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by

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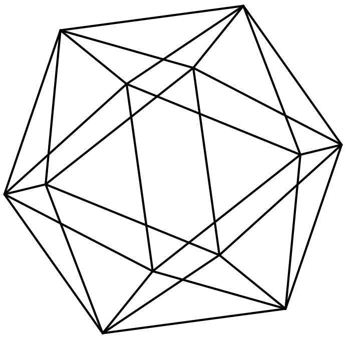

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Lie algebras in symmetric monoidal categories

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LIE ALGEBRAS IN SYMMETRIC MONOIDAL CATEGORIES

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ABSTRACT. We study algebras defined by identities in symmetric monoidal categories. Our focus is on Lie algebras. Besides usual Lie algebras, there are examples appearing in the study of knot invariants and Rozansky-Witten invariants. Our main result is a proof of Westbury's conjecture for K3-surface: there exists a Lie algebra homomorphism from Vogel's universal simple Lie algebra to the Lie algebra describing the Rozansky-Witten invariants of a K3-surface. Most of the paper involves setting up a proper language to discuss the problem and we formulate nine open questions as we proceed.

In 1996 Deligne made a conjecture that exceptional Lie algebras formed a series [4]. He proposed a possible explanation: he expected an object specializing to all exceptional Lie algebras. Influenced by this conjecture, Vogel proposed such an object in 1999 [25]. Besides exceptional Lie algebras, it specializes to all simple (complex finite dimensional) Lie algebras as well as some Lie superalgebras. The object was named Vogel's universal simple Lie algebra $\mathfrak{g}_V$. With Vogel's paper still unpublished the object remains a mystery. It is a subject of several recent studies [16, 18, 27].

The aim of this paper is to describe another specialization of $\mathfrak{g}_V$. In a private conversation Westbury has asked the author whether $\mathfrak{g}_V$ specializes to $\mathfrak{g}_X$, a certain Lie algebra associated by Kapranov [10] to an irreducible holomorphic symplectic manifold X in the study of Rozansky-Witten invariants [21]. By *Westbury's Conjecture* we understand existence of such a specialization. We give an affirmative answer to Westbury's Conjecture in the case of a K3-surface. It remains open in the general case.

Most of the paper is devoted to setting up a machinery just to ask the question rigorously. There are several levels of generality in which the setup can be laid out. As opposed to a more general operadic language of Hinich and Vaintrob [6], we choose a more elementary language in the spirit of Sawon [23] to benefit a wider audience. Here is a detailed description of the paper.

Section 1 contains all necessary definitions and facts about the categories. In Section 2 we discuss algebras in the categories. We speculate on possible utility of such algebras in the study of identities. In Section 3 we study the Lie algebras. We formulate three different notions of simplicity. We introduce Vogel's universal Lie algebra. In Section 4 we investigate the Lie algebras that appear in the study of the Rozansky-Witten invariants and give an affirmative answer to Westbury's conjecture in the case of a K3-surface.

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1. CATEGORIES

1.1. Tensor categories.

By *a tensor category over a commutative ring* $\mathbb{K}$ we understand a $\mathbb{K}$-linear[1] symmetric monoidal category

$$\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{a}, \mathbf{1}, \mathbf{l}, \mathbf{r}, \mathbf{c})$$

where $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the tensor product bifunctor, $\mathbf{a}_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is the associativity transformation, $\mathbf{c}_{A,B} : A \otimes B \to B \otimes A$ is the symmetric braiding, $\mathbf{1} \in \mathcal{C}$ is the monoidal identity, $\mathbf{l}_A : \mathbf{1} \otimes A \to A$ and $\mathbf{r}_A : A \otimes \mathbf{1} \to A$ are left and right unity transformations. We require the structures to agree: the tensor product must be $\mathbb{K}$-bilinear on morphisms and the natural morphism

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[1] The home-sets are $\mathbb{K}$-modules and the compositions are $\mathbb{K}$-bilinear, for instance, if $\mathbb{K} = \mathbb{Z}$ this means that the category is preadditive.

$\mathbb{K} \to \mathrm{end}_{\mathcal{C}}(\mathbf{1}) = \mathrm{hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ must be a ring isomorphism. We attempt to follow notations and terminology of Joyal and Street [8], although their "tensor category" is the same as a monoidal category, while our tensor categories are $\mathbb{K}$-linear and symmetric.

Observe that $\mathcal{C}$ is not required to be abelian or even additive. Additivity can be easily fixed by completing the category with respect to finite direct sums and the zero object but not being abelian is an important distinction with other "tensor categories" that appear in the literature. Our tensor categories are close to those of Hinich and Vaintrob [6] with an exception of our additional assumption that $\mathbb{K} \to \mathrm{end}_{\mathcal{C}}(\mathbf{1})$ is an isomorphism. This assumption is not very restrictive.

Proposition 1. *Suppose* $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{a}, \mathbf{1}, \mathbf{l}, \mathbf{r}, \mathbf{c})$ *is a symmetric monoidal additive category. Then the ring* $\mathbb{K}_{\mathcal{C}} = \mathrm{end}_{\mathcal{C}}(\mathbf{1})$ *is commutative and the category* $\mathcal{C}$ *is a tensor category of* $\mathbb{K}_{\mathcal{C}}$*.*

Proof. The ring $\mathbb{K}_{\mathcal{C}}$ is commutative by the Eckmann-Hilton argument [12, Prop 6.2]. Each $\mathrm{hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{K}_{\mathcal{C}}$-$\mathbb{K}_{\mathcal{C}}$-bimodule via the left action

$$\mathrm{end}_{\mathcal{C}}(\mathbf{1}) \times \mathrm{hom}_{\mathcal{C}}(X, Y) \to \mathrm{hom}_{\mathcal{C}}(\mathbf{1} \otimes X, \mathbf{1} \otimes Y) \cong \mathrm{hom}_{\mathcal{C}}(X, Y)$$

and the right action

$$\mathrm{hom}_{\mathcal{C}}(X, Y) \times \mathrm{end}_{\mathcal{C}}(\mathbf{1}) \to \mathrm{hom}_{\mathcal{C}}(X \otimes \mathbf{1}, Y \otimes \mathbf{1}) \cong \mathrm{hom}_{\mathcal{C}}(X, Y).$$

These actions coincide: given $\alpha \in \mathrm{end}_{\mathcal{C}}(\mathbf{1})$, $f \in \mathrm{hom}_{\mathcal{C}}(X, Y)$, both αf and $f\alpha$ can be read off the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathbf{l}_X^{-1}"] \arrow[d, "="] & \mathbf{1} \otimes X \arrow[r, "\alpha \otimes f"] \arrow[d, "\mathbf{c}_{\mathbf{1},X}"'] & \mathbf{1} \otimes Y \arrow[r, "\mathbf{l}_Y"] \arrow[d, "\mathbf{c}_{\mathbf{1},Y}"] & Y \arrow[d, "="] \\
X \arrow[r, "\mathbf{r}_X^{-1}"] & X \otimes \mathbf{1} \arrow[r, "f \otimes \alpha"'] & Y \otimes \mathbf{1} \arrow[r, "\mathbf{r}_Y"] & Y
\end{tikzcd}$$

as compositions from the top left corner to the bottom right corner. The diagram is commutative: the outside squares are commutative by the standard fact [8, Prop 2.1] while the middle square is commutative because the commutativity constraint $\mathbf{c}$ is a natural transformation. Hence, $\alpha f = f \alpha$.

Finally, $\mathbb{K}_{\mathcal{C}}$-bilinearity of compositions is evident. ☐

Tensor categories are ubiquitous in modern mathematics, so we spare the reader of an example now but introduce some as we go along.

1.2. **Tensor powers in a tensor category.** Given an object A in a tensor category over $\mathbb{K}$, we define its iterated tensor products by $A^{\otimes 0} := \mathbf{1}$ and $A^{\otimes n} := A^{\otimes(n-1)} \otimes A$. There is an action of the symmetric group S_n on the object $A^{\otimes n}$, i.e. a semigroup homomorphism

$$S_n \to \mathrm{end}_{\mathcal{C}}(A^{\otimes n}, A^{\otimes n}), \quad \sigma \mapsto \widetilde{\sigma}_A.$$

Using a chain of associativity constraints

$$\gamma_i : A^{\otimes n} \to A^{\otimes(i-1)} \otimes ((A \otimes A) \otimes A^{\otimes(n-i-1)}),$$

we define it on transpositions by

$$\widetilde{(i, i+1)}_A := \gamma_i^{-1} \circ (I \otimes (\mathbf{c}_{A,A}) \otimes I)) \circ \gamma_i$$

and extend to the whole group: existence of such an extension follows from the axioms of symmetric monoidal category. It gives a $\mathbb{K}S_n$-module structure on $\mathrm{hom}_{\mathcal{C}}(A^{\otimes n}, B)$ for a pair of objects A, B of $\mathcal{C}$.

