On the Density of Ratios of Chern Numbers of Algebraic Surfaces

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Recently Hirzebruch [3] gave a construction (see §1), that associates an algebraic surface $H(\Lambda,n)$ to each set $\Lambda = \{L_1,\ldots,L_k\}$ of $k \geq 3$ distinct lines in $\mathbb{P}_2(C)$ and each $n \geq 2$. These surfaces are of particular interest because they are often minimal models of general type with $c_1^2/e > 2$. This paper answers a question asked in [3] about the density of ratios of Chern numbers, studies the distribution of $\lim_{n\to\infty} c_1^2(H(\Lambda.n))/e(H(\Lambda,n))$ as Λ varies, and characterizes completely those $H(\Lambda,n)$ with ample cotangent bundle.

Let me describe this paper in detail.

In §1, I quickly review Hirzebruch's construction of the surfaces $H(\Lambda,n)$ and prove a few simple results about them that are needed in this paper.

In §2, I solve the problem raised in [3;(3.3)].

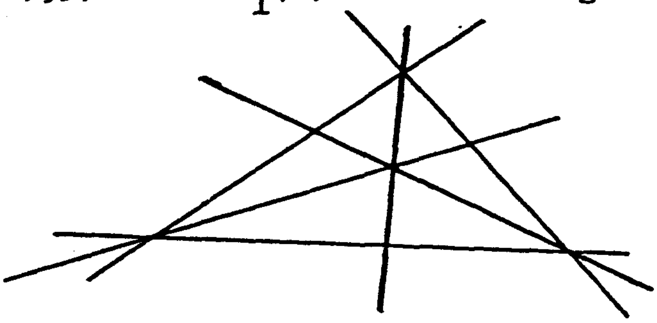
(2.1) Theorem. Every point $r \in [\frac{1}{5}, 3]$ occurs as a limit:

$$r = \lim_{n\to\infty} \frac{c_1^2\,(S_n)}{e\,(S_n)}$$

<u>for a sequence S_n of minimal surfaces of general type</u>.

Previously Persson [7] had shown the above theorem for all $r \in [\frac{1}{5}, 2]$. As F. Hirzebruch pointed out to me, an appropriate set of the $H(A,n)$ shows the theorem for $r \in [2, \frac{5}{2}]$. I refer to [3] for a fuller discussion of previously known examples with $c_1^2 > 2e$.

The method that I use is very simple minded. I take $H = H(A_1(6),5)$ where $A_1(6)$ is the arrangement:



H fibres with connected fibres over a curve R of genus 6. It is a result of Hirzebruch [3] that $c_1^2(H) = 3e(H)$. By taking appropriate branched covers C of R, the ratios:

$$c_1^2(H x_R C)/e(H x_R C)$$

are shown to be dense in [2,3]. The fact that the fibre products are minimal models of general type is trivial. A similar construction using a double cover of the rational Hirzebruch surface F_{12}, gives ratios of minimal models of general type surfaces that are dense in $[\frac{1}{5}, 2]$.

In §3, I discuss some elementary aspects of ampleness.

In §4 I characterize those $H(\Lambda,n)$ with ample cotangent bundle in the sense of Grothendieck-Hartshorne [2], i.e. those $H(\Lambda,n)$ with Grauert negative tangent bundle.

Let Λ and n be as in the first paragraph of this paper. Let t_j be the number of points of $\mathbb{P}_2(\mathbb{C})$ that lie on exactly j elements of Λ and let r_p for $p \in \mathbb{P}_2(\mathbb{C})$ be the exact number of $L_i \in \Lambda$, that contain p.

Theorem. Let Λ and n be as in the first paragraph of this paper. $T^*_{H(\Lambda,n)}$ is ample in the sense of Grothendieck-Hartshorne [2] if and only if:

a) given any line $L_i \in \Lambda$, the set $\{p \in L_i | r_p \geq 3\}$ has cardinality at least 2,

b) if $n = 3$ then $t_3 = 0$, and if $n = 2$ then $t_3 = t_4 = 0$.

Under the above conditions $H(\Lambda,n)$ immerses into a product of curves of genus ≥ 2.

In §5 I study the asymptotic distribution of these ratios, i.e. I study the characteristic number [3,4], γ, of the arrangement:

$$\gamma = \lim_{n\to\infty} \frac{c_1^2(H(\Lambda,n))}{e(H(\Lambda,n))}$$

(5.1) Theorem. Let Λ be an arrangement of k lines on $\mathbb{P}_2(\mathbb{C})$, that satisfies $t_k = t_{k-1} = 0$. Then the characteristic number, γ, of Λ satisfies:

$$\gamma \geq 2\left(\frac{k-3}{k-2}\right)$$

with equality if and only if $t_r = 0$ for $r \geq 3$.

(5.2) Corollary. Assume that Λ is an arrangement such that $t_k = t_{k-1} = 0$ and $t_r \neq 0$ for some $r \geq 3$. Then $\gamma \geq \frac{3}{2}$ with equality only if $k = 5$, $t_2 = 7$, and $t_3 = 1$. Excluding this case, $\gamma \geq \frac{8}{5}$ with equality only if $k = 6$, $t_2 = 12$, and $t_3 = 1$.

Finally we find the maximum of all γ; this answers a question first studied by Iitaka [4].

(5.3) Theorem. Let Λ be an arrangement with $t_k = t_{k-1} = 0$. Then $\gamma \leq \frac{8}{3}$. If $\gamma = \frac{8}{3}$ then $k = 9$, $t_3 = 12$ and $t_r = 0$ for $r \neq 3$; this case occurs [3,(3.3)].

I would like to express my thanks to F. Hirzebruch for explaining his examples to me and for the questions that he asked me. I would also like to thank U. Persson for going over my construction in §2 and suggesting an improvement. Finally I would like to thank the Max Planck Institut für Mathematik/Sonderforschungsbereich "Theoretische Mathematik" and the National Science Foundation for their support.

