EQUIVALIANT SURGERY THEORY: DELETING-INSERTING THEOREM OF FIXED POINT MANIFOLDS ON SPHERES

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Abstract. The paper gives a tool to delete and insert fixed point manifolds on a spheres. It is useful to classify smooth actions on spheres from the view point of fixed point data. The methods employed in the present paper are equivariant surgery and equivariant connected sum associated with elements in the Burnside ring.

1991 Mathematics Subject Classifications. 57R67, 57R85, 57R91, 19G24.

Keywords. smooth action, fixed point, equivariant surgery.

0. Introduction

A routine to construct smooth exotic actions on closed manifolds is to proceed the two steps:

(Step I) Construction of an equivariant normal map.

(Step II) Killing equivariant surgery obstructions.

This program with systematic methods was presented by T. Petrie [Pe1–4] and by his joint works with K. H. Dovermann [DPe1–2] and with J. Randall [PeR]. Briefly speaking, Petrie's methods are equivariant transverse construction of normal maps, obstruction theory to equivariant surgery, and induction theory of equivariant surgery obstructions. W. Lück and I. Madsen [LüMa] studied equivariant surgery theory with bundle data simpler than Petrie's data and provided a bridge [LüMa, Proposition (A2)] from Perie's

Partially supported by Max-Planck-Institut für Mathematik in Bonn and by Grant-in-Aid for Scientific Research Japan.

data to their data. In the classical equivariant surgery theory, the obstruction groups were Wall's L-groups. Since 1986, we have developed equivariant surgery theory involving new obstruction groups, namely L-groups with form parameters (Bak's groups) [M2–3] and those with positioning maps [BM1–3] with A. Bak. Using the induction theory [M2, §7], [BM1–2], [LM, §6] of new surgery obstruction groups and equivariant connected sum operations associated with elements in the Burnside ring, we constructed smooth one fixed point actions on spheres in [M1], [M4], [LMP] and [LM] with E. Laitinen and K. Pawałowski. In addition, we had the details of (Step I) in [M5], choosing fundamentals in [Pe1–3], [LMP] and [LM]. In the current paper, we treat (Step II) in the case of smooth actions on spheres, adopting ideas from [OPe], [BM2–3], [LMP], [LM], and obtain a basic result (Theorem 1.1).

Throughout the paper, let G be a finite group. Let $\mathcal{S}(G)$ stand for the set of all subgroups of G. If p and q are primes or 1 then $\mathcal{G}_p^q$ denotes the family of all finite groups L admitting at least one, normal series $P \triangleleft H \triangleleft L$ such that $|P|$ is a power of p (possibly $|P| = 1$), H/P is cyclic and $|L/H|$ is a power of q. We set

$$\mathcal{G}^q = \bigcup_p \mathcal{G}_p^q \quad \text{and} \quad \mathcal{G}^q(G) = \{L \leq G \mid L \in \mathcal{G}^q\}.$$

A finite group G is called an *Oliver* group if $G \notin \mathcal{G}_p^q$ for any primes p and q (cf. [O]). Clearly any finite, nonsolvable group is an Oliver group. Let $\mathcal{P}(G)$ denote the set of all subgroups P of G such that $|P|$ is a prime power. For a prime p, let G^p denote the smallest normal subgroup H of G having the property that $|G/H| = p^\ell$ for some nonnegative integer ℓ. Let $\mathcal{L}(G)$ denote the set of all subgroups H such that $G^p \subset H$ for some prime $p||G|$. Define the set $\mathcal{M}(G)$ by $\mathcal{M}(G) = \mathcal{S}(G) \setminus \mathcal{L}(G)$ (cf. [M4–5]). The set $\mathcal{M}(G)$ coincides with the set of all isotropy subgroups of G appearing on the principal $(G, \mathcal{M}(G))$-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p||G|} (\mathbb{R}[G/G^p] - \mathbb{R})$$

except the origin. G-surgery theory has been developed only on manifolds X satisfying the weak gap condition (WGC):

(WGC1) If $P \in \mathcal{P}(G)$ and $H > P$ then $2 \dim X^H \leq \dim X^P$.

(WGC2) If $P \in \mathcal{P}(G)$, $H > P$ and $2 \dim X^H = \dim X^P$ then $[H : P] = 2$, $\dim X^H > \dim X^K + 1$ for any $K > H$, and X^H is connected and orientable so that $g : X^H \to X^H$ is orientation preserving for any $g \in N_G(H)$.

(WGC3) If $P \in \mathcal{P}(G)$, $H > P$, $K > P$, $2 \dim V(G)^H = \dim V(G)^P$ and $2 \dim V(G)^K = \dim V(G)^P$ then the smallest subgroup $\langle H, K\rangle$ including H and K belongs to $\mathcal{M}(G)$.

Remark. The connectivity and orientability of X^H in (WGC2) can be removable to define a G-surgery obstruction group and a G-surgery obstruction. They are invoked here because they are assumed in [BM2–3].

If G is an Oliver group, the G-manifold $V(G) \oplus V(G)$ satisfies the weak gap condition. Condition (WGC) may look complicated but, perhaps, it is not artificial. We should remark that if X satisfies the gap condition

(GC) If $P \in \mathcal{P}(G)$ and $H > P$ then $2 \dim X^H < \dim X^P$,

then (WGC) is clearly satisfied.

A closed, smooth manifold Y is called a *homotopy sphere* if Y is homotopy equivalent to the sphere of the same dimension as Y. A smooth G-manifold Y is said to be *G-oriented* if all Y^H $(H \le G)$ are oriented in such a way that Y^H and Y^K have same orientation when $Y^H = Y^K$ and each $g \in G$ acts as an orientation preserving map $X^H \to X^{gHg^{-1}}$ for any $H \le G$. For $y \in Y$, let G_y denote the isotropy subgroup at y of G, i.e. $G_y = \{g \in G \mid gy = y\}$. Given $H \le G$, we set

$$Y^{=H} = \{y \in Y \mid G_y = H\}.$$

If A is a G-invariant smooth submanifold of Y then $\nu(A, Y)$ denote the equivariant normal bundle of A in Y.

Theorem 0.1. *Let G be a finite Oliver group and Y a G-oriented, smooth G-manifold such that its underlying manifold is a homotopy sphere of dimension $n \ge 5$ and $Y^G \ne \emptyset$. Let F_1, F_2, $\dots$, F_s denote the connected components of the G-fixed point set of Y. Suppose that Y satisfies the hypotheses* (0.1.1)–(0.1.5)*:*

(0.1.1) *Y^P is connected and simply connected for any subgroup $P \in \mathcal{P}(G)$.*

(0.1.2) *If $H \in \mathcal{L}(G)$ then for each connected component Y_α of Y^H, the number of the connected components of $Y^G \cap Y_\alpha$ is at most 1.*

(0.1.3) $\dim Y^{=H} \ge 2$ *for any $H \in \mathcal{M}(G)$.*

(0.1.4) $\dim Y^P \ge 5$ *for any $P \in \mathcal{P}(G)$.*

(0.1.5) *Y satisfies the weak gap condition* (WGC).