1.3. **Extension of scalars.** Given a tensor category $\mathcal{C}$ over $\mathbb{K}$ and ring homomorphisms $\mathbb{F} \to \mathbb{K}$ and $\mathbb{F} \to \mathbb{K}'$ of commutative rings, we can construct a new tensor category $\mathcal{C}' = \mathcal{C} \otimes_{\mathbb{F}} \mathbb{K}'$ over $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K}'$. It has the same objects as $\mathcal{C}$ but new morphisms $\mathrm{hom}_{\mathcal{C}'}(X, Y) := \mathrm{hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{F}} \mathbb{K}'$. The compositions are defined in the obvious way: $f \otimes \alpha \circ g \otimes \beta = fg \otimes \alpha\beta$. Similarly, all the natural transformations are extended to $\mathcal{C}'$. All verifications are routine and left to an interested reader.

1.4. **Tensor functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories over the same commutative ring $\mathbb{K}$. *A tensor functor* $F = (F_1, F_2, F_3) : \mathcal{C} \to \mathcal{D}$ consists of a $\mathbb{K}$-linear functor $F_1 : \mathcal{C} \to \mathcal{D}$, a natural isomorphism $F_{2,A,B} : FA \otimes FB \to F(A \otimes B)$ and an isomorphism $F_3 : \mathbf{1}_{\mathcal{D}} \to F\mathbf{1}_{\mathcal{C}}$. They have to satisfy the standard axioms [8].

2. ALGEBRAS

2.1. **Algebras.** We can talk about algebras in $\mathcal{C}$ over a $\mathbb{K}$-linear operad or even a $\mathbb{K}$-linear prop [6]. *A* $\mathbb{K}$*-linear prop* is a tensor category $\mathcal{P}$ over $\mathbb{K}$ satisfying the following five conditions:

(1) its objects form the monoid $(\mathbb{N}, +)$, i.e. $n \otimes m = n + m$,
(2) it is strict, in this context it means that all $\mathbf{a}_{n,m,k} = I_{n+m+k}$, $\mathbf{l}_n = I_n$, and $\mathbf{r}_n = I_n$,
(3) for each object $n > 0$ a semigroup embedding $S_n \hookrightarrow \hom_{\mathcal{C}}(n, n)$, $\pi \mapsto \widetilde{\pi}$ is chosen,
(4) for all $\pi \in S_n$, $\sigma \in S_m$ their tensor product is another permutation $\tau \in S_{n+m}$:

$$\widetilde{\pi} \otimes \widetilde{\sigma} = \widetilde{\tau} \text{ where } \tau(k) = \begin{cases} \pi(k) & \text{if} \quad k \leq n \\ \sigma(k-n) + n & \text{if} \quad k > n \end{cases},$$

(5) the commutativity constraint is the shuffle $\mathbf{sh}_{n,m} \in S_{n+m}$:

$$\mathbf{c}_{n,m} = \widetilde{\mathbf{sh}}_{n,m} \text{ where } \mathbf{sh}_{n,m}(k) = \begin{cases} k + m & \text{if} \quad k \leq n \\ k - n & \text{if} \quad k > n \end{cases}.$$

Now *an algebra in* $\mathcal{C}$ over a prop $\mathcal{P}$ is a tensor functor $\mathcal{P} \to \mathcal{C}$. This level of generality is not always useful as the prop needs to be constructed. A more elementary approach is more suitable for our purposes. *A signature* $I = (I, h, t)$ is a set of operations I with head and tail functions $h, t : I \to \mathbb{Z}_{\geq 0}$, so that we think of $i \in I$ as an operation on $t(i)$ variables with $h(i)$ values. *An algebra of signature* I is a triple $(\mathcal{C}, A, (m_i, i \in I))$ where $\mathcal{C}$ is a tensor category over a commutative ring $\mathbb{K}$, A is an object of $\mathcal{C}$, and $m_i \in \hom_{\mathcal{C}}(A^{\otimes t(i)}, A^{\otimes h(i)})$ is a family of morphisms. If $\mathbb{K}$ or $\mathcal{C}$ is fixed, we talk about $\mathbb{K}$-algebras or $\mathcal{C}$-algebras. We specify a signature by listing the size of I and pairs $(t(i), h(i))$, thinking that $I = \{1, 2, \dots n\}$ and the k-th pair are the tail and head values of m_k.

We would like to request axioms to hold in algebras. In the partial case of an algebra A of signature $[1 : (2, 1)]$ the axioms are just polylinear identities. Let $F = \mathbb{K}\{x_1, ...\}$ be the free nonassociative algebra in countably many variables, F_n the subset of polylinear elements of degree n, linear in fixed variables, say $x_1, \dots x_n$. There is a natural "evaluation" map of $\mathbb{K}$-modules

$$\psi_A : F_n \to \hom_{\mathcal{C}}(A^{\otimes n}, A),$$

defined recursively

$$\psi(x_1) = I_A,\ \psi(x_1 x_2) = m_1,\ \psi(v(x_{\sigma(1)}, \dots x_{\sigma(k)}) w(x_{\sigma(k+1)}, \dots x_{\sigma(n)})) = m_1 \circ (\psi(v) \otimes \psi(w)) \circ \gamma \circ \widetilde{\sigma^{-1}}_A$$

where $\gamma : A^{\otimes n} \to A^{\otimes k} \otimes A^{\otimes(n-k)}$ is the composition of associativity constraints. Now we say that A satisfies a polylinear identity f if $\psi_A(f) = 0$.

A Lie algebra is an algebra of signature $[1 : (2, 1)]$ satisfying the Jacobi and the anticommutativity identities:

$$(x_1 x_2) x_3 + (x_2 x_3) x_1 + (x_3 x_1) x_2 \quad \text{and} \quad x_1 x_2 + x_2 x_1,$$

which can be written explicitly in the tensor notation as

$$m_1 \circ (m_1 \otimes I) \circ (\widetilde{1} + \widetilde{(2,3,1)} + \widetilde{(3,1,2)}) = 0,\ m_1 \circ (\widetilde{1} + \widetilde{(1,2)}) = 0.$$

Notice that if $\mathcal{C}$ is the category of $\mathbb{K}$-modules and $\frac{1}{2} \in \mathbb{K}$ then our definition is just the usual definition of a Lie algebra over $\mathbb{K}$. However, if $\frac{1}{2} \notin \mathbb{K}$, our anticommutativity is weak, i.e., $xy + yx = 0$ but we cannot conclude that $x^2 = 0$.

An associative algebra is an algebra of signature $[2 : (2, 1), (0, 1)]$ satisfying the associativity $(x_1 x_2) x_3 = x_1 (x_2 x_3)$ and the left and right unities. The latter cannot be written in terms of ψ_A so we recourse to the tensor notation formulating the axioms:

$$m_1 \circ (m_1 \otimes I) = m_1 \circ (I \otimes m_1) \circ \mathbf{a}_{A,A,A},\ m_1 \circ (m_2 \otimes I) = \mathbf{l}_A,\ m_1 \circ (I \otimes m_2) = \mathbf{r}_A.$$

A metric object is an algebra of signature $[2:(0,2),(2,0)]$ such that both structures are symmetric:

$$m_2 = m_2 \circ \mathbf{c}_{A,A}, \quad m_1 = \mathbf{c}_{A,A} \circ m_1,$$

and A is a dual object of A in the sense of monoidal categories:

$$I_A = \mathbf{r}_A \circ (I_A \otimes m_2) \circ \mathbf{a}_{A,A,A} \circ (m_1 \otimes I_A) \circ \mathbf{l}_A^{-1}.$$

Following Hinich and Vaintrob [6], we are introducing Casimir and metric Lie algebras in the next section.