§1. Background Material

We follow the notation of [3], to which we refer for motivation and more details. An arrangement A of k lines in $P_2(\mathbb{C})$ is a set of k distincts lines given by linear forms $\ell_1,\ldots,\ell_k$. Let $L_1,\ldots, L_k$ denote the corresponding lines on $P_2(\mathbb{C})$ corresponding to $\ell_1,\ldots,\ell_k$ respectively. By t_r for $r \geq 2$ we denote the number of distinct points in $P_2(\mathbb{C})$ that lie on exactly r lines of the arrangement A. By r_p we denote the exact number of lines of A that contain a given point $p \in P_2(\mathbb{C})$; in case confusion can arise this is denoted $r_p(A)$. We define $f_0 = \sum_{r=2}^{k} t_r$ and $f_1 = \sum_{r=2}^{k} r\, t_r$. We always assume that $k \geq 3$ and $t_k = 0$. The formula:

$$k(k-1) = \sum_{r\geq 2} t_r r(r-1) \tag{1.1}$$

will be useful.

For any $n \geq 2$, and the above arrangement A, consider the function field:

$$\mathbb{C}(z_1/z_0,\ z_2/z_0)((\ell_2/\ell_1)^{\frac{1}{n}},\ldots,(\ell_k/\ell_1)^{\frac{1}{n}}),$$

where z_0,z_1,z_2 are homogenous coordinates on $P_2(\mathbb{C})$. This field determines a normal algebraic surface, X, that is a branched cover, $\pi\colon X \to P_2(\mathbb{C})$, of $P_2(\mathbb{C})$ of degree n^{k-1} with

Λ as the ramification locus. Let $H(\Lambda,n)$ denote the minimal desingularization of X with map ρ: $H(\Lambda,n) \to X$.

The Chern numbers of $H(\Lambda,n)$ are given by:

$$(1.2)\quad \begin{cases} c_1^2(H(\Lambda,n)) = \\ n^{k-3}[n^2(-5k+9+3f_1-4f_0)+4n(k+f_0-f_1)+f_1-f_0+k+t_2] \\ e(H(\Lambda,n)) = \\ n^{k-3}[n^2(3-2k+f_1-f_0)+2n(k+f_0-f_1)+f_1-t_2] \end{cases}$$

Let τ: $\hat{\mathbb{P}}_2 \to \mathbb{P}_2(\mathbb{C})$ denote $\mathbb{P}_2(\mathbb{C})$ blown up at all points where $r_p \geq 3$. There is a map σ: $H(\Lambda,n) \to \hat{\mathbb{P}}_2$ such that:

$$(1.3)\quad \begin{array}{ccc} H(\Lambda,n) & \xrightarrow{\rho} & X \\ \sigma\downarrow & & \downarrow\pi \\ \hat{\mathbb{P}}_2 & \xrightarrow{\tau} & \mathbb{P}_2(\mathbb{C}) \end{array}$$

is a fibre product; in particular σ is an n^{k-1} sheeted finite to one cover. Let π_p: $\hat{\mathbb{P}}_2 \to \mathbb{P}_1(\mathbb{C})$ denote the holomorphic map on $\hat{\mathbb{P}}_2$ gotten by composing τ with the projection to $\mathbb{P}_1(\mathbb{C})$ from p; this $\mathbb{P}_1(\mathbb{C})$ is canonically $(T_p-p)/\mathbb{C}^*$ where T_p is the tangent space of $\mathbb{P}_2(\mathbb{C})$ at p. Associated to the composition $\pi_p \circ \sigma$: $H(\Lambda,n) \to P_1(C)$ we have the Remmert-Stein factorization

$S_p \bullet R_p$, i.e. R_p: $H(\Lambda,n) \to C$ is a holomorphic surjection onto a connected normal variety with connected fibres and S_p: C $\to \mathbb{P}_1(\mathbb{C})$ is a finite to one map. Since $\mathbb{P}_1(\mathbb{C})$ is one dimensional, so also is C. Thus C is a smooth curve. The ramification locus, B, of S_p: $C \to \mathbb{P}_1(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C}) \approx (T_p - p)/\mathbb{C}^*$ is the set of r_p points that correspond to the r_p lines of Λ containing p. Using this it is easy to check that any connected component of the set $\rho^{-1}(\pi^{-1}(p))$, where ρ and π are as in (1.3), is a <u>section</u> of R_p: $H(\Lambda,n) \to C$. Therefore by the Hurwitz formula [3,(2.1)]:

$$e(C) = n^{r_p - 1}(2 - r_p) + r_p\, n^{r_p - 2} \tag{1.4}$$

where as always $e(\ )$ denotes the Euler characteristic. <u>At this point the reader only interested in §2 has all the background material needed</u>.

We need some more information about R_p. It is convenient to introduce a few concepts.

(1.5) <u>Definition</u>. An arrangement Λ is said to be <u>simple</u> if:

a) given any two points p and q that satisfy $r_p \geq 3$, $r_q \geq 3$, there is a sequence $L_{i_1}, \ldots, L_{i_t}$ of elements of Λ such that $p \in L_{i_1}$, $q \in L_{i_t}$, and such that for $j = 1, \ldots, t-1$, L_{i_j} meets $L_{i_{j+1}}$ in a point z with $r_z \geq 3$,

b) the set of points $p \in \mathbb{P}_2(\mathbb{C})$ with $r_p \geq 3$ is not collinear.

(1.5.1) <u>Lemma</u>. <u>Let Λ be a simple arrangement</u>. <u>Then the map</u> R: $H(\Lambda,n) \to \prod_{\{p, r_p \geq 3\}} C_p$ <u>given by the maps</u> $\{R_p \mid r_p \geq 3\}$ <u>is an embedding</u>.

