PROFINITE EQUIVARIANT HIGHER ALGEBRAIC K-THEORY FOR THE ACTIONS OF ALGEBRAIC GROUPS

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ABSTRACT. Let G be an algebraic group over a field F. In this paper, we study and compute equivariant higher K-groups as well as profinite equivariant higher K-groups for some G-schemes when F is a number field or p-adic field.

For example, let ${}_\gamma\mathcal{F}$ be a twisted flag variety (see 1.2.3), and B a finite dimensional separable F-algebra. When F is a number field, we prove that $K_{2n+1}({}_\gamma\mathcal{F}, B)$ is a finitely generated Abelian group; $K_{2n}({}_\gamma\mathcal{F}, B)$ is torsion (see theorem 3.1.2); $K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_l)$ is l-complete and furthermore $\operatorname{div} K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_l) = 0$ (see theorem 4.3.1). When F is a p-adic field, we prove that for all $n \geq 2$ $K_n({}_\gamma\mathcal{F}, B)_l$ is a finite group, $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_l) = K_n(({}_\gamma\mathcal{F}, B), \hat{Z}_l)$ is an l-complete profinite Abelian group and $\operatorname{div} K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_l) = 0$.

We obtain similar results for some other smooth projective varieties (see 3.1.5, 3.2.3, 4.3.5).

CONTENTS

Introduction

Let G be an algebraic group over a field F. The aim of this paper is to study equivariant K-theory as well as profinite equivariant K-theory for G-schemes with the goal of computing these K-theoretic groups for twisted flag varieties, Brauer–Severi varieties and some other smooth projective varieties over number fields and p-adic fields.

We start in section 1 by reviewing the equivariant higher algebraic K-theory for schemes (à la Thomason, see [19]) with relevant examples including those that have appeared in the works of A. Merkujev [11] and I. Panin [13]. We note, however, that the equivariant categories involved are special cases of equivariant exact categories discussed in [10], even though we have focussed in this paper on the notations and terminologies of Thomason [19].

We prove at first some finiteness results in the K-theory of twisted flag varieties. More precisely, let $\widetilde{G}$ be a semi-simple connected and simply connected F-split algebraic group over a field F, $\widetilde{P}$ a parabolic subgroup of $\widetilde{G}$, $\mathcal{F} = \widetilde{G}/\widetilde{P}$, ${}_\gamma\mathcal{F}$ the twisted form of $\mathcal{F}$ with respect to the 1-cocycle $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(F_{\mathrm{sep}})$ (see 1.2 or [13]), B a finite-dimensional separable F-algebra and $K_n({}_\gamma\mathcal{F}, B)$ the Quillen K-theory of the category $\mathcal{VB}_{\widetilde{G}}({}_\gamma\mathcal{F}, B)$ of vector bundles on ${}_\gamma\mathcal{F}$ equipped with left B-module structure. We prove that when F is a number field, $K_{2n+1}({}_\gamma\mathcal{F}, B)$ is a finitely generated abelian group and $K_{2n}({}_\gamma\mathcal{F}, B)$ is torsion and has no non-trivial divisible elements for all $n \geq 1$ (see theorem 3.1.2). When F is a p-adic field, we prove that $K_n({}_\gamma\mathcal{F}, B)_l$ is a finite group for all $n \geq 1$ (see theorem 3.1.5).

We obtain similar results for K-theory of Brauer-Severi varieties as well as for K-theory of twisted forms of some smooth projective varieties arising in the context of a motivic category constructed by I. Panin (see 3.2.3 or [13]).

In section 2 we introduce mod-l^s and profinite higher algebraic K-theory with copious examples relevant to this paper. We then prove that if F is a number field, then for all $n \geq 1$, $K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$ is l-complete and $\operatorname{div} K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) = 0$ (see theorem 4.2.1).

When F is a p-adic field, we have that for all $n \geq 1$, $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) \simeq K_n(({}_\gamma\mathcal{F}, B)\hat{\mathbb{Z}}_l)$ are l-complete profinite groups, $\operatorname{div} K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) = 0$ and the kernel and cokernel of $K_n(({}_\gamma\mathcal{F}, B)) \longrightarrow K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$ are uniquely l-divisible (see theorem 4.2.4). Similar results are obtained for Brauer-Severi varieties.

Notes on Notation. For an additive abelian group A and a positive integer m, we write A/m for A/mA, and $A[m] = \{x \in A \mid mx = 0\}$. If l is a rational prime we denote by A_l the l-primary subgroup of A, i.e. $A_l = \bigcup A[l^s] = \varinjlim A[l^s]$.

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1. Equivariant higher K-theory for schemes

In this section, we briefly review equivariant higher algebraic K-theory for schemes as defined by R.W. Thomason in [19], as well as review some relevant examples. As remarked in the Introduction, the equivariant categories involved are special cases of equivariant exact categories discussed by this author in [10], even though we shall in this paper stick to the notations and terminologies of Thomason.

1.1. **Generalities.**

1.1.1. Let G be an algebraic group over a field F and $\mathrm{Rep}_F(G)$ the category of representations of G in the category $\mathcal{P}(F)$ of finite dimensional vector spaces over F. We denote $K_0(\mathrm{Rep}_F(G))$ by $R_F(G)$ or $R(G,F)$ (or just $R(G)$ when the context is clear). Note that $R(G)$ is the free abelian group generated by the classes of irreducible representations and that $R(G)$ also has a ring structure induced by tensor product. Call $R(G)$ the representation ring.

Since $\mathrm{Rep}_F(G)$ is an exact category (see [16] or [13]) we denote $K_n(\mathrm{Rep}_F(G))$ by $K_n(G,F)$, which is also equal to $G_n(G,F)$ (see [10]). So, $G_0(G,F) = R_F(G) = K_0(G,F)$ (see 1.1.3 below).

1.1.2. Let G be a group scheme over a scheme Y (we shall mostly be interested in $Y = \mathrm{Spec}(F)$, F a field). A scheme X over Y is called a G-scheme if there is an action morphism $\theta : G \underset{Y}{\times} X \to X$ (see [19] or [11]).

A G-module M over X is a coherent O_X-module M together with an isomorphism of $O_{G\underset{Y}{\times}X}$-modules $\rho : \theta^*(M) \to p_2^*(M)$ where $p_2 : G \underset{Y}{\times} X \to X$ is the projection satisfying the cocycle condition on $G \underset{Y}{\times} G \underset{Y}{\times} X$:

$$p_{23}^*(\rho) \circ (\mathrm{id}_\rho \times \theta)^*(\rho) = (m \times \mathrm{id}_X)^*(\rho),$$

where $m : G \underset{Y}{\times} G \to G$ is the multiplication (see [11] or [19]).

1.1.3. Let $\mathcal{M}(G,X)$ denote the abelian category of G-modules over a G-scheme X. We write $G_n(G,X)$ for $K_n(\mathcal{M}(G,X))$. Note that when $X = \mathrm{Spec}(F)$ we recover $G_n(G,F)$ in 1.1.1.

Let $\mathcal{P}(G,X)$ be the full subcategory of $\mathcal{M}(G,X)$ consisting of locally free O_X-modules. We can write $K_n(G,X)$ for $K_n(\mathcal{P}(G,X))$. Note that:

(a) if G is a trivial scheme, then $G_n(G,X) \simeq G_n(X)$; $K_n(G,X) \simeq K_n(X)$.
(b) $G_n(G,-)$ is contravariant with respect to flat G-maps.
(c) $G_n(G,-)$ is covariant with respect to projective G-maps.
(d) $K_n(G,-)$ is contravariant with respect to any G-map.
(e) $G_n(-,X)$ is contravariant with respect to group homomorphisms.
(f) $K_n(-,X)$ is covariant with respect to group homomorphisms (see [19] or [11]).