2.2. **Representations.** Let $(\mathcal{C}, \mathfrak{g}, m)$ be a Lie algebra. *A representation* of $\mathfrak{g}$ is a pair (M, ρ) where M is an object in $\mathcal{C}$, $\rho \in \hom_{\mathcal{C}}(\mathfrak{g} \otimes M, M)$ and the Jacobi identity

$$\rho \circ (m \otimes I_M) = \rho \circ (I_{\mathfrak{g}} \otimes \rho) \circ \mathbf{a} - \rho \circ (I_{\mathfrak{g}} \otimes \rho) \circ (\mathbf{c}_{\mathfrak{g},\mathfrak{g}} \otimes I_M) \circ \mathbf{a} \in \hom_{\mathcal{C}}((\mathfrak{g} \otimes \mathfrak{g}) \otimes M, M)$$

holds. Completely parallel to the case of Lie algebras in vector spaces, the representations form a tensor category $\bmod_{\mathcal{C}}(\mathfrak{g})$. The morphisms $\hom_{\mathfrak{g}}(M, N)$ are those morphisms in $\hom_{\mathcal{C}}(M, N)$ that commute with the action of $\mathfrak{g}$. The tensor product is the tensor product in $\mathcal{C}$ with the usual action:

$$\rho_{M \otimes N} = (\rho_M \otimes I_N) \circ \mathbf{a}^{-1} + (I_M \otimes \rho_N) \circ (\mathbf{c}_{\mathfrak{g},M} \otimes I_M) \circ \mathbf{a}^{-1} \in \hom_{\mathcal{C}}(\mathfrak{g} \otimes (M \otimes N), M \otimes N).$$

The constraints are inherited from $\mathcal{C}$. Let us summarise this in a proposition whose proof is left to an interested reader.

Proposition 2. *Let $\mathfrak{g}$ be a Lie $\mathcal{C}$-algebra. Then the category $\bmod_{\mathcal{C}}(\mathfrak{g})$ is a tensor category over $\mathbb{K}_{\mathcal{C}}$. The multiplication in $\mathfrak{g}$ is a morphism in $\bmod_{\mathcal{C}}(\mathfrak{g})$, so $\mathfrak{g}$ is a Lie $\bmod_{\mathcal{C}}(\mathfrak{g})$-algebra.*

One could ask what sort of tensor categories one gets by iteration: consider representations of $\mathfrak{g}$ in $\bmod_{\mathcal{C}}(\mathfrak{g})$, etc. The answer is rather uninteresting: the reader can easily verify that subject to the existence of finite direct sums in $\mathcal{C}$, the category of representations of $\mathfrak{g}$ in $\bmod_{\mathcal{C}}(\mathfrak{g})$ is canonically equivalent to the category of representations of the semidirect product $\mathfrak{g} \ltimes \mathfrak{g}_a$ in $\mathcal{C}$ (where $\mathfrak{g}_a$ is the adjoint representation of $\mathfrak{g}$ treated as a Lie algebra with the zero multiplication).

Now *a Casimir Lie algebra* is a $\mathcal{C}$-algebra of signature $[2:(2,1),(0,2)]$ such that m_1 defines a Lie algebra while m_2 is a symmetric morphism in the category of $\mathfrak{g}$-modules (the action of $\mathfrak{g}$ on $\mathbf{1}$ is zero).

Similarly *a metric Lie algebra* is a $\mathcal{C}$-algebra $\mathfrak{g}$ of signature $[3:(2,1),(0,2),(2,0)]$ such that $(\mathfrak{g}, m_1)$ is a Lie $\mathcal{C}$-algebra, and $(\mathfrak{g}, m_2, m_3)$ is a metric $\bmod_{\mathcal{C}}(\mathfrak{g})$-object.

2.3. **Homomorphisms.** We distinguish two types of homomorphisms between algebras of the same signature. *A homomorphism from $(\mathcal{C}, A, m_i)$ to $(\mathcal{C}, B, m'_i)$* is a morphism $\varphi \in \hom_{\mathcal{C}}(A, B)$ which commutes with all multiplications, i.e. the diagram

$$\begin{tikzcd}
A^{\otimes h(i)} \arrow[r, "\varphi^{\otimes h(i)}"] \arrow[d, "m_i"'] & B^{\otimes h(i)} \arrow[d, "m'_i"] \\
A^{\otimes t(i)} \arrow[r, "\varphi^{\otimes t(i)}"'] & B^{\otimes t(i)}
\end{tikzcd}$$

is commutative for all i.

A specialization from $(\mathcal{C}, A, m_i)$ to $(\mathcal{D}, B, m'_i)$ is a triple $\psi = (\psi_1, \psi_2, \psi_3)$ where $\psi_1 : \mathbb{K}_{\mathcal{C}} \to \mathbb{K}_{\mathcal{D}}$ is a ring homomorphism, $\psi_2 : \mathcal{C} \otimes_{\mathbb{K}_{\mathcal{C}}} \mathbb{K}_D \to \mathcal{D}$ is a tensor functor, and ψ_3 is a homomorphism from $F(A)$ to B.

Notice that we may consider the three types of homomorphisms (or specializations) between metric Lie algebras: a Lie algebra homomorphism, a Casimir Lie algebra homomorphism, and a metric Lie algebra homomorphism. The difference is in the preserved operations.

2.4. **Identities of algebras in categories.** Varieties of algebras is a topic of an active study in Algebra. Should one study varieties of algebras in categories? In this subsection we make some easy observations and pose some questions.

At the first glance, at least if $\mathbb{K}$ is a $\mathbb{Q}$-algebra, the varieties of algebras in categories are not much different from the varieties of usual algebras. Indeed, an interested reader could easily verify that the identities of a $\mathcal{C}$-algebra is equal to the set of polylinear elements in some T-ideal of the free algebra $\mathbb{K}\{x_1, ...\}$. For instance, the following Engel-type theorem is an immediate consequence of the same theorem for usual Lie algebras [15, Th 6.4.1].

Theorem 3. *Suppose $\mathbb{K}$ is a commutative $\mathbb{Q}$-algebra, $\mathcal{C}$ is a tensor category over $\mathbb{K}$, and $\mathfrak{g}$ is a Lie $\mathcal{C}$-algebra satisfying the linearised Engel identity. Then $\mathfrak{g}$ satisfies the linearised nilpotency.*

Nevertheless, in our view, it is interesting to understand examples of $\mathcal{C}$-algebras, in particular, what identities they satisfy. For instance, if X is a rigid object in a tensor category $\mathcal{C}$ then $E_X = X \otimes X^*$ is an associative $\mathcal{C}$-algebra, so-called *the internal endomorphism algebra*. It would be interesting to understand the identities of E_X in the free associative algebra $\mathbb{K} < x_1, ... >$ (cf. [5]).

Question 1. *Given a rigid object X in $\mathcal{C}$, what is the minimal degree of an identity of E_X in $\mathbb{K} < x_1, ... >$?*

If $\mathcal{C}$ is the category of vector spaces, X is an n-dimensional vector space then E_X is the algebra of $n \times n$ matrices, whose minimal identity has degree $2n$ by the Amitsur-Levitzki theorem [1]. If $\mathcal{C}$ is the category of supervector spaces (this is the same as superbundles on a point, see Section 4.1), X is an (n, k)-dimensional supervector space then E_X is the matrix superalgebra whose minimal identity is conjectured to be of the degree $2(nk + n + k) - \min\{n, k\}$ [22].

One reason the matrix superalgebras are of interest is that in characteristic zero they generate all prime varieties of associative algebras [13]. Is it conceivable that algebras E_X generate all varieties?

Question 2. *Give an example of a commutative $\mathbb{Q}$-algebra $\mathbb{K}$ and a T-ideal in $\mathbb{K} < x_1, ... >$ that is not an ideal of identities of E_X for any rigid object X in any tensor category $\mathcal{C}$ over $\mathbb{K}$.*

We finish this section explaining the categorical analogue of the Grassmann envelope used to study the varieties of superalgebras. *The global sections functor* $\Gamma : \mathcal{C} \to \operatorname{mod}(\mathbb{K}_{\mathcal{C}})$ is defined by $\Gamma(A) := \hom_{\mathcal{C}}(\mathbf{1}, A)$. If A is an algebra in $\mathcal{C}$ then $\Gamma(A)$ is a $\mathbb{K}_{\mathcal{C}}$-algebra in $\mathbb{K}_{\mathcal{C}}$-modules. The following proposition is evident.