<u>Proof</u>. First note that the map $\pi\colon \hat{\mathbb{P}}_2 \to \prod_{\{p,r_p \geq 3\}} \mathbb{P}_1(\mathbb{C})$ given by the maps $\{\pi_p | r_p \geq 3\}$ is an embedding if and only if the set of points $p \in P_2(C)$ with $r_p \geq 3$ does not lie on a line.

Consider the commutative diagram:

$$\begin{array}{ccc} H(\Lambda,n) & \xrightarrow{R} & \prod_p C_p \\ \downarrow{\sigma} & & \downarrow{S} \\ \hat{\mathbb{P}}_2 & \xrightarrow{\pi} & \prod_p \mathbb{P}_1(\mathbb{C}) \end{array}$$

where S is the product mapping associated to $\{S_p | r_p \geq 3\}$. By construction of the R_p and S_p, the Galois group, G, of automorphisms of $H(\Lambda,n)$ over $\mathbb{P}_2$ maps homomorphically into the Galois groups of automorphisms $\tilde{G}$, of $\prod_p C_p$ over $\prod_p \mathbb{P}_1(\mathbb{C})$. Since σ is a Galois cover, i.e. since G acts transitively on the fibres of σ, the proof of (1.5.1) will be done if we show that G injects into $\tilde{G}$. To show this, it suffices to show that function field of $\prod_p C_p$ pulls back to that of $H(\Lambda,n)$.

Let M_p denote the function field of C_p. Let M be the subfield of the function field F of $H(\Lambda,n)$ generated by $\{R_p^* M_p | r_p \geq 3\}$ and $C({}^{z_1}/_{z_0}, {}^{z_2}/_{z_0})$. We will be done if we show that $M = F$.

Note that:

$M = \mathbb{C}({}^{Z_1}/_{Z_0}, {}^{Z_2}/_{Z_0})(\{({}^{\ell_i}/_{\ell_j})^{\frac{1}{n}} \mid L_i$ meets L_j in a single point p with $r_p \geq 3\})$. The hypothesis of simplicity guarantees that $({}^{\ell_i}/_{\ell_j})^{\frac{1}{n}} \in M$ for all i and j. Thus $M = F$.

□

(1.6) Definition. An arrangement Λ is a sum of the arrangements $\Lambda_1,\ldots,\Lambda_m$ if:

a) Λ is the disjoint union of the Λ_ν for $\nu \leq m$,

b) if two lines $\{L_i,L_j\} \subseteq \Lambda$ meet in a point z with $r_z \geq 3$ then $\{L_i,L_j\} \subseteq \Lambda_\nu$ for some ν.

Assume that each Λ_i satisfies the conditions of the first paragraph of this chapter. Then from (1.3) we have the diagrams:

$$\begin{array}{ccc} H(\Lambda_i,n) & \xrightarrow{\rho_i} & X_i \\ \downarrow \sigma_i & \tau_i & \downarrow \pi_i \\ \hat{\mathbb{P}}_{2,i} & \longrightarrow & \mathbb{P}_2(\mathbb{C}) \end{array}$$

and, of course

$$\begin{array}{ccc} H(\Lambda,n) & \xrightarrow{\rho} & X \\ \downarrow \sigma & & \downarrow \pi \\ \hat{\mathbb{P}}_2 & \xrightarrow{\tau} & \mathbb{P}_2(\mathbb{C}) \end{array}$$

Since the sets $\{p \mid r_p(\Lambda_i) \geq 3\}$ are disjoint, where $r_p(\Lambda_i)$ is the number of elements of Λ_i that contain p, it follows that

$\hat{\mathbb{P}}_2 \to \mathbb{P}_2(\mathbb{C})$ is the fibre product of $\{\tau_i:\hat{\mathbb{P}}_{2,i} \to \mathbb{P}_2(\mathbb{C}) | i = 1,\ldots,m\}$. Let $H_i(n)$ denote the fibre product of $\sigma_i\colon H(\Lambda_i,n) \to \hat{\mathbb{P}}_{2,i}$ and the projection $\hat{\mathbb{P}}_2 \to \hat{\mathbb{P}}_{2,i}$. Let $H(n)$ denote the fibre product of all the maps $H_i(n) \to \hat{\mathbb{P}}_2$. Since $\sigma\colon H(\Lambda,n) \to \hat{\mathbb{P}}_2$ factors through $H_i(n)$ for each $i = 1,\ldots,n$, we have a natural map $\mathcal{E}_n\colon H(\Lambda,n) \to H(n)$.

This map is onto. To see this let $\{\ell_{j,i} | j = 1,\ldots,k_i\}$ be the forms defining Λ_i for each i. Note that the function field of $H_i(n)$ as an extension of that of $\hat{\mathbb{P}}_2$ is given by:

$$\mathbb{C}({}^{z_1}/_{z_o}, {}^{z_2}/_{z_0})(({}^{\ell_{2,i}}/_{\ell_{1,i}})^{\frac{1}{n}},\ldots,({}^{\ell_{k_i,i}}/_{\ell_{1,i}})^{\frac{1}{n}})$$

Tensoring these algebras together over $\mathbb{C}({}^{z_1}/_{z_0}, {}^{z_2}/_{z_0})$ we get an algebra M over $\mathbb{C}({}^{z_1}/_{z_0}, {}^{z_2}/_{z_0})$. Since adjoining $({}^{\ell_{1,i}}/_{\ell_{1,j}})^{\frac{1}{n}}$ to M for all i and j gives the function field F, of $H(\Lambda,n)$, it follows that M is a sub-field of F.

(1.6.1) Lemma. $\mathcal{E}_n$ is an unramified cover.

