QUANTIZATION OF STRONGLY HOMOTOPY LIE BIALGEBRAS

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Abstract. Using theory of PROPs we prove a formality theorem associated with universal quantizations of (strongly homotopy) Lie bialgebras.

1. Introduction

1.1. Let V be a $\mathbb{Z}$-graded vector space over a field $\mathbb{K}$ and $\mathcal{O}_V := \widehat{\odot^{\geq 1}} V$ the completed graded commutative and cocommutative bialgebra of smooth formal functions on V. The Gerstenhaber-Schack complex of polydifferential homomorphisms,

$$\left(\mathfrak{gs}^\bullet(\mathcal{O}_V) = \bigoplus_{m,n\geq 1} \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2-m-n],\ d_{\mathfrak{gs}}\right),$$

has a L_∞-algebra structure,

$$\{\mu_n : \odot^n \mathfrak{gs}^\bullet(\mathcal{O}_V) \to \mathfrak{gs}^\bullet(\mathcal{O}_V)[2-n]\}_{n\geq 1} \quad \text{with} \quad \mu_1 = d_{\mathfrak{gs}},$$

which controls deformations of the standard bialgebra structure on $\mathcal{O}_V$ [MV]. This L_∞-algebra depends on the choice of a minimal resolution of the properad of bialgebras, but its isomorphism class is defined canonically.

On the other hand, the completed graded commutative algebra,

$$\mathcal{O}_{\mathcal{V}} := \widehat{\odot^{\geq 1}}(V[-1]) \,\widehat{\otimes}\, \widehat{\odot^{\geq 1}}(V^*[-1]),$$

of smooth formal functions on a graded vector space $\mathcal{V} := V[-1] \oplus V^*[-1]$ has a natural degree -2 Poisson structure, $\{\ ,\ \} : \mathcal{O}_{\mathcal{V}} \otimes \mathcal{O}_{\mathcal{V}} \to \mathcal{O}_{\mathcal{V}}[-2]$, given on generators by

$$\{sv, sw\} = 0, \quad \{s\alpha, s\beta\} = 0, \quad \{s\alpha, sv\} =< \alpha, v >, \quad \forall v, w \in V, \alpha, \beta \in V^*.$$

where $s : V \to V[-1]$ and $s : V^* \to V^*[-1]$ are natural isomorphisms.

1.2. **Formality Theorem.** *There exists a L_∞ quasi-isomorphism,*

$$F : (\mathcal{O}_{\mathcal{V}}[2], \{\ ,\ \}) \longrightarrow (\mathfrak{gs}^\bullet(\mathcal{O}_V), \mu_\bullet)\,.$$

As the vector space $\mathcal{O}_{\mathcal{V}}[2]$ equals cohomology of the Gerstenhaber-Schack complex $(\mathfrak{gs}^\bullet(\mathcal{O}_V), \partial_{\mathfrak{gs}})$, Theorem 1.2 asserts essentially formality of the L_∞ algebra $(\mathfrak{gs}^\bullet(\mathcal{O}_V), \mu_\bullet)$.

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1.3**. Corollary.** *Every strongly homotopy Lie bialgebra structure, γ, on a graded vector space V can be deformation quantized, i.e. there exists a strongly homotopy bialgebra structure, $\Gamma(\gamma,\hbar)$, on $\odot^\bullet V[[\hbar]]$ depending on a formal parameter $\hbar$ such that $\Gamma(\gamma,\hbar)=(\cdot,\Delta)+\hbar\nu \bmod \hbar^2$, where $(\cdot,\Delta)$ is the standard graded (co)commutative bialgebra structure on $\odot^\bullet V$.*

In the case when γ is a Lie bialgebra structure on V (i.e. higher homotopies vanish) its deformation quantization via the quasi-isomorphism F coincides with the Etingof-Kazhdan quantization [EK].

1.4. Our main technical tool in proving Theorem 1.2 is the theory of differential graded (dg, for short) PROPs. We refer to [Va] (see also [EE]) for a clear introduction into the theory of PROPs and properads, their minimal resolutions and representations. In the present context one might also be interested to look at [Me3] where very similar PROPic ideas and methods have been used to give a new proof of Kontsevich's theorem on deformation quantization of Poisson structures.

In our approach Theorem 1.2 is a corollary to a deeper statement:

1.5**. Main Theorem.** *Let LieB_∞ be a minimal resolution,*

$$\pi : \mathsf{LieB}_\infty \twoheadrightarrow \mathsf{LieB},$$

of the PROP, LieB, of Lie bialgebras. Let $\widehat{\mathsf{LieB}}_\infty$ be its completion with respect to the number of vertices, and let DefQ be the dg PROP of of quantum strongly homotopy bialgebra structures[1]*. Then there exists a* quantization *morphism of dg PROPs,*

$$\mathcal{F} : \mathsf{DefQ} \longrightarrow \widehat{\mathsf{LieB}}_\infty,$$

such that its composition, $\mathsf{DefQ} \xrightarrow{\mathcal{F}} \widehat{\mathsf{LieB}}_\infty \xrightarrow{\pi} \widehat{\mathsf{LieB}}$ is the Etingof-Kazhdan quantization morphism.

In fact the above quantization morphism $\mathcal{F}$ is induced by a *quasi-isomorphism* of certain natural extensions, $\mathcal{F}^+ : \mathsf{DefQ}^+ \to \widehat{\mathsf{LieB}}^+_\infty$, of both dg PROPs.

1.6. The paper is organized as follows. In §2 we remind key facts about the PROPs, LieB and LieB_∞, of Lie and, respectively, strongly homotopy Lie bialgebras and explain interrelations between representations of LieB_∞ in a vector space V and the Poisson algebra $(\mathcal{O}_\mathcal{V}, \{\ ,\ \})$ defined above in §1.1. In §3 we discuss the PROP, AssB, of associative bialgebras, its minimal resolution, AssB_∞, and some associated L_∞-algebras. In §4 we intoduce a subcomplex of the Gerstenhaber-Schack complex spanned by polydifferential operators, and define the dg PROP, DefQ, of quantum strongly homotopy bialgebra structures. In §5 we prove Main Theorem 1.5 and Theorem 1.2, and also explain how quasi-isomorphism of L_∞-algebras F from Theorem 1.2 can be interpreted as a quasi-isomorphism of certain dg PROPs, $\mathcal{F}^+ : \mathsf{DefQ}^+ \to \widehat{\mathsf{LieB}}^+_\infty$.

[1]By definition of DefQ, its representations in a dg vector space V are in one-to-one correspondence with strongly homotopy bialgebra structures in $\mathcal{O}_V$ which are polydifferential deformations of the standard (co)commutative bialgebra structure in $\mathcal{O}_V$ (see §4.3 for precise definition-construction).

1.7. A few words about our notations. For an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ the associated free PROP is denoted by $\Gamma\langle E\rangle$. The endomorphism PROP of a graded vector space V is denoted by $\mathsf{End}\langle V\rangle$. The one-dimensional sign representation of the permutation group $\mathbb{S}_n$ is denoted by sgn_n while the trivial representation by $\mathbb{1}_n$. All our differentials have degree +1. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$.

All our graphs are directed with flow implicitly assumed to go from bottom to top.

We work throughout over the field $\mathbb{K}$ of characteristic 0.

2. Properad of strongly homotopy Lie bialgebras

2.1. **Lie bialgebras.** A *Lie bilagbera* is, by definition [D], a graded vector space V together with two linear maps,

$$\begin{array}{rccc} \triangle\colon & V & \longrightarrow & \wedge^2 V \\ & a & \longrightarrow & \sum a_1\wedge a_2 \end{array} \quad , \quad \begin{array}{rccc} [\,,\,]: & \wedge^2 V & \longrightarrow & V \\ & a\otimes b & \longrightarrow & [a,b] \end{array}$$

satisfying,

(i) Jacobi identity: $[a,[b,c] = [[a,b],c]] + (-1)^{|b||a|}[b,[a,c]]$;

(ii) co-Jacobi identity: $(\triangle \otimes \mathrm{Id}) \triangle a + \tau(\triangle \otimes\mathrm{Id}) \triangle a + \tau^2(\triangle \otimes\mathrm{Id}) \triangle a = 0$, where τ is the cyclic permutation (123) represented naturally in $V\otimes V\otimes V$;

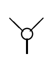
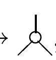
(iii) Leibniz type identity: $\triangle [a,b] = \sum a_1\wedge [a_2,b] - (-1)^{|a_1||a_2|} a_2\wedge[a_1,b] + [a,b_1]\wedge b_2 - (-1)^{|b_1||b_2|}[a,b_2]\wedge b_1$.

for any $a,b,c\in V$.

2.2. **PROP of Lie bialgebras.** It is easy to construct a PROP, LieB, whose representations,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}\langle V\rangle,$$

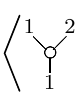
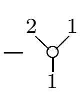
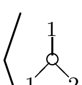
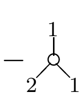
in a graded vector space V are in one-to-one correspondence with Lie bialgebra structures in V. With an association in mind,

$\triangle \leftrightarrow$  , $[\,,\,] \leftrightarrow$ ,

one can define it as a quotient,

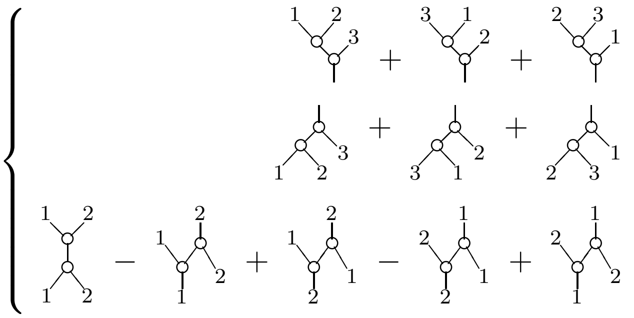
$$\mathsf{LieB} := \Gamma\langle L\rangle/(R)$$

of the free PROP, $\Gamma\langle L\rangle$, generated by the $\mathbb{S}$-bimodule $L = \{L(m,n)\}$,

$L(m,n) :=$
- $sgn_2 \otimes \mathbb{1}_1 \equiv \mathrm{span}\Big\langle$  $= -$  $\Big\rangle$ if $m=2, n=1$,
- $\mathbb{1}_1 \otimes sgn_2 \equiv \mathrm{span}\Big\langle$  $= -$  $\Big\rangle$ if $m=1, n=2$,
- 0 otherwise

modulo the ideal generated by relations

(1) $R:$



$\in \Gamma\langle E\rangle(3,1)$

$\in \Gamma\langle E\rangle(1,3)$

$\in \Gamma\langle E\rangle(2,2)$

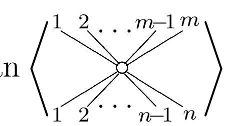
As the ideal is generated by two-vertex graphs, the properad behind LieB is quadratic.