Proposition 4. *Suppose $\mathcal{C}$ is a tensor category, A is a $\mathcal{C}$-algebra, C is an associative commutative $\mathcal{C}$-algebra. Then $\Gamma(A \otimes C)$ is a $\operatorname{mod}(\mathbb{K}_{\mathcal{C}})$-algebra of the same signature as A. Moreover, $\Gamma(A \otimes C)$ satisfies all identities of A.*

The Grassmann algebra G_∞ in countably many variables is an associative commutative algebra in the category $\mathcal{C}$ of supervector spaces. If A is a superalgebra then $\Gamma(A \otimes G_\infty)$ is the Grassmann envelope of A. The identities of $\Gamma(A \otimes G_\infty)$ are precisely the identities of A in $\mathcal{C}$. Can this be carried out in any category?

Question 3. *Let $\mathcal{C}$ be a cocomplete tensor category with split idempotents. Does there exist a commutative associative algebra G in $\mathcal{C}$ such that for any $\mathcal{C}$-algebra A all identities of $\Gamma(A \otimes G)$ hold in A?*

3. SIMPLE LIE ALGEBRAS

3.1. **Quasisimple Lie algebras.** Following Vogel [24, 25], a Lie algebra $(\mathcal{C}, \mathfrak{g}, m_1)$ is *quasisimple* if it can be extended to a metric Lie algebra $(\mathcal{C}, \mathfrak{g}, m_1, m_2, m_3)$ and the natural map $\mathbb{K}_{\mathcal{C}} \to \hom_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ is an isomorphism. We discuss how it is related to the usual simplicity in the next section.

3.2. **Simple Lie algebras.** If the $\mathbb{K}$-tensor category $\mathcal{C}$ is abelian, we can talk about subobjects, quotients, kernels, ideals, etc. in the usual way. In particular, there is the usual notion of a simple Lie algebra. However, the category being abelian is not required to introduce simplicity, although the utility of this notion is unclear. Neither is it clear whether this definition is useful for other classes of algebras.

Recall that a morphism $f \in \hom_{\mathcal{C}}(X, Y)$ is *a monomorphism* if it can be cancelled on the left, i.e., $fa = fb$ implies $a = b$ for all $a, b \in \hom_{\mathcal{C}}(W, X)$. A Lie algebra $\mathfrak{g}$ in $\mathcal{C}$ is *simple* if $\mathfrak{g} \not\cong 0$ (notice that if a zero object exists, it is unique up to a canonical isomorphism) and every algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ is a monomorphism or the zero. Notice the fine difference with the usual simple Lie algebras in vector spaces: the latter definition traditionally excludes the one-dimensional Lie algebra while our definition does not.

In general, there is no relation between simple and quasisimple Lie algebras. Let $\mathbb{K}$ be a commutative ring, not a field, $1/2 \in \mathbb{K}$. It admits a proper ideal I. Then the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ (in the category of $\mathbb{K}$-modules) is quasisimple: the trace form $\mathrm{Tr}(XY)$ is a metric. It is not simple because the quotient homomorphism $\mathfrak{sl}_2(\mathbb{K}) \to \mathfrak{sl}_2(\mathbb{K}/I)$ is neither zero, nor a monomorphism.

Here is another example of a quasisimple non-simple Lie algebra. Let X be a set of simple finite dimensional complex Lie algebras, $C = \mathbb{C}X$ the group coalgebra. A C-comodule is just an X-graded vector space $M = \oplus_{x \in X} M_x$. We consider the tensor category $\mathcal{C}$ of C-comodules with the standard tensor coproduct as a tensor product: $M \otimes N := \oplus_{x\in X} M_x \otimes_{\mathbb{C}} N_x$. Now $\mathfrak{g} = \oplus_{\mathfrak{h}\in X}\mathfrak{h}$ is a metric Lie $\mathcal{C}$-algebra under the direct sums of the products, the forms and the coforms

$$\oplus_{\mathfrak{h}\in X} m_{\mathfrak{h}} : \oplus\mathfrak{h}\otimes\mathfrak{h} \to \oplus\mathfrak{h},\ \oplus_{\mathfrak{h}\in X} m_{\mathfrak{h}} : \oplus\mathfrak{h}\otimes\mathfrak{h} \to C,\ \oplus_{\mathfrak{h}\in X} m_{\mathfrak{h}} : C \to \oplus\mathfrak{h}\otimes\mathfrak{h}.$$

Observe that $\mathfrak{g}$ is quasisimple but not simple since every $\mathfrak{h}$ is a proper ideal of $\mathfrak{g}$ resulting in the quotient homomorphism.

In the opposite direction, $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra in the category of real vector spaces but it is not quasisimple because $\hom_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) = \mathbb{C} \neq \mathbb{R}$. Nevertheless, sometimes there is a relation.

Theorem 5. *Let $\mathcal{C}$ be the category of vector spaces over an algebraically closed field $\mathbb{K}$ of characteristic zero. Then quasisimple Lie $\mathcal{C}$-algebras are exactly the finite dimensional simple[2] Lie algebras.*

Proof. A finite dimensional simple Lie algebra $\mathfrak{g}$ is quasisimple because the Killing form is non-degenerate and $\mathbb{K} \to \mathrm{end}_{\mathfrak{g}}(\mathfrak{g})$ is an isomorphism by the Schur lemma and algebraic closeness of $\mathbb{K}$.

In the opposite direction, the space of symmetric invariant form $\mathfrak{g}\otimes\mathfrak{g} \to \mathbb{K}$ is one-dimensional since it is a subspace of $(\mathfrak{g}^*\otimes\mathfrak{g}^*)^{\mathfrak{g}} \cong \hom_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) = \mathbb{K}$. By the theorem of Bajo and Benayadi [2], $\mathfrak{g}$ is simple. ☐

It would be interesting to find some other categories where an analogue of Theorem 5 holds or where the quasisimple Lie algebras admit a meaningful classification. Let us pose two precise questions.

Question 4. *Let G be an affine group scheme over a field $\mathbb{K}$ of characteristic zero. Let $\mathcal{C}$ be the category of rational G-modules. What are simple and quasisimple Lie algebras in $\mathcal{C}$?*

To get a feel of this question, let us consider an absolutely irreducible G-module V as a Lie algebra with the zero multiplication in $\mathcal{C}$. According to our definition, V is simple. On the other hand, V is quasisimple if and only if $V \cong V^*$.

Question 5. *Let $\mathcal{C}$ be the category of supervector spaces over an algebraically closed field $\mathbb{K}$ of characteristic zero. Is a quasisimple Lie $\mathcal{C}$-algebra a finite dimensional simple Lie superalgebra?*

Finite dimensional simple Lie superalgebras are classified by Kac [9]. Simple Lie algebras and simple Lie superalgebras of types $A(m,n)$, $n\neq m$, $B(m,n)$, $C(n)$, $D(m,n)$, $m-n\neq 1$, $F(4)$, and $G(3)$ have nondegenerate Killing forms [9, Th. 1]. Hence, they are quasisimple Lie $\mathcal{C}$-algebras. The remaining simple Lie superalgebras of types $A(n,n)$, $D(n+1,n)$, $P(n)$, $Q(n)$, and $D(2,1,\alpha)$ as well as Cartan type superalgebras have zero Killing forms [9, Prop. 2.4.1], although there may be an invariant form distinct from the Killing form. For instance, this happens in types $Q(n)$ and $D(2,1,\alpha)$. Her is a partial result towards this question.

Proposition 6. *Let $\mathfrak{g}$ be a quasisimple Lie algebra in the category of supervector spaces over a field $\mathbb{K}$. Then*

(1) $\mathfrak{g}$ is perfect,
(2) the centre of $\mathfrak{g}$ is zero,
(3) every minimal ideal of $\mathfrak{g}$ is abelian.

Proof. Let I be a minimal ideal. Its orthogonal complement $I^\perp$ is also an ideal. Hence, the intersection $I\cap I^\perp$ is zero or I. If $I\cap I^\perp = 0$, then $\mathfrak{g} = I\oplus I^\perp$ and we get nontrivial idempotents in $\mathbb{K} = \mathrm{end}_{\mathfrak{g}}(\mathfrak{g})$. Hence $I\cap I^\perp = I$, so $I\subseteq I^\perp$. This means that $([x,y],a) = (x,[y,a]) = 0$ for all $x,y\in I$, $a\in\mathfrak{g}$. As the form is nondegenerate, $[x,y]=0$ and I is abelian.