Proof. Let Λ_i' denote the union of the proper transforms L_j' under τ of $L_j \in \Lambda_i$ and the E_p with $r_p(\Lambda_i) \geq 3$. Note that Λ_i' and Λ_j' have no components in common if $i \neq j$. Further $\tau^{-1}(\Lambda) = \bigcup_i \Lambda_i'$

Since $H(\Lambda,n) \to \hat{\mathbb{P}}_2$ is a Galois covering and since the branching with Λ_i' as locus of ramification takes place in $H_i(n) \to \hat{\mathbb{P}}_2$, we conclude E_n: $H(\Lambda,n) \to H(n)$ is unramified.

□

(1.6.2) Theorem. Assume that Λ is a sum of simple arrangements. Then the map R: $H(\Lambda,n) \to \prod_{r_p \geq 3} C_p$ given by the maps $\{R_p | r_p \geq 3\}$ is an immersion.

Proof. Λ is a sum of simple arrangements Λ_i, $i = 1,\ldots,n$. Apply (1.5.1) and (1.6.1).

□

(1.7) Lemma. Let $p \in \mathbb{P}_2(\mathbb{C})$ be a point where $r_p \geq 3$. Let C equal the set $\rho^{-1}(\pi^{-1}(p))$ where ρ and π are as in (1.3). Then $T_{H(\Lambda,n),C} \cong T_C \oplus N_C$ where N_C is the normal bundle of C.

Proof. Since C is smooth we have the short exact sequence:

$$0 \to T_C \to T_{H(\Lambda,n),C} \to N_C \to 0$$

As we saw in showing (1.4), C is section of R_p: $H(\Lambda,n) \to C_p$. Thus dR_p: $T_{H(\Lambda,n),C} \to T_{C_p}$ is a surjection and splits the above sequence.

□

Given any line $L \subseteq \mathbb{P}_2(\mathbb{C})$ let $\gamma(L)$ denote the number of $p \in L$ such that $r_p \geq 3$. Let $\delta(L)$ equal the cardinality of the set $L \cap \{\bigcup_{L \in \Lambda} L\}$ counted without multiplicity if $L \in \Lambda$ and let

$\delta(L)$ equal the number of $p \in L$ with $r_p \geq 2$ if $L \subseteq \Lambda$.

Consider the map $\sigma\colon H(\Lambda,n) \to \hat{\mathbb{P}}_2$; here we are using the notation of (1.3). Let L' be the proper transform of any line $L \subseteq \mathbb{P}_2(\mathbb{C})$ under τ. Let C be any irreducible component of $\sigma^{-1}(L')$.

Letting $\nu(L) = 1$ or 2 depending on whether $L \notin \Lambda$ or $L \in \Lambda$, it follows from a local check and the Hurwitz formula:

$$(1.8)\quad \begin{cases} C\cdot C = [1-\gamma(L)]\cdot n^{\delta(L)-\nu(L)} \\ e(C) = \delta(L)n^{\delta(\ell)-2} - (\delta(L)-2)n^{\delta(L)-1} \end{cases}$$

To carry out the local check and in particular to see that C is smooth, note that given any singular point of $\tau^{-1}(\Lambda) \subseteq \hat{\mathbb{P}}_2$, we can choose:

a) a neighborhood V of x with coordinates z_1, z_2 satisfying $z_1(x) = z_2(x) = 0$ and $\Lambda \cap V = \{z_1 = 0\} \cup \{z_2 = 0\}$

b) a neighborhood U of $\sigma^{-1}(x)$ with coordinates a_1, a_2 satisfying $a_1(\sigma^{-1}(x)) = 0 = a_2(\sigma^{-1}(x))$ and $\sigma\colon U \to V$ given by $z_1 = a_1^n$, $z_2 = a_2^n$.

The analogue of (1.7) is very important for us.

(1.9) Lemma. Given $L \in \Lambda$ let C be as above. Then $T_{H(\Lambda,n),C} \approx T_C \oplus N_C$.

Proof. There is the exact sequence:

$$0 \to T_C \to T_{H(\Lambda,n),C} \to N_C \to 0$$

Let $y \in C$. By using the above coordinates it is immediate that there is a neighborhood U of y and a manifold V such that $\sigma: U \to \hat{P}_2$ factors $\sigma = B \circ A$ with $A: U \to V$ and $B: V \to \hat{P}_2$ where there are coordinates z_1, z_2 on V and a_1, a_2 on U so that:

a) $C \cap U = \{a_1 = 0\}$

b) $z_1 = a_1^n$, $z_2 = a_2$

Note that the annihilator of:

$$\begin{cases} \sigma^* dz_1 = n a_1^{n-1}\, da_1 \\ \sigma^* dz_2 = da_2 \end{cases}$$

the pullbacks of one forms on V, contain a unique subbundle of $T_{H(\Lambda,n),C \cap U}$ that surjects onto the normal bundle $N_{C \cap U}$.

By uniqueness we get the desired splitting of the above exact sequence.

§2. Branched Covers and the interval $[\frac{1}{5}, 3]$.

The following question is posed in [3, (3.3)] to which I refer for a detailed history of previous work.

Question. What is the set R of points (necessarily $\subseteq [\frac{1}{5}, 3]$) that are limits:

$$\lim_{n\to\infty} c_1^2(S_n)/e(S_n)$$

where the S_n are a sequence of surfaces which are minimal models of general type and all but finitely many of the $c_1^2(S_n)/e(S_n)$ are different?

Work of U. Persson [7] shows that $[\frac{1}{5}, 2] \subseteq R$ and the results of Hirzebruch [3] show that $[2, 2.5] \subseteq R$. In this section I will give a simple uniform proof that $R = [\frac{1}{5}, 3]$.

First there are a few results about branched covers. The following lemma is straightforward.