Then one can delete and insert the manifolds F_i's to the sphere of dimenision $\dim Y$ *as fixed ponit sets. Precisely to say, for arbitrary nonnegative integers n_1, n_2, ..., n_s such that at least one of them is positive, there exists a smooth G-action on the standard sphere S of dimension* $\dim Y$ *such that the G-fixed point set S^G is the disjoint union of F_i's:*

$$S^G = \coprod_{i=1}^{s} \coprod_{j=1}^{n_s} F_{i,j} \qquad (F_{i,j} = F_i),$$

and $\nu(F_{i,j}, S)$ is G-isomorphic to $\nu(F_i, Y)$.

Corollary 0.2 ([LM, Theorem A])**.** *A finite group G admits a smooth, one fixed point action on a sphere if and only if G is an Oliver group.*

Proof. The only if part is well known, actually it follows from [O, Theorem 7]. Let G be an Oliver group, $V(G)$ the principal $(G, \mathcal{M}(G))$-module and ℓ an even integer ≥ 6. Set $Y = S(\mathbb{R} \oplus \ell V(G))$. Then one gets $Y^G = F_1 \coprod F_2$, $F_1 = \{(-1, 0)\}$ and $F_2 = \{(1, 0)\}$. Apply Theorem 0.1 for $n_1 = 1$ and $n_2 = 0$ and obtain a smooth, one fixed point action on a sphere.

Theorem 0.1 (or its generalization Theorem 1.1) is useful to classify fixed point sets of smooth actions on spheres. For example, we shall see elsewhere that for a nontrivial perfect group G, a closed smooth manifold can be the fixed point set of a smooth, $\mathcal{P}(G)$-proper G-action on a sphere if and only if it can be the fixed point set of a smooth G-action on a disk. (A $\mathcal{P}(G)$-proper G-action on X means that $X^P \supsetneq X^G$ holds for every Sylow p-subgroup P of G.) The fixed point manifolds of smooth G-actions on disks are determined by Oliver [O2]. Here we remains to state a partial result.

Corollary 0.3. *Let G be a nontrivial perfect group and let F be a smooth, closed manifold. If F is the G-fixed point set of a compact, contractible, G-oriented, smooth G-manifold M such that for each $P \in \mathcal{P}(G)$, M^P is simply connected and ∂M^P is connected, then F is the G-fixed point set of a smooth G-action on a sphere.*

Proof. Since G is perfect, $\mathcal{M}(G) = \mathcal{S}(G) \setminus \{G\}$ and $V(G) = \mathbb{R}[G] - \mathbb{R}$. In particular, $V(G)$ satisfies the strong gap condition. Taking a sufficiently large even integer ℓ, let Y be the double of $M \times D(\ell V(G))$. Then $Y^G = F_1 \coprod F_2$ ($F_i = F$). Apply Theorem 0.1 for $n_1 = 1$ and $n_2 = 0$ and obtain a smooth G-action on a sphere whose fixed point set is F.

Sketch of the proof of Theorem 0.1. First, we shall construct a G-normal map (f,b) such that $X^G = \coprod_{i,j} F_{i,j}$ and $\nu(F_{i,j},X) \cong \nu(F_i,Y)$, where $f : X \to Y$ is a degree one G-map and

$$b : \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^{2m}) \to f^*(\varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \varepsilon_Y(\mathbb{R}^{2m}))$$

is a G-vector bundle isomorphism. (G-normal maps with bundle data of this kind are studied in W. Lück and I. Madsen [LüMa], although the strong gap condition is assumed. It is easier to understand the bundle data of Lück and Madsen than the systematic bundle data of Petrie. A bridge between bundle data of the two kinds is Proposition (A2) of [LüMa].) The original ideas of the construction were presented by T. Petrie [Pe1–3], and details of the construction with useful properties were given in [LM] and [M5]. This (f,b) will be modified step by step by G-surgery and G-connected sum as follows. We shall modify (f,b) so that X^P will be simply connected and be a $\mathbb{Z}_{(p)}$-homology sphere for any $P \in \mathcal{S}(G) \setminus \{1\}$, where p is the prime dividing $|P|$. This will be done inductively with respect to the order from a maximal P to a minimal P. Finally, we shall modify (f,b) so that X will be a homotopy sphere. Our G-connected sum and G-surgery won't change a tubular neighborhood $\nu(X^G,X)$ of X^G in X. Although our modification by G-connected sum won't necessary preserve the property that f is of degree one, it won't cause more difficulties than [LM] did. Here we briefly note the reason. If B is an R-homology sphere ($R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for a prime p) then the groups associated with a map $h : A \to B$

$$K_j(h;R) = \operatorname{Ker}[h_* : H_j(A;R) \to H_j(B;R)] \quad \text{and}$$

$$K^j(h;R) = \operatorname{Coker}[h^* : H^j(B;R) \to H^j(A;R)]$$

are nothing but $H_j(A;R)$ and $H^j(A;R)$ respectively ($0 < j < \dim B$). (The author learned it from T. Petrie.) Thus, even if h is not of degree one, $K_*(h;R)$ satisfies the Poincaré duality and the universal coefficient theorem within the range $0 < * < \dim B$. This enables us to use G-surgery theory without the property 'degree one'.

We give a generalization of Theorem 0.1 as Theorem 1.1. It may look very technical. However, the complicated conditions in the theorem expose where our methods have difficulties. Moreover, Theorem 1.1 is needed when we apply it to a classification problem of the fixed point sets of smooth G-actions on spheres.

The organization of the paper is as follows. Killing equivariant surgery obstructions, equivariant connected sum associated with an element in the Burnside ring plays a key role. We give the definition of it in §2. A starting point to obtain smooth exotic actions on spheres is the construction of an equivariant normal map. In §3, using results in [M5], we specify initial data of the normal map. We complete in §4 the proof of Theorem 1.1.

1. A refinement of Theorem 0.1

Let G be a finite group. For a finite G-CW-complex Y, the G-poset $\Pi_G(Y)$ (cf. [OPe]) is defined as the disjoint union of the set $\pi_0(Y^H)$ of connected components of the H-fixed point set Y^H, where H runs over $\mathcal{S}(G)$. Define the map $\rho = \rho_Y : \Pi_G(Y) \to \mathcal{S}(G)$ by assigning H to $\alpha \in \pi_0(Y^H)$. If $\alpha \in \Pi_G(Y)$ then $|\alpha|$ and Y_α denote the underlying space of α. For α, $\beta \in \Pi_G(Y)$, we say $\alpha < \beta$ if $\rho(\alpha) \supsetneqq \rho(\beta)$ and $Y_\alpha \subseteq Y_\beta$. Let $\mathcal{F}$ be a subset of $\mathcal{S}(G)$. We set

$$\Pi_G(Y, \mathcal{F}) = \rho^{-1}(\mathcal{F}).$$

Let $f : X \to Y$ be a G-map. For $\alpha \in \Pi_G(Y)$, set

$$X_\alpha = f^{-1}(Y_\alpha) \cap X^{\rho(\alpha)},$$

and let $f_\alpha : X_\alpha \to Y_\alpha$ denote the restriction of f. If $\alpha \in \Pi_G(X)$ then the element $f_*\alpha \in \Pi_G(Y)$ is specified by $\rho_Y(f_*\alpha) = \rho_X(\alpha)$ and $Y_{f_*\alpha} \supseteq f(X_\alpha)$.

