FINITE GROUPS ASSOCIATED TO MULTIPLICATIVE $\eta-$PRODUCTS

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ABSTRACT

In this article we study such finite groups that the cusp forms associated to all elements of these groups by means of some faithfull representation are modular forms with multiplicative Fourier coefficients from the special class. The Sylow subgroups of such groups of odd orders are found. We describe such metacyclic groups. The groups of order 16 and the groups of order 32 which are metacyclic or are the direct products of the group of the order 16 and the cyclic group of the order 2 are considered in detail.

Introduction.

In many modern investigations of modular forms their connections with the theory of group representations are studied. In this article we consider the problem of finding such finite groups that the modular forms associated to all elements of these groups by means of some faithfull representation belong to a special class of modular forms which are called multiplicative $\eta-$products. Modular forms are associated with the elements of finite groups by the symbol of the Frame generalized substitution ("Frame-shape"). In 1985 G.Mason has shown that the Mathieu group M_{24} belongs to the type of groups which we investigate [2]. It can be shown that there are many groups of such type which are not subgroups in M_{24}. There are multiplicative $\eta-$products which cannot be associated with elements of M_{24}. So we have a nontrivial classification problem. This problem is open: all such groups have not been found. Moreover the different variants of correspondence are possible for the same group.

In the previous works of the author it has been shown that all the groups of order 24 belong to this type, in this case we can use the regular representation as the faithful representations. The metacyclic groups without nontrivial intersections have been investigated. We give here the statement of this result. The finite subgroups in $SL(5, \mathbf{C})$ were also studied. The abelian groups of such type were described completely in the article [15].

In this article we continue the investigations.

We prove the theorem which describes all Sylow subgroups $S_p, p \neq 2$, of such groups. We also study groups of orders $2^n, n \leq 5$, in detail. The case of the groups with the order equal to a degree of the number 2 is the most important and the most difficult for this classification. We hope that the representations written out explicitely in the section will be usefull for further investigation. We give the list of metacyclic groups of the type we investigate.

1. Multiplicative $\eta-$products and representations of finite groups.

The Dedekind $\eta-$function $\eta(z)$ is determined by the formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i z},$$

z belongs to the upper complex half-plane.

We describe modular forms which are completely determined by the following conditions: they are cusp forms of integer weights with characters, they are eigenforms of Hecke algebra, all their zeroes are in the cusps and have the multiplicity one. A priory we don't suppose that they are modified products of Dedekind $\eta-$functions. But infact it is so, there are 28 such functions. We give their complete list.

Table 1.

$f(z)$	k	N	$\chi(d)$
$\eta(23z)\eta(z)$	1	23	$\left(\frac{-23}{d}\right)$
$\eta(22z)\eta(2z)$	1	44	$\left(\frac{-11}{d}\right)$
$\eta(21z)\eta(3z)$	1	63	$\left(\frac{-7}{d}\right)$
$\eta(20z)\eta(4z)$	1	80	$\left(\frac{-5}{d}\right)$
$\eta(18z)\eta(6z)$	1	108	$\left(\frac{-3}{d}\right)$
$\eta(16z)\eta(8z)$	1	128	$\left(\frac{-2}{d}\right)$
$\eta^2(12z)$	1	144	$\left(\frac{-1}{d}\right)$
$\eta^4(6z)$	2	36	1
$\eta^2(8z)\eta^2(4z)$	2	32	1
$\eta^2(10z)\eta^2(2z)$	2	20	1
$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$	2	24	1
$\eta(15z)\eta(5z)\eta(3z)\eta(z)$	2	15	1
$\eta(14z)\eta(7z)\eta(2z)\eta(z)$	2	14	1
$\eta^2(9z)\eta^2(3z)$	2	27	1
$\eta^2(11z)\eta^2(z)$	2	11	1
$\eta^3(6z)\eta^3(2z)$	3	12	$\left(\frac{-3}{d}\right)$
$\eta^6(4z)$	3	16	$\left(\frac{-1}{d}\right)$
$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$	3	8	$\left(\frac{-2}{d}\right)$
$\eta^3(7z)\eta^3(z)$	3	7	$\left(\frac{-7}{d}\right)$
$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$	4	6	1
$\eta^4(5z)\eta^4(z)$	4	5	1
$\eta^8(3z)$	4	9	1
$\eta^4(4z)\eta^4(2z)$	4	8	1
$\eta^4(4z)\eta^2(2z)\eta^4(z)$	5	4	$\left(\frac{-1}{d}\right)$
$\eta^6(3z)\eta^6(z)$	6	3	1
$\eta^{12}(2z)$	6	4	1
$\eta^8(2z)\eta^8z)$	8	2	1
$\eta^{24}(z)$	12	1	1

We add 2 cusp forms of half-integer weight to this list: $\eta(24z),\ \eta^3(8z)$.

We shall call these functions *multiplicative $\eta-$products* because they have multiplicative Fourier coefficients.

American and Canadian mathematicients J.MacCay, D.Dummit, and H. Kisilevskii obtained this list from another point of view.

They have shown that only these 30 functions have multiplicative Fourier coefficients among the functions of the kind

$$f(z)=\prod_{k=1}^{s}\eta^{t_k}(a_kz),\ \sum a_kt_k=24,\ a_k,\ t_k\in\mathbf{N}.$$

Using computer calculations (they consider about 1700 variants) they rest only such functions that have several first multiplicative Fourier coefficients Then they check that these functions generate one-dimensional spaces. The explicit expressions for the functions as products of modified Dedekind functions are used from the beginning.