(2.1) Lemma. Let $r: S \to C$ be a holomorphic surjection with connected fibres of a smooth algebraic surface S satisfying $e(S) > 0$ onto a smooth connected curve, C. Let F be a fibre of r in a neighborhood of which r is of maximal rank. Let $C' \to C$ be a finite branched cover of a smooth, connected curve C' over C. Assume that the ramification locus of

$C' \to C$ is disjoint from the image under r of the set where r is not of maximal rank. Let S' be the fibre product $S \times_C C'$. Then S' is a minimal model if S is and

$$c_1^2(S^1)/e(S') = \frac{dc_1^2(S) - 2\rho e(F)}{de(S) - \rho e(F)}$$

where d and ρ are the sheet number and ramification number respectively of $C' \to C$.

From here on S will be some fixed surface. For ease of notation e will denote $e(S)$ and c_1^2 will denote $c_1^2(S)$.

We are interested in the way that the ratios vary in the above lemma as the branched covers $C' \to C$ vary.

(2.2) Lemma. Assume that S, C, and r are as in lemma (2.1). Assume that the genus of F in lemma (2.1) is > 1 and that the genus g of C is ≥ 1. The closure of the set of ratios, $c_1^2(S')/e(S')$ obtained by considering all branched covers $C' \to C$ as in lemma (2.1), is the interval with endpoints 2 and c_1^2/e.

Proof. For definiteness we will do the case of $c_1^2/e \geq 2$; the case $c_1^2/e \leq 2$ is handled in the same manner.

$$c_1^2(S')/e(S') = c_1^2/e + (2 - c_1^2/e)\left(\frac{-\rho e(F)}{de - \rho e(F)}\right) \geq c_1^2/e + 2 - c_1^2/e = 2$$

and:

$$c_1^2(S')/e(S') = 2 + \frac{d(c_1^2 - 2e)}{d - \rho\, e(F)} \leq 2 + (c_1^2/e - 2) = c_1^2/e$$

This shows that $R \subseteq [2,\ c_1^2/e]$.

We must now show that $[2,\ c_1^2/e] \subseteq R$. To see this first choose an x sheeted unramified cover $a: C' \to C$; with C' connected this can be done for any $x \geq 1$ since $g \geq 1$. Choose a 2 sheeted branched cover $b: C'' \to C'$ with C'' connected and having 2y branch points; this can be done for any $y \geq 1$.

We have:

$$c_1^2(S'')/e(S'') = \frac{2x\, c_1^2 - 4ye(F)}{2xe - 2ye(F)} = \frac{x\, c_1^2 - 2ye(F)}{xe - ye(F)}$$

where $S'' = (S x_C C') x_{C'} C''$.

This shows that the ratios:

$$\{c_1^2/e + (2 - c_1^2/e)\frac{-ye(F)}{xe - y\, e(F)} \mid x \geq 1,\ y \geq 1\} \subseteq R$$

It suffices to show that:

$$\{ \frac{-ye(F)}{xe - ye(F)} \mid x \geq 1,\ y \geq 1 \}$$

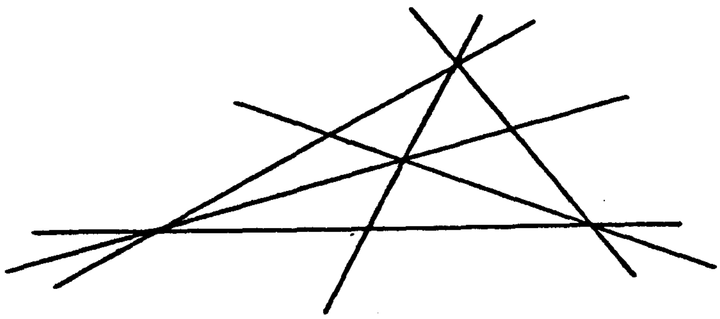
is dense in [0,1]. To see this choose any fraction p/q where p and q are integers satisfying $0 < p < q$. Then choose $x = -(q - p)e(F)$ and $y = pe$. Note then:

$$\frac{-p\cdot e\cdot e(F)}{(q - p)e(F)e - p\cdot e\cdot e(F)} = \frac{p}{q}$$

□

(2.3) Theorem. The closure, R, of the set of ratios $c_1^2/e(S)$ of the Chern numbers of the minimal models of general type surfaces is $[\frac{1}{5}, 3]$.

Proof. Let A be the arrangement of six lines:



Hirzebruch [3] showed using his formulae (1.2) that $c_1^2(H(A,5)) = 3e(H(A,5))$. By (1.4) the holomorphic surjection $R_p: H(A,5) \to C_p$ associated to any p with $r_p = 3$ has a genus 6 curve, C_p, as image. Recall that R_p has connected fibres. Since $H(A,5)$ is of general type, the genus of a generic fibre of R_p is positive (in fact 76). This by the above lemma shows that $[2,3] \subseteq R$.

Let F_{12} be the 12 th rational Hirzebruch surface, i.e. the unique holomorphic $P_1(\mathbb{C})$ bundle over $P_1(\mathbb{C})$ with a smooth irreducible holomorphic curve E of self-intersection -12. Let f be any fibre of the natural projection $F_{12} \to P_1(\mathbb{C})$. Then

$[E]^6 \otimes [f]^{76} = L$ is a very ample line bundle, e.g. [2]. Choose B, a smooth zero set of a general section of L. Let $\pi: X \to F_{12}$ be the cyclic cover of F_{12} branched over B, that is associated to the square root $\mathcal{L} = [E]^3 \otimes [f]^{38}$ of L. X is of general type and a minimal model since $K_X = \pi^*(K_{F_{12}} \otimes \mathcal{L}) = \pi^*([E] \otimes [f]^{24})$ is ample. Note that:

$$c_1^2(X) = 72$$

$$e(X) = 396$$

Let $C \to \mathbb{P}_1(\mathbb{C})$ be a double cover branched at 4 general points. Let $X' \to C$ be the fibre product of $C \to \mathbb{P}_1(\mathbb{C})$ and $X \to \mathbb{P}_1(\mathbb{C})$ where the last map is the composition of $X \to F_{12}$ and the tautological projection of $F_{12} \to \mathbb{P}_1(\mathbb{C})$. By lemma (2.1)

$$c_1^2(X') = 144 + 8 \cdot 2 = 160$$

$$e(X') = 792 + 4 \cdot 2 = 800$$

Here I have used that a general fibre of $X \to \mathbb{P}_1(\mathbb{C})$ has genus 2; this is why the $[E]^6$ was chosen in L. Note that:

$$5c_1^2(X') = e(X').$$

Use lemma (2.2).