2.3. **Minimal resolution of** LieB**.** It was shown in [Ga, Va] that the dioperad and the properad underlying the PROP LieB are both Koszul so that its minimal PROP resolution, LieB_∞, can be easily constructed. This is a dg free PROP,

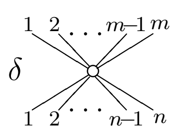
$$\mathsf{LieB}_\infty = \Gamma\langle \mathsf{L}\rangle,$$

generated by the $\mathbb{S}$–bimodule $\mathsf{L} = \{\mathsf{L}(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

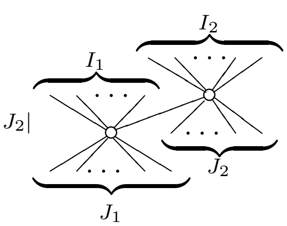
(2) $$\mathsf{L}(m,n) := sgn_m \otimes sgn_n[m+n-3] = \text{span}$$



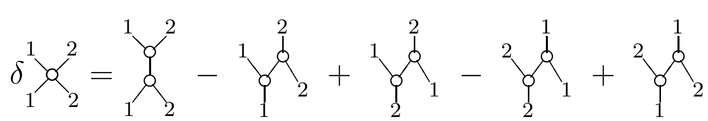
and with the differential given on generating corollas by (cf. [Me2], §5)



$$= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} (-1)^{\sigma(I_1\sqcup I_2)+\sigma(I_1\sqcup I_2)+|I_1||J_2|}$$



where $\sigma(I_1\sqcup I_2)$ and $\sigma(J_1\sqcup J_2)$ are the signs of the shuffles $[1,\ldots,m]=I_1\sqcup I_2$ and, respectively, $[1,\ldots,n]=J_1\sqcup J_2$. For example,



2.3.1**. Fact** [Me2]**.** Representations,

$$\rho : \mathsf{LieB}_\infty \longrightarrow \mathsf{End}\,\langle V\rangle\,,$$

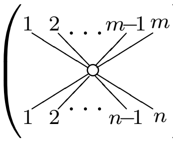
of the dg PROP $(\mathsf{LieB}_\infty, \delta)$ in a dg space (V,d) are in one-to-one correspondence with degree 3 elements, γ, in the Poisson algebra $(\mathcal{O}_{\mathcal{V}}, \{\ ,\ \})$ satisfying the equation, $\{\gamma,\gamma\}$=0.

Indeed, using natural degree $m+n$ isomorphisms,

$$s^n_m : \mathrm{Hom}\,(\wedge^n V, \wedge^m V) \longrightarrow \odot^n(V[1]) \otimes \odot^m(V^*[1]),$$

we define a degree 3 element,

$$\gamma := s_1^1(d) + \sum_{\substack{m,n\geq 1\\ m+n\geq 3}} s_n^m \circ \rho \left(\right) \in \mathcal{O}_{\mathcal{V}},$$



and then check that the equation $d\circ\rho = \rho\circ\delta$ translates precisely into the Maurer-Cartan equation $\{\gamma, \gamma\} = 0$.

In this framework Lie bialgebra structures on V get identified with homogeneous polynomials of order 3, $\gamma_3 \in \mathcal{O}_{\mathcal{V}}$, satisfying the equation $\{\gamma_3, \gamma_3\} = 0$. Indeed, every such a polynomial is equivalent to a pair $(\vartriangle\in \mathrm{Hom}(V, \wedge^2 V),\ [\,,\,] \in \mathrm{Hom}(\wedge^2 V, V))$, in terms of which the equation $\{\gamma_3, \gamma_3\} = 0$ disintegrates into relations (i)-(iii) of §2.1.

3. PROP of strongly homotopy bialgebras

3.1. **Associative bialgebras.** A *bialgebra* is, by definition, a graded vector space V equipped with two degree zero linear maps,

$$\begin{array}{rccc} \mu: & V\otimes V & \longrightarrow & V \\ & a\otimes b & \longrightarrow & ab \end{array} \quad , \qquad \begin{array}{rccc} \Delta: & V & \longrightarrow & V\otimes V \\ & a & \longrightarrow & \sum a_1\otimes a_2 \end{array}$$

satisfying,

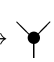
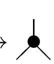
(i) the associativity identity: $(ab)c = a(bc)$;
(ii) the coassociativity identity: $(\Delta\otimes \mathrm{Id})\Delta a(\mathrm{Id}\otimes\Delta)\Delta a$;
(iii) the compatibility identity: Δ is a morphism of algebras, i.e. $\Delta(ab) = \sum (-1)^{a_2 b_1} a_1 b_1 \otimes a_2 b_2$,

for any $a, b, c \in V$. We often abbreviate "associative bialgebra" to simply "bialgebra".

3.2. **PROP of bialgebras.** There exists a PROP, AssB, whose representations,

$$\rho : \mathsf{AssB} \longrightarrow \mathsf{End}\langle V\rangle,$$

in a graded vector space V are in one-to-one correspondence with the bialgebra structures in V [EE]. With an association in mind,

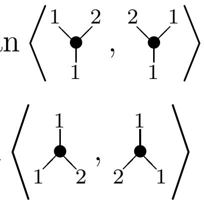
$\Delta \leftrightarrow$  , $\mu \leftrightarrow$ 

one can define it as a quotient,

$$\mathsf{AssB} := \Gamma\langle E\rangle/(R)$$

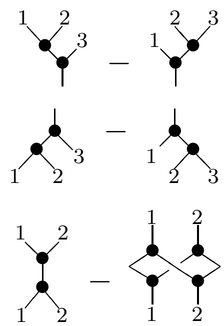
of the free PROP, $\Gamma\langle E\rangle$, generated by the $\mathbb{S}$-bimodule $E=\{E(m,n)\}$,

$$E(m,n) := \begin{cases} \mathbb{K}[\mathbb{S}_2]\otimes 1\!\!1_1 \equiv \text{span}\langle \ldots \rangle & \text{if } m=2, n=1,\\ 1\!\!1_1\otimes \mathbb{K}[\mathbb{S}_2] \equiv \text{span}\langle \ldots \rangle & \text{if } m=1, n=2,\\ 0 & \text{otherwise}\end{cases}$$



modulo the ideal generated by relations

$$R: \begin{cases} \ldots \in \Gamma\langle E\rangle(3,1)\\ \ldots \in \Gamma\langle E\rangle(1,3)\\ \ldots \in \Gamma\langle E\rangle(2,2)\end{cases}$$



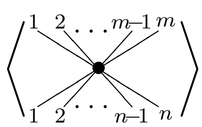
which are not quadratic in the properad sense (see [Va]).

3.3. A minimal resolution of AssB. The properad underlying the PROP of bialgebras, AssB, is not Koszul but it is homotopy Koszul [MV]. A minimal resolution of AssB,

$$(\mathsf{AssB}_\infty = \Gamma\langle \mathsf{E}\rangle, \delta) \xrightarrow{\pi} (\mathsf{AssB}, 0)$$

is freely generated by a relatively small $\mathbb{S}$-bimodule $\mathsf{E}=\{\mathsf{E}(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

$$\mathsf{E}(m,n) := \mathbb{K}[\mathbb{S}_m]\otimes\mathbb{K}[\mathbb{S}_n][m+n-3] = \text{span}\left\langle \ldots \right\rangle,$$



as was first noticed in [Ma]. The differential, δ is not quadratic, and its explicit value on generic (m,n)-corolla is not known at present. Rather surprisingly, just existence of $(\mathsf{AssB}_\infty,\delta)$ and Theorem 25 in [MV] (see Fact 3.4.2 below) are enough for our purposes.

3.4. Deformation theory of dg morphisms. Let $(\Gamma\langle\mathsf{A}\rangle,\delta)$ be a dg free PROP generated by an $\mathbb{S}$-bimodule A, and (P,d) an arbitrary dg PROP. It was shown in [MV] that the graded vector space of equivariant morphisms of $\mathbb{S}$-bimodules,

$$\mathfrak{g} := \mathrm{Hom}_{\mathbb{S}}(\mathsf{A},\mathsf{P})[-1],$$

has a canonical (filtered) L_∞-structure,

$$\{\mu_n : \odot^n(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]\}_{n\geq 1},$$

whose Maurer-Cartan elements, that is, degree 1 elements γ in $\mathfrak{g}$ satisfying the (well-defined) equation

$$\mu_1(\gamma) + \frac{1}{2!}\mu_2(\gamma,\gamma) + \frac{1}{3!}\mu_2(\gamma,\gamma,\gamma) + \ldots = 0,$$

are in one-to-one correspondence with morphisms, $\gamma : (\Gamma\langle \mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$, of dg PROPs.

If $\gamma : (\Gamma\langle \mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$ is any particular morphism of dg PROPs, then $\mathfrak{g}$ has a canonical γ-*twisted* L_∞-structure [Me1, MV],

$$\{\mu_n^\gamma : \odot^n(\mathfrak{g}[1]) \longrightarrow \mathfrak{g}[1]\}_{n\geq 1},$$

which controls deformation theory of the morphism γ. The associated Maurer-Cartan elements, Γ,

$$\mu_1^\gamma(\Gamma) + \frac{1}{2!}\mu_2^\gamma(\Gamma,\Gamma) + \frac{1}{3!}\mu_2^\gamma(\Gamma,\Gamma,\Gamma) + \ldots = 0,$$

are in one-to-one correspondence with those morphisms of dg PROPs, $(\Gamma\langle \mathsf{A}\rangle, \delta) \to (\mathsf{P}, d)$ whose values on generators are given by $\gamma + \Gamma$.