Yves Martin [6] described completely all $\eta-$quations with multiplicative coefficients. They are functions of the same kind with the condition $t_k \in \mathbf{Z}$. But we in our correspondence use only $t_k \in \mathbf{N}$.

From various points of view, these functions have been studied in recent works of American mathematiciants. [1 - 8].

We assign modular forms to elements of finite groups by the following rule. Let Φ be a representation of a finite group G by unimodular matrices in a space V whose dimension is divisible by 24. And let us suppose that for every element $g \in G$ the characteristic polynomial of the operator $\Phi(g)$ has the kind:

$$P_g(x) = \prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}, \ a_k \in \mathbf{N}, \ t_k \in \mathbf{Z}.$$

With each $g \in G$ we can associate the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z).$$

The function $\eta_g(z)$ is a cusp form of a certain level $N(g)$ and of the weight $k(g) = \frac{1}{2}\sum_{k=1}^{s} t_k$ with the character equal to the character of the quadratic field $\mathbf{Q}\sqrt{\prod_{k=1}^{s}(ia_k)^{t_k}}$.

We shall call a representation of a group as *desired* or as *a representation of a permissible type* if, by means of this representation, the multiplicative $\eta-$products are associated with all elements of this group. Permissible groups are indicated up to isomorphism. Such groups can contain only elements whose orders do not exceed 24 and are not equal to 13, 17, 19. We can immediately see that if a multiplicative $\eta-$product corresponds to an element of this group then cusp forms from the above list correspond to all powers of this element. Due to this fact, in the study of the groups it is sufficient to treat the representations only for elements that do not belong to the same cyclic group. The identity element of the group corresponds to the form $\eta^{24}(z)$.

2. Multiplicative $\eta-$products and metacyclic groups.

The metacyclic groups of the type we study are described in the following theorem which has been proved in [10, 14, 16].

Theorem 1.

Let G be a metacyclic group with the following genetic code

$$< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >,$$

such that the modular form associated with each element of this group by means of a faithful representation is a multiplicative $\eta-$product and the cyclic groups $< a >$ and $< b >$ have only trivial intersection. Then for the values of m, s, r (up to isomorphism) there are only the following possibilities:

m = 3, s = 2,4,6,8,12,18, r = 2.

m = 4, s = 2,4,6,8,10,24, r = 3.

m = 5, s = 4,8,12, r = 2; s = 2,4,6,8, r = 4.

m = 6, s = 2,4,6, r = 5.
m = 7, s = 3,6, r = 2; s = 6, 12, r = 3; s = 2, 4, 6, r = 6.
m = 8, s = 2,4, r = 3; s = 2, 4, r = 5; s = 2, 4, r = 7.
m = 9, s = 2, r = 8; s = 4, r = 8.
m = 10, s = 4,8, r = 3; s = 2, 4, r = 9.
m = 11, s = 2,4, r = 10; s = 5, r = 5; s = 10, 20, r = 2; s = 10, r = 4.
m = 12, s = 2, r = 5, 7, 11.
m = 14, s = 2, r = 13; s = 3, r = 9; s = 4, r = 3; s = 6, r = 3.
m = 15, s = 2, r = 4, 14; s = 4, r = 2.
m = 16, s = 2, r = 7, 9, 15.
m = 18, s = 2, r = 17.
m = 20, s = 2, r = 9, 19; s = 4, r = 17.
m = 21, s = 2, r = 8, 20; s = 3, r = 4; s = 6, r = 2.
m = 22, s = 2, r = 21; s = 5, r = 3; s = 10, r = 7.
m = 23, s = 2, r = 22; s = 11, r = 10; s = 22, r = 5.
m = 24, s = 2, r= 17.

3. The Sylow subgroups of odd orders of permissible groups.

Theorem 2. *Let G be a finite group such that there is a faithful representation T that for each $g \in G$ the characteristic polynomial of the operator $T(g)$ has such form $P_g(x) = \prod_{k=1}^{s}(x^{a_k} - 1)^{t_k}$ that the corresponding cusp form $\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ is a multiplicative $\eta-$product.*

Then for the Sylow $p-$subgroups $S_p, p \neq 2$, of such groups there are only the following possibilites:

$$S(3) \cong Z_3, \quad S(3) \cong Z_3 \times Z_3, \quad S(3) \cong Z_9,$$

$$S(3) \cong < a, b, c : a^3 = e, b^3 = e, c^3 = e, ab = bac, ac = ca, bc = cb >,$$

$$S(5) \cong Z_5, \quad S(7) \cong Z_7, \quad S(11) \cong Z_{11}.$$

Proof.

The permissible 3-group can contain only elements of orders 1,3 and 9.

Let T be a desired representation, T_1 be a trivial representation ($T_1(g) = 1, \forall g \in G.$) Let $\chi_T, \ \chi_1$ be their characters.

The group $Z_3 \times Z_3$.

We must consider three cases.

1. All elements of order 3 correspond to the cusp form $\eta^8(3z)$.

Then $\chi_T(e) = 24, \ \chi_T(g) = 0, \ ord(g) = 3.$

The scalar product $< \chi_T, \chi_1 >= \frac{24}{9} = \frac{8}{3}$. But this number must be integer. We obtain a contradiction and the desired representation can not be constructed.

