. By the continuity of $\mathcal{R}$, the sequence $(\mathcal{R}\sigma_n(0))$ is bounded in $B \subset \check{H}$. By the regularity of B, the sequence $(Q_{\geq}\mathcal{R}\sigma_n(0))$ has a convergent subsequence in $H^{-1/2}$ and hence in B. Therefore, (σ_n) has a convergent subsequence in W, by Lemma 2.41. Finally, since $\mathcal{D}_{B,\max}$ is dense in W_B and $(D_{B,\max})^* = D_{B^a,\max}$, we also have $(\operatorname{im} D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}$. □

We note some important consequences of Theorem 2.43.

2.44. Corollary and Definition. *If (d,V) is non-parabolic and B is elliptic, then the kernels of D_B and D_{B^a} have finite dimension, and we can define the L^2-*index *of D_B to be the number*

$$L^2\text{-}\operatorname{ind} D_B := \dim\ker D_B - \dim\ker D_{B^a}. \quad \square$$

Suppose that (d,V) is non-parabolic. For $\Lambda \in \mathbb{R}$, let $D_{<\Lambda,\max} := D_{B,\max}$ and $D_{<\Lambda,\mathrm{ext}} := D_{B,\mathrm{ext}}$, where $B = \check{H}_{<\Lambda} = H^{1/2}_{<\Lambda}$, and similarly with $\leq$ substituted for $<$. The boundary conditions $B = \check{H}_{<\Lambda}$ and $B = \check{H}_{\leq\Lambda}$ are elliptic with $B^a = H^{1/2}_{\leq-\Lambda}$ and $B^a = H^{1/2}_{<-\Lambda}$, respectively. Hence $D_{<\Lambda} = D_{<\Lambda,\max}$ and, furthermore, $D_{<\Lambda,\mathrm{ext}}$ and $D_{\leq\Lambda,\mathrm{ext}}$ are Fredholm operators with

$$(\operatorname{im} D_{\leq\Lambda,\mathrm{ext}})^\perp = \ker D_{<-\Lambda} \subset \ker D_{<-\Lambda,\mathrm{ext}}, \tag{2.45}$$

see Theorem 2.43.

2.46. Proposition. *If (d,V) is non-parabolic, then there is $\Lambda_0 \geq 0$ such that $D_{<-\Lambda,\mathrm{ext}}$ is injective and $D_{\leq\Lambda,\mathrm{ext}}$ is surjective for all $\Lambda \geq \Lambda_0$.*

Proof. For any $\Lambda \in \mathbb{R}$, $D_{<\Lambda,\mathrm{ext}}$ is a Fredholm operator. In particular,

$$E := \mathcal{R}_{\mathrm{ext}}(\ker D_{<0,\mathrm{ext}}) \subset H^{1/2}$$

has finite dimension, and hence all H^s-norms are equivalent on E for $|s| \leq 1/2$. Let $\Lambda \geq 0$, $\sigma \in \ker D_{<-\Lambda,\mathrm{ext}} \subset \ker D_{<0,\mathrm{ext}}$, and suppose that $\sigma(0) \neq 0$. Since $\sigma(0) \in E \cap \check{H}_{<-\Lambda}$, we can estimate

$$\begin{aligned} 0 \neq ||\sigma(0)||^2_{1/2} &\leq C_E^2 ||\sigma(0)||^2_{-1/2} \\ &= C_E^2 \langle (I + A_0^2)^{-1/2}\sigma(0), \sigma(0)\rangle < C_E^2(1+\Lambda^2)^{-1}||\sigma(0)||^2_{1/2}, \end{aligned}$$

a contradiction if

$$\Lambda \geq \Lambda_0 := (C_E^2 - 1)^{1/2}.$$

Therefore $\sigma(0) = 0$ if $\Lambda \geq \Lambda_0$, and then $\sigma = 0$, by the non-parabolicity of (d,V). Hence $D_{<-\Lambda,\mathrm{ext}}$ is injective for $\Lambda \geq \Lambda_0$. □

Next we would like to write the index formula in Theorem 2.43 in a way analogous to Corollary 1.82. For this, we need the *Calderón spaces*

$$\check{\mathcal{C}}_{\max} := \mathcal{R}(\ker D_{\max}) \quad \text{and} \quad \check{\mathcal{C}}_{\text{ext}} := \mathcal{R}(\ker D_{\text{ext}}). \tag{2.47}$$

Since $\mathcal{R} : \ker D_{\text{ext}} \to \check{H}$ is isometric, $\check{\mathcal{C}}_{\text{ext}}$ is a closed subspace of $\check{H}$. For $|s| \le 1/2$, we let

$$\mathcal{C}^s_{\max} := \check{\mathcal{C}}_{\max} \cap H^s \quad \text{and} \quad \mathcal{C}^s_{\text{ext}} := \check{\mathcal{C}}_{\text{ext}} \cap H^s. \tag{2.48}$$

If B is a regular boundary condition, then $\mathcal{R}$ induces isomorphisms

$$\begin{aligned} \ker D_{B,\max} &\cong B \cap \check{\mathcal{C}}_{\max} = B \cap \mathcal{C}^{1/2}_{\max}, \\ \ker D_{B,\text{ext}} &\cong B \cap \check{\mathcal{C}}_{\text{ext}} = B \cap \mathcal{C}^{1/2}_{\text{ext}}. \end{aligned} \tag{2.49}$$

We will write $\mathcal{C}_{\max}$ and $\mathcal{C}_{\text{ext}}$ instead of $\mathcal{C}^0_{\max}$ and $\mathcal{C}^0_{\text{ext}}$, respectively.

2.50. Corollary. *If (d, V) is non-parabolic and B is elliptic, then $D_{B,\text{ext}}$ is a Fredholm operator with $(\operatorname{im} D_{B,\text{ext}})^\perp = \ker D_{B^a}$ and index*

$$\begin{aligned} \operatorname{ind} D_{B,\text{ext}} &= \dim B \cap \mathcal{C}^{1/2}_{\text{ext}} - \dim B^\perp \cap \gamma\mathcal{C}^{1/2}_{\max}. \\ &= \dim B \cap \mathcal{C}_{\text{ext}} - \dim B^\perp \cap \gamma\mathcal{C}_{\max}. \end{aligned}$$

Proof. The assertions follow easily from Theorem 2.43 and Lemma 1.46, except for the last identity. Since B is elliptic, we have $B \subset H^{1/2}$ and

$$B^a = (\gamma B^\perp) \cap \check{H} \subset H^{1/2} \subset H.$$

Therefore

$$B^a \cap \mathcal{C}^{1/2}_{\max} = B^a \cap \check{\mathcal{C}}_{\max} = (\gamma B^\perp) \cap \check{\mathcal{C}}_{\max} = (\gamma B^\perp) \cap \mathcal{C}_{\max}. \qquad \square$$

2.51. Corollary. *Assume that (d, V) is non-parabolic and that P is an orthogonal elliptic projection in H. Then $D_{P,\text{ext}}$ is a Fredholm operator with $(\operatorname{im} D_{P,\text{ext}})^\perp = \ker D_{P_\gamma}$ and index*

$$\begin{aligned} \operatorname{ind} D_{P,\text{ext}} &= \dim \ker P \cap \mathcal{C}^{1/2}_{\text{ext}} - \dim \operatorname{im} P \cap \gamma\mathcal{C}^{1/2}_{\max} \\ &= \dim \ker P \cap \mathcal{C}_{\text{ext}} - \dim \operatorname{im} P \cap \gamma\mathcal{C}_{\max}. \end{aligned}$$

Proof. The boundary condition associated to P is $B_P = \ker \tilde{P} \cap \check{H}$, see (1.96). Since B_P is regular, $B_P = \ker P \cap \check{H} = \ker \hat{P}$ and therefore

$$B_P \cap \check{\mathcal{C}}_{\text{ext}} = \ker P \cap \check{\mathcal{C}}_{\text{ext}} = \ker P \cap \mathcal{C}_{\text{ext}} = \ker P \cap \mathcal{C}^{1/2}_{\text{ext}}.$$

The remaining identities follow from Corollary 2.50 since $\operatorname{im} P$ is the orthogonal complement of $\ker \hat{P}$ in H. $\square$

2.4. **Some examples.** The first two examples are Dirac systems on $\mathbb{R}_+$ which are not non-parabolic. In the first example, $\ker D_{P_{APS},\max}$ is infinite-dimensional so that $D_{P_{APS},\mathrm{ext}}$ cannot be a Fredholm operator. In the second example, the assumption of non-parabolicity would lead to the contradiction that $\ker D_{P_{APS},\mathrm{ext}}$ has infinite dimension. These examples are modelled on the Gauss-Bonnet operators of real hyperbolic spaces of even and odd dimension.

2.52. Example. For $t \in \mathbb{R}_+$ and $k \in \mathbb{Z}$, let

$$B_t(k) = \begin{pmatrix} 1 & ike^{-t} \\ -ike^{-t} & 1 \end{pmatrix},$$

and consider the evolution equation

$$\sigma' + B_t(k)\sigma = 0.$$

Solutions σ of this equation satisfy $(||\sigma||^2)' \le -2(1-|k|e^{-t})||\sigma||^2$, hence belong to $L^2(\mathbb{R}_+, \mathbb{C}^2)$. Eigenvalues and eigenvectors of $B_0(k)$ are given by

$$B_0(k)\begin{pmatrix} 1 \\ i \end{pmatrix} = (1-k)\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad B_0(k)\begin{pmatrix} 1 \\ -i \end{pmatrix} = (1+k)\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

On $L^2(\mathbb{R}_+, \mathbb{C}^2 \oplus \mathbb{C}^2)$, consider the Dirac system

$$\begin{aligned} D_k\sigma &= \begin{pmatrix} -\sigma_2' + B_t(k)\sigma_2 \\ \sigma_1' + B_t(k)\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \left(\partial_t + \begin{pmatrix} B_t(k) & 0 \\ 0 & -B_t(k) \end{pmatrix} \right) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \\ &=: \gamma(\partial_t + A_t(k))\sigma \end{aligned}$$

For any $k \in \mathbb{Z}$, let

$$\sigma_k := \begin{pmatrix} \tau_k \\ 0 \end{pmatrix} \quad \text{with} \quad \tau_k' + B_t(k)\tau_k = 0 \quad \text{and} \quad \tau_k(0) = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Then $\sigma_k \in L^2(\mathbb{R}_+, \mathbb{C}^2 \oplus \mathbb{C}^2)$, $D_k\sigma_k = 0$, and $A_0(k)\sigma_k(0) = (1-k)\sigma_k$. Hence σ_k belongs to the negative eigenspace of $A_0(k)$ for $k \ge 2$.