3.4.1. **Deformation theory of bialgebras**. In particular, if $(\Gamma\langle \mathsf{A}\rangle, \delta)$ is $(\mathsf{AssB}_\infty, \delta)$ and P is the endomorphism PROP, $\mathsf{End}\langle V\rangle$, of some dg vector space V, a morphism γ ia nothing but a strongly homotopy bialgebra structure in V. Thus there exists a natural L_∞-structure, $\{\mu_n^\gamma\}_{n\geq 1}$, on the vector space,

$$\mathfrak{gs}^\bullet(V) := \mathrm{Hom}_{\mathbb{S}}(\mathsf{E}, \mathsf{End}(V))[-1] = \bigoplus_{\substack{m,n\geq 1\\ m+n\geq 3}} \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[m+n-2],$$

which controls deformation theory of γ. In the special case of a bialgebra structure we can be precise about the initial term of the associated L_∞-structure:

3.4.2. **Fact** [MV]. If $\gamma : \mathsf{AssB} \to \mathsf{End}\langle V\rangle$ is a bialgebra structure, then the differential, $\mu_1^\gamma : \mathfrak{gs}^\bullet(V) \to \mathfrak{gs}^{\bullet+1}(V)$, of the induced L_∞ structure on $\mathfrak{gs}^\bullet(V)$ is the Gerstenhaber-Schack differential [GS],

$$d_{\mathfrak{gs}} = d_1 \oplus d_2 : \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \longrightarrow \mathrm{Hom}(V^{\otimes n+1}, V^{\otimes m}) \oplus \mathrm{Hom}(V^{\otimes n}, V^{\otimes m+1}),$$

with d_1 given on an arbitrary $f \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ by

$$\begin{aligned}(d_1 f)(v_0, v_1, \ldots, v_n) \;:=\; & \Delta^{m-1}(v_0)\cdot f(v_1, v_2, \ldots, v_n) - \sum_{i=0}^{n-1}(-1)^i f(v_1, \ldots, v_i v_{i+1}, \ldots, v_n) \\ & +(-1)^{n+1} f(v_1, v_2, \ldots, v_n)\cdot \Delta^{m-1}(v_n) \forall \;\; v_0, v_1, \ldots, v_n \in V,\end{aligned}$$

where multiplication in V is denoted by juxtaposition, the induced multiplication in the algebra $V^{\otimes m}$ by $\cdot$, the comultiplication in V by Δ, and

$$\Delta^{m-1} : (\Delta \otimes \mathrm{Id}^{\otimes m-2}) \circ (\Delta \otimes \mathrm{Id}^{\otimes m-3}) \circ \ldots \circ \Delta : V \to V^{\otimes m},$$

for $m \geq 2$ while $\Delta^0 := \mathrm{Id}$. The expression for d_2 is an obvious "dual" analogue of the one for d_1.

4. Gerstenhaber-Schack complex of polydifferential operators and PROP of quantum AssB_∞-structures

4.1. **Polydifferential operators.** For an arbitrary vector space V we consider a subspace,

$$\mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes\bullet}, \mathcal{O}_V^{\otimes\bullet}) \subset \mathrm{Hom}(\mathcal{O}_V^{\otimes\bullet}, \mathcal{O}_V^{\otimes\bullet}),$$

spanned by polydifferential operators,

$$\begin{array}{cccc} \Gamma: & \mathcal{O}_V^{\otimes m} & \longrightarrow & \mathcal{O}_V^{\otimes m} \\ & f_1\otimes\ldots\otimes f_m & \longrightarrow & \Gamma(f_1,\ldots,f_m), \end{array}$$

of the form,

$$\Gamma(f_1,\ldots,f_m) = x^{J_1}\otimes\ldots\otimes x^{J_n}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}f_1}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}f_m}{\partial x^{I_m}}\right). \tag{3}$$

where x^i are some linear coordinates in V, I and J stand for multi-indices, say, $i_1 i_2\ldots i_p$, and $j_1 j_2\ldots j_q$, and

$$x^J := x^{j_1}x^{j_2}\ldots x^{j_q}, \qquad \frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^p}{\partial x^{i_1}\partial x^{i_2}\ldots\partial x^{i_p}}.$$

As $\mathcal{O}_V$ is naturally a bialgebra, there is an associated Gerstenhaber-Schack complex [GS],

$$\left(\bigoplus_{m,n\geq 1}\mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2], d_{\mathfrak{gs}}\right), \tag{4}$$

with the differential, $d_{\mathfrak{gs}}$, given as in § 3.4.2.

4.1.1**. Lemma.** *The subspace,*

$$\bigoplus_{m,n\geq 1}\mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2] \subset \bigoplus_{m,n\geq 1}\mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[m+n-2],$$

is a subcomplex of the Gerstenhaber-Schack complex of the bialgebra $\mathcal{O}_V$, with the Gerstenhaber-Schack differential given explicitly on polydifferential operators (3) by

$$\begin{aligned} d_{\mathfrak{gs}}\Gamma \;=\; & \sum_{i=1}^{n}(-1)^{i+1}x^{J_1}\otimes\ldots\otimes\bar{\Delta}(x^{J_i})\otimes\ldots\otimes x^{J_n}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right) \\ & + \sum_{i=1}^{m}(-1)^{i+1}\sum_{I_i=I_i'\sqcup I_i''} \\ & \quad x^{J_1}\otimes\ldots\otimes x^{J_m}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1'|}}{\partial x^{I_1'}}\right)\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1''|}}{\partial x^{I_1''}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right), \end{aligned}$$

where $\bar{\Delta}:\mathcal{O}_V\rightarrow\mathcal{O}_V\otimes\mathcal{O}_V$ is the reduced diagonal.

Proof is a straightforward calculation.

4.2. **An extension of the PROP AssB_∞.** The $\mathbb{S}$-bimodule,

$$\mathsf{End}_{poly}\langle \mathcal{O}_V \rangle := \left\{ \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n}) \right\}_{m.n\geq 1},$$

has a natural PROP structure such that the natural monomorphism,

$$\mathsf{End}_{poly}\langle \mathcal{O}_V \rangle \longrightarrow \mathsf{End}\langle \mathcal{O}_V \rangle,$$

is a morphism of PROPs. If $D \in \mathrm{Hom}_{poly}(\mathcal{O}_V, \mathcal{O}_V)$ is a degree 1 differential operator such that $D^2 = 0$, then we can talk about strongly homotopy bialgebras structures,

$$\gamma : (\mathsf{AssB}_\infty, \delta) \longrightarrow (\mathsf{End}_{poly}\langle \mathcal{O}_V \rangle, D), \tag{5}$$

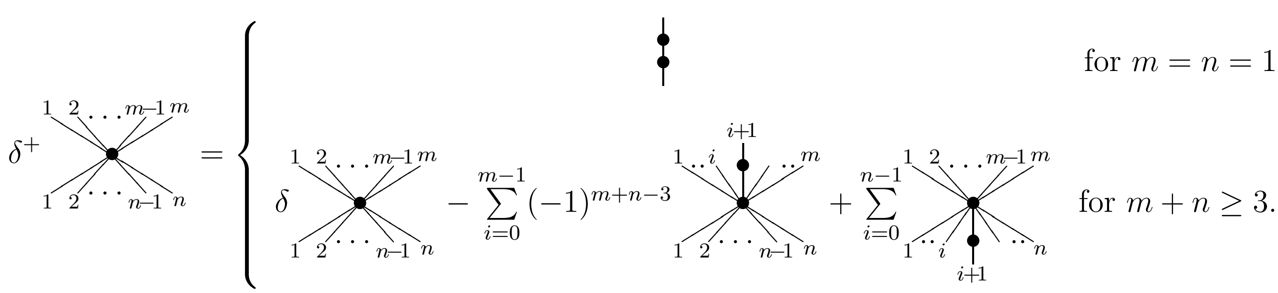
in the class of polydifferential operators with differential D. We shall reinterpret below everything we say here about the bialgebra $\mathcal{O}_V$ and its deformation theory in terms of a much more primitive underlying object — the vector space V. As operator D has a rather non-trivial internal structure,

$$D = \sum_{\substack{I,J \\ |I|,|J|\geq 1}} D_J^I x^J \frac{\partial^{|I|}}{\partial x^I},$$

in terms of V, it is not reasonable to fix D right from the beginning, but rather allow it to depend on the choice of a representation γ. Roughly speaking, we want to include "degrees of freedom" associated with a choice of D into deformation theory. There exists a simple trick to achieve that goal:

(i) we add to the list of generators of the dg free PROP AssB_∞ a new generating $(1,1)$-corolla, , in degree 1, that is, we define the free PROP $\mathsf{AssB}_\infty^+ = \Gamma\langle \mathsf{E}^+ \rangle$ generated by an $\mathbb{S}$-module $\mathsf{E}^+ = \{\mathsf{E}^+(m,n)\}_{m,n\geq 1}$ given exactly as in § 3.3 except that restriction $m+n \geq 3$ is omitted;

(ii) we define a derivation, δ^+ in AssB_∞^+ as follows,



4.2.1**. Lemma.** *The pair,* $(\mathsf{AssB}_\infty^+, \delta^+)$*, is a dg free PROP, i.e.* $(\delta^+)^2 = 0$.

Proof is an easy exercise[2].

[2]The ∞ symbol in the notation AssB_∞^+ could be misleading in the sense that AssB_∞^+ is *not* a resolution of some PROP AssB^+. In fact, the cohomology of $(\mathsf{AssB}_\infty^+, \delta^+)$ is trivial.