Now $[\mathfrak{g},\mathfrak{g}]^\perp = \{a\in\mathfrak{g} \mid \forall x,y\in\mathfrak{g}\ ([x,y],a) = (x,[y,a]) = 0\} = \{a\in\mathfrak{g} \mid \forall y\in\mathfrak{g}\ [y,a]=0\} = Z(\mathfrak{g})$. We conclude that $\mathfrak{g}$ is perfect and $Z(\mathfrak{g}) = 0$. Indeed, otherwise there exists a nonzero bilinear form on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Extend it to a form α on $\mathfrak{g}$: it is invariant, nonzero and degenerate. This contradicts the one-dimensionality of $\mathrm{end}_{\mathfrak{g}}(\mathfrak{g}) \cong (\mathfrak{g}\otimes\mathfrak{g})^{*\mathfrak{g}}$. ☐

Using Proposition 6 and the Cartan criteria, we can furnish an alternative proof of Theorem 5. Since $\mathfrak{g}$ is not solvable, the Killing form is nonzero. So the Killing form is a multiple of the non-degenerate form, and itself nondegenerate. Thus, $\mathfrak{g}$ is semisimple. It is simple since $\mathrm{end}_{\mathfrak{g}}(\mathfrak{g}) = \mathbb{K}$. This proof fails for the superalgebras because of the lack of the Cartan criteria.

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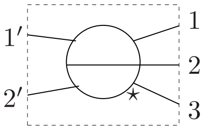
[2] Notice that this includes the one-dimensional Lie algebra according to our definition

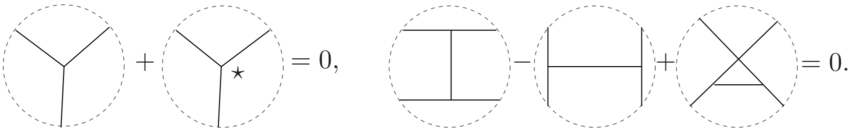
3.3. **Universal metric Lie algebra.** The universal metric Lie algebra appears in the study of Vassiliev invariants [19, 25]. The relevant symmetric monoidal category $\mathcal{C}$ is a prop. The hom-set $\hom_{\mathcal{C}}(m,n)$ is the $\mathbb{Q}$-vector space of Jacobi diagrams[3] [19]. Recall that an (m,n)-Jacobi diagram is a compact curve X such that

1. the boundary of X is the set $\{1, 2, \dots m, 1', 2' \dots n'\}$,
2. X has finitely many trivalent singular points, i.e. points x with a neighbourhood U such that $U \setminus \{x\}$ is a union of three lines,
3. for each trivalent singular point x a cyclic ordering on the components of $U \setminus \{x\}$ is fixed.

Here is an example of the Jacobi diagram in $\hom_{\mathcal{C}}(3,2)$:



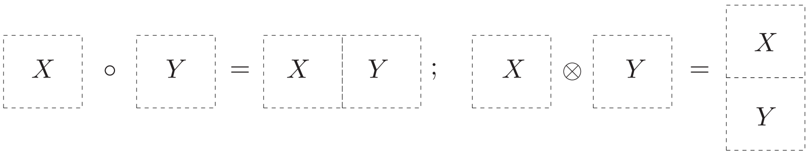
The external dashed borders have no significance. This diagram has five trivalent singular points, four of them have the standard counterclockwise ordering, while the remaining vertex marked with $\star$ has the opposite clockwise ordering. The point of transversal intersection has no significance: it is actually two distinct points of the curve which coincided after an immersion into a plane (a.k.a drawing). Now $\hom_{\mathcal{C}}(m,n)$ is the quotient space of the $\mathbb{Q}$-span of all (m,n)-Jacobi diagram subject to the AS-relation and the IHX-relation:



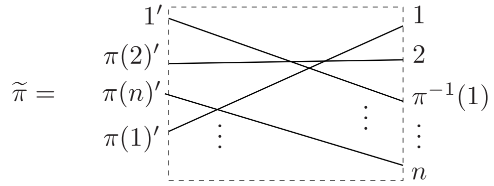
The dashed circles bound neighbourhoods in Jacobi diagrams X_1, X_2 for the first relation and Y_1, Y_2, Y_3 for the second one, which are identical except for these neighbourhoods. The relations mean that

$$X_1 + X_2 = 0 \quad \text{and} \quad Y_1 - Y_2 + Y_3 = 0.$$

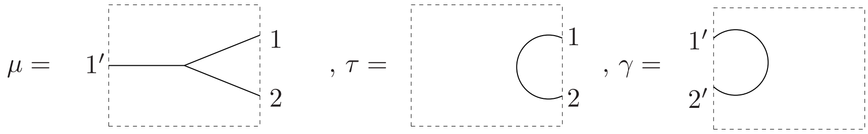
The composition and the tensor product of morphisms is done by "stacking" the boxes:



Thus, the tensor product of morphisms is just a union while the composition is gluing along the corresponding part of the boundary. The symmetric group S_n embeds in a way that $\widetilde{\pi}$ is a union of n intervals connecting k with $\pi(k)'$:



The Lie $\mathcal{C}$-algebra $\mathfrak{g}_M = (1, \mu, \tau, \gamma)$ where



is a universal metric Lie algebra. Let us state its universality property [19, 20] as a proposition.

Proposition 7. *Given a commutative $\mathbb{Q}$-algebra $\mathbb{K}$ and a metric Lie $\mathbb{K}$-algebra $(\mathcal{D}, \mathfrak{g}, m, t, c)$, there exists a unique specialization $\psi : (1, \mathfrak{g}_M, \tau) \to \mathfrak{g}$ such that $\psi_2(1) = \mathfrak{g}$, $\psi_2(\mu) = m$, $\psi_2(\tau) = t$, $\psi_2(\gamma) = c$ and $\psi_3 = I_{\mathfrak{g}}$.*

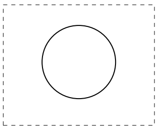
---

[3] also known as Chinese characters

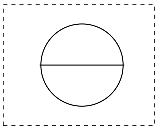
A part of the specialization ψ is a functor $\psi_2 : \mathcal{C} \otimes \mathbb{K} \to \ \text{mod}_{\mathcal{D}}(\mathfrak{g})$ that turns $\mathfrak{g}$ into an algebra over the prop $\mathcal{C} \otimes \mathbb{K}$. In the language of props $\mathcal{C} \otimes \mathbb{K}$ is the universal metric Lie algebra over $\mathbb{K}$: $\mathcal{C} \otimes \mathbb{K}$-algebras are the same as metric Lie $\mathbb{K}$-algebras.

Let us notice that $(1, \mu, \tau)$ is not quasisimple: $\mathbb{K}_{\mathcal{C}} = \text{hom}_{\mathcal{C}}(0,0) \cong \mathbb{Q}[x_1, x_2 \ldots]$, the polynomial algebra on connected $(0,0)$-Jacobi diagrams such as

$\delta =$



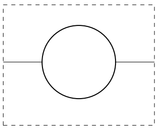
$, \theta =$



$, \cdots \in \text{hom}_{\mathcal{C}}(0,0),$

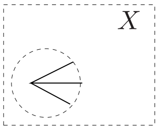
while $\text{end}_{\mathfrak{g}_M}(\mathfrak{g}_M) = \text{hom}_{\mathcal{C}}(1,1)$ with the homomorphism $\mathbb{K}_{\mathcal{C}} \to \text{hom}_{\mathcal{C}}(1,1)$ given by $v \mapsto v \otimes I_1$, so the following element ϕ is not in the image:

$\phi =$

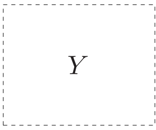


$\in \text{hom}_{\mathcal{C}}(1,1).$

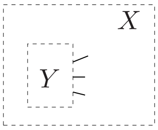
3.4. **Vogel's ring Λ.** The symmetric group S_n acts on $\text{hom}_{\mathcal{C}}(n,0)$ by permuting the inputs. As a vector space, Vogel's ring Λ is $\text{hom}_{\mathcal{C}}(3,0)^{S_3,\varepsilon}$, the skew invariants with respect to the sign character ε of S_3. The multiplication is via insertion of one diagram into any trivalent point of the second diagram:



$\cdot$

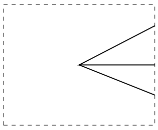


$=$

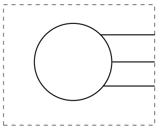


Observe that the connectedness and the skew invariance are crucial for this product to be well-defined. While the associativity of Λ is obvious, the commutativity is subtle. It has been proved by Vogel [25] who also observed that the insertion defines a Λ-module structure on $\text{hom}^s_{\mathcal{C}}(m,n) \subseteq \text{hom}_{\mathcal{C}}(m,n)$, the span of connected diagrams with at least one trivalent singular point [24, Prop 3.2]. Here are some elements of Λ:

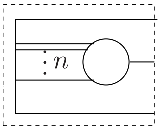
$1 =$



$, t =$



$, x_n =$



.