1.1.4. We have the following generalization of 1.1.3 (see [11], [13]):

Let A be a finite dimensional separable F-algebra, G an algebraic group over F and X a G-scheme. A G-A-module over a G-scheme X is a G-module M which is also a left $A \otimes_F O_X$-module such that $g(am) = ga \cdot gm$ for $g \in G$, $m \in M$.

Let $\mathcal{M}(G, X, A)$ be the Abelian category whose objects are G-A-modules and whose morphisms are $A \otimes_F O_X$- and G-module morphisms. We write $G_n(G, X, A)$ for $K_n(\mathcal{M}(G, X, A))$. Note that $\mathcal{M}(G, X, F) \simeq \mathcal{M}(G, X)$, and so, $G_n(G, X, F) \simeq G_n(G, X)$.

Let $\mathcal{P}(G, X, A)$ be the full subcategory of $\mathcal{M}(G, X, A)$ consisting of locally free $O_{A \otimes O_X}$-module. Write $K_n(G, X, A)$ for $K_n(\mathcal{P}(G, X, A))$. Hence $\mathcal{P}(G, X, F) \simeq \mathcal{P}(G, X)$, $K_n(G, X, F) \simeq K_n(G, X)$.

1.1.5. Let G be an affine algebraic group over F, X a G-scheme, $\mathcal{VB}_G(X)$ the category of G-equivariant vector bundles on X. If H is a closed subgroup of G, then we have an equivalence of categories

$$\mathrm{Rep}_F(H) \underset{\text{res}}{\overset{\text{ind}}{\rightleftarrows}} \mathcal{VB}_G(G/H),$$

where 'ind' and 'res' are defined as follows:

- $\triangleright$ res: For any vector bundle $E \xrightarrow{p} G/H$, $p^{-1}(\bar{e}) \in \mathrm{Rep}_F(H)$ (where $\bar{e} = eH = H$) since the stabilizer of H in $G/H = \bar{e}$.
- $\triangleright$ ind: Let $(V, \alpha : H \to \mathrm{Aut}(V)) \in \mathrm{Rep}_F(H)$. Then, one has a vector bundle $(G \times V)/H \to G/H$ where H acts on $(G \times V)/H$ by $(g, v)h = (g \cdot h, h^{-1}v)$, see [13]. We denote $(G \times V)/H$ by $\widetilde{V}$. Here $h^{-1}v := \alpha(h^{-1}v)$. So we get $K_n(\mathrm{Rep}_F(H)) \simeq K_n(\mathcal{VB}_G(G/H))$. We denote $K_n(\mathcal{VB}_G(G/H))$ by $K_n(G/H)$.

1.2. **K-theory of twisted flag varieties.** In this subsection we briefly introduce twisted flag varieties and their algebraic K-theory. Details can be found in [13]. We say enough here to develop notations for later use.

1.2.1. Let $\widetilde{G}$ be a semi-simple connected and simply connected, F-split algebraic group over a field F. Let $\widetilde{T} \subset \widetilde{G}$ be a maximal F-split torus of $\widetilde{G}$, $\widetilde{P} \subset \widetilde{G}$ a parabolic subgroup of $\widetilde{G}$ containing the torus $\widetilde{T}$. The factor variety $\mathcal{F} = \widetilde{G}/\widetilde{P}$ is smooth and projective (see [13], [2]). Call $\mathcal{F} = \widetilde{G}/\widetilde{F}$ a flag variety.

Let $N_{\widetilde{G}}(\widetilde{T})$ be the normalizer of $\widetilde{T}$ in $\widetilde{G}$, $W := N_{\widetilde{G}}(\widetilde{T})/\widetilde{T}$ the Weyl group of G — a finite group. Let $W_{\widetilde{P}} := \{w \in W \mid w\widetilde{P}w^{-1} = \widetilde{P}\}$. Put $n(\mathcal{F}) = [W : W_{\widetilde{P}}]$. Note that $R(\widetilde{P})$ is a free $R(\widetilde{G})$-module of rank $n(\widetilde{\mathcal{F}})$ (see [13]).

1.2.2. Let $\widetilde{Z}$ be the center of $\widetilde{G}$ and $\widetilde{Z}^* = \mathrm{Hom}(\widetilde{Z}, G_m)$ the group of characters of $\widetilde{Z}$. Note that $\widetilde{Z}^*$ is a finite group.

Let $x \in \widetilde{Z}^*$ and $\mathrm{Rep}^{\chi}_{\widetilde{G}}(\widetilde{P})$ be the full subcategory of $\mathrm{Rep}_F(\widetilde{P})$ consisting of those $V \in \mathrm{Rep}_F(\widetilde{P})$ such that $\widetilde{Z}$ acts on V by the character χ. The F-group scheme $\widetilde{Z}$ acts on V by the character χ and hence on every $\widetilde{V} = (\widetilde{G} \times V)/\widetilde{P} \in \mathcal{VB}_{\widetilde{G}}(\mathcal{F})$ (see 1.1.5).

Let $\mathcal{VB}_{\widetilde{G}}(\mathcal{F},\chi)$ be the full subcategory of $\mathcal{VB}_{\widetilde{G}}(\mathcal{F})$ consisting of those $\widetilde{V}$ such that $\widetilde{Z}$ acts on every fibre of $\widetilde{V}$ by the character χ. Write $K_n(\mathcal{F},\chi)$ for $K_n(\mathcal{VB}_{\widetilde{G}}(\mathcal{F},\chi)$ and $R^{\chi}(\mathcal{P})$ for $K_0(\mathrm{Rep}^{\chi}_F(\mathcal{P}))$.

1.2.3. Let $\widetilde{G},\widetilde{Z},\widetilde{T},\widetilde{P}$ be as in 1.2.1 and 1.2.2. Put $\widetilde{G}=\widetilde{G}/\widetilde{Z}$, $P=\widetilde{P}/\widetilde{Z}$, $T=\widetilde{T}/\widetilde{Z}$ and $\mathcal{F}=\widetilde{G}/\widetilde{P}=G/P$. Put $\mathfrak{g}=\mathrm{Gal}(F_{\mathrm{sep}}/F)$ where F_{sep} is the separable closure of F. Let $\gamma:\mathfrak{g}\to G(F_{\mathrm{sep}})$ be a 1-cocycle (see [13]) and ${}_{\gamma}\mathcal{F}$ the twisted form of $\mathcal{F}$ corresponding to γ (see [11] or [13]). We write $K_n({}_{\gamma}\mathcal{F})$ for $K_n(\mathcal{VB}_G({}_{\gamma}\mathcal{F}))$.

Now, for $\chi\in\widetilde{Z}^*=\mathrm{Hom}(\widetilde{Z},G_m)$, choose a non-trivial representation $V_{\chi}\in\mathrm{Rep}^{\chi}(\widetilde{G})$. Put $A_{\chi}=\mathrm{End}_F(V_{\chi})$. Then A_{χ} is an F-algebra equipped with a G-action by F-algebra automorphism (see [13]). Using the 1-cocycle γ, one gets a new $\mathfrak{g}$-action on $A_{\chi}\otimes_F F_{\mathrm{sep}}$ and hence a twisted form $A_{\chi,\gamma}$ of the algebra A_{χ} (see [13]).