We can now sum these Dirac systems to obtain a Dirac system

$$d = (\mathcal{H}, \partial_t, A_t = \oplus A_t(k), \gamma) \quad \text{on} \quad \mathcal{H} = \mathbb{R}_+ \times l^2(\mathbb{Z}, \mathbb{C}^2 \oplus \mathbb{C}^2)$$

with associated Dirac operator $D = \oplus D_k$. For this Dirac system, there is a family (σ_k) of orthogonal non-zero L^2-sections of $\mathcal{H}$ with $D\sigma_k = 0$ and $A_0\sigma_k = (1-k)\sigma_k$. Hence, with $Q_{\ge\Lambda}$ the corresponding spectral projection of A_0, the L^2-kernel of $D_{Q_{\ge\Lambda}}$ has infinite dimension, for any $\Lambda \in \mathbb{R}$. In particular, d is not non-parabolic.

2.53. Example. For $k \in \mathbb{Z}$, consider the Dirac system on $\mathbb{R}_+ \times \mathbb{C}^2$ with Dirac operator

$$\begin{aligned} D_k\sigma &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\partial_t + \begin{pmatrix} ke^{-t} & 0 \\ 0 & -ke^{-t} \end{pmatrix} \right) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \\ &=: \gamma(\partial_t + A_t(k))\sigma. \end{aligned}$$

Solutions of the equation $D_k\sigma = 0$ are obviously uniformly bounded and, therefore, admit an upper bound

$$\int_0^T ||\sigma(t)||^2 dt \le C_k T ||\sigma(0)||^2.$$

Moreover, for $k \ge 1$,

$$\sigma_k(t) = \begin{pmatrix} 0 \\ e^{-ke^{-t}} \end{pmatrix}$$

satisfies $D_k\sigma_k = 0$ and $A_0(k)\sigma_k(0) = -k\sigma_k(0)$. Again, we sum all these Dirac systems to get a Dirac system on $L^2(\mathbb{R}_+, l^2(\mathbb{Z}, \mathbb{C}^2))$ given by ∂_t, $A_t = \oplus A_t(k)$ and $\oplus\gamma$.

Let $Q_{\ge 0}$ be the spectral projection of A_0 onto the non-negative eigenspaces of A_0. We obtain that the space of $\sigma \in L^2(\mathbb{R}_+, l^2(\mathbb{Z}, \mathbb{C}^2))$ with

$$D\sigma = 0, \quad Q_{\ge 0}\sigma(0) = 0 \quad \text{and} \quad \int_0^T ||\sigma(t)||^2 dt = O(T)$$

has infinite dimension. The following lemma implies that this Dirac system is not non-parabolic.

2.54. Lemma. *Let d be a non-parabolic Dirac system. If $\sigma \in H^1_{\mathrm{loc}}(e)$ satisfies $D\sigma = 0$ and*

$$\lim_{T\to\infty} \frac{\int_0^T ||\sigma(t)||^2 dt}{T^2} = 0,$$

then $\sigma \in W$.

Proof. It suffices to find a sequence (σ_n) in $H^1_c(e^0)$ such that

$$\lim_{n\to\infty} ||D(\sigma - \sigma_n)||_{L^2(\mathcal{H})} + ||\sigma(0) - \sigma_n(0)||_{\check{H}} = 0.$$

Let ψ be a Lipschitz function on $\mathbb{R}_+$ with compact support such that $\psi = 1$ in a neighborhood of 0, and set $\psi_n(t) := \psi(t/n)$ and $\sigma_n := \psi_n\sigma$. Since $\sigma(0) = \sigma_n(0)$ and

$$D(\sigma - \sigma_n)(t) = -\frac{1}{n}\gamma\psi'(t/n)\sigma,$$

we obtain that $\|D(\sigma - \sigma_n)\|^2_{L^2(\mathcal{H})} = o(1)$ as n tends to infinity. $\square$

2.55. Example. For $\mu \in \mathbb{R}$, let d_μ be the Dirac system on $L^2([1,\infty),\mathbb{C}^2)$ with Dirac operator

$$D(\sigma_+,\sigma_-) := (-\sigma_-' + \frac{\mu}{t}\sigma_-, \sigma_+' + \frac{\mu}{t}\sigma_+).$$

Clearly, d_μ is not of Fredholm type. On the other hand, since the equation $D\sigma = \tau$ corresponds to a linear ODE in the finite dimensional space $H = \mathbb{C}^2$, d_μ is non-parabolic. We have

$$|-\sigma_-' + \frac{\mu}{t}\sigma_-|^2 = |\sigma_-'|^2 + \frac{\mu(\mu-1)}{t^2}|\sigma_-|^2 - (\frac{\mu}{t}|\sigma_-|^2)',$$

and similarly for σ_+, where all the minus signs turn into plus signs. Now W is the closure of the space of Lipschitz sections with compact support with respect to the W-norm. Hence, if $\mu > 1$ and $\sigma = (\sigma_+,\sigma_-)$ is in W, then $|\sigma/t|^2$ is integrable with integral uniformly bounded by the W-norm of σ. (This also shows non-parabolicity in the case $\mu > 1$.)

The space of solutions of the equation $D\sigma = 0$ is given by the space of sections $(at^{-\mu}, bt^{\mu})$ with $a, b \in \mathbb{C}$. For $\mu > 1$ and $b \neq 0$, $(at^{-\mu}, bt^{\mu})$ does not belong to W since $(at^{-\mu}, bt^{\mu})/t$ is not square integrable. It follows that, for $\mu > 1$, W-solutions of the equation $D\sigma = 0$ are square integrable, hence that $\mathcal{C}_{\max} = \mathcal{C}_{\mathrm{ext}}$, although d_μ is not of Fredholm type.

The above analysis can be refined. By (5.3) in [Ca2] and by what is said in the two lines above it,

$$\int_1^\infty \big(|\tau'|^2 - \frac{1}{4t^2}|\tau|^2\big) \geq \int_1^\infty \frac{1}{4t^2(\ln t)^2}|\tau|^2,$$

for all $\tau \in \mathrm{Lip}_c([1,\infty))$ with $\tau(1) = 0$. Since

$$|\tau' - \frac{\mu}{t}\tau|^2 = |\tau'|^2 - \frac{1}{4t^2}|\tau|^2 + \frac{(\mu - 1/2)^2}{t^2}|\tau|^2 - (\frac{\mu}{t}|\tau|^2)',$$

we get the following inequality

$$\int_1^\infty |\tau' - \frac{\mu}{t}\tau|^2 \geq \int_1^\infty \frac{(\mu-1/2)^2}{t^2}|\tau|^2 + \int_1^\infty \frac{1}{4t^2(\ln t)^2}|\tau|^2,$$

for all $\tau \in \mathrm{Lip}_c([1,\infty))$ with $\tau(1) = 0$. It follows that $\mathcal{C}_{\max} = \mathcal{C}_{\mathrm{ext}}$ if $|\mu| > 1/2$ (and, again, that d_μ is non-parabolic for all $\mu \in \mathbb{R}$).

3. Calderón projections and index formulas

3.1. The Calderón projections. Recall the definition of the Calderón spaces in (2.47) and (2.48).

3.1. Theorem. *Let (d,V) be non-parabolic. If $\Lambda_0 \geq 0$ is the constant from Proposition 2.46 and $\Lambda \geq \Lambda_0$, then we have a direct sum decomposition*

$$\check{\mathcal{C}}_{\text{ext}} = K_\Lambda \oplus \check{G}_\Lambda,$$

where $K_\Lambda = \{x \in \check{\mathcal{C}}_{\text{ext}} : Q_{>\Lambda}x = 0\} \subset H^{1/2}$ is of finite dimension and $\check{G}_\Lambda$ is the graph of a continuous linear map

$$T_\Lambda : H^{-1/2}_{>\Lambda} \to H^{1/2}_{\leq\Lambda},$$

where $T_\Lambda = T_{\Lambda_0}|H^{-1/2}_{>\Lambda}$. The finite rank and remainder parts $Q_{[-\Lambda,\Lambda]}T_\Lambda$ and $Q_{<-\Lambda}T_\Lambda$, respectively, satisfy

$$||Q_{[-\Lambda,\Lambda]}T_\Lambda||_s \leq C\Lambda^{-1/2-s} \quad \textit{and} \quad ||Q_{<-\Lambda}T_\Lambda||_s \leq C\Lambda^{-1},$$

where C is a constant independent of $\Lambda \geq \Lambda_0$ and $s \in [-1/2, 1/2]$. In particular,

$$\mathcal{C}^s_{\text{ext}} = K_\Lambda \oplus G^s_\Lambda,$$

where $G^s_\Lambda = \check{G}_\Lambda \cap H^s$ is the graph of $T_\Lambda|H^s_{>\Lambda}$, and hence $\mathcal{C}^s_{\text{ext}}$ is a closed subspace of H^s, for all $\Lambda \geq \Lambda_0$ and $s \in [-1/2, 1/2]$.

Proof. Throughout the proof, we assume $\Lambda \geq \Lambda_0$, where Λ_0 is the constant from Proposition 2.46.

Let $x \in \check{H}_{>\Lambda} = H^{-1/2}_{>\Lambda}$. Choose a function $\psi \in \mathrm{Lip}_c(\mathbb{R}_+)$ which is equal to 1 in a neighborhood of 0 and set $\sigma := \psi\mathcal{E}^0 x$. Then $\sigma \in \mathcal{D}_{\max} \subset W$, by Theorem 2.29. Since $D_{\leq\Lambda,\text{ext}}$ is surjective, there is $\tau \in \mathcal{D}_{\leq\Lambda,\text{ext}}$ with $D_{\text{ext}}\tau = D_{\text{ext}}\sigma$. Hence $\sigma - \tau \in \ker D_{\text{ext}}$ and

$$x = Q_{>\Lambda}((\sigma - \tau)(0)) \in Q_{>\Lambda}(\check{\mathcal{C}}_{\text{ext}}).$$

Therefore $Q_{>\Lambda} : \check{\mathcal{C}}_{\text{ext}} \to H^{-1/2}_{>\Lambda}$ is surjective. We have

$$K_\Lambda = \mathcal{R}_{\text{ext}}(\ker D_{\text{ext}}) \cap H^{1/2}_{\leq\Lambda} = \mathcal{R}_{\text{ext}}(\ker D_{\leq\Lambda,\text{ext}}),$$

hence K_Λ is of finite dimension, by Theorem 2.43. Let $\check{G}_{\Lambda_0}$ be a complement of K_{Λ_0} in $\check{\mathcal{C}}_{\text{ext}}$. Then $Q_{>\Lambda_0} : \check{G}_{\Lambda_0} \to H^{-1/2}_{>\Lambda_0}$ is an isomorphism, hence $\check{G}_{\Lambda_0}$ is the graph of a continuous linear map

$$T_{\Lambda_0} : H^{-1/2}_{>\Lambda_0} \to \check{H}_{\leq\Lambda_0} = H^{1/2}_{\leq\Lambda_0}.$$

This is the place where we gain regularity: By the very structure of $\check{H}$, T_{Λ_0} extends naturally to a smoothing operator.