Let ξ be a real G-vector bundle over Y, with a G-invariant Riemannian metric. The total space of ξ is also denoted by ξ. But, when we regard it as a set, we use $E(\xi)$ for the total space. If $H \leq G$, namely H is a subgroup of G, and A is an H-invariant subset of Y, the restriction ξ_A of ξ to A can be regarded as a real H-vector bundle. Moreover, if $\alpha \in \Pi_H(A)$ then ξ_α stands for $(\xi_A)_{|\alpha|}$. For $\alpha \in \pi_0(A^K)$, the restriction of ξ^K to the space A_α is denoted by ξ^α and its orthogonal complement is denoted by $\nu(\xi^\alpha, \xi_\alpha)$ or $\xi^{\alpha\perp}$, namely $\xi_\alpha = \xi^\alpha \oplus \xi^{\alpha\perp}$. Let

$$\boldsymbol{\mathcal{S}} = \{(A_\lambda, \mathcal{F}_\lambda) \mid \lambda \in \Lambda\}$$

be a set of pairs $(A_\lambda, \mathcal{F}_\lambda)$ consisting of an H_λ-invariant subset $A_\lambda \subseteq Y$ $(H_\lambda \leq G)$ and a subset $\mathcal{F}_\lambda \subseteq \Pi_{H_\lambda}(A_\lambda)$.

A real G-vector bundle ξ over Y is said to be $\boldsymbol{\mathcal{S}}$-*oriented* if (E1) and (E2) are satisfied:

(E1) For any $\lambda \in \Lambda$ and $\alpha \in \mathcal{F}_\lambda$, the vector bundle ξ^α is an oriented real vector bundle.

(E2) If $E((\xi^\beta)_{A_\beta \cap gA_\alpha}) = gE((\xi^\alpha)_{A_\alpha \cap g^{-1}A_\beta})$ for some λ, $\mu \in \Lambda$, $\alpha \in \mathcal{F}_\lambda$, $\beta \in \mathcal{A}_\mu$ and $g \in G$ then $g : (\xi^\alpha)_{A_\alpha \cap g^{-1}A_\beta} \to (\xi^\beta)_{A_\beta \cap gA_\alpha}$ is orientation preserving.

If ξ is $\{(Y, \Pi_G(Y))\}$-oriented (resp. $\{(Y, \Pi_G(Y, \mathcal{F}))\}$-oriented) then ξ is simply called G-*oriented* (resp. $\mathcal{F}$-*oriented*). Suppose $Y^G \neq \emptyset$ and let y_0 be a point in Y^G. Then set

$$\boldsymbol{\mathcal{S}}(\mathcal{F}, y_0) = \{(Y, \Pi_G(Y, \mathcal{F})), (\{y_0\}, \Pi_G(\{y_0\}))\}.$$

If ξ is $\boldsymbol{\mathcal{S}}(\mathcal{F}, y_0)$-oriented then we say that $(\xi; \xi_{y_0})$ is $(\mathcal{F}; G)$-*oriented.*

If Y is a smooth G-manifold then let $T(Y)$ denote the tangent bundle with the canonical G-action. A compact, smooth G-manifold Y is said to be $\boldsymbol{\mathcal{S}}$-*oriented* if so is $T(Y)$.

Theorem 1.1 (Deleting-Inserting Theorem). *Let G be a finite Oliver group and let Y be a smooth G-manifold such that its underlying manifold is a homotopy sphere of dimension $n \geq 5$ and $Y^G \neq \emptyset$. Let F_1, F_2, ..., F_s ($y_0 \in F_1$) denote the connected components of the G-fixed point set of Y and let y_0 be a point in Y^G. Suppose that Y satisfies the hypotheses* (H1)–(H6)*:*

(H1) *$(T(Y); T_{y_0}(Y))$ is $(\mathcal{P}(G); G)$-oriented.*

(H2) *For each $P \in \mathcal{P}(G)$, Y^P is connected and simply connected.*

(H3) *If $H \in \mathcal{L}(G)$ then for every connected component Y_α of Y^H, the number of the connected components of $Y^G \cap Y_\alpha$ is at most 1.*

(H4) *If $\alpha \in \Pi_G(Y, \mathcal{P}(G))$ and $K \in \mathcal{S}(G)$ satisfy that $\rho(\alpha) \triangleleft K$ and $K/\rho(\alpha)$ is a cyclic group, then* $\dim Y^{=K} \geq 2$.

(H5) *If $\alpha \in \Pi_G(Y, \mathcal{P}(G))$ then* $\dim Y_\alpha \geq 5$.

(H6) *Y satisfies the weak gap condition (WGC).*

Then, for arbitrary nonnegative integers n_1, n_2, ..., n_s such that at least one of them is positive, there exists a homotopy sphere Σ equipped with a smooth G-action such that the G-fixed point set Σ^G is the disjoint union of F_i's:

$$S^G = \coprod_{i=1}^{s} \coprod_{j=1}^{n_s} F_{i,j} \qquad (F_{i,j} = F_i),$$

and $\nu(F_{i,j}, \Sigma)$ is G-isomorphic to $\nu(F_i, Y)$.

Using [LMP, Proposition 1.3] in the context that $\mathcal{F}$ is the set of Sylow subgroups of G, we obtain the next corollary.

Corollary 1.2. *Let G, Y, F_i, n_i $(1 \leq i \leq s)$ and $F_{i,j}$ $(1 \leq i \leq s,\ 1 \leq j \leq n_i)$ be as in Theorem* 1.1*. Then there exists a smooth G-action on the standard sphere S of dimension n, $n = \dim Y$, such that the G-fixed point set S^G is the disjoint union of F_i's:*

$$S^G = \coprod_{i=1}^{s} \coprod_{j=1}^{n_s} F_{i,j} \qquad (F_{i,j} = F_i),$$

and $\nu(F_{i,j}, S)$ is G-isomorphic to $\nu(F_i, Y)$.

Theorem 0.1 immediately follows from the corollary.

2. G-Connected Sum of Framed Maps

The equivariant connected sum have been used in various contexts. Thus we specify the context of the G-connected sum of the current paper. The G-connected sum of G-manifolds are same as [LMP] and [LM], but the resulting maps here are different from those in [LMP] and [LM]. We are not care of mapping degree here, hence the definition of the resulting map of G-connected sum is simpler than those in the previous two papers. If the reader is familiar with G-connected sum then he can skip this section, and he can return here when he becomes to inquire of the definition.

We begin with recalling the G-connected sum of G-manifolds. Let X and X' be smooth G-manifolds with G-invariant Riemannian metrics, and $a \in X$ and $a' \in X$ points such that $G_a = H$ and $G_{a'} \supseteq H$, and X and X' are locally H-oriented at a and a', respectively (namely $T_a(X)$ and $T_{a'}(X')$ are H-oriented). Let U_a (resp. $U_{a'}$) be an H-slice neighborhood of a (resp. a') in X (resp. X'). We fix H-equivariant identification maps $U_a \cong T_a(X)$ and $U_{a'} \cong T_{a'}(X')$, and regard U_a and $U_{a'}$ as vector spaces with inner products via the identification maps. Let $\varphi : U_{a'} \to U_a$ be a metric preserving, linear H-diffeomorphism satisfying the conditions

(2.1.1) $\varphi^K : U_{a'}^K \to U_a^K$ is orientation reversing for any $K \leq H$, and

(2.1.2) $\varphi_K : U_{a'K} \to U_{aK}$ is orientation preserving for any $K \leq H$, where $U_a = U_a^K \oplus U_{aK}$.