4.2.2. **Proposition.** *There is a one-to-one correspondence between representations,*

$$\gamma^+ : (\mathsf{AssB}_\infty^+, \delta^+) \longrightarrow (\mathsf{End}_{poly}\langle \mathcal{O}_V\rangle, 0),$$

and pairs, (D, γ), consisting of a differential $D \in \mathrm{Hom}_{poly}(\mathcal{O}_V, \mathcal{O}_V)$ and a representation,

$$\gamma : (\mathsf{AssB}_\infty, \delta) \longrightarrow (\mathsf{End}_{poly}\langle \mathcal{O}_V\rangle, D).$$

Proof. ($\Rightarrow$): A representation γ^+ is a collection of polydifferential operators,

$$\{\Gamma^m_n \in \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes m}), \quad |\Gamma^m_n| = 3 - m - n\}_{m,n\geq 1},$$

such that

$$(\Gamma^1_1)^2 = 0,$$

and all the rest operators, $\{\Gamma^m_n\}_{m,n\geq 1, m+n\geq 3}$, assemble into a representation, $\gamma : (\mathsf{AssB}_\infty, \delta) \rightarrow (\mathsf{End}_{poly}\langle \mathcal{O}_V\rangle, D = \Gamma^1_1)$.

The implication ($\Leftarrow$) is analogous. □

4.2.3. **Remark.** The above extension of $(\mathsf{AssB}_\infty, \delta)$ to $(\mathsf{AssB}_\infty^+, \delta^+)$ works, of course, for an arbitrary dg free PROP with the corresponding analogue of Proposition 4.2.2 holding true. Moreover that construction is functorial.

A similar extension of the dg PROP $(\mathsf{LieB}_\infty, \delta)$ we denote by $(\mathsf{LieB}_\infty^+, \delta^+)$. Thus LieB_∞^+ is a free PROP generated by the $\mathbb{S}$-bimodule $\mathsf{L}^+ = \{\mathsf{L}(m,n)\}_{m,n\geq 1}$ defined exactly as in (2) except that the restriction $m + n \geq 3$ is dropped.

4.3. **PROP of quantum strongly homotopy bialgebra structures.** The vector space $\mathcal{O}_V$ has a natural graded commutative and cocommutative bialgeba structure. Let

$$\alpha : (\mathsf{AssB}_\infty^+, \delta^+) \longrightarrow (\mathsf{End}_{poly}\langle \mathcal{O}_V\rangle, 0),$$

be the associated morphism of dg PROPs.

In accordance with § 3.4, the vector space,

$$\mathfrak{gs}^\bullet(\mathcal{O}_V) := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{E}^+, (\mathsf{End}\langle \mathcal{O}_V\rangle) = \bigoplus_{m,n\geq 1} \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes m})[m+n-2]$$

has a canonical L_∞ structure, $\{\mu^\alpha_n\}_{n\geq 1}$, whose Maurer-Cartan elements,

$$\mu^\alpha_1(\Gamma) + \frac{1}{2!}\mu^\alpha_2(\Gamma,\Gamma) + \frac{1}{3!}\mu^\alpha_3(\Gamma,\Gamma,\Gamma) + \ldots = 0,$$

describe deformed AssB_∞-structures on $\mathcal{O}_V$. By Theorem 25 in [MV], the initial term, μ^α_1, of this L_∞-structure is precisely the Gerstenhaber-Schack differential $d_{\mathfrak{gs}}$ as described in Lemma 4.1.1.

Thus we have a sheaf of L_∞-algebras, $(\mathfrak{gs}^\bullet(\mathcal{O}_V), \mu^\alpha_\bullet)$, on the formal manifold V whose operations $\mu^\alpha_\bullet$ are compositions of differentiation, multiplication and comultiplication in $\mathcal{O}_V$. Hence we can apply a trick from ([Me3], §2.5) and construct a dg PROP out of the data $(\mathfrak{gs}^\bullet(\mathcal{O}_V), \mu^\alpha_\bullet)$.

4.3.1. **Proposition.** *There exists a (completed) dg free PROP,* $(\mathsf{DefQ}^+, \mathsf{d})$, *whose representations,*

$$(\mathsf{DefQ}^+, \mathsf{d}) \longrightarrow (\mathsf{End}\langle V\rangle, d),$$

in a dg vector space (V, d) *are in one-to-one correspondence with Maurer-Cartan elements,* Γ, *in the* L_∞*-algebra* $(\mathfrak{gs}^\bullet(\mathcal{O}_V), \mu^\alpha_\bullet)$, *i.e. with strongly homotopy bialgebra deformations of the standard bialgebra structure in* $\mathcal{O}_V$.

Proof. An arbitrary degree 1 element Γ in $\mathfrak{gs}^\bullet(\mathcal{O}_V)$ has the form,

$$\Gamma = \sum_{m,n\geq 1} \sum_{\substack{I_1,\dots I_n\\ J_1,\dots J_m}} \Gamma^{I_1\dots I_m}_{J_1\dots J_n} x^{J_1}\otimes\dots\otimes x^{J_n}\cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right), \tag{6}$$

where $\Gamma^{I_1\dots I_m}_{J_1\dots J_n}$ are the coordinate components of some linear map,

$$V^{\odot|J_1|}\otimes\dots\otimes V^{\odot|J_1|} \longrightarrow V^{\odot|I_1|}\otimes\dots\otimes V^{\odot|I_m|},$$

of degree $3-m-n$.

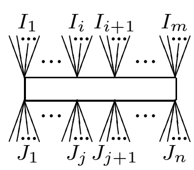
Thus there is a one-to-one correspondence between degree 1 elements, $\Gamma\in\mathfrak{gs}^\bullet(\mathcal{O}_V)$, and representations in $\mathsf{End}\langle V\rangle$ (not in $\mathsf{End}\langle\mathcal{O}_V\rangle$!) of the free PROP, $\mathsf{DefQ} := \Gamma\langle\mathsf{D}\rangle$, generated by the $\mathbb{S}$-bimodule, $\mathsf{D}=\{\mathsf{D}(p,q)\}_{p,q\geq 1}$, where

$$\mathsf{D}(p,q) := \begin{cases} (\mathsf{D}(1)\otimes\mathsf{D}(1)/\mathbb{K})[-1] & \text{for } p=q=1\\ \mathsf{D}(p)\otimes\mathsf{D}(q)[p+q-3] & \text{for } p+q\geq 3\end{cases}$$

and

$$\mathsf{D}(p) := \bigoplus_{m\geq 1}\bigoplus_{\substack{[p]=I_1\sqcup\dots\sqcup I_m\\ |I_1|,\dots,|I_m|\geq 1}} \mathrm{Ind}^{\mathbb{S}_m}_{\mathbb{S}_{|I_1|}\times\dots\times\mathbb{S}_{|I_m|}} \mathbf{1}_{|I_1|}\otimes\dots\otimes\mathbf{1}_{|I_m|}.$$

Thus the generators of DefQ can be pictorially represented by degree $3-m-n$ directed planar corollas of the form,

(7) 

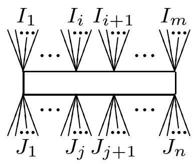
$\simeq \quad \Gamma^{I_1,\dots,I_m}_{J_1,\dots,J_n},$

where

- the input legs are labeled by the set $[q]=[1,2,\dots,q]$ partitioned into n disjoint non-empty subsets,
$$[q]=J_1\sqcup\dots\sqcup J_j\sqcup J_{j+1}\sqcup\dots\sqcup J_n,$$
and legs in each J_j-bunch are symmetric (so that it does not matter how labels from the set J_j are distributed over legs in j-th bunch);
- the output legs are labeled by the set $[p]$ partitioned into m disjoint non-empty subsets,
$$[p]=I_1\sqcup\dots\sqcup I_i\sqcup I_{i+1}\sqcup\dots\sqcup I_k,$$
and legs in each I_i-bunch are symmetric.

Our next movement is encoding of the L_∞-structure, $\{\mu^\alpha_\bullet\}$, on $\mathfrak{gs}^\bullet(\mathcal{O}_V)$ into a differential d in DefQ^+. The idea is simple [Me3]: we replace coefficients $\Gamma^{I_1,\dots,I_m}_{J_1,\dots,J_n}$ in the formal series (6) by the associated generating corollas (7) getting thereby a well-defined element

$$\bar{\Gamma} := \sum_{m,n\geq 1} \sum_{\substack{I_1,\dots I_n\\ J_1,\dots J_m}}$$



$$\otimes x^{J_1}\otimes\dots\otimes x^{J_n}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right),$$

in the tensor product $\mathsf{DefQ}^+\otimes\mathfrak{gs}^\bullet(\mathcal{O}_V)$. We extend by linearity L_∞-operations $\{\mu^\alpha_\bullet\}$ on $\mathfrak{gs}^\bullet(\mathcal{O}_V)$ to the tensor product $\mathsf{DefQ}^+\otimes\mathfrak{gs}^\bullet(\mathcal{O}_V)$ and then define a derivation, d, in DefQ^+ by comparing coefficients of the monomials, $x^{J_1}\otimes\dots\otimes x^{J_n}\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_m}}\right)$, in the equality,

$$\mathsf{d}\bar{\Gamma} = \sum_{n=1}^{\infty}\frac{1}{n!}\mu^\alpha_n(\bar{\Gamma},\bar{\Gamma},\dots,\bar{\Gamma}). \tag{8}$$

The quadratic equations which hold for $\{\mu^\alpha_\bullet\}$ imply then $\mathsf{d}^2=0$ [3]. As the r.h.s. of (8) can contain graphs with arbitrary large genus, *we assume from now on that* DefQ^+ *is completed with respect to the natural genus filtration.*

By the very construction of $(\mathsf{DefQ}^+,\mathsf{d})$, its representations

$$(\mathsf{DefQ}^+,\mathsf{d})\to(\mathsf{End}\langle V\rangle, d),$$

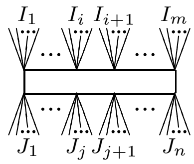
in a dg vector space V are in one-to-one correspondence with Maurer-Cartan elements Γ in the L_∞-algebra $(\mathfrak{gs}^\bullet(\mathcal{O}_V),\mu^\alpha_\bullet)$. The proof-construction is completed. □