□

(2.4) Remark. If U is a quasi-projective smooth connected surface, the logarithmic Chern numbers $\bar{c}_1^2$ and $\bar{e}$ can be defined [4,8]. F. Sakai [8] showed the analogue of the Miyaoka inequality [6], $\bar{c}_1^2 \le 3\bar{e}$. The number, $\delta(U)$, of connected components of $\bar{U} - U$, where $\bar{U}$ is any connected projective manifold containing U as a Zariski open set, is independent of $\bar{U}$. It is natural to ask about the density of $\bar{c}_1^2(U)/\bar{e}(U)$ as U ranges over quasi-projective surfaces with $\delta(U)$ fixed.

Note that in the examples we used we could increase the magnitude of c_1^2 and e, while keeping the ratio c_1^2/e unchanged, by going to a cover of the base curve. Further note that the smooth fibres of the maps from our examples to the base curves always had the same Euler characteristics, i.e. -150 and -2 respectively. Further if we chose U by pulling out δ smooth fibres from a given example, X, with e(F) as Euler characteristic of a smooth fibre we get:

$$\bar{c}_1^2(U) = c_1^2(X) - 2\delta(U)\, e(F)$$

$$\bar{e}(U) = e(X) - \delta(U)\, e(F)$$

As an immediate corollary of (2.3) and the considerations of this last paragraph, we obtain the following corollary of (2.3).

(2.4.1) Corollary. Let $\mathcal{U}_t$ be the set of all smooth connected quasi-projective surfaces U such that:

a) $\delta(U) = t$

b) _given any projective surface_ $\bar{U}$ _containing_ U _as a Zariski open set and letting_ $D = \bar{U} - U$, _it follows that_ $(K_{\bar{U}} \otimes [D])^N$ _has at least one non-trivial section for some_ $N > 0$.

Then the set of limits of the numbers $\bar{c}_1^2(U)/\bar{e}(U)$, _where_ $U \subseteq \mathcal{U}_t$, _contains_ $[\frac{1}{5}, 3]$.

§3. Ampleness

In this section we discuss ampleness in the sense of Grothendieck-Hartshorne [1,2]. We follow the now standard convention of not notationally distinguishing between a holomorphic vector bundle and its locally free sheaf of germs of holomorphic sections.

Let E be a holomorphic vector bundle on a projective variety X. Let $\mathbb{P}(E)$ denote $(E^* - X)/\mathbb{C}^*$ and let $\pi_E: \mathbb{P}(E) \to X$ denote the natural projection. There is a tautological line bundle ξ_E on $\mathbb{P}(E)$ with the properties:

(3.11) $\pi_{E*}(\xi_E) \approx E$ where π_{E*} denotes the direct image functor.

(3.1.2) $\xi_{E,F} \approx \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(1)$ for any fibre F of π_E.

(3.2) Definition. A vector bundle E is ample if there exists an N > 0 and an embedding $\phi: P(E) \to \mathbb{P}(\mathbb{C})$ such that $\phi^*\mathcal{O}_{P(C)}(1) \approx \xi_E^N$.

Some basic facts [1,2] we will use over and over are:

(3.2.1) If E is ample and $g: \mathcal{D} \to X$ is a finite to one map from a projective variety $\mathcal{D}$, then g^*E is ample,

(3.2.2) If E is ample then given any irreducible curve $\mathcal{D} \subseteq X$, and any line bundle L on $\mathcal{D}$ such that $E_{\mathcal{D}} \to L \to 0$, it follows that deg $L > 0$,

(3.2.3) If E is spanned by global sections then E is ample if there exists no irreducible curve $D \subseteq X$ with a trivial line bundle quotient:

$$E_D \to O_D \to 0.$$

(3.3) Lemma. Let $R: X \to \prod_{c=1}^{m} C_i$ be an immersion of a smooth connected projective surface X into a product of curves of genus ≥ 2. Let $D \subseteq$ be an irreducible curve such that there is a trivial sub-bundle:

$$*) \quad 0 \to O_D \to T_{X,D}.$$

If there is a factor map $R_i: X \to C_i$ such that dR_i is non-trivial on O_D, then $R_i(D)$ is a point.

Proof. If $R_i(D)$ is not a point, then $R_{i,D}$ is finite to one. Since C_i has genus ≥ 2, $T^*_{C_i}$ is ample and therefore by (3.2.1) $R^*_{i,D}\, T^*_{C_i}$ is ample. Thus composing with the dual of *) we get what must be the trivial map $R^*_{i,D}\, T^*_{C_i} \to O_D$. This contradiction establishes the lemma.

□

§4. Characterization of $H(\Lambda,n)$ with Ample Cotangent Bundles

(4.1) Theorem. Let Λ be an arrangement of $k \geq 3$ lines. Then $T^*_{H(\Lambda,n)}$ is ample in the sense of Grothendieck-Hartshorne [2] if any only if:

(4.1.1) if $L \in \Lambda$, then the set $\{p \in L | r_p \geq 3\}$ has cardinality at least 2, and

(4.1.2) if $n = 3$ then $t_3 = 0$ and if $n = 2$ then $t_3 = t_4 = 0$.