Observe that $x_1 = 2t$ and $x_2 = t^2$ [25]. The ring Λ is naturally graded: Λ_n is spanned by the diagrams with $2n+1$ trivalent singular points. The Poincare series of Λ is not known.

The ring Λ is closely related to a subalgebra $\widetilde{\Lambda} = \mathbb{Q}[\sigma_1] \oplus \omega\mathbb{Q}[z_1, z_2, z_3]^{S_3}$ of $\mathbb{Q}[z_1, z_2, z_3]^{S_3}$ where $\omega = (\sigma_1 + z_1)(\sigma_1 + z_2)(\sigma_1 + z_3) = \sigma_3 + \sigma_2\sigma_1 + 2\sigma_1^3$ and σ_1, σ_2, σ_3 are the elementary symmetric polynomials. Kneissler constructs further elements $\chi_n \in \widetilde{\Lambda}$ by

$$\chi_0 = 0,\ \chi_1 = 2\sigma_1,\ \chi_2 = \sigma_1^2,\ \chi_{n+3} = \sigma_1\chi_{n+2} - \sigma_2\chi_{n+1} + \sigma_3\chi_n + \frac{\sigma_2\sigma_1^{n+1}}{2} - \frac{\sigma_3\sigma_1^n}{2} - \sigma_3(2\sigma_1)^n$$

and a homomorphism of graded algebras [14]

$$\varphi : \widetilde{\Lambda} \to \Lambda, \quad \varphi(\chi_n) = x_n.$$

Vogel constructs a polynomial $\pi \in \widetilde{\Lambda}$ such that $\varphi(\pi) = 0$ and conjectures that $\widetilde{\Lambda}/(\pi) \cong \Lambda$ [24]. Thus we have homomorphisms of graded rings

$$\mathbb{Q}[z_1, z_2, z_3] \hookleftarrow \mathbb{Q}[z_1, z_2, z_3]^{S_3} \hookleftarrow \widetilde{\Lambda} \to \Lambda$$

and their projective spectra

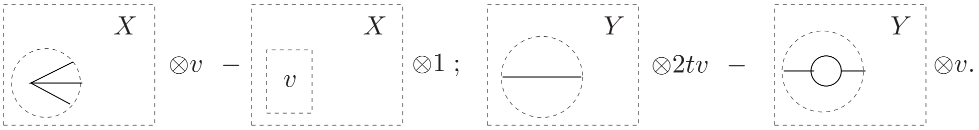
$$\mathbb{P}^2 \to \mathbb{P}^2_{1,2,3} \xrightarrow{f} \widetilde{\mathbb{V}} \xleftarrow{g} \mathbb{V}.$$

A quasisimple Lie $\mathbb{C}$-superalgebra $\mathfrak{g}$ defines a ring homomorphism

$$\Theta_{\mathfrak{g}} : \Lambda \to \mathbb{C}, \quad \Theta_{\mathfrak{g}}(x)[a,b] = \sum_i \psi_2(x)(a \otimes b \otimes e_i)e^i$$

where $x \in \Lambda$, $a, b, e^i, e_i \in \mathfrak{g}$ and the summation is over some dual bases [24, 25]. It depends on the choice of the form on $\mathfrak{g}$. The dependence is controlled by the natural action of the multiplicative group $\mathbb{G}_m$, so the character $\Theta_{\mathfrak{g}}$ is a well-defined point of $\mathbb{V}(\mathbb{C})$. The point $g(\Theta_{\mathfrak{g}}) \in \widetilde{\mathbb{V}}(\mathbb{C})$ and the set $f^{-1}(g(\Theta_{\mathfrak{g}})) \in \mathbb{P}^2_{1,2,3}(\mathbb{C})$ are also of interest. All three varieties $\mathbb{P}^2_{1,2,3}$, $\widetilde{\mathbb{V}}$ or $\mathbb{V}$ can be called *Vogel's plane*.

3.5. **Vogel's universal Lie algebra.** A quasisimple Lie superalgebra $\mathfrak{g}$ defines a character $\Theta_{\mathfrak{g}}$ of the ring Λ and a specialization of $\mathfrak{g}_M$. Vogel has found a way to combine these two into a single structure by forcing the action of Λ on the category $\mathcal{C}$ [25]. The new tensor category $\mathcal{C}'$ is a quotient of $\mathcal{C} \otimes_{\mathbb{Q}} \Lambda$ by the Vogel relations:



Thus, $\mathcal{C}'$ is also a prop with morphisms $\hom_{\mathcal{C}'}(m,n) := \hom_{\mathcal{C}}(m,n) \otimes_{\mathbb{Q}} \Lambda / I_{m,n}$ where $I_{m,n}$ is the $\mathbb{Q}$-span of all

$$X_1 \otimes v - X_2 \otimes 1 \ , \quad Y_1 \otimes 2tv - X_2 \otimes v$$

where $v \in \Lambda$, X_1 and X_2 differ as on the diagrams in the first relation, Y_1 and Y_2 differ as on the diagrams in the second relation.

Theorem 8. *(1) $\mathcal{C}'$ is a tensor category over $\Lambda[\delta]$.*
(2) $\mathfrak{g}_V = (1, \mu \otimes 1, \tau \otimes 1, \gamma \otimes 1)$ is a quasisimple Lie $\mathcal{C}'$-algebra.

Proof. Observe that the tensor product and the composition of cosets

$$(f + I_{m,n}) \circ (g + I_{n,k}) = fg + I_{m,k}, \ \ (f + I_{m,n}) \otimes (g + I_{m,n}) = f \otimes g + I_{m,n}$$

are well defined. It follows that $\mathcal{C}'$ is a tensor category.

To compute the scalars, we consider $\hom^s_{\mathcal{C}}(n,m) \subseteq \hom^c_{\mathcal{C}}(n,m) \subseteq \hom_{\mathcal{C}}(n,m)$ where $\hom^c$ is the span of the connected diagrams and $\hom^s$ is the span of the connected diagrams with at least one singular point. As observed above, $\hom^s_{\mathcal{C}}(n,m)$ is a Λ-module. Furthermore,

$$\mathbb{K}_{\mathcal{C}} = \hom_{\mathcal{C}}(0,0) \cong \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}(\hom^s_{\mathcal{C}}(0,0)) \cong \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}(\Lambda \cdot \theta)$$

where $\mathrm{Sym}_{\mathbb{Q}}$ is the symmetric algebra of a vector space. The element θ is a free generator of the Λ-module [24, Cor 4.6]. Since the first relation in $I_{0,0}$ asserts Λ-linearity on $\mathrm{Sym}_{\mathbb{Q}}(\Lambda \cdot \theta)$,

$$\mathbb{K}_{\mathcal{C}'} = \hom_{\mathcal{C}}(0,0) \otimes_{\mathbb{Q}} \Lambda / I_{0,0} \cong \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} \mathrm{Sym}_{\Lambda}(\Lambda \cdot \theta)/(\theta - 2t\delta) \cong \Lambda[\delta],$$

proving the first statement. Since $\mathfrak{g}_M$ is a metric Lie algebra, $\mathfrak{g}_V$ is a metric Lie algebra. Furthermore,

$$\mathrm{end}_{\mathfrak{g}_M}(\mathfrak{g}_M) = \hom_{\mathcal{C}}(1,1) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\hom^c_{\mathcal{C}}(1,1) \oplus (\hom^c_{\mathcal{C}}(0,1) \otimes_{\mathbb{Q}} \hom^c_{\mathcal{C}}(1,0))) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi)$$

because ϕ is a free generator of the Λ-module $\hom^s_{\mathcal{C}}(1,1)$ [24, Cor 4.6] and $\hom^c_{\mathcal{C}}(0,1) = 0$, $\hom^c_{\mathcal{C}}(1,0) = 0$ [24, Prop. 4.3]. Finally,

$$\mathrm{end}_{\mathfrak{g}_V}(\mathfrak{g}_V) = \hom_{\mathcal{C}'}(1,1) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi) \otimes_{\mathbb{Q}} \Lambda / I_{1,1} \cong \mathbb{K}_{\mathcal{C}'} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi)/(\phi - 2tI_1) \cong \mathbb{K}_{\mathcal{C}'}.$$

□

We say that a Lie $\mathcal{C}$-algebra $\mathfrak{g}$ is *V-simple* if it is quasisimple and a specialization $\mathfrak{g}_M \to \mathfrak{g}$ extends to a specialization $\mathfrak{g}_V \to \mathfrak{g}$.