4.3.2. **Definition.** We call $(\mathsf{DefQ}^+,\mathsf{d})$ the dg *PROP of quantum strongly homotopy bialgebra structures.*

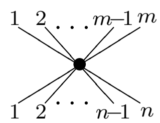
4.4. **Normalized Gerstenhaber-Schack complex.** We defined the complex $(\mathfrak{gs}^\bullet(\mathcal{O}_V), d_{\mathfrak{gs}})$ as the one which is spanned by polydifferential operators (3) with multi-indices I_i and J_i allowed to have cardinalities $|I_i|=0$ and/or $|J_i|=0$ (so that the standard multiplication and comultiplication in $\mathcal{O}_V$ belong to the class of polydifferential operators). However, this complex is quasi-isomorphic (cf. [Lo]) to the *normalized* one, i.e. the one which is spanned by operators which vanish if at least one input function is constant, and never take constant value in each tensor factor. As L_∞-structures are transferable via quasi-isomorphisms, we can assume without loss of generality that the L_∞-algebra $(\mathfrak{gs}^\bullet(\mathcal{O}_V),\mu^\alpha_\bullet)$ is spanned by *normalized* polydifferential operators, or, equivalently, that the PROP DefQ is generated by corollas (7) with all $|I_i|\geq 1$ and all $|J_j|\geq 1$.

[3]See also §2.5 and §2.5.1 in [Me3] for a detailed explanation of this implication in the case when $\mu_\bullet$ is a dg Lie algebra; below in § 4.5 we illustrate definition (8) with some explicit computations.

4.5. **Examples.** The generators,

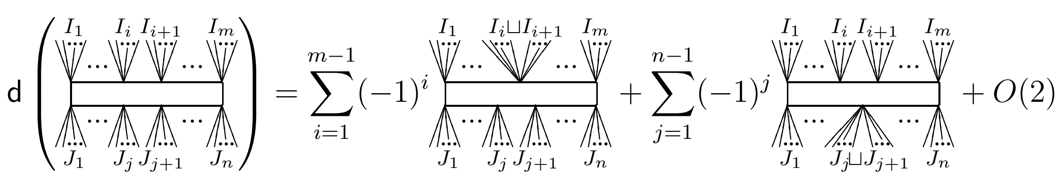


are kind of "thickenings" of the corresponding generators,

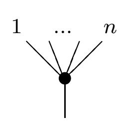


of AssB_∞^+ (mimicking thickening of V into $\mathcal{O}_V$), and the differential d in DefQ is also a kind of "thickening" (and twisting by α) of the differential δ^+ in AssB_∞^+. Once an explicit formula for δ^+ is known it is only a straightforward but tedious calculation to compute the associated expression for d using (8). Below we show such expressions for those components of δ^+ which we know explicitly.

4.5.1. **Fact**. As μ_1^α is given explicitly by Lemma 4.1.1, we immediately get from (8) the following expression for d modulo terms, $O(2)$, in $\Gamma\langle\mathsf{D}\rangle$ with number of vertices ≥ 2,

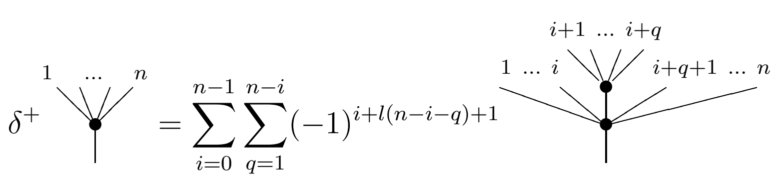
(9) 

4.5.2. **Fact**. As on the generators,

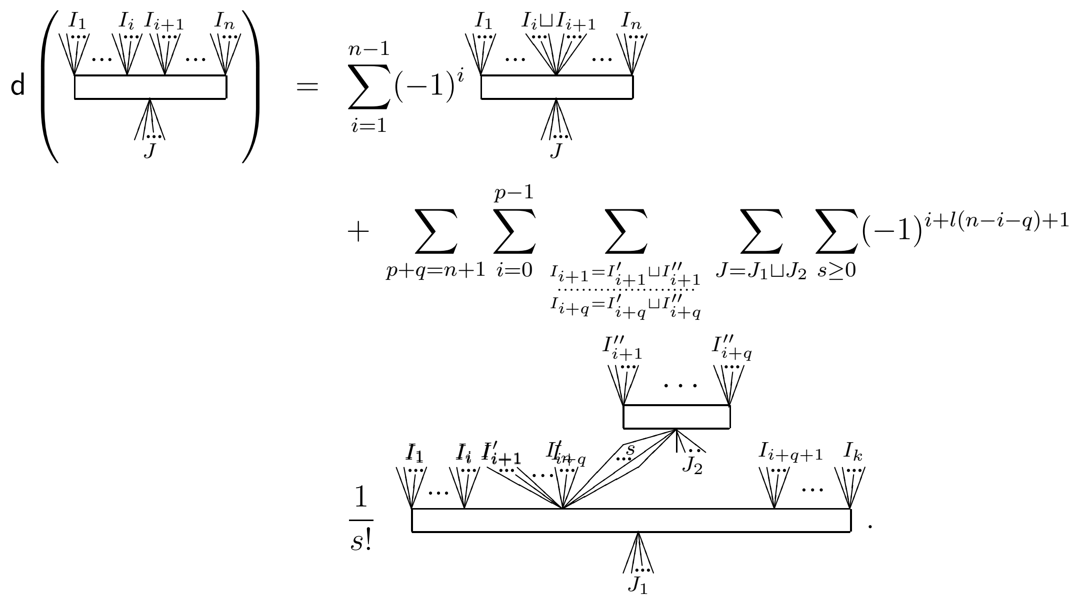


$, \quad n \geq 1,$

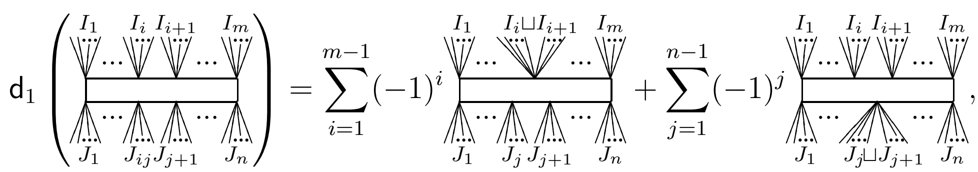
of AssB_∞^+ the differential is given by



we get after some tedious calculation (cf. [Me3]) the following expression for the value of d on the associated "thickened" generators of DefQ,



4.6. **DefQ^+ versus LieB^+_∞.** It is easy to check that a derivation, d_1, of DefQ^+, given on generators by

(10) 

is a differential. By (9), the differential d_1 is the linear in number of vertices part of the full differential d. As a first approximation to the formality interrelation between $(\mathsf{DefQ}^+, \mathsf{d})$ and $(\mathsf{LieB}^+_\infty, \delta^+)$, we notice the following

4.6.1**. Theorem.** *The cohomology, $H^\bullet(\mathsf{DefQ}^+, \mathsf{d}_1)$, is isomorphic to the completed PROP, $\widehat{\mathsf{LieB}}^+_\infty$, of strongly homotopy Lie bialgebras.*

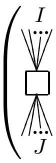
Proof. The complex $(\mathsf{DefQ}^+, \mathsf{d}_1)$ is essentially the graph incarnation of a well-studied complex which appeared, for example, in the proof of Konstant-Hochschild-Rosenberg theorem on the Hochschild cohomology of the ring $\mathcal{O}_V$ [Ko]. We omit the details and refer instead to Section 4.6.1.1 in [Ko]. □

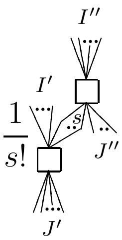
4.6.2**. Corollary.** *The cohomology of the Gerstenhaber-Schack complex, $H(\mathfrak{gs}^\bullet(\mathcal{O}_V), d_{\mathfrak{gs}})$, is isomorphic to the vector space $\mathcal{O}_{\mathcal{V}} := \widehat{\odot^{\geq 1}(V[1])} \widehat{\otimes} \widehat{\odot^{\geq 1}(V^*[1])}$.*

Proof. This Corollary is a straightforward extension of the Konstant-Hohcshild-Rosenberg theorem from the Hochschild complex of the algebra $\mathcal{O}_V$ to the Gertenhaber-Schack complex of the bialgebra $\mathcal{O}_V$. It is equivalent to Theorem 4.6.1 as differential d_1 is precisely the graph incarnation of the Gertsenhaber-Schack differential. □

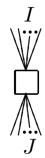
4.7. **Quotient of DefQ^+.** It follows from Fact 4.5.2 that

$$\mathsf{d}\left(\quad\right) = \sum_{I=I'\sqcup I'' \atop J=J'\sqcup J''} \sum_{s\geq 1} \frac{1}{s!}$$





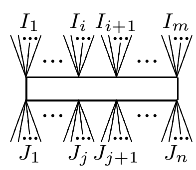
which in turn implies that the ideal, I, in DefQ generated by corollas



is differential. Hence the quotient,

$$\mathsf{DefQ} := \frac{\mathsf{DefQ}^+}{\mathsf{I}},$$

is a *differential* PROP. The induced differential we denote by d. It is a free PROP generated by corollas



with $m+n\geq 3$. Proposition 4.3.1 then implies obviously the following

4.7.1**. Proposition.** *There is a one-to-one correspondence between representations,*

$$(\mathsf{DefQ},\mathsf{d}) \longrightarrow (\mathsf{End}\langle V\rangle, d),$$

in a dg vector space (V,d) and Maurer-Cartan elements, $\Gamma=\{\Gamma_n^m\in \mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes n},\mathcal{O}_V^{\otimes m})\}_{m,n\geq 1}$, in the L_∞-algebra $(\mathfrak{gs}^\bullet(\mathcal{O}_V),\mu_\bullet^\alpha)$, whose summand Γ_1^1 is the first order linear differential operator on $\mathcal{O}_V$ induced by the differential d in V.