Let $|s| \leq 1/2$. Since the image of T_{Λ_0} is contained in $H^{1/2}$, $x = y + T_{\Lambda_0} y \in \check{G}_{\Lambda_0}$ is in H^s if and only if $y = Q_{>\Lambda_0} x$ is in H^s, i.e.,

$$G^s_{\Lambda_0} = \check{G}_{\Lambda_0} \cap H^s = \{y + T_{\Lambda_0} y \, : \, y \in H^s_{>\Lambda_0}\}.$$

For $\Lambda \geq \Lambda_0$, we define

$$\begin{aligned} \check{G}_\Lambda :&= \{x \in \check{G}_{\Lambda_0} \, : \, Q_{(\Lambda_0,\Lambda]} x = 0\} \\ &= \{y + T_{\Lambda_0} y \, : \, y \in H^{-1/2}_{>\Lambda}\}. \end{aligned}$$

Let $|s| \leq 1/2$. Then $G^s_\Lambda = \check{G}_\Lambda \cap H^s$ is the graph of $T_\Lambda := T_{\Lambda_0} | H^s_{>\Lambda}$. Hence G^s_Λ is a closed subspace of H^s.

We show next that G^s_Λ is a complement of K_Λ in $\mathcal{C}^s_{\mathrm{ext}} = \check{\mathcal{C}}_{\mathrm{ext}} \cap H^s$. Since $K_{\Lambda_0} \subset K_\Lambda$ and, clearly, $K_\Lambda \cap G^s_\Lambda = 0$, it is enough to show that $G^s_{\Lambda_0} \subset K_\Lambda + G^s_\Lambda$. Now for $y \in G^s_{\Lambda_0}$ there is $z \in \check{G}_{\Lambda_0}$ with $Q_{>\Lambda_0} z = Q_{(\Lambda_0,\Lambda]} y$, by the surjectivity of $Q_{>\Lambda_0} | \check{G}_{\Lambda_0}$. It follows that $z \in K_\Lambda$ and $y - z \in G^s_\Lambda$.

For $x \in H^{-1/2}_{>\Lambda}$,

$$||Q_{[-\Lambda,\Lambda]} T_\Lambda x||_{1/2} \leq ||T_\Lambda x||_{1/2} \leq C||x||_{-1/2},$$

where $C = ||T_{\Lambda_0}||_{\check{H}}$, and similarly for $Q_{<-\Lambda} T_\Lambda x$. For $r < t$ and $y \in H^t$ with $Q_{(-\Lambda,\Lambda)} y = 0$, we have

$$||y||_r \leq \Lambda^{r-t} ||y||_t. \tag{3.2}$$

Hence

$$\begin{aligned} ||Q_{[-\Lambda,\Lambda]} T_\Lambda x||_s &\leq ||Q_{[-\Lambda,\Lambda]} T_\Lambda x||_{1/2} \\ &\leq C||x||_{-1/2} \leq C\Lambda^{-1/2-s} ||x||_s, \end{aligned} \tag{3.3}$$

$$\begin{aligned} ||Q_{<-\Lambda} T_\Lambda x||_s &\leq \Lambda^{s-1/2} ||Q_{<-\Lambda} T_\Lambda x||_{1/2} \\ &\leq C\Lambda^{s-1/2} ||x||_{-1/2} \leq C\Lambda^{-1} ||x||_s, \end{aligned} \tag{3.4}$$

for all $|s| \leq 1/2$ and $x \in H^s_{>\Lambda}$. □

3.5. Definition. The orthogonal projections in H onto (the closure of) $\mathcal{C}_{\max} = \mathcal{C}^0_{\max}$ and onto $\mathcal{C}_{\mathrm{ext}} = \mathcal{C}^0_{\mathrm{ext}}$ will be called the *Calderón projection* and the *extended Calderón projection* associated to the Dirac-Schrödinger system (d, V) and will be denoted by $C_{\max}$ and C_{ext}, respectively.

3.6. Theorem. *Let (d, V) be non-parabolic. Then there are constants $\Lambda_0, C \geq 0$ such that, for $\Lambda \geq \Lambda_0$,*

$$C_{\mathrm{ext}} = Q_> + R_\Lambda + S_\Lambda,$$

where R_Λ and S_Λ are smoothing, R_Λ has finite rank, and

$$||S_\Lambda||_s + ||S^*_\Lambda||_s \leq C\Lambda^{-1}$$

for all $|s| \le 1/2$. In particular, $C_{\rm ext}$ is $1/2$-smooth, $C_{\rm ext}-Q_>$ is compact in H^s for all $|s| \le 1/2$, and $C_{\rm ext}$ is elliptic. Furthermore, $D_{C_{\rm ext},{\rm ext}} : W_{C_{\rm ext}} \to L^2(\mathcal{H})$ is an isomorphism.

Proof. We use notation and results from Theorem 3.1. Since T_Λ maps $H^{-1/2}_{>\Lambda}$ to $H^{1/2}_{\le\Lambda}$, the dual operator of T_Λ maps $H^{-1/2}_{\le\Lambda}$ to $H^{1/2}_{>\Lambda}$. We recall that the dual operator of T_Λ is the extension of the adjoint T^*_Λ of $T_\Lambda|H_{>\Lambda}$. In particular, T^*_Λ is smoothing as well and, considered as a linear map from $H^{-1/2}_{\le\Lambda}$ to $H^{1/2}_{>\Lambda}$, it satisfies

$$||T^*_\Lambda|| = ||T_\Lambda|| \le ||T_{\Lambda_0}|| = C.$$

Arguing as in (3.4), we obtain that $T^*_\Lambda T_\Lambda : H^{-1/2}_{>\Lambda} \to H^{1/2}_{>\Lambda}$ satisfies

$$\begin{aligned} ||T^*_\Lambda T_\Lambda x||_s &\le \Lambda^{s-1/2}||T^*_\Lambda T_\Lambda x||_{1/2} \\ &\le C\Lambda^{s-1/2}||x||_{-1/2} \le C\Lambda^{-1}||x||_s, \end{aligned}$$

for all $|s| \le 1/2$ and $x \in H^s_{>\Lambda}$. Hence $||T^*_\Lambda T_\Lambda||_s \le C\Lambda^{-1}$ for all $|s| \le 1/2$. In particular, if I denotes the identity of $H_{>\Lambda}$, then $I+T^*_\Lambda T_\Lambda$ is invertible with $1/2$-smooth inverse as soon as $\Lambda > C$, and for $\Lambda \ge 2C$ we find

$$||(I + T^*_\Lambda T_\Lambda)^{-1}||_s \le 2.$$

Clearly,

$$(I + T^*_\Lambda T_\Lambda)^{-1} = I - T^*_\Lambda T_\Lambda (I + T^*_\Lambda T_\Lambda)^{-1} =: I + T^x_\Lambda,$$

where T^x_Λ is smoothing with $||T^x_\Lambda||_s \le 2C\Lambda^{-1}$ and the superscript x means that this object will not survive the end of the proof.

In accordance with with our convention $H = H^0$, we let $G_\Lambda = G^0_\Lambda$. Then G_Λ is the graph of the restriction of T_Λ to $H_{>\Lambda}$, for short also denoted by T_Λ. We recall that

$$G^\perp_\Lambda = \{(-T^*_\Lambda y, y) \,:\, y \in H_{\le\Lambda}\}.$$

Hence the orthogonal projection P_Λ onto G_Λ in H is given by

$$P_\Lambda = \begin{pmatrix} (I + T^*_\Lambda T_\Lambda)^{-1} & (I + T^*_\Lambda T_\Lambda)^{-1} T^*_\Lambda \\ T_\Lambda (I + T^*_\Lambda T_\Lambda)^{-1} & T_\Lambda (I + T^*_\Lambda T_\Lambda)^{-1} T^*_\Lambda \end{pmatrix},$$

where the operator matrix arises from the decomposition $H_{>\Lambda} \oplus H_{\le\Lambda}$ of H and I denotes the identity of $H_{>\Lambda}$ as above. We now get a representation

$$P_\Lambda = Q_{>\Lambda} + R^x_\Lambda + S_\Lambda$$

analogous to the asserted representation for C_{ext}, where

$$R^x_\Lambda = \begin{pmatrix} 0 & (I+T^x_\Lambda)(Q_{[-\Lambda,\Lambda]}T_\Lambda)^* \\ Q_{[-\Lambda,\Lambda]}T_\Lambda(I+T^*_\Lambda T_\Lambda)^{-1} & Q_{[-\Lambda,\Lambda]}T_\Lambda(I+T^*_\Lambda T_\Lambda)^{-1}T^*_\Lambda \end{pmatrix},$$

$$S_\Lambda = \begin{pmatrix} T^x_\Lambda & (I+T^x_\Lambda)(Q_{<-\Lambda}T_\Lambda)^* \\ Q_{<-\Lambda}T_\Lambda(I+T^*_\Lambda T_\Lambda)^{-1} & Q_{<-\Lambda}T_\Lambda(I+T^*_\Lambda T_\Lambda)^{-1}T^*_\Lambda \end{pmatrix}.$$

Obviously, R^x_Λ and S_Λ are smoothing, R^x_Λ has finite rank, and the operator norms of S_Λ satisfy the desired inequalities.

The orthogonal complement of G_Λ in $\mathcal{C}_{\mathrm{ext}}$ is $(I-P_\Lambda)(K_\Lambda) \subset H^{1/2}$ so that $C_{\mathrm{ext}} - P_\Lambda$ is smoothing of finite rank. This implies the asserted formula for C_{ext} with $R_\Lambda = R^x_\Lambda + C_{\mathrm{ext}} - P_\Lambda - Q_{(0,\Lambda]}$.

By Proposition 2.46, $D_{\mathrm{ext}} : W \to L^2(\mathcal{H})$ is surjective. By definition, the kernel of $D_{C_{\mathrm{ext}},\mathrm{ext}}$ is trivial. The theorem follows. □

3.7. Theorem. *Assume that (d,V) is non-parabolic. Then*

$$C_{\max} = C_{\mathrm{ext},\gamma} = \gamma^*(I - C_{\mathrm{ext}})\gamma.$$

In particular, $C_{\max}$ is elliptic, $C_{\max} - Q_>$ is compact in H^s for all $|s| \le 1/2$, and $\operatorname{ind} D_{C_{\max},\mathrm{ext}} = \operatorname{rk}(C_{\mathrm{ext}} - C_{\max})$.