2. All elements of order 3 corresponds to the cusp form $\eta^6(3z)\eta^6(z)$.

In this case the group is permissible. The desired representation contains T_1 with the multiplicity 8, all other representations are contained in it with the multiplicity 2.

3. In this case u elements of order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z), v$ elements corresponds $\eta^8(3z)$, $0 \leq u, 0 \leq v$.

The group is permissible.

Because g and g^2 correspond to the same modular form then the numbers u and v are even. The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{9} \cdot (24 + 6u) = \frac{1}{3} \cdot (8 + 2u).$$

Because this number must be integer then $u = 2$. It is a suitable variant. Let elements f and f^2 correspond to the form $\eta^6(3z)\eta^6(z)$. Let T_k be one-dimensional representation of our group, m_k be its multiplicity in the desired representation in T. If $T_k(f) = 1$ then $m_k = 4$. For all other Для других one-dimensional representations $m_k = 2$.

The group $Z_3 \times Z_3 \times Z_3$.

We shall show that this group is not permissible. We must consider two cases.

1. All elements of order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z)$.

The scalar product

$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 26 \cdot 6) = \frac{20}{3}$.

We obtain a contradiction because this number must be integer.

2. In this case u elements correspond to the cusp form $\eta^6(3z)\eta^6(z), v$ elements correspond to the cusp form $\eta^8(3z)$, $0 \leq u, 0 \leq v$.

The numbers u and v are even.

The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 6u) = \frac{1}{9} \cdot (8 + 2u).$$

Because this number must be integer then $u = 14, m_1 = 4$. Let T_k be an one-dimensional representation of our group, m_k be its multiplicity in the desired representation T.

Let u_1 be a number of such elements g that correspond to $\eta^6(3z)\eta^6(z)$ and $T_k(g) = 1$;
u_2 be a number of such elements g that correspond to $\eta^6(3z)\eta^6(z)$ and $T_k(g) = \zeta_3$;
u_3 be a number of such elements g that correspond to $\eta^6(3z)\eta^6(z)$ and $T_k(g) = \zeta_3^2$.
Then $u_2 = u_3, u_1 + 2u_2 = 14$.

$$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 6(u_1 + \zeta_3 \cdot u_2 + \zeta_3^2 \cdot u_3)) =$$

$$= \frac{1}{27} \cdot (24 + 6(u_1 - u_2)) = \frac{1}{9} \cdot (8 + 2(u_1 - u_2)) =$$

If T_k is not trivial, then $u_1 \neq 14$, and $u_1 - u_2 = 5$. We obtain $u_1 = 8, u_2 = 3$ and $Ker(T_k)$ does not contain elements, corresponding to $\eta^8(3z)$. If the element h correspond to $\eta^8(3z)$ then $T_k(h) \neq 1, \forall k \neq 1$. In this case there are only 4 eigenvalues,equal to 1, among the eigenvalues of the operator $T(h)$ and the characteristic polynomial of the operator $T(h)$ can not be equal to $(x^3 - 1)^8$. We obtain a contradiction.

The group $Z_9 \times Z_3$.

We show that this group is not permissible.

In this group there are 8 elements of the order 3, 18 elements of the order 9 and the element e.

$$\chi_T(e) = 24;\ \chi_T(g) = 0,\ ord(g) = 9;\ \chi_T(g) = 6,\ ord(g) = 3.$$

The number $< \chi_T, \chi_1 >= \frac{8}{3}$. We obtain a contradiction.

The group $S(3) \cong < a, b, c : a^3 = e, b^3 = e, c^3 = e, ab = bac, ac = ca, bc = cb. >$

This group has the order 27, there are 11 conjugacy classes in it.

1.e 2.c 3.c^2 4.a, ac, ac^2 5.b, bc, bc^2 6.ab, abc, abc^2 7.a^2b, a^2bc, a^2bc^2
8.a^2, a^2c, a^2c^2 9.b^2, b^2c, b^2c^2 10.ab^2, ab^2c, ab^2c^2 11.$a^2b^2, a^2b^2c, a^2b^2c^2$

The commutant of the group is generated by the element c. $G/G' \cong Z_3 \times Z_3$.

This group has the following irreducible representations:
$T_k(a) = \zeta_3^k, T_k(b) = \zeta_3, T_k(c) = 1, k = \overline{1,3}$,
$T_k(a) = \zeta_3^k, T_k(b) = \zeta_3^2, T_k(c) = 1, k = \overline{4,6}$,
$T_k(a) = \zeta_3^k, T_k(b) = 1, T_k(c) = 1, k = \overline{7,9}$,

$$T_{10}(a) = T_{11}(b) = \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{10}(b) = T_{11}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$T_{10}(c) = \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \quad T_{11}(c) = T_{10}(c^2).$$

The desired reprentation contains the representations T_1 and T_2 with the multiplicity 1, other irreducible representations are included with the multiplicity 2.

The elements $a, b, a^2, b^2, ac, bc, a^2c, b^2c, ac^2, bc^2, a^2c^2, b^2c^2$ correspond to the cusp form $\eta^8(3z)$ other elements of order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z)$.

In the article [16] it has been proved that the group $< a, b : a^9 = e, b^3 = e, b^{-1}ab = a^4 >$ is not permissible.

The Sylow $p-$subgroups, $p = 5, 7, 11$.