1.2.4. As in 1.2.3, let $\gamma:\mathfrak{g}\to G(F_{\mathrm{sep}})$ be a 1-cocycle and let ${}_{\gamma}\mathcal{F}$ be the twisted form of $\mathcal{F}$ corresponding to the cocycle γ. Assume that $\mathrm{char}(F)=0$ or $\mathrm{char}(P)$ is prime to the order of $\widetilde{Z}^*$. Now consider the exact sequence

$$\{1\}\longrightarrow\widetilde{Z}\longrightarrow\widetilde{G}\longrightarrow\widetilde{G}/\widetilde{Z}\longrightarrow\{1\}$$

and the boundary map $\partial:H^1(F,G)\to H^2(F,\widetilde{Z})$. Then we have an element $\partial\gamma\in H^2(F,\widetilde{Z})$. Now, any $\chi\in\bar{Z}^*=\mathrm{Hom}(\bar{Z},G_m)$ induces a map $\chi_*:H^2(F,\widetilde{Z})\to H^2(F,G_m)=\mathrm{Br}(F)$. Hence we now have a map

$$\begin{aligned}\beta:\widetilde{Z}^*&\longrightarrow\mathrm{Br}(F)\\ \chi&\longmapsto\chi_*(\partial\gamma)\end{aligned}$$

1.2.5. **Lemma** (Tits, [20]). *Assume that* $\mathrm{char}(F)=0$ *or that* $\mathrm{char}(F)$ *is prime to the order of* $\widetilde{Z}^*$*, then* $[A_{\chi,\gamma}]=\beta(\gamma)\in\mathrm{Br}(F)$.

1.2.6. *Remarks.*

(a) Note from 1.2.5, that $A_{\chi,\gamma}$ is a central simple F-algebra.

(b) We give one example of the structure above. Other examples can be found in [13]. Take $\widetilde{G}=\mathrm{SL}_n$, $G=\mathrm{PGL}_n$, $\widetilde{Z}=\mu_n$, the group scheme of n^{th} roots of unity, $\widetilde{Z}^*=Z/nZ$ whose generator is the embedding $\mu_n\overset{\chi}{\hookrightarrow}G_m$. Let V_n be the regular n-dimensional representation of $\widetilde{G}$. Then $V_n\in\mathrm{Rep}^{\chi}(\widetilde{G})$. Take $V_{\chi^i}:=V_n^{\otimes i}\in\mathrm{Rep}^{\chi^i}(\widetilde{G})$, $A_i:=\mathrm{End}_F(V_{\chi^i})$. Then $A_{\chi,\gamma}$ is a central simple F-algebra of degree n corresponding to γ, and $A_{\chi^i,\gamma}\simeq A_{\chi,\gamma}^{\otimes i}$ (for $i=0,1,\ldots,n-1$). Put $P=\left\{\left(\begin{smallmatrix}\underline{a}&\underline{b}\\0&\underline{c}\end{smallmatrix}\right)\middle|\,\det(\underline{a})\det(\underline{b})=1\right\}$, $\underline{a}\in\mathrm{GL}_k$, $\underline{c}\in\mathrm{GL}_{n-k}$. Then $\widetilde{G}/\widetilde{P}=\mathrm{Gr}(k,n)$ is the Grassmannian variety of k-dimensional linear subspaces of a fixed n-dimensional space.

1.2.7. Let B be a finite dimensional separable F-algebra, X a smooth projective variety equipped with the action of an affine algebraic group G over F, ${}_{\gamma}X$ the twisted form of X via a 1-cocycle γ. Let $\mathcal{VB}_G({}_{\gamma}X,B)$ be the category of vector bundles on ${}_{\gamma}X$ equipped with left B-module structure. We write $K_n({}_{\gamma}X,B)$ for $K_n(\mathcal{VB}_G({}_{\gamma}X,B))$.

2. Profinite Higher K-Theory for Schemes — Definitions and Relevant Examples

In this section we briefly introduce mod-l^s and profinite K-theory for exact categories with examples relevant to this paper. More details and examples can be found in [10, chapter 8] or [8].

2.1. Mod-l^s K-theory of $\mathcal{C}$.

2.1.1. Let $\mathcal{C}$ be an exact category, l a rational prime, s a positive integer, $M^{n+1}_{l^s}$ the $(n+1)$-dimensional mod-l^s-space i.e. the space obtained from S^n by attaching an $(n+1)$-cell via a map of degree l^s (see [3], [12]).

If X is any H-space, write $\pi_{n+1}(X, \mathbb{Z}/l^s)$ for $[M^{n+1}_{l^s}, X]$, the set of homotopy classes of maps from $M^{n+1}_{l^s}$ to X. If $\mathcal{C}$ is an exact category and $X = BQ\mathcal{C}$, write $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ for $\pi_{n+1}(BQ\mathcal{C}, \mathbb{Z}/l^s)$ for $n \geq 1$ and $K_0(\mathcal{C}, \mathbb{Z}/l^s)$ for $K_0(\mathcal{C}) \otimes \mathbb{Z}/l^s$. Call $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ mod-l^s K-theory of $\mathcal{C}$.

2.1.2. Note from [10, 8.1.12] or [8] that the exact sequence

$$\cdots \longrightarrow K_n(\mathcal{C}) \xrightarrow{l^s} K_n(\mathcal{C}) \xrightarrow{\rho} K_n(\mathcal{C}, \mathbb{Z}/l^s) \xrightarrow{\beta} K_n(\mathcal{C}) \longrightarrow K_n(\mathcal{C}) \longrightarrow \cdots$$

induces a short exact sequence for all $n \geq 2$

$$0 \longrightarrow K_n(\mathcal{C})/l^s \longrightarrow K_n(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K_n(\mathcal{C})[l^s] \longrightarrow 0.$$

2.1.3. *Examples.*

(i) if A is a ring with identity, and $\mathcal{C} = \mathcal{P}(A)$ the category of finitely generated projective A-modules, write $K_n(A, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(A), \mathbb{Z}/l^s)$. Note that $K_n(A, \mathbb{Z}/l^s)$ is also $\pi_n(BGL(A)^+, \mathbb{Z}/l^s)$.

(ii) If Y is a scheme and $\mathcal{C} = \mathcal{P}(Y)$, the category of locally free sheaves of O_Y-modules, write $K_n(Y, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(Y), \mathbb{Z}/l^s)$. Note that for $Y = \mathrm{Spec}(A)$, A commutative, we recover $K_n(A, \mathbb{Z}/l^s)$.