Proof. Let $x \in \check{\mathcal{C}}_{\max}$ and $y \in \check{\mathcal{C}}_{\mathrm{ext}}$. Choose $\sigma \in \ker D_{\max}$ with $\sigma(0) = x$ and $\tau \in \ker D_{\mathrm{ext}}$ with $\tau(0) = y$. Let (τ_n) be a sequence in $\mathcal{L}_c(e^0)$ which converges to τ in W. Then $D\tau_n \to 0 = D_{\mathrm{ext}}\tau$ in $L^2(\mathcal{H})$ and $\tau_n(0) \to \tau(0)$ in $\check{H}$. By Theorem 2.30.5,

$$\omega(x,y) \leftarrow \omega(\sigma(0),\tau_n(0)) = (D_{\max}\sigma,\tau_n)_{L^2(\mathcal{H})} - (\sigma, D\tau_n)_{L^2(\mathcal{H})} \to 0.$$

We conclude that $\check{\mathcal{C}}_{\mathrm{ext}} \subset (\check{\mathcal{C}}_{\max})^a$ and hence that $\mathcal{C}_{\mathrm{ext}} \subset \gamma(\mathcal{C}_{\max})^\perp$.

Suppose now that $\mathcal{C}_{\mathrm{ext}}$ is not equal to $\gamma(\mathcal{C}_{\max})^\perp$. Then there is a vector z of norm 1 in $\gamma(\mathcal{C}_{\max})^\perp$ which is perpendicular to $\mathcal{C}_{\mathrm{ext}}$. Choose $y \in H^{1/2}$ with $||y - z||_H \le 1/2$ and set $x := (I - C_{\mathrm{ext}})y$. Then x is non-zero, $x \notin \gamma\mathcal{C}^{1/2}_{\max}$, and is perpendicular to $\mathcal{C}_{\mathrm{ext}}$. Furthermore, $x \in H^{1/2}$ since C_{ext} is 1/2-smooth. Let $P := C_{\mathrm{ext}} + R$, where R is the orthogonal projection onto $\mathbb{C}x$ in H. Then P is an elliptic orthogonal projection, by Lemma 1.88, since C_{ext} is elliptic and x is in $H^{1/2}$. By Corollary 2.50,

$$\begin{aligned} \operatorname{ind} D_{P,\mathrm{ext}} &= \dim(\ker P \cap \mathcal{C}^{1/2}_{\mathrm{ext}}) - \dim(\operatorname{im} P \cap \gamma\mathcal{C}^{1/2}_{\max}) \\ &= -\dim(\operatorname{im} P \cap \gamma\mathcal{C}^{1/2}_{\max}). \end{aligned}$$

Let $y \in \mathcal{C}_{\mathrm{ext}}$ and $\alpha \in \mathbb{C}$, and suppose that $y + \alpha x \in \gamma\mathcal{C}^{1/2}_{\max}$. Since $\mathcal{C}_{\mathrm{ext}}$ is perpendicular to $\gamma\mathcal{C}^{1/2}_{\max}$ and x, we get $y = 0$. This implies that $\alpha x \in \gamma\mathcal{C}^{1/2}_{\max}$ and hence that $\alpha = 0$, by the choice of x. Hence $\operatorname{im} P \cap \gamma\mathcal{C}^{1/2}_{\max} = 0$ and, therefore, $\operatorname{ind} D_{P,\mathrm{ext}} = 0$.

On the other hand, the inclusion $i_P : \mathcal{D}_{P,\mathrm{ext}} \to \mathcal{D}_{C_{\mathrm{ext}},\mathrm{ext}}$ is a Fredholm operator of index -1. Since $D_{P,\mathrm{ext}} = D_{C_{\mathrm{ext}},\mathrm{ext}} \circ i_P$ and $D_{C_{\mathrm{ext}},\mathrm{ext}}$ is an isomorphism, we get $\operatorname{ind} D_{P,\mathrm{ext}} = -1$, a contradiction. We conclude that $\mathcal{C}_{\mathrm{ext}} = \gamma(\mathcal{C}_{\max})^{\perp}$ and hence that $C_{\max} = C_{\mathrm{ext},\gamma}$.

Since C_{ext} is elliptic $C_{\max} = C_{\mathrm{ext},\gamma}$ is elliptic as well. Moreover,

$$C_{\max} - Q_{>} = \gamma^*(I - C_{\mathrm{ext}} - Q_{<})\gamma = \gamma^*(Q_{\geq} - C_{\mathrm{ext}})\gamma,$$

hence $C_{\max} - Q_{>}$ is compact, by Theorem 3.6.

Finally, since $C_{\max}$ is elliptic, $D_{C_{\max},\mathrm{ext}}$ is a Fredholm operator. Now $\operatorname{im} C_{\max} \subset \mathcal{C}_{\mathrm{ext}}$, hence

$$\begin{aligned}\operatorname{ind} D_{C_{\max},\mathrm{ext}} &= \dim(\ker C_{\max} \cap \mathcal{C}_{\mathrm{ext}}) - \dim(\operatorname{im} C_{\max} \cap \gamma\mathcal{C}^{1/2}_{\max}) \\ &= \dim(\ker C_{\max} \cap \mathcal{C}_{\mathrm{ext}}) = \operatorname{rk}(C_{\mathrm{ext}} - C_{\max}). \qquad \square\end{aligned}$$

3.8. Corollary. *If (d, V) is of Fredholm type, then $C_{\mathrm{ext}} = C_{\mathrm{ext},\gamma}$, that is,* $\ker \hat{C}_{\mathrm{ext}}$ *is an elliptic self-adjoint boundary condition.*

Proof. Since (d, V) is of Fredholm type, we have $W = \mathcal{D}_{\max}$ and hence $C_{\mathrm{ext}} = C_{\max}$. $\square$

3.9. Theorem. *Assume that (d, V) is non-parabolic. Then*

$$\mathcal{C}^{1/2}_{\max} = \operatorname{im} \hat{C}_{\max} = \operatorname{im} C_{\max} \cap H^{1/2}. \tag{1}$$

If B is an elliptic boundary condition and $B^{(s)}$ denotes the closure of B in H^s, where $|s| \leq 1/2$, then $(B^{(s)}, \mathcal{C}^s_{\mathrm{ext}})$ is a Fredholm pair in H^s with nullity and deficiency independent of s. More precisely, we have

$$\operatorname{null}(B^{(s)}, \mathcal{C}^s_{\mathrm{ext}}) = \dim(B \cap \mathcal{C}^{1/2}_{\mathrm{ext}}), \tag{2}$$

$$\operatorname{def}(B^{(s)}, \mathcal{C}^s_{\mathrm{ext}}) = \dim(B^a \cap \mathcal{C}^{1/2}_{\max}). \tag{3}$$

Proof. Clearly, $\mathcal{C}^{1/2}_{\max} \subset \operatorname{im} \hat{C}_{\max} \cap H^{1/2}$. If they are not equal, there is a vector $y \in \operatorname{im} \hat{C}_{\max} \setminus \mathcal{C}^{1/2}_{\max}$, and then $x = \gamma y \in H^{1/2}$ is non-zero, $x \notin \gamma\mathcal{C}^{1/2}_{\max}$, and is perpendicular to $\mathcal{C}_{\mathrm{ext}}$. Arguing as in the proof of Theorem 3.7, we arrive at a contradiction.

Let B be an elliptic boundary condition and $|s| \leq 1/2$. Choose Λ_0 according to Theorem 3.1 and let $\Lambda \geq \Lambda_0$. Write

$$B = \{x + y + by \,:\, x \in F, y \in U \cap H^{1/2}\}$$

as in Proposition 1.65, where $F \subset H^{1/2}_{>\Lambda}$ is of finite dimension, $U \subset H_{\leq\Lambda}$ is the orthogonal complement of a subspace $E \subset H^{1/2}_{\leq\Lambda}$ of finite dimension, and $b : U \to V = F^{\perp} \cap H_{>\Lambda}$ is 1/2-smooth. In particular,

$$B^{(s)} = \{x + y + by \,:\, x \in F, y \in U^{(s)}\},$$

where $U^{(s)}$ is the closure of $U \cap H^{1/2}$ in H^s and, simultaneously, the annihilator of E in $H^s_{\le\Lambda}$. By Theorem 3.1 we have, on the other hand,

$$\check{\mathcal{C}}_{\mathrm{ext}} = \{u + v + Tv \,:\, u \in K_\Lambda, v \in H^{-1/2}_{>\Lambda}\},$$

where $K_\Lambda \subset H^{1/2}_{\le\Lambda}$ and $T : H^{-1/2}_{>\Lambda} \to H^{1/2}_{\le\Lambda}$. In particular, $Q_{\le\Lambda} z \in H^{1/2}$ for any $z \in \check{\mathcal{C}}_{\mathrm{ext}}$. We conclude that $B^{(s)} \cap \mathcal{C}^s_{\mathrm{ext}}$ is contained in $H^{1/2}$ and hence that

$$B^{(s)} \cap \mathcal{C}^s_{\mathrm{ext}} = B \cap \mathcal{C}^{1/2}_{\mathrm{ext}}. \tag{3.10}$$

By the above characterization of $B^{(s)}$, $(B^{(s)}, H^s_{>\Lambda})$ is a Fredholm pair. Now $I - C_{\mathrm{ext}} - Q_{\le\Lambda}$ is compact, by Theorem 3.6, hence $(B^{(s)}, \mathcal{C}^s_{\mathrm{ext}})$ is a left-Fredholm pair, by Proposition A.13. By Theorem 3.7, we have

$$(\mathcal{C}^s_{\mathrm{ext}})^{\mathrm{pol}} = \gamma \operatorname{im} C^{-s}_{\max},$$

where the superscript 'pol' indicates the annihilator of a subset of H^s in H^{-s}. Using (A.6), we obtain

$$\begin{aligned}(B^{(s)} + \mathcal{C}^s_{\mathrm{ext}})^{\mathrm{pol}} &= (B^{(s)})^{\mathrm{pol}} \cap (\mathcal{C}^s_{\mathrm{ext}})^{\mathrm{pol}} \\ &= (B^{(s)})^{\mathrm{pol}} \cap \gamma \operatorname{im} C^{-s}_{\max} \\ &= \gamma\big(\gamma(B^{(s)})^{\mathrm{pol}} \cap \operatorname{im} C^{-s}_{\max}\big),\end{aligned}$$

We also have

$$\operatorname{im} C^{-s}_{\max} \subset \operatorname{im} C^{-s}_{\mathrm{ext}} = \mathcal{C}^{-s}_{\mathrm{ext}} \subset \check{\mathcal{C}}_{\mathrm{ext}} \subset \check{H}.$$

By the ellipticity of B,

$$\gamma(B^{(s)})^{\mathrm{pol}} \cap \check{H} \subset \gamma B^0 \cap \check{H} = B^a \subset H^{1/2},$$

where B^0 denotes the annihilator of B in $H^{-1/2}$. In conclusion,

$$(B^{(s)} + \mathcal{C}^s_{\mathrm{ext}})^{\mathrm{pol}} = \gamma(B^a \cap \operatorname{im} C^{-s}_{\max}) = \gamma(B^a \cap \mathcal{C}^{1/2}_{\max}). \tag{3.11}$$ □

3.2. **Some index formulas.** Theorem 3.9 and Corollary 2.50 have the following consequence:

3.12. Theorem. *If (d, V) is non-parabolic and B is elliptic, then*

$$\operatorname{ind}_{\mathrm{ext}} D_B = \operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind}(\bar{B}, \mathcal{C}_{\mathrm{ext}}),$$

where $\bar{B}$ denotes the closure of B in H. □

3.13. Theorem. *If (d, V) is non-parabolic and B is elliptic, then*

$$\operatorname{ind} D_{B,\mathrm{ext}} + \operatorname{ind} D_{B^a,\mathrm{ext}} = \dim(\mathcal{C}_{\mathrm{ext}} / \operatorname{im} C_{\max}),$$

where $\operatorname{im} C_{\max}$ *is the closure of* $\mathcal{C}_{\max}$ *in H.*

Proof. Since $C_{\text{ext}} - C_{\max}$ is compact in H and $\mathcal{C}_{\text{ext}} = \operatorname{im} C_{\text{ext}}$, we have

$$\operatorname{ind}(\bar{B}, \mathcal{C}_{\text{ext}}) = \operatorname{ind}(\bar{B}, \operatorname{im} C_{\max}) + \operatorname{ind}(\ker C_{\max}, \mathcal{C}_{\text{ext}}),$$

by Proposition A.13. Since $\mathcal{C}_{\text{ext}} = \gamma(\operatorname{im} C_{\max})^{\perp}$, we have, by Theorem IV.4.8 in [Ka],

$$\begin{aligned}\operatorname{ind}(\bar{B}, \operatorname{im} C_{\max}) &= -\operatorname{ind}(B^{\perp}, (\operatorname{im} C_{\max})^{\perp}) \\ &= -\operatorname{ind}(\gamma B^{\perp}, \operatorname{im} C_{\text{ext}}) = -\operatorname{ind} D_{B^a,\text{ext}}.\end{aligned}$$

Furthermore, $\operatorname{im} C_{\max} = \bar{\mathcal{C}}_{\max} \subset \mathcal{C}_{\text{ext}}$, hence

$$\operatorname{ind}(\ker C_{\max}, \mathcal{C}_{\text{ext}}) = \dim(\mathcal{C}_{\text{ext}} / \operatorname{im} C_{\max}). \qquad \square$$

Theorem 3.12 implies the following index formula of Agranovič-Dynin type, which corresponds to Theorem 23.1 in [BW].

3.14. Theorem. *If (d, V) is non-parabolic, B is elliptic, and $\Lambda \in \mathbb{R}$, then*

$$\operatorname{ind} D_{B,\text{ext}} = \operatorname{ind} D_{\leq\Lambda,\text{ext}} + \operatorname{ind}(\bar{B}, H_{>\Lambda}).$$

Proof. Since $C_{\text{ext}} - Q_{>\Lambda}$ is compact, we can apply Theorem 3.12 and Proposition A.13 and get

$$\operatorname{ind} D_{B,\text{ext}} = \operatorname{ind}(\bar{B}, \operatorname{im} C_{\text{ext}}) = \operatorname{ind}(\bar{B}, H_{>\Lambda}) + \operatorname{ind}(H_{\leq\Lambda}, \operatorname{im} C_{\text{ext}}) \qquad \square$$

Note that in the notation of Proposition 1.65,

$$\operatorname{ind}(\bar{B}, H_{>\Lambda}) = \dim F - \dim E. \tag{3.15}$$

In the corresponding form, the index formula in Theorem 3.14 was also observed in [BäB] (in the case of Dirac operators on smooth manifolds).

3.16. Corollary (Discontinuity formula). *If (d, V) is non-parabolic and $\Lambda \in \mathbb{R}$, then*

$$\operatorname{ind} D_{\leq\Lambda,\text{ext}} = \operatorname{ind} D_{<\Lambda,\text{ext}} + \dim H_{\Lambda}. \qquad \square$$

In one of its versions, the Cobordism Theorem for Dirac operators states that the index of the Dirac operator D^+ of a closed spin manifold M of even dimension vanishes if M bounds a compact spin manifold. As an application of our results, we derive a general form of this. Let (d, V) be a Dirac-Schrödinger system. Set

$$H^{\pm} := \{x \in H \,:\, i\gamma x = \pm x\}. \tag{3.17}$$

Since γ and A anti-commute,

$$B^{\pm} := H^{\pm} \cap \check{H} = H^{\pm} \cap H^{1/2}. \tag{3.18}$$

Since H^+ is the orthogonal complement of H^- in H, we conclude that B^+ and B^- are mutually adjoint elliptic boundary conditions.

3.19. Cobordism Theorem. *If the system (d, V) is of Fredholm type, then the restriction $A^+ : H_A^+ \to H^-$ of $A = A_0$ satisfies* $\operatorname{ind} A^+ = 0$.

Proof. Since (d, V) is of Fredholm type, $\ker C_{\text{ext}}$ is an elliptic self-adjoint boundary condition, by Corollary 3.8. Now Theorem 1.83 implies $\operatorname{ind} A^+ = 0$ (compare also Corollary 1.84). $\square$

We now consider Dirac-Schrödinger systems together with a boundary value problem which models the decomposition of a manifold M into two pieces M_1 and M_2 along a closed hypersurface $N = M_1 \cap M_2$. This requires the transmission boundary condition for sections of bundles over M and Dirac-Schrödinger operators acting on them; compare Example 1.85.

Let (d_1, V_1) and (d_2, V_2) be Dirac-Schrödinger systems with the same initial Hilbert space H at $t = 0$ (after some appropriate identification). Suppose that, at $t = 0$,

$$A_{1,0} = -A_{2,0} =: A \quad \text{and} \quad \gamma_{1,0} = -\gamma_{2,0} =: \gamma. \tag{3.20}$$

We consider the Dirac-Schrödinger system $(d, V) = (d_1, V_1) \oplus (d_2, V_2)$ with the boundary condition

$$B = \{(x, x) \,:\, x \in H^{1/2}\}, \tag{3.21}$$

where we use A to define $H^{1/2}$. We already observed in Example 1.85 that B is elliptic and self-adjoint. The Calderón space of d is the direct sum of the Calderón spaces of d_1 and d_2,

$$\check{\mathcal{C}}_{\text{ext}} = \check{\mathcal{C}}_{1,\text{ext}} \oplus \check{\mathcal{C}}_{2,\text{ext}} \quad \text{and} \quad \mathcal{C}_{\text{ext}} = \mathcal{C}_{1,\text{ext}} \oplus \mathcal{C}_{2,\text{ext}}. \tag{3.22}$$

We then arrive at the following index formula of Bojarski type.

3.23. Theorem. *If (d_1, V_1) and (d_2, V_2) are non-parabolic, then (d, V) is non-parabolic, $(\mathcal{C}_{1,\text{ext}}, \mathcal{C}_{2,\text{ext}})$ is a Fredholm pair in H, and*

$$\operatorname{ind} D_{B,\text{ext}} = \operatorname{ind}(\mathcal{C}_{1,\text{ext}}, \mathcal{C}_{2,\text{ext}}).$$

Proof. The first assertion is clear. By Theorem 3.9, $(C_{1,\text{ext}}, H_{\le})$ is a Fredholm pair, where we use spectral projections and spaces associated to A. By Theorem 3.6, $C_{2,\text{ext}} - Q_{\le}$ is a compact operator. Hence $(\mathcal{C}_{1,\text{ext}}, \mathcal{C}_{2,\text{ext}})$ is a Fredholm pair, by Proposition A.13. As for the index formula, we note that

$$\begin{aligned}
\bar{B} \cap \mathcal{C}_{\text{ext}} &= \{(x, x) \in H \oplus H \,:\, \ x \in \mathcal{C}_{1,\text{ext}} \text{ and } x \in \mathcal{C}_{2,\text{ext}}\} \\
&\cong \mathcal{C}_{1,\text{ext}} \cap \mathcal{C}_{2,\text{ext}} \\
B^{\perp} \cap \mathcal{C}_{\text{ext}}^{\perp} &= \{(x, -x) \in H \oplus H \,:\, \ x \perp \mathcal{C}_{1,\text{ext}} \text{ and } x \perp \mathcal{C}_{2,\text{ext}}\} \\
&\cong (\mathcal{C}_{1,\text{ext}} + \mathcal{C}_{2,\text{ext}})^{\perp}.
\end{aligned}$$

Therefore

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind}(\bar{B}, \mathcal{C}_{\mathrm{ext}}) = \operatorname{ind}(\mathcal{C}_{1,\mathrm{ext}}, \mathcal{C}_{2,\mathrm{ext}}). \qquad \square$$

Using Theorem 3.14 and Corollary 3.16, we get a splitting formula for the index, which generalizes Theorems 23.3 of [BW] and 4.3 of [BL1].

3.24. Theorem (Splitting formula). *If (d_1, V_1) and (d_2, V_2) are non-parabolic, B_1 is an elliptic boundary condition with respect to A, and B_2 is an elliptic boundary condition with respect to $-A$, then*

$$\begin{aligned} \operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind} D_{1,B_1,\mathrm{ext}} &+ \operatorname{ind} D_{2,B_2,\mathrm{ext}} \\ &- \operatorname{ind}(H_>, \bar{B}_1) - \operatorname{ind}(H_\leq, \bar{B}_2). \end{aligned}$$

In particular, if B_1 is any elliptic boundary condition with respect to A and $B_2 = B_1^\perp \cap H^{1/2}$, then

$$\operatorname{ind} D_{B,\mathrm{ext}} = \operatorname{ind} D_{1,B_1,\mathrm{ext}} + \operatorname{ind} D_{2,B_2,\mathrm{ext}}.$$

Proof. By Theorem 3.14 and Corollary 3.16,

$$\begin{aligned} \operatorname{ind} D_{B,\mathrm{ext}} &= \operatorname{ind} D_{1,\leq,\mathrm{ext}} + \operatorname{ind} D_{2,\geq,\mathrm{ext}} + \operatorname{ind}(\bar{B}, H_> \oplus H_<) \\ &= \operatorname{ind} D_{1,\leq,\mathrm{ext}} + \operatorname{ind} D_{2,>,\mathrm{ext}} \\ &= \operatorname{ind} D_{1,B_1,\mathrm{ext}} - \operatorname{ind}(H_>, \bar{B}_1) + \operatorname{ind} D_{2,B_2,\mathrm{ext}} - \operatorname{ind}(H_\leq, \bar{B}_2). \end{aligned}$$

If $B_2 = B_1^\perp \cap H^{1/2}$, then the second and last term on the right hand side cancel each other. $\square$

Besides modeling the case mentioned in the beginning of this section, the above results also apply to a Dirac-Schrödinger system d defined over the whole real line, decomposed into pieces $d_1 := d|\mathbb{R}_+$ and $d_2 := d|\mathbb{R}_-$, where we need to turn the latter into a Dirac-Schrödinger system over $\mathbb{R}_+$ in the appropriate and obvious way.