Clearly, all this definition requires is a ring homomorphism $\Theta_{\mathfrak{g}} : \Lambda[\delta] \to \mathbb{K}_{\mathcal{C}}$ satisfying the Vogel relations. Existence of such a homomorphism is not clear to us, in general. Vogel gives a categorical criterion for a quasisimple Lie algebra to be V-simple [25] but it is of limited use. To prove that a quasisimple Lie superalgebra $\mathfrak{g}$ is V-simple [25] it is easier to use the homomorphism $\Theta_{\mathfrak{g}}$ constructed at the end of Section 3.4 (notice that $\Theta_{\mathfrak{g}}(\delta) = \dim_{\mathcal{C}} \mathfrak{g}$ in any category). In the final chapter we use the same strategy of constructing a ring homomorphism explicitly while the Vogel categorical criterion fails. So far, all known to us quasisimple Lie algebras are V-simple, hence, the following question is interesting.

Question 6. *Let $\mathcal{C}$ be a tensor category. Is a quasisimple Lie $\mathcal{C}$-algebra V-simple?*

4. Rozansky-Witten invariants

4.1. **The category of vector superbundles.** A holomorphic manifold X admits an associated tensor category, crucial for Kapranov's approach to Rozansky-Witten invariants [10, 20]. We attach a slightly smaller category than Kapranov. We explain the difference after we explain the construction.

The objects in the category $\mathrm{SV}(X)$ of vector superbundles are locally free coherent sheaves $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. The tensor product is usual:

$$(\mathcal{F} \otimes \mathcal{G})_0 = (\mathcal{F}_0 \otimes \mathcal{G}_0) \oplus (\mathcal{F}_1 \otimes \mathcal{G}_1),\ (\mathcal{F} \otimes \mathcal{G})_1 = (\mathcal{F}_1 \otimes \mathcal{G}_0) \oplus (\mathcal{F}_0 \otimes \mathcal{G}_1).$$

A locally free coherent sheaf $\mathcal{F}$ gives to two objects: even $\mathcal{F}^+ = \mathcal{F} \oplus 0$ and odd $\mathcal{F}^- = 0 \oplus \mathcal{F}$. The unit object is the even trivial line bundle $\mathcal{O}_X^+$. The symmetric braiding on $\mathrm{SV}(X)$ is the usual superbraiding

$$\tau(v_i \otimes w_j) = (-1)^{ij} w_j \otimes v_i$$

where v_i, w_i are homogeneous of degree i or j, i.e. local sections of $\mathcal{F}_i$ and $\mathcal{G}_j$, correspondingly.

The hom-sets are slightly unusual:

$$\hom_{\mathrm{SV}(X)}(\mathcal{F}_0 \oplus \mathcal{F}_1, \mathcal{G}_0 \oplus \mathcal{G}_1) = \oplus_{i,j,n} \mathrm{Ext}^{(i-j)+2n}(\mathcal{F}_i, \mathcal{G}_j).$$

The composition is the cup-product of extensions. The tensor product of two morphisms $f = (f_n)$ and $g = (g_n)$ is defined by $(f \otimes g)_n = f_n \otimes g_n$.

Usually in the literature, "larger" categories are considered. First, one can get more morphisms in by considering all extensions

$$\hom_{\mathrm{SV}_1(X)}(\mathcal{F}_0 \oplus \mathcal{F}_1, \mathcal{G}_0 \oplus \mathcal{G}_1) = \oplus_{i,j} \mathrm{Ext}^*(\mathcal{F}(V_i), \mathcal{F}(\mathcal{G}_j))$$

which gives a category $\mathrm{SV}_1(X)$: $\mathrm{SV}(X)$ is a subcategory of $\mathrm{SV}_1(X)$ with the same objects. One can add more objects by considering all coherent (super)-sheaves and their extensions: $\mathrm{SV}_1(X)$ is a full subcategory of the resulting category $\mathrm{SV}_2(X)$. One can further extend to the derived bounded category of coherent sheaves $\mathrm{SV}_3(X)$. Finally, there is a well-defined subcategory $\mathrm{SV}_4(X)$ of $\mathrm{SV}_3(X)$ of "C_2-equivariant" complexes [11]. For us all these larger categories are irrelevant since all the necessary objects and morphisms for the Kapranov theorem are in $\mathrm{SV}(X)$.

4.2. **Atiyah classes.** Let $\mathcal{T}_X$ be the tangent sheaf on X, $\mathcal{D}_X^{<n}$ the sheaf of differential operators with holomorphic coefficients of order less than n. We have an exact sequence of $\mathcal{O}_X - \mathcal{O}_X$-bimodules

$$0 \to \mathcal{O}_X \longrightarrow \mathcal{D}_X^{<2} \longrightarrow \mathcal{T}_X \to 0.$$

Notice that on both $\mathcal{O}_X$ and $\mathcal{T}_X$ the right and the left actions of $\mathcal{O}_X$ coincide but on $\mathcal{D}_X^{<2}$ they are different. Given a locally free coherent sheaf $\mathcal{F}$, we get a new exact sequence by tensoring with it

$$0 \to \mathcal{F} \longrightarrow \mathcal{D}_X^{<2} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to 0.$$

This is extension is the Atiyah class

$$A_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{F}) \subseteq \hom_{\mathrm{SV}(X)}(\mathcal{T}_X^- \otimes \mathcal{F}, \mathcal{F}).$$

Let us remark that the standard Atiyah class is actually $-A_{\mathcal{F}}$ [10]. The following theorem has essentially been proved by Kapranov [10], but a reader may benefit by looking at later reviews [19, 20].

Theorem 9. *(1)* $\mathfrak{g}_X = (\mathrm{SV}(X), \mathcal{T}_X^-, A_{\mathcal{T}_X})$ *is a Lie algebra.*
(2) Every superbundle $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ *is a representation of* $\mathfrak{g}_X$ *with the action* $A_{\mathcal{F}}$*.*
(3) If ω *is a holomorphic symplectic form on* X *then* $\mathfrak{g}_X$ *is a metric Lie algebra with a form* ω*.*

Proof. Kapranov's original proof establishes this theorem in the bigger category $\mathrm{SV}_1(X)$ [10]. We just need to point out that everything in Kapranov's proof happens in either odd or even extensions, so the theorem actually holds in $\mathrm{SV}(X)$. In particular, observe that ω is skew-symmetric and $\mathcal{T}_X^-$ is odd, hence ω is symmetric in $\mathrm{SV}(X)$. □

At the moment, the difference between $\mathrm{SV}(X)$ and $\mathrm{SV}_1(X)$ looks purely cosmetic. It becomes crucial when one addresses quasisimplicity.

4.3. **Quasisimplicity.** Let $S^n\mathcal{F}$ (or $T^n\mathcal{F}$, or $\Lambda^n\mathcal{F}$) be the n-th symmetric (or tensor, or exterior) power of a locally free coherent sheaf $\mathcal{F}$. Let us make a couple of general observations before addressing quasisimplicity of $\mathfrak{g}_X$. First, observe that $H^m(X,\mathcal{F})=0$ for all $m>\dim_{\mathbb{C}} X$ [26, Cor 4.39]. Secondly, observe[4] that for a holomorphic symplectic manifold X

$$\mathrm{end}(\mathcal{T}_X)\cong T^2\mathcal{T}_X\cong S^2\mathcal{T}_X\oplus\Lambda^2\mathcal{T}_X\cong S^2\mathcal{T}_X\oplus(\mathcal{L}_X\oplus\mathcal{O}_X)$$

with the trace map splitting the morphism $\mathcal{O}_X\to\Lambda^2\mathcal{T}_X$.