We show that the group $Z_5 \times Z_5$ is not permissible. There are 24 elements of order 5 in it. $\chi_T(e) = 24, \chi_T(g) = 4$ if $ord(g) = 5$. The number $< \chi_T, \chi_1 >= \frac{120}{25} = \frac{24}{5}$. But this number must be integer. The elements of order 25 do not correspond to multiplicative $\eta-$products. In any group of order $5^k, 2 \leq k$ there is a subgroup of order 25. The group of order 25 is isomorphic to $Z_5 \times Z_5$ or to Z_{25}. So we have the only possibility: $S(5) \cong Z_5$.

The cases $p = 7, 11$ are considered analogously.

Theorem 3. *There is no such finite solvable group G that one can assign with all elements of G by a faithful representation all multiplicative $\eta-$products and only them.*

Proof. The order of this group must be equal to $2^k \cdot 3^m \cdot 5 \cdot 7 \cdot 11$. According to the theorem of Ph.Hall [9] there is a subgroup of order 35 in this group. There is only one group of order 35 : Z_{35}. But the elements of order 35 do not correspond to multiplicative $\eta-$products. The theorem is proved.

We shall formulate one open problem:

To find such algebraic structure that we can associate with its elements **all** *multiplicative $\eta-$products and* **only** *them in according to some rule.*

4. The groups of order 2^k and multiplicative $\eta-$products.

Let us note as u the number of elements corresponding to the cusp form $\eta^8(2z)\eta^8(z)$, as v the number of elements corresponding to the cusp form $\eta^{12}(2z)$.

In the article [15] the group $Z_2 \times Z_2 \times Z_2$ has been considered in details. It has been shown that the desired representations can be found by such way that the numbers u and v are equal to any values from 0 to 7. The sum of these numbers is equal to 7.

All groups of order 16 are permissible.

In this section we consider in detail the groups of the order 16 and the order 32 which are metacyclic or are direct products of the group of the order 16 and the group Z_2. We shall point out the unique possible variant for the realization of the elementary abelian group of order 32.

4.1. Группа $Z_2 \times Z_2 \times Z_2 \times Z_2$ and multiplicative $\eta-$products.

In the article [14] it was considered only one of possible variants .

Here we consider in details some other possibilities.

Let T be a desired representation, χ is its character.

4.1.1. The case $u = 1, v = 14$.

Let the element g correspond to $\eta^8(2z)\eta^8(z)$,other elements of order 2 correspond to $\eta^{12}(2z)$. The desired representation contains the one-dimensional representations which send the element g to 1 with the multiplicity 2. It contains the one-dimensional representations which send the element g to -1 with the multiplicity 1.

4.1.2. The case $u = 3, v = 12$.

Let the elements g_1, g_2, g_3 correspond to the modular form $\eta^8(2z)\eta^8(z)$, other elements of order 2 correspond to $\eta^{12}(2z)$.

Here we must consider two different cases:

1) The elements g_1, g_2, g_3, g_4 are generators for the group G. The element g_4 corresponds to $\eta^{12}(2z)$. In the following table we point out the multiplicity of all one-dimensional representations in a desired permissible representation. The values of one-dimensional representations on generators are pointed out in the columns of the table. The multiplicities are written in the last row.

Table 2.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	3	3	2	2	2	2	1	1	2	2	1	1	1	1	0	0

2) The element $g_3 = g_1 \cdot g_2$. In this case four representations which send the elements g_1 and g_2 to 1 are included in the desired representation with the multiplicity 3. Other one-dimensional representations are included in it with the multiplicity 1.

4.1.3. The case $u = 5, v = 10$.

Let the generators g_1, g_2, g_3, g_4 correspond to $\eta^8(2z)\eta^8(z)$. In this case there is one more element corresponding to this form. We have three different cases.

1) This element is $g_1g_2g_3$. Let Φ be such one-dimentional representation that $\Phi(g_k) = -1, k = 1, 2, 3, 4$. Let us calculate its multiplicity in the desired representation. This number may be equal to zero but it may not be fractional or negative. But we obtain:

$$< \chi, \chi_\Phi >= \frac{1}{16}(24 - 8 \cdot 5) = -1.$$

This contradiction shows that this case is not permissible. Two other cases are permissible.

2) The cusp form $\eta^8(2z)\eta^8(z)$ correspond to the elements $g_1, g_2, g_3, g_4, g_1g_2$. The table of multiplicities is:

Table 3.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	4	3	3	2	2	1	1	0	2	1	1	0	2	1	1	0

3) The cusp form $\eta^8(2z)\eta^8(z)$ correspond to the elements $g_1, g_2, g_3, g_4, g_1g_2g_3g_4$. The table of multiplicities is:

Table 4.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	4	2	2	2	2	2	2	0	2	2	2	0	2	0	0	0

4.1.4. The case $u = 6, v = 9$.

This group is permissible only in two different cases.

1) The $g_1, g_2, g_3, g_4, g_1g_2, g_3g_4$ correspond to the form $\eta^{12}(2z)$. The table of multiplicities is:

Table 5.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	6	0	0	0	0	2	2	2	0	2	2	2	0	2	2	2

2) The elements $g_1, g_2, g_3, g_4, g_1g_2, g_1g_2g_3g_4$ correspond to the cusp form $\eta^{12}(2z)$.