(iii) Let A be a Noetherian ring and $\mathcal{M}(A)$ the category of finitely generated A-modules. We write

$$G_n(A, \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{M}(A), \mathbb{Z}/l^s).$$

(iv) If Y is a Noetherian scheme, $\mathcal{C} = \mathcal{M}(Y)$ the category of coherent sheaves of O_Y-modules, write

$$G_n(Y, \mathbb{Z}/l^s) \quad \text{for} \quad G_n(\mathcal{M}(Y), \mathbb{Z}/l^s).$$

(v) Let G be an algebraic group over a field F, X a G-scheme and $\mathcal{C} = \mathcal{M}(G, X)$ as defined in 1.1.3. Write

$$G_n((G, X), \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{M}(G, X), \mathbb{Z}/l^s).$$

(vi) If $\mathcal{C} = \mathcal{P}(G, X)$ as defined in 1.1.3, write

$$K_n((G, X), \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{P}(G, X), \mathbb{Z}/l^s).$$

(vii) If $\mathcal{C} = \mathcal{VB}_G({}_\gamma X, B)$ as in 1.2.7 we write

$$K_n(({}_\gamma X, B), \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{VB}_G({}_\gamma X, B); \mathbb{Z}/l^s).$$

(viii) If $\mathcal{C} = \mathcal{M}(G, X, A)$ as defined in 1.1.4, write

$$G_n((G, X, A), \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{M}(G, X, A), \mathbb{Z}/l^s).$$

(ix) If $\mathcal{C} = \mathcal{P}((G, X, A), \mathbb{Z}/l^s)$ as in 1.1.4, we write

$$K_n((G, X, A), \mathbb{Z}/l^s) \quad \text{for} \quad K_n(\mathcal{P}(G, X, A), \mathbb{Z}/l^s).$$

2.2. **Profinite K-theory.**

2.2.1. Let $\mathcal{C}$ be an exact category, l a rational prime, s a positive integer. Put $M^{n+1}_{l^\infty} = \varinjlim M^{n+1}_{l^s}$. We define the profinite K-theory of $\mathcal{C}$ by $K^{\text{pr}}_n(\mathcal{C}, \hat{\mathbb{Z}}_l) := [M^{n+1}_{l^\infty}; BQ\mathcal{C}]$. We also write $K_n(\mathcal{C}, \hat{\mathbb{Z}}_l)$ for $\varprojlim_s(\mathcal{C}, \mathbb{Z}/l^s)$. Note that for all $n \geq 1$, we have an exact sequence

$$0 \longrightarrow \varprojlim_s{}^1 K_{2n+1}(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K^{\text{pr}}_n(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow K_n(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow 0.$$

For more information see [10] or [8].

2.2.2. *Examples.*

(i) If $\mathcal{C} = \mathcal{P}(A)$ as in 2.1.3(i), we write $K^{\text{pr}}_n(A, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{P}(A), \hat{\mathbb{Z}}_l)$ and $K_n(A, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{P}(A), \hat{\mathbb{Z}}_l)$.

(ii) If $\mathcal{C} = \mathcal{P}(Y)$ as in 2.1.3(ii) we write $K^{\text{pr}}_n(Y; \hat{\mathbb{Z}}^l)$ for $K^{\text{pr}}_n(\mathcal{P}(Y), \hat{\mathbb{Z}}_l)$ and $K_n(Y, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{P}(Y), \hat{\mathbb{Z}}_l)$.

(iii) If $\mathcal{C} = \mathcal{M}(A)$ as in 2.1.3(iii) we write $G_n(A, \hat{\mathbb{Z}}_l)$ for $G^{\text{pr}}_n(\mathcal{M}(A), \hat{\mathbb{Z}}_l)$ and $G_n(A, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{M}(A), \hat{\mathbb{Z}}_l)$.

(iv) If $\mathcal{C} = \mathcal{M}(Y)$ as in 2.1.3(iv) write

$$G^{\text{pr}}_n(Y, \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{M}(Y), \hat{\mathbb{Z}}_l).$$

(v) If $\mathcal{C} = \mathcal{M}(G, X)$ as in 2.1.3(v) write

$$G^{\text{pr}}_n((G, X), \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{M}(G, X), \hat{\mathbb{Z}}_l).$$

(vi) If $\mathcal{C} = \mathcal{P}(G, X)$ as in 2.1.3(vi) write

$$K^{\text{pr}}_n((G, X), \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{P}(G, X), \hat{\mathbb{Z}}_l).$$

(vii) If $\mathcal{C} = \mathcal{VB}_G({}_\gamma X, B)$ as in 2.1.3(vii), write

$$K^{\text{pr}}_n(({}_\gamma X, B), \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{VB}_G({}_\gamma X, B), \hat{\mathbb{Z}}_l).$$

(viii) If $\mathcal{C} = \mathcal{M}(G, X, A)$ as in 2.1.3(viii) write

$$G^{\text{pr}}_n((G, X, A), \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{M}(G, X, A), \hat{\mathbb{Z}}_l).$$

(ix) If $\mathcal{C} = \mathcal{P}(G, X, A)$ as in 2.1.3(ix) write

$$K^{\text{pr}}_n((G, X, A), \hat{\mathbb{Z}}_l) \quad \text{for} \quad K^{\text{pr}}_n(\mathcal{P}(G, X, A), \hat{\mathbb{Z}}_l).$$

3. Some Finiteness Results in Higher K-Theory of Twisted Smooth Projective Varieties

In this section, we prove some finiteness results in the K-theory of twisted flag varieties as well as K-theory of twisted forms of some other smooth projective varieties over number fields and $p-adic$ fields.

3.1. **Finiteness results for twisted flag varieties.**

3.1.1. Let $\widetilde{G}$ be a semi-simple, simply connected and connected F-split algebraic group over a field F, $\widetilde{P}$ a parabolic subgroup of G, γ the 1-cocycle $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \widetilde{G}(F_{\mathrm{sep}})$, ${}_{\gamma}\mathcal{F}$ the twisted form of $\mathcal{F}$. Let B be a finite dimensional separable F-algebra. We write $K_n({}_{\gamma}\mathcal{F}, B)$ for K_n of the category $\mathcal{VB}_G({}_{\gamma}\mathcal{F}, B)$ of vector bundles on ${}_{\gamma}\mathcal{F}$ equipped with left B-module structure. We prove the following result.

3.1.2. **Theorem.** *Let F be a number field. Then for all $n \geq 1$,*

(a) *$K_{2n+1}({}_{\gamma}\mathcal{F}, B)$ is a finitely generated Abelian group.*
(b) *$K_{2n}({}_{\gamma}\mathcal{F}, B)$ is a torsion group and has no non-trivial divisible elements.*

In order to prove 3.1.2, we first prove the following

3.1.3. **Theorem.** *Let Σ be a semi-simple algebra over a number field F. Then for all $n \geq 1$*

(a) *$K_{2n+1}(\Sigma)$ is finitely generated Abelian group.*
(b) *$K_{2n}(\Sigma)$ is torsion and has no non-zero divisible elements.*

Proof. (a) Let R be the ring of integers of F. It is well-known that any semi-simple F-algebra contains at least one maximal R-order (see [10], [16] or [4]). So let Γ be a maximal order in Σ. From the localization sequence

$$\cdots \to \bigoplus_{\underline{p}} G_{2n+1}(\Gamma/\underline{p}\Gamma) \to G_{2n+1}(\Gamma) \to G_{2n+1}(\Sigma) \to \bigoplus_{\underline{p}} G_{2n}(\Gamma/\underline{p}\Gamma) \to \cdots \tag{I}$$

(whose $\underline{p}$ ranges over all prime ideals of R) we have

$$G_{2n}(\Gamma/\underline{p}\Gamma) \simeq K_{2n}((\Gamma/\underline{p}\Gamma)/\operatorname{rad}(\Gamma/\underline{p}\Gamma))$$

where $(\Gamma/\underline{p}\Gamma)/\operatorname{rad}(\Gamma/\underline{p}\Gamma)$ is a finite semi-simple ring which is a direct product of matrix algebras over finite fields. So, $G_{2n}(\Gamma/\underline{p}\Gamma) = 0$. Note that since Γ and Σ are regular, $K_n(\Gamma) \simeq G_n(\Gamma)$ and $K_n(\Sigma) \simeq G_n(\Sigma)$ for all $n \geq 0$. But $K_{2n+1}(\Gamma)$ is finitely generated (see [10, theorem 7.1.13] or [7]). Hence $K_{2n+1}(\Sigma)$ is finitely generated as a homomorphic image of $G_{2m+1}(\Gamma)$. $\square$