Let $S(\varphi) : S(U_{a'}) \to S(U_a)$ denote the restriction of φ, where $S(U_a)$ is the unit sphere of U_a. Then the G-connected sum $X\#(G \times_H X')$ of isotropy type (H) with attaching map φ is the G-space

$$\{X \setminus G \cdot \mathrm{Int}(D(U_a))\} \bigcup_{G \times_H S(\varphi)} \{G \times_H (X' \setminus \mathrm{Int}(D(U_{a'})))\},$$

where $D(U_a)$ is the unit disk of U_a.

In order to fix the smooth structure of $X\#(G \times_H X')$, we give an alternative definition of $X\#(G \times_H X')$. Let $\zeta : \mathbb{R} \to \mathbb{R}^2$ $(\zeta(t) = (\zeta_1(t), \zeta_2(t)))$ be a smooth curve such that

$$\begin{cases} \zeta_1(t) = t, \quad \zeta_2(t) = \frac{1}{2} & (\text{if } t \leq \frac{1}{2}), \\ \zeta_1'(t) > 0, \quad \zeta_2'(t) > 0 & (\text{if } 0 < t < 1), \quad \text{and} \\ \zeta_1(t) = 1, \quad \zeta_2(t) = t & (\text{if } t \geq 1). \end{cases}$$

Let V be a finite dimensional, real vector space with inner product. Define a submanifold Connector(V) of $\mathbb{R} \times V$ as the union of the three subspaces:

$$\begin{cases} \{(-1, v) \mid v \in V \text{ and } \|v\| \geq 1\}, \\ \{(t, v) \mid -1 < t < 1, \, v \in V, \, \|v\| = \zeta_2(|t|)\}, \quad \text{and} \\ \{(1, v) \mid v \in V \text{ and } \|v\| \geq 1\}. \end{cases}$$

Set

$$C_{(-1,1)} = \{(t, v) \in \mathbb{R} \times U_a \mid \ (t, v) \in \text{Connector}(U_a) \text{ and } -1 < t < 1\}.$$

The smooth G-manifold $X \#(G \times_H X')$ is the identification space of the three manifolds

$$\begin{cases} X \setminus G \cdot \text{Int}(D(U_a)), \\ G \times_H \text{Connector}(U_a), \quad \text{and} \\ G \times_H (X' \setminus \text{Int}(D(U_{a'}))), \end{cases}$$

where $gu \in G \cdot (U_a \setminus \text{Int}(D(U_a)))$ $(\subset X)$ is identified with $[g, (-1, u)] \in G \times_H \text{Connector}(U_a)$, and $[g, u] \in G \times_H (U_{a'} \setminus \text{Int}(D(U_{a'})))$ is identified with $[g, (1, \varphi(u))] \in G \times_H \text{Connector}(U_a)$. The smooth structure of $X\#(G \times X')$ is obtained from the smooth structures of the three submanifolds.

Let $f : X \to Y$ and $f' : X' \to Y$ be G-maps satisfying the condition

(2.2) $\quad f \cdot \varphi = f'|_{U_{a'}}$.

Then $f'' = f\#(G \times_H f') : X\#(G \times_H X') \to Y$ is the G-map such that

$$\begin{cases} f''(x) = f(x) \text{ for } x \in X \setminus G \cdot \text{Int}(D(U_a))), \\ f''([g, (t, u)]) = gf(u) \text{ for } g \in G, \, (t, u) \in C_{(-1,1)}, \ \text{and} \\ f''([g, x']) = gf'(x') \text{ for } [g, x'] \in G \times_H (X' \setminus \text{Int}(D(U_{a'}))). \end{cases}$$

Clearly, $f\#(G \times_H f')$ is smooth whenever f and f' are smooth.

Set $e = f(a)$ $(= f'(a') \in Y^H)$. Let U_e be a G-slice neighborhood of e in Y and fix an H-equivariant identification map $U_e \cong T_e(Y)$. Suppose that

(2.3) $\quad f(U_a) \subseteq U_e$ and $f(U_{a'}) \subseteq U_e$.

The total space of $T(X)_{U_a}$ $(= T(U_a))$ is canonically identified with $U_a \times T_a(X)$. Let η be a real G-vector bundle over Y and let

$$b : \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^{2m}) \to f^*(\varepsilon_Y(\mathbb{R}) \oplus \eta \oplus \varepsilon_Y(\mathbb{R}^{2m}))$$

be a G-vector bundle isomorphism such that the restriction over U_a satisfies

$$b(u,(t,v,w))_1 = (u,t) \tag{2.4.1}$$

for any $u \in U_a$, $t \in \mathbb{R}$, $v \in T_a(X)$ and $w \in \mathbb{R}$, where

$$b(u,(t,v,w)) = (b(u,(t,v,w))_1, b(u,(t,v,w))_2) \in \varepsilon_X(\mathbb{R}) \oplus (f^*\eta \oplus \varepsilon_X(\mathbb{R}^{2m})).$$

In addition, let

$$b' : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus \varepsilon_{X'}(\mathbb{R}^{2m}) \to f'^*(\varepsilon_Y(\mathbb{R}) \oplus \eta \oplus \varepsilon_Y(\mathbb{R}^{2m}))$$

be a G-vector bundle isomorphism such that the restriction over $U_{a'}$ satisfies

$$b'(u,(t,v,w)) = ((u,-t), b(\varphi(u),(t,\varphi(v),w))_2) \tag{2.4.2}$$

for any $u \in U_{a'}$, $t \in \mathbb{R}$, $v \in T_{a'}(X')$ and $w \in \mathbb{R}$. The tangent bundle $T(X'')$ of $X'' = X\#(G \times_H X')$ is the union of the three bundles

$$\begin{cases} T(X \setminus G \cdot \mathrm{Int}(D(U_a)), \\ T(G \times_H C_{(-1,1)}), \quad \text{and} \\ G \times_H T(X' \setminus \mathrm{Int}(D(U_{a'})). \end{cases}$$

Set $\mathrm{Band}(U_a) = \mathbb{R} \times U_a$. The total space of $T(\mathrm{Band}(U_a))$ is canonically identified with $(\mathbb{R} \times U_a)) \times (\mathbb{R} \times T_a(X))$. Take an H-equivariant orthogonal decomposition

$$T(\mathrm{Band}(U_a))_{\mathrm{Connector}(U_a)} \overset{\iota}{\cong} \varepsilon_{\mathrm{Connector}(U_a)}(\mathbb{R}) \oplus T(\mathrm{Connector}(U_a))$$

such that

$$\begin{cases} \iota(((1,u),(t,v))) = (((1,u),-t),((1,u),v)) \quad \text{and} \\ \iota((-1,u),(t,v)) = (((-1,u),t),((-1,u),v)) \end{cases}$$

for any $u \in U_a$, $t \in \mathbb{R}$, $v \in T_a(X)$.