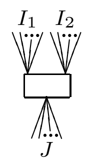
4.7.2**. Remark.** There is a canonical epimorphism of dg free PROPs, $(\mathsf{DefQ}^+,\mathsf{d}) \stackrel{p}{\longrightarrow} (\mathsf{DefQ},\mathsf{d})$.

4.7.3**. Remark.** Any morphism, $\mathcal{F}^+:\mathsf{DefQ}^+\longrightarrow \mathsf{P}$, from the PROP DefQ into a *non-positively* graded PROP P factors through the composition

$$\mathcal{F}^+:\mathsf{DefQ}^+ \stackrel{p}{\longrightarrow} \mathsf{DefQ} \stackrel{\mathcal{F}}{\longrightarrow} \mathsf{P}$$

for some uniquely defined morphism of PROPs, $\mathcal{F}:\mathsf{DefQ}\rightarrow\mathsf{P}$. Indeed, as the morphism $\mathcal{F}^+$ is uniquely defined by its values on the generators, all the positively graded corollas in DefQ^+ must be mapped to zero. The only such corollas in DefQ are precisely the ones which generate the ideal I. Thus the ideal I lies in the kernel of $\mathcal{F}^+$.

4.7.4**. Remark.** To prove Formality Theorem 1.2 we need to construct a morphism of dg PROPs, $(\mathsf{DefQ}^+, \mathsf{d}) \to (\widehat{\mathsf{LieB}}_\infty, \delta)$. As $(\widehat{\mathsf{LieB}}_\infty, \delta)$ is non-positively graded, then, by Remark 4.7.3, this problem is equivalent to the problem of constructing a morphism $\mathcal{F} : (\mathsf{DefQ}, \mathsf{d}) \to (\widehat{\mathsf{LieB}}_\infty, \delta)$. Thus, if we were only interested in proving Theorem 1.2 we could do it without introducing the extended PROP DefQ^+ and working only with DefQ (see §5 below). If, however, one is also interested in complete PROP encoding of the formality morphism F of L_∞-algebras in Theorem 1.2, then the dg PROP $(\mathsf{DefQ}^+, \mathsf{d})$ is unavoidable. The reason is that a corresponding analogue of Theorem 4.6.1 for the PROP $(\mathsf{DefQ}, \mathsf{d})$ can not be true. The cohomology $H^\bullet(\mathsf{DefQ}, \mathsf{d}_1)$ is much larger than $\widehat{\mathsf{LieB}}_\infty$, e.g. all the generators



are d_1-cycles, and their symmetrizations over $I_1 \cup I_2$ determine non-trivial cohomology classes.

5. Formality morphism of L_∞-algebras via morphism of PROPs

5.1. **Etingof-Kazhdan quantizations.** There are two universal quantizations of Lie bialgebras which were constructed in [EK]: the first one involves universal formulae with traces and hence is applicable only to finite-dimensional Lie bialgebras, while the second one avoids traces and hence is applicable to infinite-dimensional Lie bialgebras as well. In terms of PROPs, one can reinterpret these results (cf. [EE] for a similar but different viewpoint) as existence of two morphisms of dg PROPs,

$$\mathcal{EK}^{\circlearrowright} : (\mathsf{DefQ}, \mathsf{d}) \longrightarrow (\widehat{\mathsf{LieB}}^{\circlearrowright}, 0),$$

and

$$\mathcal{EK} : (\mathsf{DefQ}, \mathsf{d}) \longrightarrow (\widehat{\mathsf{LieB}}, 0),$$

where $\mathsf{LieB}^{\circlearrowright}$ is the wheeled completion of the PROP LieB introduced and studied in §3.13 of [Me3]. We do not work with $\mathcal{EK}^{\circlearrowright}$ in this paper and hence refer the interested reader to [Me3] for a precise definition of $\mathsf{LieB}^{\circlearrowright}$.

5.1.1. **Planck constant in the PROP approach to quantization**. The Etingof-Kazhdan morphism $\mathcal{EK}$ is a morphism into a *completed* PROP of Lie bialgebras. The value of $\mathcal{EK}$ on a generator of DefQ is a well-defined (probably, infinite) linear combination of graphs from $\widehat{\mathsf{LieB}}$. Thus one does *not* need a formal parameter $\hbar$ to make morphism $\mathcal{EK}$ rigorously defined[4].

However, a formal parameter $\hbar$ becomes unavoidable once we want to switch from PROPs to their representations in a particular dg space V. Let us explain its necessity in more detail. A bialgebra structure on V is a morphism of PROPs,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}\langle V\rangle,$$

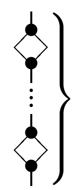
with

$$\rho\left(\begin{smallmatrix}2 & & 1\\ & \bullet & \\ & 1 & \end{smallmatrix}\right) =: \triangle \in \mathrm{Hom}(V, V\otimes V), \quad \rho\left(\begin{smallmatrix} & 1 & \\ & \bullet & \\ 1 & & 2\end{smallmatrix}\right) =: [\, ,\,] \in \mathrm{Hom}(V\otimes V, V)$$

being co-Lie and, respectively, Lie brackets in V. However, the morphism ρ can *not* be automatically extended to a representation,

$$\hat{\rho} : \widehat{\mathsf{LieB}} \longrightarrow \mathsf{End}\langle V\rangle,$$

of the completed PROP as, for example, a well-defined infinite sum of graphs in $\widehat{\mathsf{LieB}}$,

$$\sum_{n=1}^{\infty}$$

n times

gets mapped into an infinite sum of elements of $\mathrm{Hom}(V, V)$ which is divergent in general.

Thus we interpret instead a bialgebra structure in V as a morphism of PROPs,

$$\rho : \mathsf{LieB} \longrightarrow \mathsf{End}\langle V\rangle[[\hbar]],$$

by setting

$$\rho\left(\begin{smallmatrix}2 & & 1\\ & \bullet & \\ & 1 & \end{smallmatrix}\right) =: \hbar\, \triangle \in \mathrm{Hom}(V, V\otimes V)[[\hbar]], \quad \rho\left(\begin{smallmatrix} & 1 & \\ & \bullet & \\ 1 & & 2\end{smallmatrix}\right) =: \hbar[\, ,\,] \in \mathrm{Hom}(V\otimes V, V)[[\hbar]].$$

This morphism extends to a morphism

$$\hat{\rho} : \widehat{\mathsf{LieB}} \longrightarrow \mathsf{End}\langle V\rangle[[\hbar]],$$

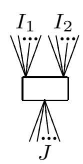
with no divergency problems. Then the composition of $\hat{\rho}$ with the Etingof-Kazhdan quantization morphism $\mathcal{EK}$ gives us a well-defined morphism of dg PROPs,

$$\mathcal{F}_\rho : (\mathsf{DefQ}, \mathsf{d}) \xrightarrow{\hat{\rho}\circ\mathcal{EK}} \mathsf{End}\langle V\rangle[[\hbar]],$$

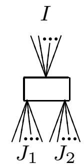
and hence, by Proposition 4.3.1, a strongly homotopy bialgebra structure on V which depends on the formal parameter $\hbar$ and equals the standard graded (co)commutative bialgebra structure

[4]at this point we differ from the approach advocated in [EE] where PROPs and morphisms are defined over formal power series ring $\mathbb{K}[[\hbar]]$ rather than over $\mathbb{K}$ as in our case.

at $\hbar = 0$. Since PROP LieB is concentrated in degree 0, the morphism $\mathcal{F}_\rho$ must take zero values on all generators of DefQ except these two families (the only ones in DefQ which have degree 0),



and



and hence defines a depending on $\hbar$ bialgebra structure in $\mathcal{O}_V$, which deforms the original bialgebra structure.

In a similar way one extends representations,

$$\rho : (\mathsf{LieB}_\infty, \delta) \longrightarrow (\mathsf{End}\langle V\rangle, d),$$

of the dg PROP of strongly homotopy bialgebra structures, to representations,

$$\hat{\rho} : (\widehat{\mathsf{LieB}}_\infty, \delta) \longrightarrow (\mathsf{End}\langle V\rangle, d),$$

of its completion.

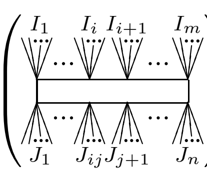
5.2. **Theorem.** *There exists a morphism of dg PROPs, $\mathcal{F}$, making the diagram*

$$\begin{tikzcd} & (\widehat{\mathsf{LieB}}_\infty, \delta) \arrow[d, "qis"] \\ (\mathsf{DefQ}, \delta) \arrow[ur, "\mathcal{F}"] \arrow[r, "\mathcal{EK}"'] & (\widehat{\mathsf{LieB}}, 0) \end{tikzcd}$$

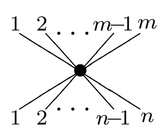
commutative. Moreover, such an $\mathcal{F}$ satisfies the following conditions:

(i) *for any generating corolla e in DefQ, the composition $p_k \circ \mathcal{F}(\mathsf{e})$ is a* finite *linear combination of graphs from LieB_∞, where p_k is the projection from $\widehat{\mathsf{LieB}}_\infty$ to its subspace spanned by decorated graphs with precisely k vertices.*

(ii) $p_1 \circ \mathcal{F}$ (



) $=$ {

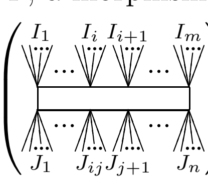


for $|I_1| = \ldots = |I_m| = |J_1| = \ldots = |J_n| = 1$

0 *otherwise.*

Proof. As DefQ is a free PROP, a morphism $\mathcal{F}$ is completely determined by its values,

$\mathcal{F}$ (

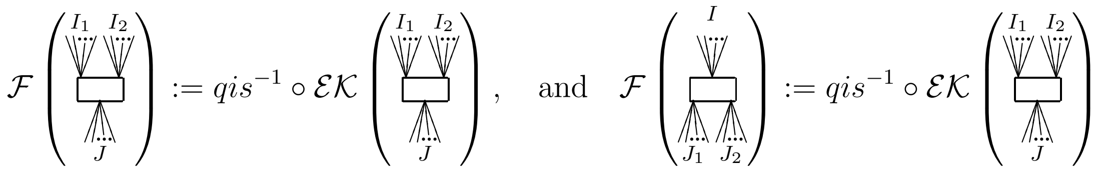


) $\in \widehat{\mathsf{LieB}}_\infty, \quad m + n \geq 3,$

on the generating corollas. We shall construct $\mathcal{F}$ by induction[5] on the "weight", $\mathfrak{w} := m+n-3$, associated to such corollas. For $\mathfrak{w} = 0$ we set $\mathcal{F}$ to be an arbitrary lift along the surjection qis

[5]this induction is an almost literal analogue of the Whitehead lifting trick in the theory of CW-complexes in algebraic topology.

of the Etingof-Kazhdan morphism $\mathcal{EK}$, i.e. we begin our induction by setting



where qis^{-1} is an arbitrary section of the quasi-isomorphism qis, i.e. an arbitrary lifting of cohomology classes into cycles.