(b) Recall from the proof of (a) that $G_{2n}(\Gamma/\underline{p}\Gamma) = 0$. Hence Quillen's localization sequence yields

$$0 \to G_{2n}(\Gamma) \to G_{2n}(\Sigma) \to \bigoplus_{\underline{p}} G_{2n-1}(\Gamma/\underline{p}\Gamma) \to SK_{2n-1}(\Gamma) \to 0. \tag{II}$$

Also recall that since Γ, Σ are regular, $K_n(\Gamma) \simeq G_n(\Gamma)$ and $K_n(\Sigma) \simeq G_n(\Sigma)$ for all $n \geq 0$. But $G_{2n}(\Gamma) \simeq K_{2n}(\Gamma)$ is a finite group for all $n \geq 1$ (see [10] theorem 7.1.12 or [6]). Also, $\bigoplus G_{2n+1}(\Gamma/\underline{p}\Gamma)$ is a torsion group as a direct sum of finite groups, see [10, 7.1.12]. Hence it follows from the diagram (II) above that $G_{2n}(\Sigma) \simeq K_{2n}(\Sigma)$ is a torsion group.

Also from one sequence (II), $\bigoplus G_{2n-1}(\Gamma/\underline{p}\Gamma)$, as a direct sum of finite groups has no non-trivial divisible elements. So any divisible element in $K_{2n}(\Sigma)$ must come from $G_{2n}(\Gamma) \simeq K_{2n}(\Gamma)$. But $K_{2n}(\Gamma)$ is a finite group and also has no non-trivial divisible elements. Hence $G_{2n}(\Sigma)$ has no non-trivial divisible elements.

Proof of 3.1.2. (a) It was proved in [13] that for all $n \geq 0$ $K_n(A_{\chi,\gamma} \otimes_F B) \simeq K_n({}_\gamma\mathcal{F}, B)$. So, it suffices to prove that $K_{2n+1}(A_{\chi,\gamma} \otimes_F B)$ is finitely generated. Now, as discussed in 1.2.4–1.2.6, $A_{\chi,\gamma}$ is a central simple F-algebra. Also B being separable is also semi-simple. So, $A_{\chi,\gamma} \otimes_F B$ is a semi-simple F-algebra (see [14, p. 136]). Hence by theorem 3.1.3(a) $K_{2n+1}(A_{\chi,\gamma} \otimes_F B)$ is finitely generated. Hence $K_{2n-1}(\gamma\mathcal{F}, B)$ is finitely generated.

(b) follows from theorem 3.1.3(b) by substituting $A_{\chi,\gamma} \otimes_F B$ for Σ. □

3.1.4. *Remarks.*

(a) One can also see that $K_{2n+1}({}_\gamma\mathcal{F})$ is finitely generated as a special case of 3.1.2(a). However one can also prove it directly as follows: Since $\bigoplus_1^{n(\mathcal{F})} K_{2n+1}(F) = K_{2n+1}({}_\gamma\mathcal{F})$ (see [13]), we only have to see that $K_{2n+1}(F)$ is finitely generated (since we have a finite direct sum of $K_{2n+1}(F)$). Now by Quillen's result, $K_{2n+1}(R)$ is finitely generated and by Soule's result $K_{2n+1}(R) \simeq K_{2n+1}(F)$ is finitely generated.

(b) To see that $K_{2n}({}_\gamma\mathcal{F})$ is torsion it suffices to show that $K_{2n}(F)$ is torsion since $\bigoplus_1^{n(\mathcal{F})} K_{2n}(F) \simeq K_{2n}({}_\gamma\mathcal{F})$. The arguments are similar to the proof of 3.1.3(b) applied to the short exact sequence

$$0 \longrightarrow K_{2n}(R) \longrightarrow K_{2n}(F) \longrightarrow \bigoplus_{\underline{p}} K_{2n-1}(R/\underline{p}) \longrightarrow 0$$

of Soule, realizing that $K_{2n}(R)$ is finite and each $K_{2n-1}(R/\underline{p})$ is also finite.

We now turn attention to the local structure.

3.1.5. **Theorem.** *Let F be a p-adic field, l a rational prime such that $l \neq p$. Then for all $n \geq 1$ and any separable F-algebra B, $K_n({}_\gamma\mathcal{F}, B)_l$ is a finite group.*

Proof. As noted before, $A_{\chi,\gamma} \otimes_F B$ is a semi-simple F-algebra and so, it suffices to prove that for any semi-simple F-algebra Σ, $K_n(\Sigma)_l$ is a finite group for any $n \geq 1$. To do this, it suffices to show that for any central division algebra D over some p-adic field F, $K_n(D)_l$ is a finite group.

Now, D has at least one maximal order Γ, say (see [4]). Let $\underline{m}$ be the unique maximal ideal of Γ. Then, from the localization sequence

$$\cdots \to K_n(\Gamma/\underline{m}, \mathbb{Z}/l^s) \to K_n(\Gamma, \mathbb{Z}/l^s) \to K_n(D, \mathbb{Z}/l^s) \to K_{n-1}(\Gamma/\underline{m}, \mathbb{Z}/l^s) \to \cdots \tag{III}$$

we know that $K_n(\Gamma, \mathbb{Z}/l^s) \simeq K_n(\Gamma/\underline{m}, \mathbb{Z}/l^s)$ for all $n \geq 1$. (See [18, corollary 2 to theorem 2]).

Now, the groups $K_n(\Gamma/\underline{m}, \mathbb{Z}/l^s)$, $n \geq 1$ are finite groups with uniformly bounded orders (see [18]). Hence, so are the groups $K_n(D, \mathbb{Z}/l^s)$ and $K_n(\Gamma, \mathbb{Z}/l^s)$ (from the exact sequence (III)). Also from 2.1.2, we have an exact sequence

$$0 \longrightarrow K_{n+1}(D)/l^s \longrightarrow K_n(D, \mathbb{Z}/l^s) \longrightarrow K_n(D)[l^s] \longrightarrow 0 \tag{IV}$$

where $K_{n+1}(D, \mathbb{Z}/l^s)$ is finite group having uniformly bounded orders (as shown above). So the groups $K_n(D)[l^s]$ are equal for $s \geq$ some s_0. But $K_n(D)_l = \bigcup_{n=1}^{\infty} K_n(D)[l^s]$. Hence $K_n(D)_l$ is finite. □

3.1.6. *Remarks.* Let V be a Brauer-Severi variety over a field F, and A the finite dimensional central division F-algebra associated to V. D. Quillen shows in [15] that

$$K_n(V) = \bigoplus_{s=0}^{\dim V} K_n(A^{\otimes s}),$$

for all $n \geq 1$.

(a) Suppose that F is a number field, then $K_{2n+1}(V)$ is a finitely generated Abelian group. Again, this follows from theorem 3.1.3.
(b) If F is a p-adic field then for all $n \geq 1$ $K_n(V)_l$ is a finite group if l is a prime $\neq p$.

3.2. **Finiteness results for some objects of the motivic category $\mathcal{C}(G)$.**

3.2.1. Let G be an algebraic group over a field F. By considering a smooth projective G-scheme as an object of a category $\mathcal{C}(G)$ defined below, we have similar finiteness results to those of 3.1 for $K_n({}_\gamma X, B)$ where γ is a 1-cocycle, ${}_\gamma X$ is the γ-twisted form of X and B is a separable F-algebra.