Under the above conditions $H(\Lambda,n)$ immerses a product of curves of genus ≥ 2.

Proof. Let H denote $H(\Lambda,n)$. Assume that T^*_H is ample.

To see that (4.1.1) must hold, assume otherwise. Let $L \in \Lambda$ be such that there is at most one point $p \in L$ with $r_p \geq 3$. Letting C be as in (1.8) and (1.9) we see that:

$$T^*_{H,C} \approx T^*_C \oplus N^*_C$$

$$\text{degree } N^*_C = - C\cdot C \leq 0$$

By (3.2.2), T^*_H is not ample.

To see that (4.1.2) must hold, let $p \in \mathbb{P}_2(\mathbb{C})$ with $r_p \geq 3$. As we saw earlier before (1.4), there is a smooth connected curve C on H with:

$$*)\quad e(C) = n^{r_p - 1}(2 - r_p) + r_p\, n^{r_p - 2}$$

If (4.1.2) failed there would by *) be a curve C on H with degree $T_C^* \leq 0$. Since T_C^* is a quotient of $T_{H,C}^*$, it would follow from (3.2.2) that T_H^* is not ample.

Before we prove the converse we need a lemma.

(4.1.3) Lemma. Let Λ be an arrangement of $k \geq 3$ lines. Assume that it is a sum of simple arrangements [see(1.5),(1.6)]. Then the following are equivalent:

a) $T_{H(\Lambda,n)}^*$ is spanned by global sections,

b) $H(\Lambda,n)$ immerses into a product of curves of positive genus,

c) if $n = 2$, then $t_3 = 0$.

Proof. By (1.6.2) there is an immersion:

$$R\colon H(\Lambda,n) \to \prod_{\{p|r_p \geq 3\}} C_p .$$

To see that c) ⇒ b) it suffices to show that $e(C_p) \leq 0$ for all p with $r_p \geq 3$. By (1.4) this follows from c). That b) ⇒ a) is clear. To see that a) ⇒ c) note that $T_{H(\Lambda,n)}^*$ being spanned implies all smooth $C \subseteq H(\Lambda,n)$ have $e(C) \leq 0$. Using the fact that each factor map $R_p\colon H(\Lambda,n) \to C_p$ has a section, we can use (1.4) to conclude c).

□

Assume (4.1.1) and (4.1.2) hold. It is a simple check that Λ is a sum of simple arrangements. Thus by the above lemma and (4.1.2) we have an immersion:

$$R: H \to \Pi C_p$$

where $e(C_p) \leq 0$. Using the full strength of (4.1.2) and the same argument as the lemma we see that $e(C_p) < 0$ for all p with $r_p \geq 3$.

If T_H^* is not ample then there exists an irreducible curve $\mathcal{D} \subseteq H$ and a quotient map:

$$*) \qquad T_{H,\mathcal{D}}^* \to \mathcal{O}_{\mathcal{D}} \to 0.$$

$\mathcal{D} \subseteq \sigma^{-1}(\tau^{-1}(\Lambda))$ in the notation of (1.3). To see this assume otherwise. Note that by (1.7) and (1.9):

$$T_{H,\mathcal{D}}^* \approx T_{\mathcal{D}}^* \oplus N_{\mathcal{D}}^*$$

By (4.1.2) combined with (1.8) and [3, (2.1)], this implies that $T_{H,\mathcal{D}}^*$ is a direct sum of ample line bundles and thus can't have a trivial quotient line bundle.

Since $\sigma(\mathcal{D}) \subseteq \tau^{-1}(\Lambda)$ we can use the condition (4.1.1) to see that there are two distinct $p,q \in \mathbb{P}_2(\mathbb{C})-\mathcal{D}$. Since $\sigma(\mathcal{D}) \subseteq \tau^{-1}(\Lambda)$, it follows that σ is étale in a neighborhood of a general point neighborhood of a general point of $\mathcal{D}$. Since $\pi_p: \hat{\mathbb{P}}_2 \to \mathbb{P}_1(\mathbb{C})$ and $\pi_q: \hat{\mathbb{P}}_2 \to \mathbb{P}_1(\mathbb{C})$ give an embedding $(\pi_p,\pi_q): \hat{\mathbb{P}}_2 - \tau^{-1}(\Lambda) \to \mathbb{P}_1(\mathbb{C}) x \mathbb{P}_1($

and since neither π_p or π_q collapse σ $(\mathcal{D})$ to a point, we can conclude that:

1) Neither $R_p: H \to C_p$ nor $R_q: H \to C_q$ collapse $\mathcal{D}$ to a point,

2) $(R_p, R_q): H - \sigma^{-1}(\Lambda) \to C_p \times C_q$ is an immersion.

This implies we have a factor map which contradicts (3.3).

$\square$

(4.2) Question. Is $H(\Lambda,n)$ a $K(\pi,1)$ if $T^*_{H(\Lambda,n)}$ is ample and Λ is simple?

(4.3) Question. What are the irregularities of $H(\Lambda,n)$ with ample cotangent bundle?

Note that in general there are more than the holomorphic one forms coming from surjections $R_p: H(\Lambda,n) \to C_p$ with $r_p \geq 3$. To see this note that Ishida [5] has shown for $n = 5$ and Gläser has shown for $n \geq 6$ that $H(\Lambda,n)$ where Λ is as in (2.3) has $5\,\frac{(n-1)(n-2)}{2}$ holomorphic one forms even though the curves C_p account for only $2(n-1)\cdot(n-2)$ holomorphic one forms. In this case K. Zuo (at Bonn) has shown that the extra holo- one forms can be accounted for by the pencil of conics passing through the four points with $r_p \geq 3$. Gläser has made computations of the number of independent holomorphic one forms for the $H(\Lambda,n)$ associated to some other Λ.