Let F denote the composition

$$\mathbb{R} \times U_a \xrightarrow{proj} U_a \subset X \xrightarrow{f} Y.$$

Let B denote the G-vector bundle map

$$\begin{array}{ccc} T(\mathrm{Band}(U_a)) \oplus \varepsilon_{\mathrm{Band}(U_a)}(\mathbb{R}^{2m}) & \to & F^*(\varepsilon_Y(\mathbb{R}) \oplus \eta \oplus \varepsilon_Y(\mathbb{R}^{2m})) \\ (((s,u),(t,v)),((s,u),w)) & \mapsto & ((s,u), b(u,(t,v,w))), \end{array}$$

where $s \in \mathbb{R}$, $u \in U_a$, $t \in \mathbb{R}$, $v \in T_a(X)$, $w \in \mathbb{R}^{2m}$. Then the G-vector bundle isomorphism

$$b\#(G \times_H b') : \varepsilon_{X''}(\mathbb{R}) \oplus T(X'') \oplus \varepsilon_{X''}(\mathbb{R}^{2m}) \to (f\#(G \times_H f'))^*(\varepsilon_Y(\mathbb{R}) \oplus \eta \oplus \varepsilon_Y(\mathbb{R}^{2m}))$$

is defined to be the union of the three maps

$$\begin{cases} b_{X \setminus G\mathrm{Int}(D(T_a(X)))}, \\ G \times_H B_{\mathrm{Connector}(U_a)}, \quad \text{and} \\ G \times_H b_{X' \setminus \mathrm{Int}(D(T_{a'}(X')))}. \end{cases}$$

The case where $X' = X$ will be often treated in the current paper, and there $(1 + [G/H])X$ stands for $X\#(G \times_H X)$ and $(1 + [G/H])f$ does for $f\#[G/H]f$. If Condition (2.1.1) is replaced by the condition

(2.1.1′) $\varphi^K : U_{a'}^K \to U_a^K$ is orientation preserving for any $K \le H$,

then we can define $X\#(G \times_H (-X'))$. If $X' = X$, this space is denoted by $(1 - [G/H])X$. In general, for a formal sum

$$\beta = \sum_{(H) \in \mathcal{S}(G)/G} b(H)[G/H] \qquad (b(H) \in \mathbb{Z})$$

the G-connected sum $(1+\beta)X$ is obtained under certain hypotheses. Similarly, notation $f\#(G \times_H (-f'))$, $(1 - [G/H])f$, $(1 + \beta)f$, $b\#(G \times_H (-b'))$, $(1 - [G/H])b$, and $(1 + \beta)b$ will be used.

3. Construction of an Initial G-Normal Map

We shall obtain a G-normal map (f, b) using the G-transversal construction originated by T. Petrie [Pe1–3]. In the construction, a special element β in the Burnside ring $\Omega(G)$ plays a key role. This β is very useful to killing various obstructions of a G-normal map (f, b).

Lemma 3.1 ([LM, Proposition 1.2 and Theorem 1.3], [M5, Theorems 2.7 and 2.12]). *Let G be an Oliver group. Then there exists an element β in the Burnside ring $\Omega(G)$ such that $\chi_G(\beta) = 0$ and $\chi_K(\beta) = 1$ for all $K \in \mathcal{M}(G)$.*

We fix an element β as in the lemma and write β in the form

$$\beta = \sum_{(H)\in\mathcal{S}(G)/G} b_H[G/H] \tag{3.2.1}$$

with coefficients $b_H \in \mathbb{Z}$ such that $b_G = 0$. Set

$$|\beta| = \sum_{(H)\in\mathcal{S}(G)/G} |b_H|[G/H]. \tag{3.2.2}$$

Take positive integers c_H for $(H) \in \mathcal{S}(G)/G$ with $H \neq G$ such that

$$c_H \equiv -b_H \mod 2|G||\widetilde{K}_0(\mathbb{Z}[G])|$$

and set

$$(-\beta)^{\%} = \sum_{(H)\in\mathcal{S}(G)/G} c_H[G/H], \tag{3.2.3}$$

where $c_G = 0$.

If A and B are smooth manifold, B is oriented, $h : A \to B$ is a map, and

$$c : \varepsilon_A(\mathbb{R}) \oplus T(A) \oplus \varepsilon_A(\mathbb{R}^{2m}) \to h^*(\varepsilon_B(\mathbb{R}) \oplus T(B) \oplus \varepsilon_B(\mathbb{R}^{2m}))$$

is a vector bundle isomorphism, then the three vector bundles $\varepsilon_A(\mathbb{R})$, $\varepsilon_A(\mathbb{R}^{2m})$ and $h^*(\varepsilon_B(\mathbb{R}) \oplus T(B) \oplus \varepsilon_B(\mathbb{R}^{2m}))$ have canonical orientations, hence one can specify the orientation of $T(A)$ so that c is an orientation preserving isomorphism.

Lemma 3.3. *Let Y be a smooth G-manifold as in Theorem 1.1. Then there exists a pair (f, b) of a degree one, G-map $f : X \to Y$ and a G-vector bundle isomorphism*

$$b : \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^{2m}) \to f^*(\varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \varepsilon_Y(\mathbb{R}^{2m}))$$

for a sufficiently large integer m satisfying the following conditions. The orientation of a connected component X_α, $\alpha \in \Pi_G(X)$, (equivalently of $\varepsilon_{X_\alpha}(\mathbb{R}) \oplus T(X_\alpha) \oplus \varepsilon_{X_\alpha}(\mathbb{R}^{2m})$) is induced by $b_{X_\alpha}^{\rho(\alpha)}$ whenever $Y_{f_*\alpha}$ is oriented.

(3.3.1) *For each* $1 \leq i \leq s$, $f^{-1}(F_i) = \coprod_{j=1}^{m_i} E_{i,j}$ *(connected components).*

(3.3.2) *For each* $1 \leq i \leq s$, $f^{-1}(F_i)^G = \coprod_{j=1}^{n_i} E_{i,j}$.

(3.3.3) *For each* $1 \leq i \leq s$ *and* $1 \leq j \leq m_i$, *the isotropy subgroups (with respect to the* G*-action) appearing all over* $E_{i,j}$ *coincide with each other. Let* $H_{i,j}$ *denote the isotropy subgroup, i.e.* $\mathrm{Iso}(G, E_{i,j}) = \{H_{i,j}\}$.

(3.3.4) *For each* $1 \leq i \leq s$ *and* $1 \leq j \leq m_i$, $E_{i,j} \cap gE_{i,j} = \emptyset$ *whenever* $g \in G \setminus H_{i,j}$.

(3.3.5) *For each* $1 \leq i \leq s$ *and* $1 \leq j \leq m_i$, *the restriction of* f *to* $E_{i,j}$ *is a diffeomorphism from* $E_{i,j}$ *to* F_i. *Let* $f_{i,j}$ *denote the restriction of* f.

(3.3.6) *For each* $1 \leq i \leq s$ *and for* $1 \leq j \leq m_i$, *the restriction of* f *to the* $H_{i,j}$*-tubular neighborhood* $\nu(i,j) = \nu(E_{i,j}, X)$ *of* $E_{i,j}$ *in* X *is a diffeomorphism from* $\nu(i,j)$ *to the* G*-tubular neighborhood* $\nu(F_i, Y)$ *that has the structure of an* $H_{i,j}$*-vector bundle map covering* $f_{i,j} : E_{i,j} \to F_i$. *Let* $f_{\nu(i,j)}$ *denote the restriction of* f.