4. Supersymmetric systems

Our treatment so far does not allow to treat the usual index theorems since $D_{0,c}$ is symmetric. To adjust this we formulate a further axiom, introducing a supersymmetry, i.e. an involution which anticommutes with $D_{\max}$.

V. Axiom. There is a section

$$\alpha \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{L}(H_A)),$$

such that the following relations hold:

$$\alpha_t = \alpha_t^* = \alpha_t^{-1} \quad \text{on } H_t, \tag{1}$$

$$\alpha_t\gamma_t + \gamma_t\alpha_t = 0 \quad \text{on } H_A, \tag{2}$$

$$[\partial, \alpha] = 0 \quad \text{on } \mathrm{Lip}_{\mathrm{loc}}(\mathcal{H}), \tag{3}$$

$$[A_t, \alpha_t] = 0 \quad \text{on } H_A, \tag{4}$$

$$\alpha_t V_t + V_t\alpha_t = 0 \quad \text{on } H_t. \tag{5}$$

A *supersymmetric Dirac-Schrödinger system* is a Dirac-Schrödinger system (d, V) together with a supersymmetry α as in Axiom V.

Let (d, V, α) be a supersymmetric Dirac-Schrödinger system. Then we have, for each $t \geq 0$, an orthogonal decomposition

$$H_t = H_t^+ \oplus H_t^-, \quad H_t^\pm := \{x \in H \, : \, \alpha_t x = \pm x\}. \tag{4.1}$$

Since A_t commutes with α_t, we get an associated decomposition

$$H_A = H_{A,t}^+ \oplus H_{A,t}^-, \quad H_{A,t}^\pm := H_A \cap H_t^\pm, \tag{4.2}$$

which is orthogonal with respect to the graph norm of A_t and such that A_t maps $H_{A,t}^\pm$ to $H_t^\pm$. There are analogous decompositions of the associated Sobolev and function spaces. We also have

$$\alpha D + D\alpha = 0 \tag{4.3}$$

on $\mathcal{L}_{\mathrm{loc}}(e^0)$. It follows that D is an *odd operator*, that is, maps locally Lipschitz sections of $\mathcal{H}^\pm$ to locally essentially bounded measurable sections of $\mathcal{H}^\mp$. We let $D^\pm$ be the corresponding parts of D so that D is represented by the matrix

$$\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{4.4}$$

with respect to the above decomposition of $\mathcal{L}_{\mathrm{loc}}(e^0)$. We obtain orthogonal decompositions

$$\mathcal{D}_{\max} = \mathcal{D}_{\max}^+ \oplus \mathcal{D}_{\max}^- \quad \text{and} \quad W = W^+ \oplus W^-, \tag{4.5}$$

and $D_{\max}$ and D_{ext} are odd operators with respect to these with corresponding parts $D^{\pm}_{\max}$ and $D^{\pm}_{\text{ext}}$, respectively. Since $\ker D_{\max}$ and $\ker D_{\text{ext}}$ are α-invariant, we have

$$\begin{aligned} \ker D_{\max} &= \ker D^{+}_{\max} \oplus \ker D^{-}_{\max}, \\ \ker D_{\text{ext}} &= \ker D^{+}_{\text{ext}} \oplus \ker D^{-}_{\text{ext}}, \end{aligned} \tag{4.6}$$

respectively. Since $\mathcal{R}$ commutes with α_0, $\check{C}_{\max}$ and $\check{C}_{\text{ext}}$ are α_0-invariant and hence decompose accordingly,

$$\check{\mathcal{C}}_{\max} = \check{\mathcal{C}}^{+}_{\max} \oplus \check{\mathcal{C}}^{-}_{\max} \quad \text{and} \quad \check{\mathcal{C}}_{\text{ext}} = \check{\mathcal{C}}^{+}_{\text{ext}} \oplus \check{\mathcal{C}}^{-}_{\text{ext}}. \tag{4.7}$$

We are interested in boundary value problems that are compatible with the supersymmetry. That is, we require that boundary conditions B are α_0-invariant, and then we have a decomposition $B = B^{+} \oplus B^{-}$ as above. In other words, we pose the boundary conditions separately for the $+$ and $-$ parts of the elements in the corresponding domains and get corresponding domains and operators

$$D^{\pm}_{B^{\pm},\text{ext}} : W^{\pm}_{B^{\pm}} \to L^2(\mathcal{H}^{\mp}), \tag{4.8}$$

and similarly for D and $D_{\max}$.

4.9. Proposition. *Let (d, V) be a non-parabolic supersymmetric Dirac-Schrödinger system with supersymmetry α and B be an α_0-invariant elliptic boundary condition. Then*

$$\operatorname{ind} D_{B,\text{ext}} = \operatorname{ind} D^{+}_{B^{+},\text{ext}} + \operatorname{ind} D^{-}_{B^{-},\text{ext}}. \quad \square$$

If C is an α_0-invariant subspace of H, then $C^{\perp}$ and γC are invariant under α_0 as well and we have

$$(\gamma C^{\perp})^{\pm} = \gamma(C^{\perp,\mp}) = \gamma((C^{\mp})^{\perp} \cap H^{\mp}). \tag{4.10}$$

In particular, from Theorem 3.7,

$$(\operatorname{im} C_{\max})^{\mp} = (\gamma \mathcal{C}^{\perp}_{\text{ext}})^{\mp} = \gamma(\mathcal{C}^{\perp,\pm}_{\text{ext}}). \tag{4.11}$$

If P is a projection in H, then $\ker P$ and $\operatorname{im} P$ are invariant under α_0 if and only if $[P, \alpha_0] = 0$, and then P decomposes as

$$P = \frac{1}{2}(\alpha + I)P + \frac{1}{2}(\alpha - I)P =: P^{+} + P^{-}. \tag{4.12}$$

Clearly $[P_{\gamma}, \alpha_0] = 0$ if $[P, \alpha_0] = 0$, and then

$$P^{\pm}_{\gamma} = \gamma^{*}(I^{\mp} - P^{*,\mp})\gamma. \tag{4.13}$$

The following index formulas are immediate from Theorems 3.12, 3.13, and 3.14.

4.14. THEOREM. *Let (d,V) be a non-parabolic supersymmetric Dirac-Schrödinger system with supersymmetry α and B be an α_0-invariant elliptic boundary condition. Then*

$$\begin{aligned} \operatorname{ind} D^+_{B^+,\mathrm{ext}} &= \operatorname{ind}(\bar{B}^+, \mathcal{C}^+_{\mathrm{ext}}) \\ &= \operatorname{ind} D^+_{H^+_{\leq},\mathrm{ext}} + \operatorname{ind}(\bar{B}^+, H^+_{>}), \\ \operatorname{ind} D^+_{B^+,\mathrm{ext}} + \operatorname{ind} D^-_{B^{a,-},\mathrm{ext}} &= \dim(\mathcal{C}^+_{\mathrm{ext}} / \operatorname{im} C^+_{\max}) \\ &= \dim(\mathcal{C}^-_{\mathrm{ext}} / \operatorname{im} C^-_{\max}). \quad \square \end{aligned}$$

Recall the setup in Theorems 3.23 and 3.24. Let α_1 and α_2 be supersymmetries of the Dirac-Schrödinger systems (d_1, V_1) and (d_2, V_2), respectively, that agree at $t = 0$. Consider the Dirac-Schrödinger system $(d, V) = (d_1, V_1) \oplus (d_2, V_2)$ with the induced supersymmetry (α_1, α_2). The boundary condition B from (3.21) is (α_1, α_2)-invariant with

$$B^{\pm} = \{(x, x) \,:\, x \in H^{\pm}\} \cap H^{1/2}. \tag{4.15}$$

We also have

$$\check{\mathcal{C}}^{\pm}_{\mathrm{ext}} = \check{\mathcal{C}}^{\pm}_{1,\mathrm{ext}} \oplus \check{\mathcal{C}}^{\pm}_{2,\mathrm{ext}}. \tag{4.16}$$

Arguing as in the proofs of Theorems 3.23 and 3.24 we get the following index formulas.

4.17. THEOREM. *Assume that (d_1, V_1) and (d_2, V_2) are non-parabolic. Then*

$$\operatorname{ind} D^+_{B^+,\mathrm{ext}} = \operatorname{ind}(\mathcal{C}^+_{1,\mathrm{ext}}, \mathcal{C}^+_{2,\mathrm{ext}}).$$

If B_1 is any α_1-invariant elliptic boundary condition for d_1 and B_2 any α_2-invariant elliptic boundary condition for d_2, then

$$\begin{aligned} \operatorname{ind} D^+_{B^+,\mathrm{ext}} = \operatorname{ind} D^+_{1,B^+_1,\mathrm{ext}} &+ \operatorname{ind} D^+_{2,B^+_2,\mathrm{ext}} \\ &- \operatorname{ind}(H^+_{>}, \bar{B}^+_1) - \operatorname{ind}(H^+_{\leq}, \bar{B}^+_2). \end{aligned}$$

In particular, if B_1 is any α_1-invariant elliptic boundary condition for d_1 and $B_2 = B_1^{\perp} \cap H^{1/2}$, then

$$\operatorname{ind} D^+_{B^+,\mathrm{ext}} = \operatorname{ind} D^+_{1,B^+_1,\mathrm{ext}} + \operatorname{ind} D^+_{2,B^+_2,\mathrm{ext}}. \quad \square$$

5. Manifolds with boundary

In this last chapter, we explain how our results can be applied to obtain formulas for the index of Dirac type operators on manifolds with boundary. Such formulas are well known in the case of compact manifolds with smooth boundary and Dirac operators with smooth coefficients, see for instance [BW]. However, in applications one often faces the problem that the boundary of the manifold is not smooth or that the coefficients of the operator are not smooth. We will encounter such a situation in a forthcoming article on L^2-index formulas on manifolds with finite volume and pinched negative curvature in which we extend the results of [BB2]. Here we concentrate on a rather general case which sets the stage for the applications we have in mind, but should also be useful in other situations.

5.1. **The geometric setup.** Let M be a $C^{1,1}$ manifold with compact boundary $N = \partial M$ and with a Lipschitz continuous Riemannian metric. Let $E \to M$ be a $C^{0,1}$ Hermitian vector bundle and D be a differential operator on E of order one with L^∞_{loc} coefficients. Then we obtain a linear operator

$$D : \mathrm{Lip}_{\mathrm{loc}}(M,E) \to L^\infty_{\mathrm{loc}}(M,E). \tag{5.1}$$

Let $\mathrm{Lip}_{0,c}(M,E)$ be the space of Lipschitz sections of E with compact support in M, which vanish along the boundary N, and set $D_{0,c} := D|\,\mathrm{Lip}_{0,c}(M,E)$, considered as an unbounded operator on $L^2(M,E)$. We assume that $D_{0,c}$ is symmetric, that is,

$$(D\sigma_1, \sigma_2)_{L^2(M,E)} = (\sigma_1, D\sigma_2)_{L^2(M,E)} \tag{5.2}$$

for all $\sigma_1, \sigma_2 \in \mathrm{Lip}_{0,c}(M,E)$. We let $D_{\min}$ be the closure of $D_{0,c}$ and $D_{\max}$ be the adjoint of $D_{0,c}$ in $L^2(M,E)$. We denote by $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ the domains of $D_{\min}$ and $D_{\max}$, respectively.