Table 6.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	6	0	0	0	1	1	1	3	1	1	1	3	0	2	2	2

4.1.5. The case $u = 7, v = 8$.

1) We consider the case when in the group there is such subgroup of the order 8 that all its elements not equal to identity correspond to $\eta^8(2z)\eta^8(z)$. This variant is suitable: the irreducible representations identical on this subgroup are included in the desired representation with the multiplicity 5, other 14 representations - with the multiplicity 1.

2) In this group there is a subgroup of the order 8 which contains 5 elements corresponding to the cusp form $\eta^8(2z)\eta^8(z)$. For example the group will be permissible in the case when the elements $g_1g_2, g_1g_3, g_2g_3, g_3g_4, g_1g_2g_3, g_3, g_4$ correspond to the cusp form $\eta^8(2z)\eta^8(z)$ and other elements of order 2 correspond to $\eta^{12}(2z)$.

Table 7.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	5	3	0	0	2	0	1	1	2	0	1	1	3	1	2	2

In the subgroup $< g_1 > \times < g_2 > \times < g_1g_2g_3 >$ there are exactly 3 elements corresponding to $\eta^8(2z)\eta^8(z)$; in the subgroup $< g_1 > \times < g_2 > \times < g_1g_2 >$ there are 2 elements corresponding to this form; in the subgroup which consists of products of two or four generators there are4 such elements.

We can also consider the second variant.

The elements $g_1, g_2, g_3, g_4, g_1g_4, g_2g_4, g_1g_3, g_1g_2$ correspond to the cusp form $\eta^8(2z)\eta^8(z)$ and other elements of the order 2 correspond to $\eta^{12}(2z)$.

Table 8.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	5	2	4	1	2	1	1	0	2	1	1	0	1	2	0	2

3) In the group there is a subgroup F of the order 8 which contains only one element g_1 corresponding to $\eta^8(2z)\eta^8(z)$. Let the elements $g_2,\ g_3$ do not belong to F and correspond to $\eta^{12}(2z)$. The group G is permissible only in the case when $g_1 \neq g_2g_3$. In this case we may assume that the elements g_1, g_2, g_3, g_4 are the generators.

Table 9.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	5	1	1	2	1	2	3	3	0	0	1	1	0	1	1	2

4.1.6. The case $u = 9, v = 6$.

This group is permissible only in two different cases.

1) The elements $g_1, g_2, g_3, g_4, g_1g_2, g_3g_4$correspond to $\eta^{12}(2z)$.

Table 10.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	6	0	0	0	0	2	2	2	0	2	2	2	0	2	2	2

2) The elements $g_1, g_2, g_3, g_4, g_1g_2, g_1g_2g_3g_4$ correspond to $\eta^{12}(2z)$.

Table 11.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	6	0	0	0	1	1	1	3	1	1	1	3	0	2	2	2

4.1.7. The case $u = 11, v = 4$.

The group is permissible only in the case when all elements corresponding to $\eta^{12}(2z)$ are generators for G.

Table 12.

g_1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
g_3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
g_4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
m	7	0	0	1	1	0	0	2	0	1	1	2	0	2	2	3

4.1.8. The case $u = 13, v = 2$.

We may assume that among four generators the first two g_1 and g_2 correspond to $\eta^{12}(2z)$. Then the identical representation is included in the desired one with the multiplicity 8, irreducible representations which are not equal on the elements g_1 and g_2 are included with the multiplicity 1, nonidentical representations which are equal on the elements g_1 and g_2 to 1 are not included in the desired representation. Other representations are included with the multiplicity 2.

4.1.9. The case $u = 15, v = 0$.

Then the identical representation is included in the desired one with the multiplicity 9, Other representations are included with the multiplicity 1.

4.2. The group $Z_4 \times Z_4$ and multiplicative $\eta-$products.

$Z_4 \times Z_4 \cong < f > \times < h >$.

We consider all possible cases.

1) All elements of the order 2 correspond to $\eta^{12}(2z)$. All elements of the order 4 correspond to $\eta^6(4z)$.

This case is not permissible because the multiplicity of the identical representation m_1 is fractional in this case.

2) The element f^2 corresponds to $\eta^{12}(2z)$, the elements h^2, f^2h^2 correspond to $\eta^8(2z)\eta^8(z)$.

This case is not permissible because the multiplicity m_Φ is fractional in this case. Here $\Phi(f) = i, \Phi(h) = i$.

3) The elements f^2, h^2 correspond to $\eta^{12}(2z)$, the element f^2h^2 corresponds to $\eta^8(2z)\eta^8(z)$.

Let s elements correspond to $\eta^4(4z)\eta^4(2z)$, t elements correspond to $\eta^4(4z)\eta^2(2z)\eta^4(2z)$. The number $0 \le t \le 4$. We obtain

$$m_1 = \frac{1}{16} \cdot (24 + 8 + 4t).$$

Hence, $t = 0$ or $t = 4$. Both variants are permissible.

The case $t = 0$.

In this case 8 representations which send f^2h^2 into 1 are included with the multiplicity 2, other irreducible representations are included with the multiplicity 1.

The case $t = 4$.

In this case the elements fh, f^3h, fh^3, f^3h^3 correspond to $\eta^4(4z)\eta^2(2z)\eta^4(2z)$.

Table 13.

f	1	1	1	1	i	i	i	i	-1	-1	-1	-1	-i	-i	-i	-i
h	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i
m	3	1	1	1	1	2	1	2	1	1	3	1	1	2	1	2

4) All elements of the order 2 correspond to $\eta^8(2z)\eta^8(z)$.