(3.3.7) *For each* $H \in \mathcal{S}(G) \setminus \{G\}$, *there exist* $a(H,+) \in E_{1,j_{H,+}}$ *and* $a(H,-) \in E_{1,j_{H,-}}$ *such that* $H_{1,j_{H,+}} = H = H_{1,j_{H,-}}$, $f(a(H,+)) = f(a(H,-)) = y_0$, *and*

$$b_{a(i,H,+)} : \mathbb{R} \oplus T_{a(H,+)}(X) \oplus \mathbb{R}^{2m} \to \mathbb{R} \oplus T_{y_0}(Y) \oplus \mathbb{R}^{2m}$$
$$(\textit{resp.}\quad b_{a(H,-)} : \mathbb{R} \oplus T_{a(H,-)}(X) \oplus \mathbb{R}^{2m} \to \mathbb{R} \oplus T_{y_0}(Y) \oplus \mathbb{R}^{2m}\ )$$

coincides with

$$id_{\mathbb{R}} \oplus df_{a(H,+)} \oplus id_{\mathbb{R}^{2m}}$$
$$(\textit{resp.}\quad (-id_{\mathbb{R}}) \oplus df_{a(H,-)} \oplus id_{\mathbb{R}^{2m}}\ ).$$

(3.3.8) *The restriction of* b *to the tubular neighborhood* $\nu(i,j)$ *coincides with the induced map from the stabilization of* $df_{\nu(i,j)}$ $(1 \leq i \leq s,\ 1 \leq j \leq n_i)$.

(3.3.9) *For each* $H \in \mathcal{M}(G)$, *there exists an* H*-normal cobordism from* $\mathrm{Res}^G_H(f,b)$ *to the identity normal map* $\mathrm{Res}^G_H(id_Y, id_{T(Y)})$.

Remark 3.4. It follows from (3.3.7) that for each $K \leq H$, df^K is orientation preserving at $a(H,+)$ and reversing at $a(H,-)$, and df_K is orientation preserving at $a(H,+)$ and $a(H,-)$.

Proof of Lemma 3.3. Let

$$Z = \{y \in Y \mid G_y \in \mathcal{L}(G)\}.$$

We denote the connected components of Z/G by $A_1, \ldots, A_\ell$ and set

$$Z_i = \pi^{-1}(A_i) \quad (i = 1, \ldots, \ell) ,$$

where $\pi : Z \to Z/G$ is the canonical projection. Clearly, it holds that $Z = Z_1 \amalg \cdots \amalg Z_\ell$ (disjoint union) and $\mathrm{Iso}(G, Y \setminus Z) \subseteq \mathcal{M}(G)$. Reordering $Z_1, \ldots, Z_\ell$ if necessary, we may assume $Z_1 \supseteq F_1, \ldots, Z_s \supseteq F_s$ $(s \le \ell)$. In order to employ [M5, THeorem 4.4], we define $\gamma_i \in \Omega(G)$, $i = 1, \ldots, \ell$, in the expression of [M5, (4.1)] by

$$\gamma_i = \begin{cases} (n_i - 0)[G/G] & (1 \le i \le s) \\ (0 - 0)[G/G] & (s + 1 \le i \le \ell). \end{cases}$$

Then the equivariant transversality construction [M5, Theorem 4.4] with the help by the modification theorem of bundle data [M5, Lüeck–Madsen's Theorem 3.6] gives a pare (f, b) desired in the lemma. □

4. Proof of Theorem 1.1

Let G be an Oliver group.

Lemma 4.1. *Let Y be as in Theorem 1.1. Then there are the following properties.*

(4.1.1) *For a prime p dividing $|G|$ and a p-subgroup of G, Y^P is a $\mathbb{Z}_{(p)}$-homology sphere.*

(4.1.2) *If $\dim Y_\alpha$ is odd then the Euler characteristic $\chi(Y_\alpha)$ of Y_α is equal to 0.*

(4.1.3) *Let $H \in \mathcal{G}^1(G)$. If α, $\beta \in \Pi_G(Y, \{H\})$ then $\dim Y_\alpha \equiv \dim Y_\beta \mod 2$.*

(4.1.4) *Let $H \in \mathcal{G}^1(G)$. If $\dim Y^H$ is even then $\chi(Y^H)$ is equal to 2.*

Proof. (4.1.1): This follows from Smith's theory.

(4.1.2): This follows from the Poincaré duality over $\mathbb{Z}_2$.

(4.1.3): Take a subgroup $P \in \mathcal{P}(G)$ such that $P \triangleleft H$ and H/P is cyclic. By (4.1.1), Y^P is connected. The group H/P acts on Y^P and Y_α and Y_β are connected components of the H/P-fixed point set. Since the H/P action preserves the orientation and H/P is cyclic, $\dim Y_\alpha \equiv \dim Y^P \equiv \dim Y_\beta \mod 2$.

(4.1.4): By pinching the outside of an open disk neighborhood of y_0, we get a degree one G-map $k : Y \to S(\mathbb{R} \oplus V)$. This k is a homotopy equivalence and the mapping cone M_k of k is contractible. By [O1, Proposition 2], we get $\chi({M_f}^H) = 1$. Note that

$$\chi({M_f}^H) = \chi(S(\mathbb{R} \oplus V)^H) - \chi(Y^H) + 1 = 3 - \chi(Y^H).$$

Thus, we conclude $\chi(Y^H) = 2$. $\square$

Let Y be the homotopy sphere in Theorem 1.1. Take a G-normal map (f, b) as in Lemma 3.3. But in the following modification of (f, b), the property that $f : X \to Y$ is of degree one won't be kept.

Recall that our goal is to kill the homotopy groups $\pi_j(X)$ $(0 < j < \dim X)$. By Smith's theory we must kill the homology groups $H_j(X^P; \mathbb{Z}_{(p)})$ $(P \in \mathcal{P}(G),\ 0 < j < \dim X^P)$. To this end, by Lefschetz's fixed point formula, we must modify X so that $\chi(X^H) = 2$ or 0 according to $\dim Y^H \equiv 0$ or $1 \mod 2$ for any $H \triangleright P$ such that H/P is a nontrivial cyclic group.

Let $\mathcal{U}$ be a subset of $\mathcal{G}^1(G)$ satisfying the following (4.2.1)–(4.2.4):

(4.2.1) If $H \leq K \in \mathcal{G}^1(G)$ and $H \in \mathcal{U}$ then K belongs to $\mathcal{U}$.

(4.2.2) If $H \in \mathcal{U}$ then $\chi(X^H) = \chi(Y^H)$.

(4.2.3) If $P \in \mathcal{P}(G) \cap \mathcal{U}$ and $P \neq \{1\}$ then X^P is a simply connected, $\mathbb{Z}_{(p)}$-homology sphere, where p is the prime dividing $|P|$.

(4.2.4) If $\{1\} \in \mathcal{U}$ then X is a homotopy sphere.

