AMPLE LINE BUNDLES ON BLOWN UP SURFACES

Oliver Küchle

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

MPI / 94-119

AMPLE LINE BUNDLES ON BLOWN UP SURFACES

Oliver Küchle

Abstract. Given a smooth complex projective surface S and an ample divisor H on S, consider the blow up of S along k points in general position. Let H' be the pullback of H and $E_1, ..., E_k$ be the exceptional divisors. We show that $L = nH' - E_1 - ... - E_k$ is ample if and only if L^2 is positive provided the integer n is at least 3.

Introduction.

In this note we give an answer to the following question: Given a smooth projective surface S over $\mathbb{C}$ and an ample divisor H on S, consider the blow up $f : S' \longrightarrow S$ of S along k points in general position. Let $H' = f^*H$ and $E_1, \ldots, E_k$ be the exceptional divisors. When is the divisor

$$L = nH' - \sum_{i=1}^{k} E_i$$

ample ?

We show that the condition $L^2 > 0$, which clearly is necessary, is also sufficient provided the integer n is at least 3. Note that the answer to this question has been unknown even in the case of $S = \mathbb{P}^2$ (cf. [Fuj]). The basic idea is to study the situation on the surface S with variational methods.

Shortly after this work has been completed the author learned that Geng Xu obtained a similar result in the case of $S = \mathbb{P}^2$ independently.

It's a pleasure to thank Rob Lazarsfeld, who introduced me to this circle of ideas.

Proofs.

The main technical tool is an estimate on the self-intersection of moving singular curves established by Ein, Lazarsfeld and Xu in the context of Seshadri constants of ample line bundles on smooth surfaces (cf. [EL],1.2, and [Laz], 5.16). The precise statement is:

Proposition. *Let $\{C_t\}_{t \in \Delta}$ be a 1-parameter family of reduced irreducible curves on a smooth projective surface X, and $y, y_1, \ldots, y_r \in X$ be distinct points such that*

Supported by Max-Planck-Gesellschaft

$mult_{y_i}C_t \geq m_i$ for all $t \in \Delta$ and $i = 1, \ldots, r$. Suppose there exist t, t' with $mult_y C_t = m > 0$ and $y \notin C_{t'}$. Then

$$(C_t)^2 \geq m(m-1) + \sum_{i=1}^{r} m_i^2.$$

□

Since ampleness is an open condition in a flat family of line bundles, it is enough to show the existence of one k-tuple $p_1, \ldots, p_k$ of distinct points such that L is ample on the blow up S' along these points; then the same will hold for k points in general position, i.e. outside a Zariski closed proper subset of $S \times \cdots \times S$.

Using the Proposition we can prove:

Theorem. *Let $a > 2$ be a rational number. Then there exists a surface S' as above such that for the $\mathbb{Q}$-divisor*

$$M = aH' - \sum_{i=1}^{k} E_i$$

the following hold:

(1) *If $M^2 = a^2H^2 - k \geq 2$, then M is ample on S'.*

(2) *If $M^2 = a^2H^2 - k \geq 1$, then M is positive on all curves $C' \subset S'$ for which j exists with $C'.E_j \geq 2$.*

Proof. Suppose the theorem is not true, and choose an irreducible curve $C' \subset S'$ such that $M.C' \leq 0$. Consider $C = f(C')$. Defining $m_i = mult_{p_i}(C)$, we may suppose that $m_1 \geq \cdots \geq m_k$. Since $M.C' \leq 0$, we have

$$\sum_{i=1}^{k} m_i \geq a(H.C). \qquad (*)$$

Now we may assume that

- C passes through all the points p_i, i.e. $m_i \geq 1$
- C is irreducible and reduced
- C moves

Here C moves even in the strong sense, that is, fixing $p_1, \ldots, p_{k-1}$, the curve C still moves in a family of curves satisfying $(*)$. To see this simply observe that any curve on S lies in one of countably many families, but no neighbourhood of p_k is covered by countably many curves.

Finally we claim that a general member of this family has sufficiently big multiplicity at $p_1, \ldots, p_{k-1}$. But any member satisfies $(*)$, so this follows from semicontinuity.