Let s elements correspond to $\eta^4(4z)\eta^4(2z)$, t elements correspond to $\eta^4(4z)\eta^2(2z)\eta^4(2z)$. Число $0 \le t \le 12$. We obtain

$$m_1 = \frac{1}{16} \cdot (24 + 24 + 4t).$$

Hence, t can be equal to one of the numbers 0, 4,8, 12. All these variants are permissible.

The case $t = 0$.

In this case 4 representations which send the elements f and h into в 1 or -1 are included in the desired representation with the multiplicity 3, other irreducible representations are included with the multiplicity 1.

The case $t = 4$.

In this case we may assume that the elements f, f^3, fh^2, f^3h^2 correspond to $\eta^4(4z)\eta^2(2z)\eta^4(2z)$.

Table 14.

f	1	1	1	1	i	i	i	i	-1	-1	-1	-1	-i	-i	-i	-i
h	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i
m	4	1	4	1	1	1	1	1	2	1	2	1	1	1	1	1

The case $t = 8$.

In this case we may assume that the elements f, f^3, fh^2, f^3h^2 correspond to $\eta^4(4z)\eta^4(2z)$.

Table 15.

f	1	1	1	1	i	i	i	i	-1	-1	-1	-1	-i	-i	-i	-i
h	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i
m	5	1	1	1	1	1	1	1	3	1	3	1	1	1	1	1

The case $t = 12$.

This variant is permissible. The table of multiplicities is:

Table 16.

f	1	1	1	1	i	i	i	i	-1	-1	-1	-1	-i	-i	-i	-i
h	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i	1	i	-1	-i
m	6	1	2	1	1	1	1	1	2	1	2	1	1	1	1	1

The groups $Z_4 \times Z_2 \times Z_2$ и $Z_8 \times Z_2$ have been considered in the article [15].

4.3. The group $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$ and multiplicative $\eta-$products.

The group is permissible only in the case when $u = 21, v = 10$.

Let the elements g_1, g_2, g_3, g_4, g_5 be the generators for the group G. This group is permissible if the elements

$$g_1,\ g_2,\ g_3,\ g_4,\ g_1g_2,\ g_1g_2g_3g_4,\ g_1g_3g_5,\ g_1g_2g_5,\ g_3g_4g_5,\ g_1g_4g_5$$

correspond to the cusp form $\eta^{12}(2z)$.

The table of multiplicities is:

Table 17.

g_1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
g_2	1	-1	-1	1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	-1
g_3	1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1
g_4	1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	-1
g_5	1	-1	1	1	1	-1	1	1	1	1	-1	1	-1	1	-1	-1
m	6	1	1	1	2	1	1	1	1	1	2	1	1	1	1	2

Other irreducible representations are not included in the desired representation.

The permissible representation can not be constructed for other vallues of numbers u and v.

Let us show how to prove it for $u = 1$. Let g correspond to $\eta^8(2z)\eta^8(z)$. Let $\Phi(g) = -1$. Then $m_\Phi = \frac{1}{2}$.

In other cases we can give the proofs by an analogous way.

4.4. Nonabelian groups of the order 32 and multiplicative η–products.

In the our examples we shall meet all nonabelian groups of the order 16 as subgroups in groups of the order 32.

The groups D_{16} and D_8 are permissible [10].

4.4.1. Группа $D_4 \times Z_4$.

The genetic code of the group is:

$< a, b, c : a^4 = b^2 = c^4 = e, b^{-1}ab = a^3, ac = ca, bc = cb. >$

All irreducible representations of the group are:

$$T_k(a) = (-1)^k, T_k(b) = 1, k = \overline{1,8};\ T_k(a) = (-1)^k, T_k(b) = -1, k = \overline{8,16};$$

$$T_k(c) = i^k, k = \overline{1,16}.$$

$$T_k(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ T_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ k = \overline{17,20};$$

$$T_k(c) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^k, k = \overline{17,20}.$$

The permissible representation is a direct sum of all irreducible representations. The elements a, a^3, ac^2, a^3c^2 correspond to $\eta^4(4z)\eta^4(2z)$, other elements of the order 4 correspond to $\eta^6(4z)$, the element a^2 correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 2 correspond to $\eta^{12}(2z)$.

This group contains a subgroup $D_4 \times Z_2$.

4.2. The group $Q_8 \times Z_4$.

The genetic code of the group is:

$< a, b, c : a^4 = b^4 = c^4 = e, b^{-1}ab = a^3, a^2 = b^2, ac = ca, bc = cb. >$

The one-dimensional irreducible representations are:

$$T_k(a) = (-1)^k, T_k(b) = 1, k = \overline{1,8};\ T_k(a) = (-1)^k, T_k(b) = -1, k = \overline{8,16};$$

$$T_k(c) = i^k, k = \overline{1,16}.$$

The two-dimensional irreducible representations are:

$$T_k(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ T_k(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ k = \overline{17,20};$$

$$T_k(c) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^k, k = \overline{17,20}.$$

The permissible representation is a direct sum of all irreducible representations. The elements $a, a^3, b, b^3, ab, a^3b, ac^2, a^3c^2, bc^2, b^3c^2, a^2c^2, abc^2, a^3bc^2$ correspond to $\eta^4(4z)\eta^4(2z)$, other elements of the order 4 correspond to $\eta^6(4z)$, the element a^2 correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 2 to $\eta^{12}(2z)$.