If another subset $\mathcal{U}'$ of $\mathcal{G}^1(G)$ satisfies Conditions (4.2.1)–(4.2.4), then the set $\mathcal{U} \cup \mathcal{U}'$ also satisfies the conditions. Let $\mathcal{G}^1(G, X)$ denote the unique maximal subset of $\mathcal{G}^1(G)$ satisfying Conditions (4.2.1)–(4.2.4). We shall modify X (keeping X^G unchanged) to X' such that $\mathcal{G}^1(G, X') = \mathcal{G}^1(G)$. Then X' is a desired homotopy sphere. The modification will be done inductively according to the order of elements of $\mathcal{G}^1(G)$ from the biggest to the smallest. If $\mathcal{G}^1(G, X) = \mathcal{G}^1(G)$ then we can take X as X'. Thus, in the following we assume $\mathcal{G}^1(G, X) \neq \mathcal{G}^1(G)$.

Let $H \in \mathcal{G}^1 \setminus \mathcal{G}^1(G, X)$ be a maximal subgroup. We describe how to modify (f, b) at the level H in the three cases: (1) $H \notin \mathcal{G}^1(G) \setminus \mathcal{P}(G)$, (2) $H \in \mathcal{P}(G) \setminus \{1\}$, and (3) $H = \{1\}$.

Case 1 ($H \notin \mathcal{P}(G)$). If $\dim X^H = \dim Y^H$ is odd, then $\chi(X^H) = 0 = \chi(Y^H)$. This implies $H \in \mathcal{G}(G, X)$, which is a contradiction. Thus $\dim X^H = \dim Y^H$ must be even. We are now required to modify X (more precisely (f, b)) so that $\chi(X^H) = 2$. If one perform G-connected sum operation on (f, b) and obtain $((1 + [G/K])f, (1 + [G/K])b)$ $(K < G)$, then for $L \in \mathcal{S}(G)$

$$\chi_L([(1 \pm [G/K])X]) - 2 = \chi_L(1 + [G/K])(\chi_L([X]) - 2).$$

Thus one gets

$$\chi_L([(1 + (-\beta)^{\%})X]) - 2 = \chi_L(1 + (-\beta)^{\%})(\chi_L([X]) - 2).$$

If $\chi(X^L) = 2$ then $\chi(((1 + (-\beta)^{\%})X)^L) = 2$. Since $\chi_H((-\beta)^{\%}) \equiv \chi_H(-\beta) = -1 \mod 2|G|$, it follows that $\chi_L([(1+(-\beta)^{\%})X]) - 2 \equiv 0 \mod 2|G|$. Thus one can perform 0- or 1-dimensional G-surgery on X of isotropy type (H) so that $\chi(X^H) = 2$. This completes the proof in Case 1.

Before beginning the other two cases, we prepare notation.

Let $H \in \mathcal{P}(G)$. If $H = \{1\}$ then set $R = \mathbb{Z}$ and if $|H|$ is divisible by a prime p then set $R = \mathbb{Z}_{(p)}$.

Now suppose $\dim X^H$ is odd (say $\dim X^H = 2k+1$). Take mutually disjoint $N_G(H)/H$-embeddings $\Psi_i : N_G(H)/H \times S^k \times D^{k+1} \to X^{=H}$ $(i = 1, \ldots, t)$ such that one can perform $N_G(H)/H$-surgery on (f^H, b^H) using these Ψ_i's and the cores $\psi_i = \Psi_i(H/H, -, 0)$ span $H_k(X^H; R)$ over R. Set

$$U = \bigcup_{i=1}^{t} \mathrm{Im}(\Psi_i) \quad \text{and} \quad X_0^H = X^H \setminus \mathrm{Int}(U).$$

Then we obtain the butterfly diagram:

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & \begin{array}{c} H_{k+1}(X^H, X_0^H; R) \\ \| \\ H_{k+1}(U, \partial U; R) \end{array} \arrow[dr] \arrow[rr, bend left] & & H_k(X_0^H; R) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& H_{k+1}(X^H; R) \arrow[ur] \arrow[dr] & & H_k(\partial U; R) \arrow[ur] \arrow[dr] & & H_k(X^H; R) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & \begin{array}{c} H_{k+1}(X^H, U; R) \\ \| \\ H_k(X_0^H, \partial U; R) \end{array} \arrow[ur] \arrow[rr, bend right] & & H_k(U; R) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

Diagram 4.3

Case 2 ($H \in \mathcal{P}(G)$ and $H \neq \{1\}$). Let p be the prime dividing $|H|$. By G-surgery on X (more precisely (f, b)) of isotropy type (H), we can make X^H $(k-1)$-connected.

Lemma 4.4. *Suppose that* $\dim X^H$ *is even (say* $\dim X^H = 2k$*). Then* $H_k(X^H; \mathbb{Z}_{(p)})$ *is* $\mathbb{Z}_{(p)}[N_G(H)/H]$-*free.*

Proof. Since $X^G \neq \emptyset$, we can take a point $x_0 \in X^G$. Set $V = T_{x_0}(X)$ and let $h : X \to S := S(\mathbb{R} \oplus V)$ be the map pinching $X \setminus \mathrm{Int}(D(T_{x_0}(X)))$ to a point. Let M_h be the mapping cone of h. Then $H_{k+1}({M_h}^H;\mathbb{Z}_{(p)}) \cong H_k(X^H;\mathbb{Z}_{(p)})$. If P is a p-subgroup such that $P \triangleright H$ and $P \neq H$ then $h^P : X^P \to S^P$ is a $\mathbb{Z}_{(p)}$-homology equivalence, and hence ${M_h}^P$ is $\mathbb{Z}_{(p)}$-acyclic. Clearly, $\widetilde{H}_i({M_h}^H;\mathbb{Z}_{(p)}) = 0$ for $i \neq k+1$. By [OP, Lemma 2.1], $H_{k+1}({M_h}^H;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}[N_G(H)/H]$-projective. For each subgroup K such that $K \triangleright H$, $K \neq H$, and K/H is cyclic, $\chi({M_h}^K) = 1$ follows from $\chi(X^K) = 2$. Thus, by [OP, Lemma 2.5], $H_{k+1}({M_h}^H;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}[N_G(H)/H]$-free. □

Lemma 4.5. *Suppose that* $\dim X^H$ *is odd (say* $\dim X^H = 2k+1$*). Then the modules* $H_k(X_0^H;\mathbb{Z}_{(p)})$ *and* $H_k(X^H,U;\mathbb{Z}_{(p)})$ *in Diagram* 4.3 *is* $\mathbb{Z}_{(p)}[N_G(H)/H]$*-free.*

Proof. Similarly to Proof of Lemma 4.4. □

By virtue of Lemmas 4.4 and 4.5, one can define an $N_G(H)/H$-surgery obstruction $\sigma(f^H,b^H)$ analogously to [M2] if $\dim X^H$ is odd and to [BM2], [BM3] if $\dim X^H$ is even. The obstruction group $\mathcal{O}(N_G(H)/H, Y^H, \mathbb{Z}_{(p)})$ where $\sigma(f^H,b^H)$ lies is

$$(4.6)\qquad \begin{aligned} &W_{2k+1}(\mathbb{Z}_{(p)}[N_G(H)/H], \Gamma N_G(H)/H(Y^H); triv.) \text{ of [M2, Theorem A]} \\ &\qquad\qquad \text{when } \dim Y^H = 2k+1, \text{ or} \\ &W_{2k}(\mathbb{Z}_{(p)}, N_G(H)/H, \emptyset, S, S) \text{ of [BM2, Theorem 6.5]} \\ &\qquad\qquad \text{when } \dim Y^H = 2k. \end{aligned}$$