Assume we constructed values of $\mathcal{F}$ on all corollas of weight $\mathfrak{w} \leq N$. Let e be a generating corolla of DefQ with non-zero weight $\mathfrak{w} = N + 1$. Note that de is a linear combination of graphs whose vertices are decorated by corollas of weight $\leq N$ (as $\mathfrak{w}$ is the precisely minus of the degree of e, and d increases degree by $+1$). By induction, $\mathcal{F}(\mathsf{de})$ is a well-defined element in $\widehat{\mathsf{LieB}}_\infty$. As $\mathcal{EK}(\mathsf{e}) = 0$, the element,

$$\mathcal{F}(\mathsf{de})$$

is a closed element in $\widehat{\mathsf{LieB}}_\infty$ which projects under qis to zero. Since the surjection qis is a quasi-isomorphism, this element must be exact. Thus there exists $\mathfrak{e} \in \widehat{\mathsf{LieB}}_\infty$ such that

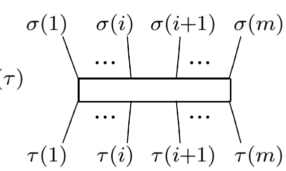
$$\delta\mathfrak{e} = \mathcal{F}(\mathsf{de}).$$

We set $\mathcal{F}(\mathsf{e}) := \mathfrak{e}$ completing thereby inductive construction of $\mathcal{F}$.

Next, if e is a generating corolla in Defq of degree $3 - m - n$ then $p_k \circ \mathcal{F}(\mathsf{e})$ is a linear combination of graphs built from k generating corollas in LieB_∞ with total number of half edges attached to these k vertices being equal $3(k-1)+m+n$. There is only a *finite* number of graphs in LieB_∞ satisfying these conditions. This proves Claim (i).

Claim (ii) is obvious in the part *otherwise*. Let

$$e_n^m := \sum_{\sigma\in\mathbb{S}_m}\sum_{\tau\in\mathbb{S}_n}(-1)^{sgn(\sigma)+sgn(\tau)}$$



be the skewsymmetrization of the generating corolla in DefQ with $|I_1| = \ldots = |I_m| = |J_1| = \ldots = |J_n| = 1$. To prove Claim (ii) it is enough to show that $p_1 \circ \mathcal{F}(e_n^m) \neq 0$ for all $m, n \geq 1$, $m + n \geq 3$. We shall show this by induction on the weight $\mathfrak{w} = m + n - 3$.

Denote the composition $p_k \circ \mathcal{F}$ by $\mathcal{F}_k$.

If $m + n = 3$, then $\mathcal{F}_1(e_n^m) \neq 0$ by the construction of $\mathcal{F}$.

Assume that $\mathcal{F}_1(e_p^q) \neq 0$ for all e_p^q with weight $\mathfrak{w} \leq N$ and consider e_n^m with non-zero weight $\mathfrak{w} = N + 1$. Then

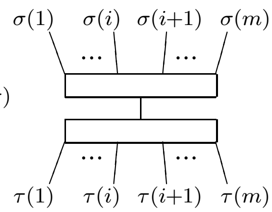
$$\mathcal{F}_1(e_n^m) \neq 0 \quad \Leftrightarrow \delta\left(\mathcal{F}_1(e_n^m)\right) \neq 0.$$

By our construction of $\mathcal{F}$, we have

$$\delta(\mathcal{F}_1(e_n^m)) = \mathcal{F}_2(\mathsf{d}e_n^m) = \mathcal{F}_2(\mathsf{d}_1 e_n^m) + \mathcal{F}_1 \boxtimes \mathcal{F}_1(\mathsf{d}_2 e_n^m),$$

where d_1 is the linear in number of vertices part of the differential d in DefQ and is given by (10), d_2 stands for the quadratic (i.e. spanned by two-vertex graphs) part of d, and $\mathcal{F}_1 \boxtimes \mathcal{F}_1$ means the morphism $\mathcal{F}_1$ applied to decoration of each of the two vertices in every graph summand of $\mathsf{d}_2 e_m^n$. Now $\mathsf{d}_1 e_n^m = 0$, while $\mathsf{d}_2 e_m^n$ contains, for example, the following linear combination of graphs,

$$\sum_{\sigma\in\mathbb{S}_m}\sum_{\tau\in\mathbb{S}_n}(-1)^{sgn(\sigma)+sgn(\tau)}$$



,

as irreducible summands coming from the genus zero part of the differential δ in AssB_∞ (see [Ma]). Then, by the induction assumption, $\mathcal{F}_1 \boxtimes \mathcal{F}_1(\mathsf{d}_2 e_n^m)$ can not be zero implying $\mathcal{F}_1(e_n^m) \neq 0$. This completes the proof of Claim (ii) and hence of the Theorem. □

5.3. **Proof of Corollary 1.3.** Let $\gamma : (\mathsf{LieB}_\infty, \delta) \to (\mathsf{End}\langle V\rangle, d)$ be a strongly homotopy Lie bialgebra structure on a dg space V, and let $\hat{\gamma} : (\widehat{\mathsf{LieB}}_\infty, \delta) \to (\mathsf{End}\langle V\rangle[[\hbar]], d)$ be the associated extension of γ to the completed PROP (see §5.1.1). Then the composition of the morphisms,

$$\hat{\gamma} \circ \mathcal{F} : (\mathsf{DefQ}, \mathsf{d}) \longrightarrow (\mathsf{End}\langle V\rangle[[\hbar]], d)$$

provides us with the required quantum strongly homotopy structure on $\mathcal{O}_V$ which depends on the parameter $\hbar$ and is equals at $\hbar = 0$ to the standard graded (co)commutative bialgebra structure.

Another proof can be given using Theorem 1.2. □

5.4. **Proof of Theorem 1.2.** We are going to construct a sequence of linear maps,

$$F_k : \wedge^k(\mathcal{O}_\mathcal{V}[2]) \longrightarrow \mathfrak{gs}^\bullet, \quad k \geq 1,$$

of degree $1-k$ satisfying quadratic relations of an L_∞-morphism,

$$\text{(11)} \qquad \sum_{\sigma\in Sh(2)} F_{k-1}\left(\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}, \ldots, f_{\sigma(k)}\right) =$$

$$= \sum_{i=1}^{k} \sum_{k=k_1+\ldots+k_i} \sum_{\sigma\in Sh(k_1,\ldots,k_r)} \pm\mu_r(F_{k_1}\left(f_{\sigma(1)}, \ldots, f_{\sigma(k_1)}\right), \ldots, F_{k_r}(f_{\sigma(k-k_r+1)}, \ldots, f_{\sigma(k)}\right),$$

where $Sh(k_1, \ldots, k_r)$ stands for the subgroup of $(k_1, \ldots, k_r)$-shuffles in $\mathbb{S}_k$, and $f_1, \ldots, f_k$ are arbitrary elements in $\mathcal{O}_\mathcal{V}[2]$. In a linear coordinate system $\{x^j\}$ on V (and the dual coordinate system $\{p_i\}$ on V^*) such an element f is a formal power series,

$$f = \sum_{m,n\geq 1} f_{j_1\ldots j_n}^{i_1\ldots i_m} x^{j_1} \wedge \ldots \wedge x^{j_n} \wedge p_{j_1} \wedge \ldots \wedge p_{j_m}.$$

We define a degree 0 linear map

$$\begin{array}{rccc} F_1: & \mathcal{O}_{\mathcal{V}}[2] & \longrightarrow & \mathfrak{gs}^\bullet(\mathcal{O}_V) \\ & f & \longrightarrow & F_1(f) \end{array}$$

by setting

$$F_1(f) := \sum_{m,n\geq 1} f^{i_1\ldots i_m}_{j_1\ldots j_n} x^{j_1}\otimes\ldots\otimes x^{j_n}\cdot\Delta^{n-1}\left(\frac{\partial}{\partial x^{i_1}}\right)\cdot\ldots\cdot\Delta^{n-1}\left(\frac{\partial}{\partial x^{i_m}}\right).$$

Clearly, $d_{\mathfrak{gs}}\circ F_1 = 0$ so that this map sends $\mathcal{O}_{\mathcal{V}}[2]$ into cycles in $\mathfrak{gs}^\bullet(\mathcal{O}_V)$.