This group contains a subgroup $Q_8 \times Z_2$.

4.4.3. The group $< a, b : a^8 = e, b^4 = e, b^{-1}ab = a^3 >$.

The one-dimensional irreducible representations are:

$$T_k(a) = 1, T_k(b) = i^k, 1, k = \overline{1,4};\ T_k(a) = -1, T_k(b) = i^k, k = \overline{5,8};$$

The two-dimensional irreducible representations are:

$$T_9(a) = T_{10}(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^3 \end{pmatrix},\quad T_{11}(a) = T_{12}(a) = \begin{pmatrix} \zeta_8^5 & 0 \\ 0 & \zeta_8^7 \end{pmatrix},$$

$$T_{13}(a) = T_{14}(a) = \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & \zeta_8^6 \end{pmatrix},\quad T_9(b) = T_{11}(b) = T_{13}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_{10}(b) = T_{12}(b) = T_{14}(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The permissible representation is the direct sum which contains the representations T_{13} and T_{14} with the multiplicity 2, other representations are included with the multiplicity 1. All elements of the order 8 correspond to $\eta^2(8z)\eta^2(4z)$, the elements a^2, a^6, a^2b^2, a^6b^2 correspond to $\eta^4(4z)\eta^4(2z)$, other elements of the order 4 correspond to $\eta^6(4z)$, the element a^4 correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 4 correspond to $\eta^{12}(2z)$.

This group contains the subgroup with the genetic code
$< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^3 >$.

4.4.4. The group $< a, b : a^8 = e, b^4 = e, b^{-1}ab = a^5 >$.

All irreducible representations of the group are:
$T_k(a) = 1, T_k(b) = i^k, 1, k = \overline{1,4}$;
$T_k(a) = -1, T_k(b) = i^k, k = \overline{5,8}$;
$T_k(a) = i, T_k(b) = i^k, 1, k = \overline{9,12}$;
$T_k(a) = -i, T_k(b) = i^k, k = \overline{13,16}$.

$$T_{17}(a) = T_{18}(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^5 \end{pmatrix},\quad T_{19}(a) = T_{20}(a) = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^7 \end{pmatrix},$$

$$T_{17}(b) = T_{19}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_{18}(b) = T_{20}(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The permissible representation is a direct sum of all irreducible representations. The correspodence between modular forms and elements of the group is as in 4.4.3.

This group contains the subgroup with the genetic code
$< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^5 >$ of the order 16.

4.4.5. The group $< a, b : a^8 = e, b^4 = e, b^{-1}ab = a^7 >$.

All irreducible representations of the group are:

$$T_k(a) = 1, T_k(b) = i^k, 1, k = \overline{1,4};\ T_k(a) = -1, T_k(b) = i^k, k = \overline{5,8};$$

$$T_9(a) = T_{10}(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, \quad T_{11}(a) = T_{12}(a) = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix},$$

$$T_{13}(a) = T_{14}(a) = \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & \zeta_8^6 \end{pmatrix}, \quad T_9(b) = T_{11}(b) = T_{13}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_{10}(b) = T_{12}(b) = T_{14}(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The permissible representation is a direct sum which contains the representations T_{13} and T_{14} with the multiplicity 2, other representations are included with the multiplicity 1. The correspodence between modular forms and elements of the group is as in 4.4.3.

This group contains the subgroup with the genetic code $< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^7 >$.

4.4.6. The group $< 2, 2, 2 >_2 \times Z_2$.

The genetic code of the group is
$< a, b, c, d : a^2 = b^2 = c^2 = d^2 = e, abc = bca = cab, ad = da, bd = db, cd = dc >$.

There are 20 conjugacy classes in the group:

$$1.\{e\}2.\{abc\}3.\{acb\}4.\{(ba)^2\}5.\{d\}6.\{a, bab\}7.\{b, aba\}8.\{c, aca\}9.\{ab, ba\}10.\{bc, cb\}$$

$$11.\{ac, ca\}12.\{abcd\}13.\{acbd\}14.\{d(ba)^2\}15.\{ad, babd\}16.\{bd, abad\}17.\{cd, acad\}$$

$$18.\{abd, bad\}19.\{bcd, cbd\}20.\{acd, cad\}$$

The commutant of the group is $G' \cong < (bc)^2 >$.

$G/G' \cong Z_2 \times Z_2 \times Z_2 \times Z_2$.

This group has 16 one-dimensional irreducible representations (the values 1 and -1 alternate on the elements a, b, c, d.)

The two-dimensional irreducible representations:

$$T_1(a) = T_2(b) = T_3(a) = T_4(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T_1(c) = T_2(c) = T_3(c) = T_4(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T_2(a) = T_1(b) = T_4(a) = T_3(b) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

$$T_1(d) = T_2(d) = E, T_3(d) = T_4(d) = -E.$$

The permissible representation is a direct sum of all irreducible representations. All elements of the order 4 correspond to $\eta^4(4z)\eta^4(2z)$, the element $(ab)^2$ correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 2 correspond to $\eta^{12}(2z)$.