F. Hirzebruch using a generalization of his proportionality principle has shown that $c_1^2(H(\Lambda,n)) = 3e(H(\Lambda,n))$ implies that n = 2, 3, 5 and that in these cases there are strong constraints.

§5. Asymptotic Properties of $c_1^2(H(\Lambda,n)/e(H(\Lambda,n)$

In this section we find the lower and upper bound of the characteristic number, γ, of an arrangement [3,4]. This solves a question first looked at by Iitaka [4], whose paper was very helpful to me.

(5.1) Theorem. Let Λ be an arrangement of k lines of $\mathbb{P}_2(\mathbb{C})$, that satisfies $t_k = t_{k-1} = 0$. Then the characteristic number, γ, of Λ satisfies:

$$\gamma \geq \frac{2k - 5}{k - 2}$$

with equality if and only if $t_r = 0$ for $r \geq 3$.

Proof. Recall [3,(3.3)] that:

*) $$\gamma = \frac{5}{2} - \frac{(3f_0 - f_1 - 3)}{2(3 - 3k + f_1 - f_0)}$$

It suffices to show that $\gamma \leq \frac{2k - 6}{k - 2}$ implies that $t_r = 0$ for $r \geq 3$. By *), $\gamma \leq \frac{2k - 6}{k - 2}$, is equivalent to:

$$kf_1 \leq (2k - 2)f_0 + k^2 - k$$

or

**) $$\sum_{r=2}^{k-2} (rk - 2k + 2)t_r \leq k^2 - k$$

using the fact that $t_{k-1} = t_k = 0$.

Rewriting and using the fact that $t_k = t_{k-1} = 0$

we get:

$$\sum_{r=2}^{\infty}(r^2 - r)t_r + \sum_{r=3}^{k-2}(r - r^2 + rk - 2k + 2)t_r \leq k^2 - k$$

or by (1.1):

$$\sum_{r=3}^{k-2}(r - r^2 + rk - 2k + 2)t_r \leq 0.$$

Note that $r - r^2 + rk - 2k + 2 = r - r^2 + (r - 2)k + 2$ $\geq r - r^2 + (r - 2)(r + 2) + 2 = r - 2 \geq 1$ for $3 \leq r \leq k - 2$. This implies that each t_r in this range is 0. Combined with $t_k = t_{k-1} = 0$, we get our theorem.

□

(5.2) Corollary. Assume that Λ is an arrangement such that $t_k = t_{k-1} = 0$ and $t_r \neq 0$ for all $r \geq 3$, then the characteristic number γ is $\geq \frac{3}{2}$ with equality if and only if $k = 5$, $t_2 = 7$, and $t_3 = 1$. Excluding this case, $\gamma \geq \frac{8}{5}$ with equality if and only if $k = 6$, $t_2 = 12$, and $t_3 = 1$.

Proof. By elementary projective geometry and the formula for γ we get the following table for the numbers attached to arrangements with $k \leq 6$ and $t_k = t_{k-1} = 0$.

k	t_2	t_3	t_4	γ
5	7	1	0	3/2
5	4	2	0	2
6	12	1	0	8/5
6	9	0	1	5/3
6	9	2	0	7/4
6	6	3	0	2
6	6	1	1	2
6	3	4	0	5/2

Since $\frac{2\cdot 7-6}{7\cdot 2} = 8/5$ the corollary follows from the table and theorem (5.1)

□

For any $\varepsilon > 0$ and by ever more tedious enumeration of Λ with $t_k = t_{k-1} = 0$, the reader can use theorem (5.1) as in the proof of corollary (5.2) to enumerate all of the finite number of characteristic numbers that are $\leq 2 - \varepsilon$. F. Hirzebruch has constructed Λ to get γ dense in $[2, \frac{5}{2}]$.

He has also pointed out [3,(3.3)] that among the possible γ are:

$$\frac{5}{2} + \frac{3m - 6}{2(2m^2 - 3m)} \quad m \geq 2$$

Note that the largest is $\frac{8}{3}$ with $k = 9$, $t_3 = 12$, and $t_r = 0$ for $r \neq 3$.

(5.3) <u>Theorem</u>. <u>Let Λ be an arrangement with</u> $t_k = t_{k-1} = 0$. <u>Then</u> $\gamma \leq \frac{8}{3}$. <u>If</u> $\gamma = \frac{8}{3}$ <u>then</u> $k = 9, t_3 = 12$ <u>and</u> $t_r = 0$ <u>for</u> $r \neq 3$.

<u>Proof</u>. If $\gamma \geq \frac{8}{3}$ then by the formula for γ we get:

$$3t_2 + t_3 \leq \sum_{r\geq 4} (r - 4)t_r + k + 3$$

Using the Hirzebruch-Sakai inequality [3; remark 2 added in proof] we get:

$$\frac{5}{3} t_2 + \frac{4}{3}(k + t_5 + 2t_6 + \ldots) \leq \sum_{r\geq 4} (r - 4)t_r + k + 3$$

or

$$5t_2 + k + \sum_{r\geq 4} (r - 4)t_r \leq 9$$

This inequality and the table from the corollary (5.2) we get:

$$\text{*)} \qquad 7 \leq k \leq 9, \ t_2 = 0$$

We argue case by case. If $k = 9$ then we conclude from the Hirzebruch-Sakai inequality and *) that $t_3 \geq 12$. Using (1.1) we conclude that $t_3 = 12$ and $t_r = 0$ for $r \neq 3$. In this case $\gamma = \frac{8}{3}$.

If $k = 8$ then *) and the Hirzebruch-Sakai inequality implies that $t_3 \geq 11$. By (1.1) we get $8^2 - 8 \geq 66$ which is absurd.

If $k = 7$ then *) and the Hirzebruch-Sakai inequality implies that $t_3 \geq 10$. By (1.1) we get the absurdity $7^2 - 7 \geq 60$

□

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