Theorem 10. *Let X be a holomorphic symplectic manifold. Then $\mathfrak{g}_X$ is quasisimple if and only if $H^{2*}(X,S^2\mathcal{T}_X\oplus\mathcal{L}_X)=0$.*

Proof. The scalars of the category $\mathrm{SV}(X)$ are

$$\mathbb{K}_{\mathrm{SV}(X)}=\mathrm{end}_{\mathrm{SV}(X)}(\mathcal{O}_X^+)=\mathrm{Ext}^{2*}(\mathcal{O}_X,\mathcal{O}_X)=H^{2*}(X,\mathcal{O}_X).$$

The naturality of the Atiyah class means that every homomorphism in $\mathrm{SV}(X)$ is a homomorphism of representations of $\mathfrak{g}_X$. Hence,

$$\mathrm{end}_{\mathfrak{g}_X}(\mathfrak{g}_X)=\mathrm{Ext}^{2*}(\mathcal{T}_X,\mathcal{T}_X)=H^{2*}(T^2\mathcal{T}_X)=H^{2*}(X,S^2\mathcal{T}_X)\oplus H^{2*}(X,\mathcal{L}_X)\oplus H^{2*}(X,\mathcal{O}_X)$$

and the natural map $\mathbb{K}_{\mathrm{SV}}(X)\to\mathrm{hom}_{\mathfrak{g}_X}(\mathfrak{g}_X,\mathfrak{g}_X)$ is the identity on the third component. □

The even cohomology is easy to control in the case of K3-surfaces. This is in sharp contrast to the general behaviour of the symmetric plurigenus $Q_n(X):=H^0(X,S^n\mathcal{T}_X^*)$ that is not a topological invariant of a complex manifold X [3]. The crucial argument for the next theorem has been explained to us by A. Kuznetsov.

Theorem 11. *If X is a K3-surface then $\mathfrak{g}_X$ is quasisimple.*

Proof. By Theorem 10 it suffices to prove that $H^0(X,S^2\mathcal{T}_X)$ vanishes because $\mathcal{L}_X=0$ for a surface. Suppose $H^0(X,S^2\mathcal{T}_X)\neq 0$. Then we have two sections of

$$\mathrm{end}(\mathcal{T}_X)\cong T^2\mathcal{T}_X=\Lambda^2\mathcal{T}_X\oplus S^2\mathcal{T}_X:$$

the skew symmetric identity map $I\in\mathrm{end}(\mathcal{T}_X)$ and some symmetric section $S\in\mathrm{end}(\mathcal{T}_X)$. The determinant $\det(I+\lambda S)$ is a global function on X for each $\lambda\in\mathbb{C}$. Thus, it is constant and one can choose $\lambda_0\in\mathbb{C}$ such that $\det(I+\lambda_0 S)=0$. We conclude that the rank of $S':=I+\lambda_0 S$ is 1 at a generic point.

Furthermore, since I is skew-symmetric and S is symmetric, S' does not vanish, so the rank is 1 at each point. Thus, the image of S' is an invertible sheaf $\mathcal{L}$. Let us show that $\mathcal{L}$ must be trivial.

If X is not algebraic then the only line bundle on X is trivial. If X is algebraic then $\mathcal{T}_X$ is semistable [17] (cf. [7, Ch. 7.4]) and $\mu_H(\mathcal{L})\leq\mu_H(\mathcal{T}_X)=0$ for any ample divisor H. The dual of the natural map $\mathcal{T}_X\to\mathcal{L}$ is an embedding $\mathcal{L}^*\hookrightarrow\mathcal{T}_X^*\cong\mathcal{T}_X$, hence $-\mu_H(\mathcal{L})=\mu_H(\mathcal{L}^*)\leq\mu_H(\mathcal{T}_X)=0$. Thus, $\mathcal{L}\cdot H=\mu_H(\mathcal{L})=0$ for any ample divisor H, proving that $\mathcal{L}$ is trivial.

Since $\mathcal{L}$ is trivial, $\mathcal{L}$ and consequently $\mathcal{T}_X$ has a nonzero section. This contradicts $H^0(X,\mathcal{T}_X)=0$. □

4.4. **V-simplicity.** The specialization $\mathrm{RW}:\mathfrak{g}_M\to\mathfrak{g}_X$ is known as the Rozansky-Witten invariants (a.k.a weight system) of a holomorphic symplectic manifold X. It is computed up to various degrees of explicitness for many concrete X [19]. Using the holomorphic symplectic form ω, one gets natural embeddings $\Lambda^{n+m}\mathcal{T}_X^*\hookrightarrow\mathrm{hom}(T^n\mathcal{T}_X,T^m\mathcal{T}_X)$ so that for $\gamma\in\mathrm{hom}_{\mathcal{C}}(\mathfrak{g}_M^{\otimes n},\mathfrak{g}_M^{\otimes m})$ with k singular trivalent points[5]

$$\mathrm{RW}_2(\gamma)\in H^k(X,\Lambda^{n+m}\mathcal{T}_X^*)\hookrightarrow H^k(X,\mathrm{hom}(T^n\mathcal{T}_X,T^m\mathcal{T}_X))\hookrightarrow\mathrm{hom}_{\mathrm{SV}(X)}(\mathfrak{g}_X^{\otimes n},\mathfrak{g}_X^{\otimes m}).$$

In particular, if X is a K3-surface, then $RW_2(\gamma)=0$ whenever γ has at least three trivalent singular points. For the diagrams with $RW_2(\gamma)\neq 0$ we have [19]

$$\mathrm{RW}_2(\delta)=-2,\ \mathrm{RW}_2(\phi)=-24[\overline{\omega}],\ \mathrm{RW}_2(\theta)=48[\overline{\omega}]\in H^2(X,\mathcal{O}_X).$$

Theorem 12. *If X is a K3-surface then $\mathfrak{g}_X$ is a V-simple Lie algebra.*

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[4] This corresponds to $T^2L(\omega_1)\cong L(2\omega_1)\oplus L(\omega_2)\oplus L(\omega_0)$ for the representations of $\mathfrak{sp}(n)$.

[5] Observe that $k+n+m$ is always even.

Proof. By Theorem 11 the Lie algebra $\mathfrak{g}_X$ is quasisimple and

$$\mathrm{end}_{\mathfrak{g}_X}(\mathfrak{g}_X) \cong H^{2*}(X, \mathcal{O}_X) = \mathbb{C}[z]/(z^2) \quad \text{where} \quad z = [\varpi] \in H^2(X, \mathcal{O}_X).$$

A graded homomorphism

$$\Theta_X : \Lambda \to \mathbb{C}[z]/(z^2), \quad \Theta_X(t) = -24z$$

is uniquely determined since $\Lambda_0 = \mathbb{Q}$ and $\Lambda_1 = \mathbb{Q}t$. We claim that Θ_X defines a required specialization. Indeed, most of the relations defining $\mathcal{C}'$ hold in $\mathrm{SV}(X)$ for the trivial reason: both sides are zero. The relations[6] $\theta = 2t\delta$ and $\phi = 2tI$ hold as both sides become $48z$ and $-24z$. □

Theorem 12 proves Westbury's conjecture for a K3-surface. Westbury has conjectured it for any compact irreducible holomorphic symplectic manifold Y. We finish by stating this conjecture more carefully as a series of questions. The quasisimplicity of $\mathfrak{g}_Y$ can be established by Theorem 10 and boils down to the following question.

Question 7. *Which compact irreducible holomorphic symplectic manifolds* Y *satisfy* $H^{2*}(Y, S^2\mathcal{T}_Y \oplus \mathcal{L}_Y) = 0$*?*

If $\dim_{\mathbb{C}} Y = n$ then the scalars of $\mathrm{SV}(Y)$ are $\mathbb{K}_{\mathrm{SV}(Y)} \cong H^{2*}(Y, \mathcal{O}_Y) = \mathbb{C}[z]/(z^n)$ where $z = [\varpi] \in H^2(Y, \mathcal{O}_Y)$. Thus, the V-simplicity of $\mathfrak{g}_Y$ boils down to some explicit identities on the Rozansky-Witten invariants.

Question 8. *Does there exist a graded homomorphism* $\Theta_Y : \Lambda \to \mathbb{C}[z]/(z^n)$ *that gives rise to a specialization* $\mathfrak{g}_V \to \mathfrak{g}_Y$*?*

It would be interesting if these homomorphisms are compatible for different Y or, at least, for the Hilbert schemes.

Question 9. *Does there exists a* V*-simple Lie algebra* $\mathfrak{g}_H$ *over* $\mathbb{C}[z]$ *that specializes to* $\mathfrak{g}_{X^{[n]}}$ *for all* X *and* n *where* $X^{[n]}$ *is the Hilbert scheme of* n *points on a K3-surface* X*, so that the specialization* $\mathfrak{g}_M \to \mathfrak{g}_{X^{[n]}}$ *factorises as* $\mathfrak{g}_M \to \mathfrak{g}_H \to \mathfrak{g}_{X^{[n]}}$*?*

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