This group contains the subgroup with the genetic code
$< a, b, c : a^2 = b^2 = c^2 = e, abc = bca = cab >$.
4.4.7. The group $G \cong < a, b, c : a^4 = b^4 = c^2 = (ab)^2 = (a^3b)^2 = e, ca = ac, bc = cb >$
.
There are 20 conjugacy classes in the group:

$$1.\{e\}2.\{a^2\}3.\{b^2\}4.\{a^2b^2\}5.\{c\}6.\{b, a^2b^3\}7.\{b^3, a^2b\}8.\{a, a^3b^2\}9.\{a^3, ab^2\}10.\{ab, a^3b^3\}$$

$$11.\{ab^3, a^3b\}12.\{a^2c\}13.\{b^2c\}14.\{a^2b^2c\}15.\{bc, a^2b^3c\}16.\{b^3c, a^2bc\}17.\{ac, a^3b^2c\}$$

$$18.\{a^3c, ab^2c\}19.\{abc, a^3b^3c\}20.\{ab^3c, a^3bc\}$$

The commutant of the group is $G' \cong < a^2b^2 >$.
$G/G' \cong Z_2 \times Z_2 \times Z_4$.
This group has 16 one-dimensional irreducible representations.
The two-dimensional irreducible representations:

$$T_1(a) == T_3(a) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \; T_2(a) = T_4(a) = T_1(b) = T_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$T_2(b) == T_4(b) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$T_k(c) = (i)^k E, k = \overline{1,4}.$$

The permissible representation is a direct sum of all irreducible representations. All elements of the order 4 correspond to $\eta^6(4z)$, all elements of the order 2 correspond to $\eta^{12}(2z)$.

4.4.8. The group $< a, b, : b^2 = a^4 = (ab)^2, a^8 = e, b^{-1}ab = a^7, ac = ca, bc = cb >$.

The group is the direct product of the group Z_2 and the group of the generalized quaternions.

The commutant of the group $G' \cong < (a)^2 >$.
$G/G' \cong Z_2 \times Z_2 \times Z_2$.
This group has 8 one-dimensional irreducible representations.
The two-dimensional irreducible representations:

$$T_1(a) = T_4(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, \; T_2(a) = T_5(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$T_3(a) = T_6(a) = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}, \; T_1(b) = T_3(b) = T_4(b) = T_6(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$T_2(b) = T_5(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

The permissible representation is a direct sum which contains the representations T_2 and T_5 with the multiplicity 2, other representations are included with the multiplicity 1. All elements of the order 8 correspond to $\eta^2(8z)\eta^2(4z)$, all elements of the order 4 to $\eta^4(4z)\eta^4(2z)$, the element a^4 correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 2 correspond to форме $\eta^{12}(2z)$.

This group contains the subgroup with the genetic code

$< a, b : b^2 = a^4 = (ab)^2, a^8 = e, b^{-1}ab = a^7 >$.

4.4.9. The group $< a, b : a^{16} = e, b^2 = e, b^{-1}ab = a^7 >$.

All irreducible representations of the group:

$$T_k(a) = \begin{pmatrix} \zeta_{16}^k & 0 \\ 0 & \zeta_{16}^{7k} \end{pmatrix}, \quad k = \overline{1,4}; \quad T_5(a) = \begin{pmatrix} \zeta_{16}^6 & 0 \\ 0 & \zeta_{16}^{10} \end{pmatrix},$$

$$T_6(a) = \begin{pmatrix} \zeta_{16}^9 & 0 \\ 0 & \zeta_{16}^{15} \end{pmatrix}, \quad T_7(a) = \begin{pmatrix} \zeta_{16}^{11} & 0 \\ 0 & \zeta_{16}^{13} \end{pmatrix},$$

$$T_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \overline{1,7}.$$

$T_k(a) = 1, T_k(b) = (-1)^k, k = 8, 9;$

$T_k(a) = -1, T_k(b) = (-1)^k, k = 10, 11.$

The permissible representation is a direct sum which contains the representations T_2, T_4 and T_5 with the multiplicity 2, other representations are included with the multiplicity 1. All elements of the order 16 correspond to $\eta(16z)\eta(8z)$, all elements of the order 8 correspond to $\eta^2(8z)\eta^2(4z)$, all elements of the order 4 to $\eta^4(4z)\eta^4(2z)$, the element a^8 correspond to $\eta^8(2z)\eta^8(z)$, other elements of the order 2 correspond to $\eta^{12}(2z)$.

This group contains the subgroup with the genetic code

$< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^7 >$.

4.4.10. The group $< a, b : a^{16} = e, b^2 = e, b^{-1}ab = a^9 >$.

All irreducible representations of the group are:

$T_k(a) = \zeta_8^k, T_k(b) = 1, k = \overline{1,8};$

$T_k(a) = \zeta_8^k, T_k(b) = -1, k = \overline{9,16};$

$$T_{17}(a) = \begin{pmatrix} \zeta_{16} & 0 \\ 0 & \zeta_{16}^9 \end{pmatrix}, \quad T_{18}(a) = \begin{pmatrix} \zeta_{16}^3 & 0 \\ 0 & \zeta_{16}^{11} \end{pmatrix},$$

$$T_{19}(a) = \begin{pmatrix} \zeta_{16}^5 & 0 \\ 0 & \zeta_{16}^{13} \end{pmatrix}, \quad T_{20}(a) = \begin{pmatrix} \zeta_{16}^7 & 0 \\ 0 & \zeta_{16}^{15} \end{pmatrix},$$

$$T_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \overline{17,20}.$$

The permissible representation is a direct sum of all irreducible representations. The correspondence between modular forms and elements of the group is as in 4.4.9.

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