3.2.2. The category $\mathcal{C}(G)$ is constructed as follows (the construction is due to I. Panin, see [13], or [11]):

The objects of $\mathcal{C}(G)$ are pairs (X, A) whose X is a smooth projective G-scheme and A is a finite dimensional separable F-algebra on which G acts by F-algebra automorphisms. Define

$$\mathrm{Hom}_{\mathcal{C}(G)}((X, A), (Y, B)) := K_0(G, X \times Y, A^{\mathrm{op}} \otimes_F B).$$

Composition of morphisms is defined as follows: If $u : (X, A) \to (Y, B)$, $v : (Y, B) \to (Z, c)$ are two morphisms, then the composite is defined by

$$v \circ u := p_{13}^*(p_{23}^*(v) \otimes_B p_{12}^*(u)),$$

where $p_{12} : X \otimes Y \otimes Z \longrightarrow X \otimes Y$, $p_{13} : X \otimes Y \otimes Z \longrightarrow X \otimes Z$, and $p_{23} : X \otimes Y \otimes Z \longrightarrow Y \otimes Z$.

The identity endomorphism of (X, A) in $\mathcal{C}(G)$ is the class $[A \otimes_F O_\Delta]$ (where $\Delta \subset X \times X$ is the diagonal) in $K_o(G, X \times X, A^{(\gamma)} \otimes_F A) = \mathrm{End}_{\mathcal{C}(G)}(X, A)$.

We now have the following results.

3.2.3. **Theorem.** *Let $\alpha : C \xrightarrow{\sim} X$ be an isomorphism in the category $\mathcal{C}(G)$, i.e., $\alpha : (\mathrm{Spec}(F), C) \xrightarrow{\sim} (X, F)$. For every 1-cocycle $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G_{F_{\mathrm{sep}}}$ and any finite dimensional separable F-algebra B, let $K_n({}_\gamma Y, B)$ be as defined in 1.2.3.*

(a) *If F is a number field, then for $n \geq 1$,*
  (i) *$K_{2n+1}({}_\gamma X, B)$ is a finitely generated Abelian group and has no non-trivial divisible elements.*
  (ii) *$K_{2n}({}_\gamma X, B)$ is a torsion group and has no non-trivial divisible elements.*
(b) *If F is a p-adic field, l a rational prime such that $l \neq p$, then for all $n \geq 1$ and any separable F-algebra B, $K_n(\gamma X, B)_l$ is a finite group.*

Proof. From [13], we have that for all $n \geq 1$ $K_n(C_\gamma \otimes_F B) \simeq K_n({}_\gamma X, B)$ where F is any field and $C_\gamma \otimes_F B$ is a semi-simple F-algebra Σ, say.

If F is a number field, (a)(i),(ii) follows from 3.1.3(a),(b). IF F is a p-adic field it suffices to prove that for all $n \geq 1$, $K_n(\Sigma)_l$ is a finite group. But this is done already in the proof of 3.1.5. □

4. Profinite Equivariant K-Theory for G-Schemes

4.1. **A general result.** We first prove the following general result for later use

4.1.1. **Theorem.** *Let $\mathcal{C}$, $\mathcal{C}'$ be exact categories and $f : \mathcal{C} \to \mathcal{C}'$ an exact functor which induces an Abelian group homomorphism $f_* : K_n(\mathcal{C}) \to K_n(\mathcal{C}')$, for each $n \geq 0$. Let l be a rational prime, s a positive integer*

(a) *Suppose that f_* is injective (resp. surjective, resp. bijective), then so are the induced maps*

$$\hat{f}_* : K_n(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K_n(\mathcal{C}', \mathbb{Z}/l^s) \quad \textit{and}$$
$$f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow K_n^{\mathrm{pr}}(\mathcal{C}', \hat{\mathbb{Z}}_l).$$

(b) *If f_* is split surjective (resp. split injective) then so is*

$$\hat{f}_* : K_n(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K_n(\mathcal{C}, \mathbb{Z}/l^s).$$

Proof. Consider the following commutative diagram (I) where the rows are exact and the vertical arrows are induced from f_*.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n(\mathcal{C})/l^s & \xrightarrow{\delta} & K_n(\mathcal{C}, \mathbb{Z}/l^s) & \xrightarrow{\eta} & K_{n-1}(\mathcal{C})[l^s] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{f}_*} & & \downarrow{\scriptstyle \hat{f}_*} & & \downarrow{\scriptstyle f'_*} & & \\ 0 & \longrightarrow & K_n(\mathcal{C}')/l^s & \xrightarrow{\delta'} & K_n(\mathcal{C}', \mathbb{Z}/l^s) & \xrightarrow{\eta'} & K_{n-1}(\mathcal{C}')[l^s] & \longrightarrow & 0 \end{array} \qquad \text{(I)}$$

□

Now, f_* injective (resp. surjective, resp. bijective) implies that $\bar{f}_*$, f'_* are injective (resp. surjective, resp. bijective). So by applying the five lemma to diagram (I), we have that $\bar{f}_*, f'_*$ injective (resp. surjective, resp. biyective) imply that $\hat{f}_*$ is injective (resp. surjective, resp. biyective). Hence f_* injective (resp. surjective, resp. bijective) implies that $\hat{f}_*$ is injective (resp. surjective, resp. biyective). This proves the first part of (a).

Now consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim_s{}^1 K_{n+1}(\mathcal{C}, \mathbb{Z}/l^s) & \xrightarrow{\delta} & K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l) & \xrightarrow{\eta} & K_n(\mathcal{C}, \hat{\mathbb{Z}}_l) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat{f}'_*} & & \downarrow{\scriptstyle f_*^{\mathrm{pr}}} & & \downarrow{\scriptstyle \hat{f}''_*} & & \\ 0 & \longrightarrow & \varprojlim_s K_{n+1}(\mathcal{C}', \mathbb{Z}/l^s) & \xrightarrow{\delta'} & K_n^{\mathrm{pr}}(\mathcal{C}', \hat{\mathbb{Z}}_l) & \xrightarrow{\eta'} & K_n(\mathcal{C}', \hat{\mathbb{Z}}_l) & \longrightarrow & 0 \end{array} \qquad \text{(II)}$$

where $\hat{f}'_*$ and $\hat{f}''_*$ are induced by $\hat{f}_*$ in diagram (I).

Note that,

$$K_n(\mathcal{C}, \hat{\mathbb{Z}}_l) := \varprojlim_s K_n(\mathcal{C}, \mathbb{Z}/l^s) \quad \text{and} \quad K_n(\mathcal{C}', \hat{\mathbb{Z}}_l) := \varprojlim_s K_n(\mathcal{C}, \mathbb{Z}/l^s).$$

Now if $\hat{f}_*$ is injective (resp. surjective, resp. biyective) in diagram (I), then $\hat{f}'_*$ and $\hat{f}_*$ are both injective (resp. surjective, resp. bijective) in diagram (II).

Also by applying the five lemma to diagram (II) we find that if $\hat{f}'_*$ and $\hat{f}''_*$ are both injective (resp. surjective, resp. bijective), then f_*^{pr} is injective (resp. surjective, resp. bijective). Hence if f_* is injective (resp. surjective, resp. bijective) then so is $\hat{f}_*$ and this implies that f_*^{pr} is injective (resp. surjective, resp. bijective) as required.