VI. Axiom. There is a Lipschitz function $\rho : M \to \mathbb{R}_+$ and a constant $r > 0$ such that $N = \rho^{-1}(0)$ and $O := \rho^{-1}([0,r))$ is relatively compact in M. Moreover, there is a Dirac-Schrödinger system $(d,V) = (\mathcal{H}, \partial, A, \gamma, V)$ with Lipschitz coefficients, and a unitary isomorphism $U : L^2(O,E) \to L^2(\mathcal{H}|[0,r))$ such that

(1) $U((\varphi \circ \rho)\sigma) = \varphi U\sigma$ for all $\sigma \in L^2(O,E)$ and $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$.
(2) $(1 - \varphi \circ \rho)\sigma \in \mathcal{D}_{\min}$ for all $\sigma \in \mathcal{D}_{\max}$ and $\varphi \in \mathrm{Lip}(\mathbb{R}_+)$ with compact support in $[0,r)$ and equal to one close to zero.
(3) $U(\mathrm{Lip}_c(O,E))$ is contained and dense in $\mathcal{L}_c(\mathcal{H}|[0,r))$ with respect to the graph norm of D^d.

(4) $U(\mathrm{Lip}_{0,c}(O,E))$ is contained and dense in $\mathcal{L}_{0,c}(\mathcal{H}|[0,r))$ with respect to the graph norm of D^d.
(5) $D^d(U\sigma) = U(D\sigma)$ for all $\sigma \in \mathrm{Lip}_c(O,E)$.

Here $\mathcal{L}_c(\mathcal{H}|[0,r))$ denotes the space of sections in $\mathcal{L}_c(\mathcal{H})$ with compact support in $[0,r)$. We also use a superscript d to distinguish quantities belonging to (d,V) if necessary.

5.3. Remark. Axiom VI is tailored to fit the geometric examples which we will discuss in our next article, notably the case of Dirac-Schrödinger operators over the ends of complete Riemannian manifolds with finite volume and pinched negative sectional curvature, see [BB2].

For $\sigma \in \mathrm{Lip}_{\mathrm{loc}}(M,E)$, let $\mathcal{R}\sigma := \mathcal{R}^d(U((\varphi\circ\rho)\sigma))$, where $\mathcal{R}^d$ denotes the restriction map of d and $\varphi \in \mathrm{Lip}(\mathbb{R}_+)$ has compact support in $[0,r)$ and is equal to one close to zero. By Axiom VI.1 above, $\mathcal{R}\sigma$ does not depend on the choice of φ. As before, we also write $\sigma(0) = \mathcal{R}\sigma$. Using (2.14), (5.2), and Axiom VI.5 we get

$$(D\sigma_1,\sigma_2) - (\sigma_1, D\sigma_2) = \omega^d(\sigma_1(0),\sigma_2(0)) =: \omega(\sigma_1(0),\sigma_2(0)), \tag{5.4}$$

for all $\sigma_1,\sigma_2 \in \mathrm{Lip}(M,E)$ with compact support.

5.5. Lemma. *Suppose $\sigma \in L^2(M,E)$ has compact support in O. Then $\sigma \in \mathcal{D}_{\max}$ if and only if $U\sigma \in \mathcal{D}^d_{\max}$, and then $D^d_{\max}(U\sigma) = UD_{\max}\sigma$.*

Proof. We need only to test against Lipschitz sections of E with compact support in O and vanishing along N respectively Lipschitz sections of $\mathcal{H}$ with compact support in $[0,r)$ and vanishing at 0. To these, (4) and (5) of Axiom VI apply, and the lemma follows. □

Using Axiom VI and Lemma 5.5, we get the following characterization of the maximal domain $\mathcal{D}_{\max}$.

5.6. Corollary. *For any $\varphi \in \mathrm{Lip}(\mathbb{R}_+)$ with compact support in $[0,r)$ and equal to one close to zero and any $\sigma \in L^2(M,E)$,*

$$\sigma \in \mathcal{D}_{\max} \Longleftrightarrow \varphi U\sigma \in \mathcal{D}^d_{\max} \text{ and } (1-(\varphi\circ\rho))\sigma \in \mathcal{D}_{\min}. \quad \square$$

5.7. Proposition (Regularity). *The maximal domain $\mathcal{D}_{\max}$ satisfies:*
(1) *$\mathrm{Lip}_c(M,E)$ is dense in $\mathcal{D}_{\max}$.*
(2) *The restriction map on $\mathrm{Lip}_c(M,E)$ extends to a continuous surjective map $\mathcal{R} : \mathcal{D}_{\max} \to \check{H}$.*
(3) *For $\sigma_1,\sigma_2 \in \mathcal{D}_{\max}$ we have*

$$(D_{\max}\sigma_1,\sigma_2) - (\sigma_1, D_{\max}\sigma_2) = \omega(\sigma_1(0),\sigma_2(0)).$$

Proof. Apply Proposition 2.30, Corollary 5.6, and Axiom VI. □

For a boundary condition $B \subset \check{H}$, we set

$$\mathcal{D}_{B,\max} := \{\sigma \in \mathcal{D}_{\max} : \mathcal{R}\sigma \in B\}, \tag{5.8}$$
$$D_{B,\max} := D_{\max}|\mathcal{D}_{B,\max}.$$

Then $D_{B,\max}$ is closed with adjoint $D_{B^a,\max}$, see Section 1.4.

5.2. **Fredholm properties.** We now discuss Fredholm properties of and index formulas for the operators D_B. As in the case of Dirac-Schrödinger systems, we need the non-parabolicity condition of the third named author:

VII. Axiom. We say that D is *non-parabolic* if for any compact subset $K \subset M$ there is a positive constant C_K such that any $\sigma \in \mathcal{D}_{\max}$ satisfies

$$\|\sigma\|_{L^2(K)} \leq C_K \left(\|\mathcal{R}\sigma\|_{\check{H}} + \|D_{\max}\sigma\|_{L^2(M,E)}\right).$$

Assume from now on that D is non-parabolic. Let W be the completion of $\mathcal{D}_{\max}$ with respect to the norm appearing on the right hand side of the equation in Axiom VII. There is the following analogue of Lemma 2.39.

5.9. Lemma. *If D is non-parabolic, then we have:*

(1) *The restriction map $\mathcal{R}$ and $D_{\max}$ extend to continuous maps*
$$\mathcal{R}_{\text{ext}} : W \to \check{H} \quad \textit{and} \quad D_{\text{ext}} : W \to L^2(M, E),$$
respectively; $\mathcal{R}_{\text{ext}}$ induces an isometry from $\ker D_{\text{ext}}$ into $\check{H}$.
(2) *If $\psi \in \operatorname{Lip}_c(M)$ and $\sigma \in W$, then $\psi\sigma \in \mathcal{D}_{\max} \subset W$. Moreover, there is a constant C_ψ such that*
$$||\psi\sigma||_{D_{\max}} \leq C_\psi ||\sigma||_W.$$
In particular, W can be viewed as a space of locally integrable functions and $W \cap L^2(M, E) = \mathcal{D}_{\max}$.
(3) *$W = \mathcal{D}_{\max}$ if and only if there is a constant C such that*
$$||\sigma||_{L^2(\mathcal{H})} \leq C||\sigma||_W \quad \textit{for all } \sigma \in \operatorname{Lip}_c(M, E).$$

Proof. (1) and (3) are clear. As for (2), use Lemma 5.5 and argue as in the proof of (2) of Lemma 2.39. □

Similarly, there is an analogue of Lemma 2.41:

5.10. Lemma. *Let V be a bounded subset of W. Then V is precompact if and only if $D_{\text{ext}}(V) \subset L^2(M, E)$ and $Q_{\geq}\mathcal{R}_{\text{ext}}(V) \subset \check{H}$ are both precompact.*

Proof. It is easy to adapt the arguments in the proof of Lemma 2.41 to the present situation. □

For a boundary condition $B \subset \check{H}$, set

$$W_B := \{\sigma \in W \, : \, \sigma(0) \in B\} \quad \text{and} \quad D_{B,\mathrm{ext}} := D_{\mathrm{ext}}|W_B. \tag{5.11}$$

We arrive at the following generalization of Theorem 2.43, Corollary 2.44, and Proposition 2.46

5.12. THEOREM. *Assume that D is non-parabolic and that B is regular. Then $D_{B,\mathrm{ext}} : W_B \to L^2(\mathcal{H})$ is a left-Fredholm operator with* $(\operatorname{im} D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}$ *and extended index*

$$\operatorname{ind}_{\mathrm{ext}} D_B := \operatorname{ind} D_{B,\mathrm{ext}} = \dim\ker D_{B,\mathrm{ext}} - \dim\ker D_{B^a,\max}.$$

If B is elliptic, then the kernels of D_B and D_{B^a} have finite dimension and the L^2-index *of D_B is well defined,*

$$L^2\text{-}\operatorname{ind} D_B := \dim\ker D_B - \dim\ker D_{B^a}.$$

Moreover, there is $\Lambda_0 \geq 0$ such that $D_{<-\Lambda,\mathrm{ext}}$ is injective and $D_{\leq\Lambda,\mathrm{ext}}$ is surjective for all $\Lambda \geq \Lambda_0$. □

We define Calderón spaces and projections as in the case of Dirac-Schrödinger systems, see (2.47), (2.48), and Definition 3.5. If B is a regular boundary condition, then $\mathcal{R}$ induces isomorphisms

$$\begin{aligned} \ker D_{B,\max} &\cong B \cap \check{\mathcal{C}}_{\max} = B \cap \mathcal{C}^{1/2}_{\max}, \\ \ker D_{B,\mathrm{ext}} &\cong B \cap \check{\mathcal{C}}_{\mathrm{ext}} = B \cap \mathcal{C}^{1/2}_{\mathrm{ext}}. \end{aligned} \tag{5.13}$$

As before, we write $\mathcal{C}_{\max}$ and $\mathcal{C}_{\mathrm{ext}}$ instead of $\mathcal{C}^0_{\max}$ and $\mathcal{C}^0_{\mathrm{ext}}$, respectively. We have the following analogue of Corollary 2.50:

5.14. COROLLARY. *Assume that D is non-parabolic and that B is elliptic. Then $D_{B,\mathrm{ext}}$ is a Fredholm operator with* $(\operatorname{im} D_{B,\mathrm{ext}})^\perp = \ker D_{B^a,\max}$ *and index*

$$\begin{aligned} \operatorname{ind} D_{B,\mathrm{ext}} &= \dim B \cap \mathcal{C}^{1/2}_{\mathrm{ext}} - \dim B^\perp \cap \gamma\mathcal{C}^{1/2}_{\max}. \\ &= \dim B \cap \mathcal{C}_{\mathrm{ext}} - \dim B^\perp \cap \gamma\mathcal{C}_{\max}. \quad \square \end{aligned}$$

It is a routine matter to check that the arguments developed in Chapter 3 also work under Axioms VI and VII imposed here; hence all the results obtained there have their analogues here. We arrive at the following version of Theorems 3.6, 3.7, and 3.9.