Therefore we can apply the Proposition and obtain the estimate

$$C \cdot C \geq m_1^2 + \cdots + m_{k-1}^2 + m_k(m_k - 1),$$

and hence combined with the Hodge-Index-Theorem

$$\left(\sum_{i=1}^{k} m_i\right)^2 \geq a^2(H.C)^2 \geq a^2 H^2 \cdot C^2 \geq a^2 H^2 \Big(\sum_{i=1}^{k} m_i^2 - m_k\Big). \qquad (**)$$

By $(*)$, $(**)$ and the assumption $a > 2$ we may assume $k \geq 2$ in the following.

Suppose for the time being that C is not smooth at one of the p_j, which is the case if and only if $C'.E_j \geq 2$. Then $m_1 \geq 2$, and $(**)$ contradicts the following Lemma:

Lemma. *Let $k \geq 2$ and $x_1, \ldots, x_k \in \mathbb{Z}$ be integers with $x_1 \geq \cdots \geq x_k \geq 1$ and $x_1 \geq 2$. Then we have*

$$(k+1)\sum_{i=1}^{k} x_i^2 > \Big(\sum_{i=1}^{k} x_i\Big)^2 + x_k(k+1).$$

Proof of the Lemma. We argue by induction on $k \geq 2$.

For $k = 2$ one proves

$$3(x_1^2 + x_2^2) - (x_1 + x_2)^2 - 3x_2 > 0$$

by minimizing this expression with respect to x_2. From the inductive hypothesis, we then obtain

$$(k+1)\sum_{i=1}^{k} x_i^2 > kx_k^2 + \sum_{i=1}^{k} x_i^2 + \left(\sum_{i=1}^{k-1} x_i\right)^2 + kx_k$$

$$= \left(\sum_{i=1}^{k} x_i\right)^2 + x_k(k+1) - x_k^2 - 2 \cdot \sum_{i=1}^{k-1} x_i x_k - x_k + kx_k^2 + \sum_{i=1}^{k} x_i^2$$

$$= \left(\sum_{i=1}^{k} x_i\right)^2 + x_k(k+1) + \sum_{i=1}^{k-1}(x_i - x_k)^2 + x_k^2 - x_k.$$

So what we need to show is

$$\sum_{i=1}^{k-1}(x_i - x_k)^2 + x_k^2 \geq x_k,$$

but this is obvious.

□

This proves the second part of the Theorem. To prove the first part it remains to exclude the case $m_1 = \cdots = m_k = 1$. But then $(**)$ reads

$$k^2 \geq H^2 \cdot a^2(k-1),$$

contradicting the assumptions on a.

□

Corollary. *Let L be as in the introduction. Then L is ample if and only if $L^2 > 0$.*

Proof. It clearly suffices to prove the if–part. So suppose $L^2 > 0$ and that L is not ample. Then by the Theorem we know that $L^2 = 1$, i.e. $n^2H^2 = k + 1$, and that there exists an irreducible reduced curve $C \subset S$ which is smooth at all the p_i satisfying $k \geq n(H.C)$.

We claim that $k = n(H.C)$ holds. Otherwise we have $L.C' < 0$. Consider the surface $\hat{S}$ obtained from S' by contracting the exceptional divisor E_j, where j is an index such that C passes smoothly through p_j. The image $\hat{L}$ of L then satisfies $\hat{L}^2 = L^2 + 1 = 2$, hence it is ample by the Theorem. But this contradicts $L.C' + 1 = \hat{L}.\hat{C} \leq 0$ for the image $\hat{C}$ of C'.

Therefore we conclude $k + 1 = n^2H^2 = n(H.C) + 1$, but this is impossible since besides $n \neq 1$ also H^2 and $(H.C)$ are integers.

□

Remark. The example of a line in $\mathbb{P}^2$ through any two points shows that we cannot drop the assumption $n \geq 3$ in general. On the other hand an analysis of the proof shows that the Corollary still holds in the case $n \geq 2$ if two general points on S can not be joined by a curve C with $(H.C) = 1$, which is true e.g. whenever $H^2 \geq 2$.

References

[EL] L. Ein and R. Lazarsfeld, *Seshadri constants on smooth surfaces*, Journées de Géometrie Algébrique d'Orsay, Astérisque **218** (1993), 177-186.

[Fuj] T. Fujita, *Problems*, Birational geometry of algebraic varieties, open problems. Taniguchi foundation, 23rd international Symposium (1988), 42-45.

[Laz] R. Lazarsfeld, *Lectures on linear series*, Park City/IAS Math. Series vol. 3 (to appear).

Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 53225 Bonn

E-mail address: kuechle@mpim-bonn.mpg.de