(b) We prove here only that f_* split surjective implies that $\hat{f}_*$ is split surjective since proving that f_* split injective implies that $\hat{f}_*$ is split injective is similar.

First observe that the horizontal sequences in diagram (I) are split exact (see [10] or [1]) since l is an odd prime. Hence there exist a map $\hat{\delta} : K_n(\mathcal{C}, \mathbb{Z}/l^s) \to K_n(\mathcal{C})/l^s$ such that $\hat{\delta}\delta = 1_{K_n(\mathcal{C}/l^s)}$, as well as a map $\hat{\delta}' : K_n(\mathcal{C}', \mathbb{Z}/l^s) \to K_n(\mathcal{C}')/l^s$ such that $\hat{\delta}'\delta' = 1_{K_n(\mathcal{C}'/l^s)}$. Also, f_* split surjective implies that $\bar{f}_*$ is split surjective. So, there exists $\bar{f}'_*$ such that $\hat{f}_*\hat{f}'_* = 1_{K_n(\mathcal{C})/l^s}$. Put $\hat{f}'_* = \delta\bar{f}'_*\hat{\delta}'$. Then for any $x \in K_n(\mathcal{C}', \mathbb{Z}/l^s)$,

$$\begin{aligned}\hat{f}_*\hat{f}'_*(x) &= \hat{f}_*\delta\bar{f}'_*\hat{\delta}'(x)\\ &= \delta'\bar{f}_*\bar{f}'_*\hat{\delta}'(x), \quad \text{by the commutativity of the left-hand square,}\\ &= x.\end{aligned}$$

Hence $\hat{f}_*\hat{f}'_* = \mathrm{id}_{K_n(\mathcal{C}', \mathbb{Z}/l^s)}$ i.e. $\hat{f}_*$ is split surjective.

4.1.2. *Remark.* This author is not able to use the procedure above to show that f_* split surjective (resp. split injective) implies that

$$f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l)$$

is split surjective (resp. split injective). This is because it is not known (to the author) that the sequence

$$0 \longrightarrow \varprojlim_s{}^1 K_{n+1}(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow K_n(\mathcal{C}, \hat{\mathbb{Z}}_l) \longrightarrow 0$$

is split.

4.2. **Remarks and examples.** Theorem 4.1.1 applies notably in the following situations

(a) Let B be a split solvable group, $T \subset B$ a split maximal torus, X a B-scheme. Then, by [11], $G_n(B, X) \longrightarrow G_n(T, X)$ is an isomorphism. So, by 4.1.1, $G_n^{\mathrm{pr}}((B, X), \hat{\mathbb{Z}}_l) \longrightarrow G_n^{\mathrm{pr}}((T, X), \hat{\mathbb{Z}}_l)$ is an isomorphism.

(b) Let G be an algebraic group over a field F, H a closed subgroup of G such that $G/H \simeq \mathbb{A}_F^1$ and X a G-scheme. It is known (see [11]) that $G_n(G, X) \simeq G_n(H, X)$. Hence $G_n^{\mathrm{pr}}((G, X), \hat{\mathbb{Z}}_l) \simeq G_n^{\mathrm{pr}}((H, X), \hat{\mathbb{Z}}_l)$.

(c) Let G be a split reductive group with $\pi_1(G)$ torsion free and X a smooth projective G-scheme. Then the restriction homomorphism $G_n(G, X) \longrightarrow G_n(X)$ is surjective (see [11]). Hence, by 4.1.1 follows that $G_n^{\mathrm{pr}}((G, X), \hat{\mathbb{Z}}_l) \longrightarrow G_n^{\mathrm{pr}}(X, \hat{\mathbb{Z}}_l)$ is surjective.

(d) Let G be a reductive group defined over a field F such that G is factorial (i.e. for any finite field extension E/F, $\mathrm{Pic}(G_E)$ is trivial). Let X be a smooth projective G-scheme over F. Then the restriction homomorphism $G_n(G,X) \longrightarrow G_n(X)$ is split surjective (see [11]). Hence by 4.1.1 $G_n((G,X),\mathbb{Z}/l^s) \longrightarrow G_n(X,\mathbb{Z}/l^s)$ is split surjective and so $G_n((G,X),\hat{\mathbb{Z}}_l) \longrightarrow G_n(X,\hat{\mathbb{Z}}_l)$ is split surjective. (Recall that $G_n(\mathcal{C},\hat{\mathbb{Z}}_l) = \varprojlim_s G_n(\mathcal{C},\mathbb{Z}/l^s)$.)

(e) Let G be an algebraic group over F and X a quasi-projective smooth G-scheme. Then $K_n(G,X,A) \simeq G_n(G,X,A)$ (see [11]). Hence by 4.1.1 $K_n^{\mathrm{pr}}((G,X,A),\hat{\mathbb{Z}}_l) \simeq G_r^{\mathrm{pr}}((G,X,A),\hat{\mathbb{Z}}_l)$.

(f) Let U be a split unipotent group over F, X a U-scheme. Then the restriction homomorphism $G_n(U,X) \longrightarrow G_n(X)$ is an isomorphism (see [11]). Hence by 4.1.1, $K_n^{\mathrm{pr}}((U,X),\hat{\mathbb{Z}}_l) \simeq K_n^{\mathrm{pr}}(X,\hat{\mathbb{Z}}_l)$

4.3. **Some computations.** In this subsection, we obtain some l-completeness and other results for some twisted flag varieties as well as Brauer–Severi varieties over number fields and p-adic fields. Recall that if l is a rational prime, an Abelian group H is said to be l-complete if $H = \varprojlim_s H/l^sH$.

4.3.1. **Theorem.** *Let F be a number field, $\widetilde{G}$ a semi-simple, connected, simply connected split algebraic group over F, $\widetilde{P}$ a parabolic subgroup of $\widetilde{G}$, $\mathcal{F} = \widetilde{G}/\widetilde{P}$, γ a 1-cocycle $\mathrm{Gal}(F_{\mathrm{sep}}/F) \longrightarrow \widetilde{G}(F_{\mathrm{sep}})$, $\gamma\mathcal{F}$ the γ-twisted form of $\mathcal{F}$, B a finite dimensional separable F-algebra. Then for all $n \geq 1$,*

(1) *$K_{2n}^{\mathrm{pr}}((\mathcal{F},B),\hat{\mathbb{Z}}_l)$ is an l-complete Abelian group.*
(2) $\mathrm{div}\, K_{2n}^{\mathrm{pr}}((\mathcal{F},B),\hat{\mathbb{Z}}_l) = 0$.