5.15. THEOREM. *Assume that D is non-parabolic. Then:*

(1) *The Calderón projections C_{ext} and $C_{\max}$ are elliptic with $C_{\max} = C_{\mathrm{ext},\gamma}$.*
(2) *$C_{\max} - Q_>$ and $C_{\mathrm{ext}} - Q_>$ are compact in H^s for all $|s| \leq 1/2$.*

(3) *If B is an elliptic boundary condition, then $(\bar{B}, \mathcal{C}_{\text{ext}})$ is a Fredholm pair in H and*

$$\bar{B} \cap \check{\mathcal{C}}_{\text{ext}} = B \cap \mathcal{C}_{\text{ext}}^{1/2} \quad \textit{and} \quad (\bar{B} + \mathcal{C}_{\text{ext}})^{\perp} = B^{\perp} \cap \gamma C_{\max}^{1/2}. \quad \square$$

With the same arguments as in Chapter 3, we get the analogues of the index formulas in Theorems 3.12, 3.13, and 3.14:

5.16. Theorem. *Assume that D is non-parabolic and that B is an elliptic boundary condition. Then*

$$\begin{aligned} \operatorname{ind} D_{B,\text{ext}} &= \operatorname{ind}(\bar{B}, \mathcal{C}_{\text{ext}}) \\ &= \operatorname{ind} D_{H_{\le},\text{ext}} + \operatorname{ind}(H_{>}, \bar{B}), \\ \operatorname{ind} D_{B,\text{ext}} + \operatorname{ind} D_{B^a,\text{ext}} &= \dim(\mathcal{C}_{\text{ext}} / \operatorname{im} C_{\max}). \quad \square \end{aligned}$$

5.17. Remark. The further results from Chapter 3 and Chapter 4 are consequences of the results on the Calderón projections and the index formulas from Theorems 3.12, 3.13, and 3.14. Therefore they have their exact analogs here, and we refrain from repeating the corresponding statements.

Appendix A. Fredholm pairs

T. Kato has developed the notion of *Fredholm pairs of closed subspaces*, cf. [Ka, Ch.IV, Section 4]. Consider a Banach space E and a pair of closed subspaces F and G. Introduce *nullity* and *deficiency* of the pair (F, G),

$$\mathrm{null}(F, G) := \dim(F \cap G), \tag{A.1}$$

$$\mathrm{def}(F, G) := \mathrm{codim}(F + G), \tag{A.2}$$

and recall that $\mathrm{def}\,(F, G) < \infty$ implies that $F + G$ is closed. We say that the pair (F, G) is a *left-* or *right-Fredholm pair*, respectively, if

$$F + G \text{ is closed} \tag{A.3}$$

and

$$\mathrm{null}\,(F, G) < \infty \quad \text{or} \quad \mathrm{def}\,(F, G) < \infty, \tag{A.4}$$

respectively. We say that (F, G) is a *semi-Fredholm pair* if it is a left- or right-Fredholm pair, and that it is a *Fredholm pair* if it is a left- and right-Fredholm pair. For any semi-Fredholm pair (F, G), its *index*,

$$\mathrm{ind}(F, G) := \mathrm{null}\,(F, G) - \mathrm{def}\,(F, G), \tag{A.5}$$

is well defined as an extended real number. The index of (F, G) is a rough measure of the non-complementarity of F and G in E.

Let E' be the dual space of E and $F^0, G^0 \subset E'$ be the annihilators (or polar sets) of F and G, respectively. By [Ka, Theorem IV.4.8], $F^0 + G^0$ is closed if and only if $F + G$ is closed,

$$(F \cap G)^0 = F^0 + G^0, \quad (F + G)^0 = F^0 \cap G^0, \tag{A.6}$$

$$\mathrm{null}\,(F^0, G^0) = \mathrm{def}\,(F, G), \quad \mathrm{def}\,(F^0, G^0) = \mathrm{null}\,(F, G). \tag{A.7}$$

For Banach spaces E_1, E_2 and an operator $T \in \mathcal{L}(E_1, E_2)$, we recover the Fredholm properties of T by considering

$$E = E_1 \times E_2, \quad F = E_1 \times \{0\}, \quad G = \mathrm{graph}\, T. \tag{A.8}$$

To that end we note that $F + G$ is closed in E if and only if $\mathrm{im}\, T$ is closed in E_2 and that the canonical inclusions $E_1 \to E$ and $E_2 \to E$ induce isomorphisms

$$\ker T \cong F \cap G \quad \text{and} \quad \mathrm{coker}\, T \cong E/(F + G). \tag{A.9}$$

In particular, if T is semi-Fredholm, then the index of T is

$$\mathrm{ind}\, T = \dim \ker T - \dim \mathrm{coker}\, T = \mathrm{ind}(F, G), \tag{A.10}$$

where F and G are as above. Next we quote a criterion for left-Fredholmness of T which is used several times in this work; for a proof, see for example [Hö, Proposition 19.1.3].

A.11. Lemma. *The following conditions are equivalent:*

(1) $T \in \mathcal{L}(E_1, E_2)$ *is a left-Fredholm operator.*
(2) *If* (x_n) *is a bounded sequence in* E_1 *with* (Tx_n) *convergent in* E_2, *then* (x_n) *possesses a convergent subsequence.* □

Traditionally, the results on Fredholm pairs we have mentioned are applied to subspaces with topological complements, i,e. to pairs of spaces of the form $F = \operatorname{im} P$, $G = \operatorname{im} Q$, where P, Q are projections (continuous idempotents) in E. We need the more general case of a pair formed by a closed subspace and the image of a projection.

A.12. Proposition. *Let* B *be a closed subspace and* P *be a projection in* E. *Then*

$$(I - P)(B) = \ker P \cap (B + \operatorname{im} P),$$

and $(I - P)(B)$ *is closed in* E *if and only if* $B + \operatorname{im} P$ *is closed in* E. *Furthermore, the codimension of* $(I - P)(B)$ *in* $\ker P$ *is equal to the codimension of* $B + \operatorname{im} P$ *in* E. *In particular,* $(I - P) : B \to \ker P$ *is a left-Fredholm operator if and only if* $(B, \operatorname{im} P)$ *is a left-Fredholm pair, and then*

$$\operatorname{ind}((I - P) : B \to \ker P) = \operatorname{ind}(B, \operatorname{im} P).$$

Proof. Let $x \in \ker P$ and suppose that $x = y + Pz$ for some $y \in B$. Then $x = (I-P)x = (I-P)y \in (I-P)(B)$. Conversely, if $x = (I-P)y$ for some $y \in B$, then $x = y - Py \in B + \operatorname{im} P$. This shows the first assertion.

If $B+\operatorname{im} P$ is closed in E, then also $(I-P)(B) = \ker P \cap (B+\operatorname{im} P)$. Vice versa, suppose that $(I - P)(B)$ is closed and let $(x_n = y_n + z_n)$ be a sequence in $B + \operatorname{im} P$ converging to $x \in E$. Then

$$(I - P)y_n = (I - P)x_n \to (I - P)x,$$

hence there is a $y \in B$ with $(I-P)y = (I-P)x$, by assumption. Hence

$$x = (I - P)y + Px = y + P(x - y) \in B + \operatorname{im} P.$$

It follows that $(I - P)(B)$ is closed if and only if $B + \operatorname{im} P$ is closed.

The natural linear map $\ker P \to E/(B + \operatorname{im} P)$ is surjective with kernel $\ker P \cap (B + \operatorname{im} P) = (I - P)(B)$. Hence the codimension of $(I-P)(B)$ in $\ker P$ is equal to the codimension of $B+\operatorname{im} P$ in E. The remaining assertions follow. □

A.13. Proposition (Stability). *Let* P, Q *be projections in* E *such that* $P - Q$ *is compact. Then* $(\operatorname{im} P, \ker Q)$ *is a Fredholm pair.*

If B *is a closed subspace of* E, *then* $(B, \operatorname{im} P)$ *is a left-Fredholm pair if and only if* $(B, \operatorname{im} Q)$ *is a left-Fredholm pair, and then*

$$\operatorname{ind}(B, \operatorname{im} P) = \operatorname{ind}(B, \operatorname{im} Q) + \operatorname{ind}(\ker Q, \operatorname{im} P).$$

Proof. It is immediate from Lemma A.11 that $(I-P) : \ker Q \to E$ is a left-Fredholm operator. Applying Proposition A.12 to $B = \ker Q$ we get that $\operatorname{im} P + \ker Q$ is closed in E. The annihilator of $\operatorname{im} P + \ker Q$ in the dual space E' is $\ker P' \cap \operatorname{im} Q'$. Now $P' - Q'$ is compact, hence $\ker P' \cap \operatorname{im} Q'$ is of finite dimension. It follows that $(\ker Q, \operatorname{im} P)$ is a Fredholm pair. We also have

$$(I-P)(I-Q) = (I-P) - (I-P)Q = (I-P) + C,$$

where $C = (P-Q)Q$ is compact. Now Proposition A.12 applies. □

Appendix B. An inequality

In the proof of Lemma 1.28, we need a special case of the Sobolev inequality, cf. Theorem 3.9 in [Ag]. For the sake of completeness, we give a very simple proof here.

Let σ be a complex valued Lipschitz function on some interval $I \subset \mathbb{R}$. Then

$$(|\sigma|^2)' = 2\operatorname{Re}(\sigma'\bar{\sigma}). \tag{B.1}$$

Hence, if $I = [s,\infty)$ and σ has compact support, then

$$a|\sigma(s)|^2 \le ||\sigma'||^2_{L^2([s,\infty))} + a^2||\sigma||^2_{L^2([s,\infty))}, \tag{B.2}$$

for any constant $a > 0$. A corresponding estimate holds for bounded intervals: Let $s < t$ and σ be a complex valued Lipschitz function on $[s,t]$. Then, for any constant $a > 0$,

$$a|\sigma(s) - \sigma(t)|^2 \le 2||\sigma'||^2_{L^2([s,t])} + 2a^2||\sigma||^2_{L^2([s,t])}. \tag{B.3}$$

Proof of (B.3). By shifting $[s,t]$ if necessary we can assume $s = -t$. Since even functions are perpendicular to odd functions in $H^1([-t,t])$, we can assume that σ is odd. Then the left hand side of the inequality is equal to $4a|\sigma(t)|^2$. Using (B.1) and $\sigma(0) = 0$, we derive the asserted estimate. □

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