Next we shall read off the maps F_k for $k\geq 2$ from the components, $\mathcal{F}_k$, of the morphism $\mathcal{F}$ we constructed in the proof of Theorem 5.2. As the morphism $\mathcal{F}$ lands in the world which does not contain the $(1,1)$-corolla , we are forced to set

$$F_k(f_1,\ldots,f_k) = 0,\quad k\geq 2,$$

if at least one input f_i lies in $V[1]\otimes V^*[1]$. Thus F_k for $k\geq 2$ must factor through the projection,

$$\wedge^k(\mathcal{O}_{\mathcal{V}}[2])\longrightarrow\wedge^k\mathfrak{g}\xrightarrow{F_k}\mathfrak{gs}^\bullet(\mathcal{O}_V),$$

where

$$\mathfrak{g} := \bigoplus_{\substack{m,n\geq 1\\ m+n\geq 3}}\wedge^m V\otimes\wedge^n V^*[2-m-n]\subset\mathcal{O}_{\mathcal{V}}[2].$$

From now on we identify the latter with the vector space of $\mathbb{S}$-equivariant linear maps,

$$\mathfrak{g}\equiv \mathrm{Hom}(\mathsf{L},\mathsf{End}\langle V\rangle)[-1],$$

where L is the $\mathbb{S}$-bimodule of generators of the PROP LieB_∞ (see (2) for the definition).

The maps F_k will be defined once we define the compositions,

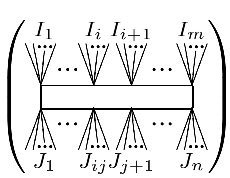
$$F(k)^{|I_1|,\ldots,|I_m|}_{|J_1|,\ldots,|J_n|}:\wedge^k\mathfrak{g}\xrightarrow{F_k}\mathfrak{gs}^\bullet(\mathcal{O}_V)\xrightarrow{proj}\mathrm{Hom}_{poly}(\mathcal{O}_V^{\otimes n},\mathcal{O}_V^{\otimes m})\xrightarrow{proj}$$

$$\xrightarrow{proj}\mathrm{Hom}\left(\odot^{|J_1|}V\otimes\ldots\otimes\odot^{|J_n|}V,\odot^{|I_1|}\otimes\ldots\otimes\odot^{|I_m|}V\right).$$

for all $k\geq 2$, $m+n\geq 3$, and $|J_1|,\ldots,|J_n|,|I_1|,\ldots,|I_m|\geq 1$.

We define $F(k)^{|I_1|,\ldots,|I_m|}_{|J_1|,\ldots,|J_n|}(f_1,\ldots,f_k)$ for arbitrary $f_1,\ldots,f_k\in\mathrm{Hom}(\mathsf{L},\mathsf{End}\langle V\rangle)[-1]$ in three steps:

Step 1. Consider

$\mathcal{F}_k$ 

$$= \sum_G G\langle l_1,l_2,\ldots,l_k\rangle_{Aut(G)}\in\mathsf{LieB}_\infty$$

where the sun runs over a family of graphs with k vertices, $Vert(G) = \{v_1, \ldots, v_k\}$, decorated with some elements $l_i \in \mathsf{L}(|In(v_i)|, |Out(v_i)|$, $i = 1, \ldots, k$, where $|In(v_i)|$ (resp. $|Out(v_i)|$) stands for the number of input (resp. output) half edges attached to the vertex v_i and

$$G\langle l_{v_1}, l_{v_2}, \ldots, l_{v_k}\rangle := \left(\sum_{s:[k]\rightarrow Vert(G)} l_{s(1)} \otimes \ldots \otimes l_{s(k)}\right)_{\mathbb{S}_k}$$

stands for the *unordered* tensor product of l_i over the set $Vert(G)$.

Step 2. Define next

$$\mathcal{F}_k(f_1, \ldots, f_k) := \sum_G \sum_{\sigma\in\mathbb{S}_k} (-1)^{Koszul\ sgn} G\langle f_{\sigma(1)}(l_1), f_{\sigma(2)}(l_2), \ldots, f_{\sigma(k)}(l_k)\rangle_{Aut(G)}.$$

This is a sum of graphs whose vertices are decorated by elements of the endomorphism PROP $\mathsf{End}\langle V\rangle$. Hence we can apply to $\mathcal{F}_k(f_1, \ldots, f_k)$ the vertical and horizontal $\mathsf{End}\langle V\rangle$-compositions to get a well defined element,

$$comp_{\mathsf{End}\langle V\rangle}\left(\mathcal{F}_k(f_1, \ldots, f_k)\right) \in \mathsf{End}\langle V\rangle,$$

which, in fact, lies by the construction in the subspace

$$\mathrm{Hom}\left(\odot^{|J_1|}V \otimes \ldots \otimes \odot^{|J_n|}V, \odot^{|I_1|}V \otimes \ldots \otimes \odot^{|I_m|}V\right) \subset \mathsf{End}\langle V\rangle.$$

Step 3. Finally set

$$F(k)^{|I_1|,\ldots,|I_m|}_{|J_1|,\ldots,|J_n|}(f_1, \ldots, f_k) := comp_{\mathsf{End}\langle V\rangle}\left(\mathcal{F}_k(l_1, \ldots, l_k)\right).$$

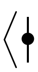
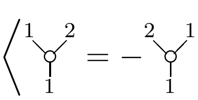
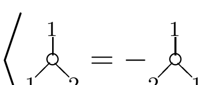
It is easy to check that the constructed map F_k has degree $1-k$. It is also a straightforward untwisting of the definitions of differentials d in Defq and δ in LieB_∞ to show that equations (11) follow directly from the basic property, $\delta \circ \mathcal{F} = \mathcal{F} \circ \mathsf{d}$, of the morphism $\mathcal{F}$. □

5.5. **Homotopy equivalence of PROPs.** The functor $(\ )^+$ introduced in § 4.2.3 for dg free PROPs can be easily extended to quotients of free PROPs, $\Gamma\langle E\rangle/I$, modulo some ideal I. For example, let $\mathsf{LieB} = \Gamma\langle L\rangle/(R)$ be the PROP of Lie bialgebras as defined in § 2.2. We construct its dg extension,

$$\mathsf{LieB}^+ = \Gamma\langle L^+\rangle/(R),$$

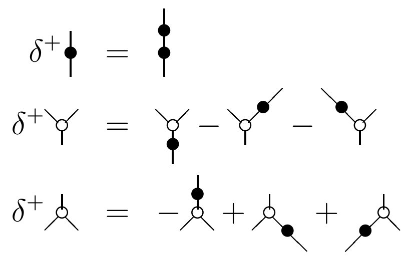
as the quotient of the free PROP on the bimodule, $L^+ = \{L^+(m,n)\}_{m,n\geq 1}$,

$L^+(m,n) :=$

- $\mathbb{1}_1 \otimes \mathbb{1}_1[-1] \equiv \mathrm{span}\langle$  $\rangle$ if $m=1, n=1$,
- $sgn_2 \otimes \mathbb{1}_1 \equiv \mathrm{span}\langle$  $\rangle$ if $m=2, n=1$,
- $\mathbb{1}_1 \otimes sgn_2 \equiv \mathrm{span}\langle$  $\rangle$ if $m=1, n=2$,
- 0 otherwise

modulo the ideal, (R), generated by the same relations as in (1).

5.5.1. **Lemma.** *The data,*



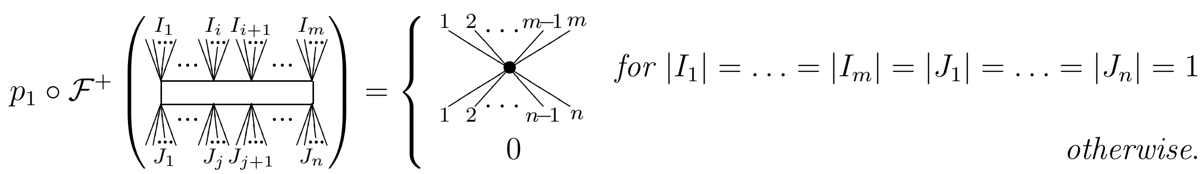
makes LieB^+ *into a dg PROP with trivial cohomology.*

Proof. It is easy to check that $(\delta^+)^2 = 0$ and that $\delta^+ R \subset (R)$ so that $(\mathsf{LieB}^+, \delta^+)$ is indeed a dg PROP. The statement about triviality of cohomology is also nearly obvious. □

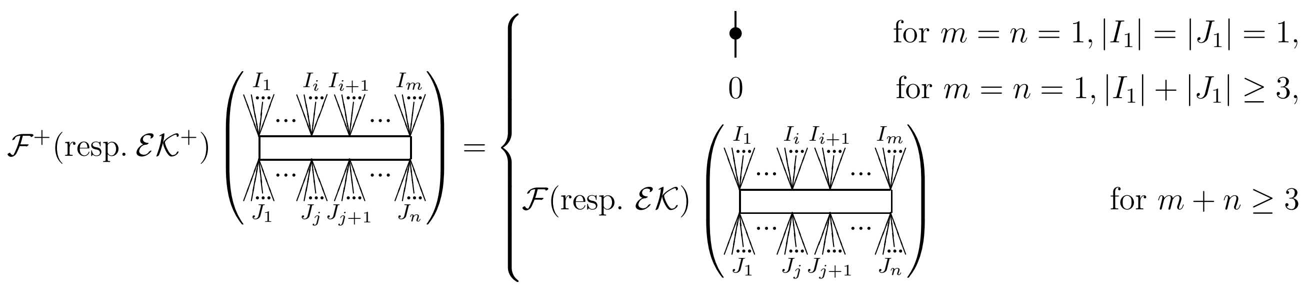
5.5.2. **Proposition.** *There exists a commutative diagram,*

$$\begin{array}{ccc} & & (\widehat{\mathsf{LieB}}^+_\infty, \delta^+) \\ & \overset{\mathcal{F}^+}{\nearrow} & \downarrow qis \\ (\mathsf{DefQ}^+, \delta^+) & \xrightarrow[\mathcal{EK}^+]{} & (\widehat{\mathsf{LieB}}^+, \delta^+) \end{array}$$

with every morphism being a homotopy equivalence, i.e. a quasi-isomorphism. Moreover,



Proof. We define morphisms $\mathcal{F}^+$ and, respectively, $\mathcal{EK}^+$ by their values on the generators,



It is easy to check that they commute with the differentials. All the other claims are almost obvious corollaries of Theorems 4.6.1 and 5.2. □

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