Here $\Gamma N_G(H)/H(Y^H)$ is the form parameter

$$\langle x - (-1)^k \overline{x},\ K/H \mid x \in \mathbb{Z}_{(p)}[N_G(H)/H],\ K \triangleright H,\ |K/H| = 2,\ \dim Y^K = k\rangle_{\mathbb{Z}_{(p)}}$$

and $S = \{K \mid K \triangleright H,\ |K/H| = 2,\ \dim Y^K = k\}$. The term $N_G(H)/H$-surgery obstruction means the property that if $\sigma(f^H,b^H) = 0$ then one can perform $(k-1)$- and k-dimensional $N_G(H)/H$-surgery on (f^H,b^H) relatively to $X^{>H}$ so that $H_j(X^H;\mathbb{Z}_{(p)}) = 0$ for all $0 < j \leq k$ (consequently, X^H becomes a simply connected, $\mathbb{Z}_{(p)}$-homology sphere). But if such $N_G(H)/H$-surgery on (f^H,b^H) is possible then it is automatically thickened to G-surgery on (f,b) of isotropy type (H). Thus $\sigma(f^H,b^H)$ is called the G-surgery obstruction of isotropy type (H).

Recall β in Lemma 3.1. Let κ denote the element in $\Omega(G)$ such that $1 - \kappa = (1-\beta)^{|G|}$. Similarly to (3.2.1), write κ in the form. Using the expression of κ, we consider the G-connected sum $(f',b') =$

$((1-\kappa)f,(1-\kappa)b)$. (Note this is possible by virtue of Condition (3.2.7).) There is a canonical ring homomorphism $\mathrm{Fix}_H : \Omega(G) \to \Omega(N_G(H)/H)$ such that $\mathrm{Fix}_H([A]) = [A^H]$ for finite G-CW complexes A. It follows that

$$\sigma({f'}^H,{b'}^H) = \mathrm{Fix}_H(1-\kappa)\sigma(f^H,b^H).$$

If $\dim X^H$ is even then $\sigma({f'}^H,{b'}^H)=0$ follows by the same argument as in [LM, Proof of Lemma 7.1, Case $H \in \mathcal{P}(G)\setminus\{1\}$]. For reader's convenience, we give an outline of the proof. Set $G' = N_G(H)$. Then the correspondence

$$H' \longmapsto \mathcal{O}(G', \mathrm{Res}^{G'}_{H'} Y^H; \mathbb{Z}_{(p)})$$

(where H' runs over all subgroups of G') is a Mackey functor on G' and moreover a Green module over the Green functor

$$H' \longmapsto SGW_0(\mathbb{Z}, H', S(H')) \quad \text{of [BM2]}$$

where $S(H') = \{K/H \in H' \mid K \rhd H, |K/H| = 2, \dim Y^K = k\}$. Using [BM2, Corollary 4.3] or [LM, Corollary 6.8], we can check $\sigma({f'}^H,{b'}^H)=0$.

If $\dim X^H$ is odd, then by [B, §12], $W_{2k+1}(\mathbb{Z}_{(p)}[N_G(H)/H], \Gamma N_G(H)/H(Y^H); triv.)$ is a module over $GW_0(\mathbb{Z}, N_G(H)/H)$. By [LM, Proposition 6.3],

$$\mathrm{Fix}_H(1-\kappa)GW_0(\mathbb{Z}, N_G(H)/H) = 0.$$

Thus, $\sigma({f'}^H,{b'}^H)=0$.

In the both cases, one can perform G-surgery on (f,b) of isotropy type (H) making X^H a $\mathbb{Z}_{(p)}$-homology sphere. This completes the proof in Case 2.

Case 3 ($H=\{1\}$). First make X $[\dim X/2]$-connected by G-surgery on (f,b) of isotropy type $\{1\}$.

Suppose $\dim X$ is even (say $\dim X = 2k$). From Nakayama's theory [N, Lemma 2], a finitely generated $\mathbb{Z}[G]$-module M is $\mathbb{Z}[G]$-projective if for each prime q and q-subgroup Q of G, M is $\mathbb{Z}_{(q)}[Q]$-projective. Thus by the argument in Proof of Lemma 4.4, $H_k(X;\mathbb{Z})$ is $\mathbb{Z}[G]$-projective. Now take the G-connected sum $((1+(-\beta)^{\%})f, ((1+(-\beta)^{\%})b)$ with respect to (3.2.3). Then

$$[H_k((1+(-\beta)^{\%})X;\mathbb{Z})] = (1+(-\beta)^{\%})[H_k(X;\mathbb{Z})] \quad \text{in } \widetilde{K}_0(\mathbb{Z}[G]).$$

The group $\widetilde{K}_0(\mathbb{Z}[G])$ is a module over the representation ring $R(G,\mathbb{Q})$. Since $\mathrm{Res}^G_C(1-\beta)=0$ for any cyclic subgroup C of G, $(1-\beta)R(G,\mathbb{Q})=0$, and hence $(1-\beta)\widetilde{K}_0(\mathbb{Z}[G])=0$. Recall

$$1+(-\beta)^{\%} \equiv 1-\beta \mod |\widetilde{K}_0(\mathbb{Z}[G])|.$$

This implies $(1+(-\beta)^{\%})\widetilde{K}_0(\mathbb{Z}[G])=0$. Thus we can perform G-connected sum operation on (f,b) so that $H_k(X;\mathbb{Z})$ becomes stably $\mathbb{Z}[G]$-free. Performing $(k-1)$-dimensional G-surgery of isotropy type $\{1\}$ on (f,b), we may assume $H_k(X;\mathbb{Z})$ is $\mathbb{Z}[G]$-free.

Next suppose $\dim X$ is odd (say $\dim X = 2k+1$). Then the modules $H_k(X_0;\mathbb{Z})$ and $H_k(X,U;\mathbb{Z})$ in Diagram 4.3 are $\mathbb{Z}[G]$-projective. Performing the G-connected sum $(1+(-\beta)^{\%})(f,b)$, we can modify the modules to being stably $\mathbb{Z}[G]$-free. Adding trivial embeddings to the family $\{\Psi_1,\ldots,\Psi_t\}$, we may assume the modules in Diagram 4.3 are $\mathbb{Z}[G]$-free.

Thus in the both cases one can define the G-surgery obstruction $\sigma(f,b)\in\mathcal{O}(G,Y,\mathbb{Z})$ as in Case 2, where $\mathcal{O}(G,Y,\mathbb{Z})$ is the group given in (4.6) by substituting G, Y, and $\mathbb{Z}$ to $N_G(H)/H$, Y^H and $\mathbb{Z}_{(p)}$, respectively. Then $\sigma((1-\kappa)(f,b))=0$ follows by the argument in Case 2. Consequently, we can kill $H_i((1-\kappa)X;\mathbb{Z})$ for all $0<i\le k$ by $(k-1)$- and k-dimensional surgery on $(1-\kappa)(f,b)$ relatively to the singular set. Then the resulting G-manifold is a homotopy sphere. □

Acknowledgements

This research was motivated by B. Oliver's work [O2]. The author wishes to express his gratitude to E. Laitinen, W. Lück, B. Oliver and K. Pawałowski for their kind suggestions.

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