Proof. From [13] we have an isomorphism $K_n(A_{\chi,\gamma} \otimes_F B) \simeq K_n({}_\gamma\mathcal{F},B)$ for all $n \geq 0$. Hence by 4.1.1 we also have $K_n^{\mathrm{pr}}((A_{\chi,\gamma}\otimes_F B),\hat{\mathbb{Z}}_l) \simeq K_n^{\mathrm{pr}}((\mathcal{F},B),\hat{\mathbb{Z}}_l)$. So, it suffices to show that $K_{2n}^{\mathrm{pr}}((A_{\chi,\gamma} \otimes_F B),\hat{\mathbb{Z}}_l)$ is l-complete for all $n \geq 1$. As earlier explained in the proof of 3.1.2, $A_{\chi,\gamma} \otimes_F B$ is a semi-simple F-algebra and so, by theorem 3.1.3, $K_{2n+1}(\Sigma)$ is a finitely generated Abelian group. Now it is proved in [10, lemma 2.8] or [8], that for all $m \geq 2$ and any exact category $\mathcal{C}$,

$$\varprojlim_s (K_m^{\mathrm{pr}}(\mathcal{C},\hat{\mathbb{Z}}_l)/l^s \simeq K_m(\mathcal{C},\hat{\mathbb{Z}}_l)$$

Hence for any $m \geq 2$

$$\varprojlim_s K_m^{\mathrm{pr}}(\Sigma,\hat{\mathbb{Z}}_l)/l^s \simeq K_m(\Sigma,\hat{\mathbb{Z}}_l). \tag{III}$$

Also, for any $m \geq 2$ and any exact category $\mathcal{C}$ we have from [10, lemma 8.2.1] or [8] an exact sequence

$$0 \longrightarrow \varprojlim_s{}^1 K_{m+1}(\mathcal{C},\mathbb{Z}/l^s) \longrightarrow K_m^{\mathrm{pr}}(\mathcal{C},\hat{\mathbb{Z}}_l) \longrightarrow K_m(\mathcal{C},\hat{\mathbb{Z}}_l) \longrightarrow 0.$$

Hence we have an exact sequence (for $m \geq 2$)

$$0 \longrightarrow \varprojlim_s{}^1 K_{n+1}(\Sigma,\mathbb{Z}/l^s) \longrightarrow K_m^{\mathrm{pr}}(\Sigma,\hat{\mathbb{Z}}_l) \longrightarrow K_n(\mathcal{C},\hat{\mathbb{Z}}_l) \longrightarrow 0. \tag{IV}$$

Since $K_{2n+1}(\Sigma)$ is finitely generated for $n \geq 1$ then $K_{2n+1}(\Sigma, \mathbb{Z}/l^s)$ is a finite group and so, $\varprojlim_s{}^1 K_{2n+1}(\Sigma, \mathbb{Z}/l^s) = 0$. Hence from (IV),

$$K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l) \simeq K_{2n}(\Sigma, \hat{\mathbb{Z}}_l). \tag{V}$$

Also from (III),

$$\varprojlim_s K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l)/l^s \simeq K_{2n}(\Sigma, \hat{\mathbb{Z}}_l). \tag{VI}$$

From (V) and (VI) we now have

$$\varprojlim_s K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l)/l^s \simeq K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l).$$

So, $K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l)$ is l-complete. Hence $K_{2n}^{\text{pr}}(r\mathcal{F}, \hat{\mathbb{Z}}_l)$ is l-complete.

(b) From [10, theorem 8.2.2(ii)] or [8], we have that for all $m \geq 2$ and any exact category $\mathcal{C}$,

$$\varprojlim_s{}^1 K_{m+1}(\mathcal{C}, \mathbb{Z}/l^s) = \operatorname{div} K_m^{pr}(\mathcal{C}, \hat{\mathbb{Z}}_l)$$

Hence for all $m \geq 2$,

$$\varprojlim_s{}^1 K_{m+1}(\Sigma, \mathbb{Z}/l^s) = \operatorname{div} K_m^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l).$$

If $m = 2n$, then $K_{2n+1}(\Sigma)$ is finitely generated and so, $K_{2n+1}(\Sigma, \mathbb{Z}/l^s)$ is a finite group. Hence, $\varprojlim{}^1 K_{2n+1}(\Sigma, \mathbb{Z}/l^s) = 0$. Hence $\operatorname{div} K_{2n}^{\text{pr}}(\Sigma, \hat{\mathbb{Z}}_l) = 0$ and so, $\operatorname{div} K_{2n}^{\text{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) = 0$. $\square$

4.3.2. *Remarks.* The following results can be proved by procedures similar to those above.

(a) If F is a number field, ${}_\gamma\mathcal{F}$ as in 4.2.1, then $K_{2n}^{\text{pr}}(\mathcal{F}, \hat{\mathbb{Z}}_l)$ is an l-complete Abelian group and $\operatorname{div} K_{2n}^{\text{pr}}({}_\gamma\mathcal{F}, \hat{\mathbb{Z}}_l) = 0$. The proof in this case is easier.

(b) If V is a Brauer-Severi variety over a number field F, then for all $n \geq 2$, $K_{2n}^{\text{pr}}(V, \hat{\mathbb{Z}}_l)$ is l-complete and $\operatorname{div} K_{2n}^{\text{pr}}(V, \hat{\mathbb{Z}}_l) = 0$.

4.3.3. Our next aim is to consider the situation when F is a p-adic field. Before doing this, we make some general observations. Note that for any exact category $\mathcal{C}$, the natural map $M_{l^\infty}^{n+1} \to S^{n+1}$ induces a map

$$[S^{n+1}, BQ\mathcal{C}] \xrightarrow{\varphi} [M_{l^\infty}^{n+1}, BQ\mathcal{C}]$$

i.e.,

$$K_n(\mathcal{C}) \xrightarrow{\varphi} K_n^{\text{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l) \tag{VII}$$

and hence maps

$$K_n(\mathcal{C})/l^s \longrightarrow K_n^{\text{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l)/l^s \tag{VIII}$$

and

$$K_n(\mathcal{C})[l^s] \longrightarrow K_n^{\text{pr}}(\mathcal{C}, \hat{\mathbb{Z}}_l)[l^s]. \tag{IX}$$

We shall denote the maps in (VIII) and (IX) also by φ by abuse of notation.

We now prove the following result.

4.3.4. **Theorem.** *Let p be a rational prime, F a p-adic field, $\widetilde{G}$ a semisimple connected and simply connected split algebraic group over F, $\widetilde{P}$ a parabolic subgroup og $\widetilde{G}$, γ a 1-cocycle $\mathrm{Gal}(F_{\mathrm{sep}}/F) \longrightarrow \widetilde{G}(F_{\mathrm{sep}})$, ${}_\gamma\mathcal{F}$ the γ-twisted form of $\mathcal{F}$, B a finite dimensional separable F-algebra, l a rational prime such that $l \neq p$. Then for all $n \geq 2$*

(a) $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$ *is an l-complete profinite Abelian group.*
(b) $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) \simeq K_n(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$.
(c) *The map* $\varphi : K_n({}_\gamma\mathcal{F}, B) \longrightarrow K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$ *induces isomorphisms*
  (1) $K_n({}_\gamma\mathcal{F}, B)[l^s] \simeq K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)[l^s]$,
  (2) $K_n({}_\gamma\mathcal{F}, B)/l^s \simeq K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)/l^s$.
(d) *Kernel and cokernel of* $K_n(({}_\gamma\mathcal{F}, B)) \longrightarrow K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l)$ *are uniquely l-divisible.*
(e) $\mathrm{div}\, K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) = 0$ *for* $n \geq 2$.

Proof. (a), (b). Since $K_n(A_{\chi,\gamma} \otimes_F B) \simeq K_n(({}_\gamma\mathcal{F}, B))$ and $A_{\chi,\gamma} \otimes_F B$ is a semi-simple F-algebra Σ, say, it suffices for the proof of (a) to show that $K_n^{\mathrm{pr}}(\Sigma, \hat{\mathbb{Z}}_l)$ is l-complete profinite Abelian group. To do this it suffices to prove that $K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l)$ is an l-complete profinite Abelian group for a central division algebra over a p-adic field F. From the proof of 3.1.5, we saw already that $K_n(D, \mathbb{Z}/l^s)$ is a finite group. Hence, in the exact sequence

$$0 \longrightarrow \varprojlim_s{}^1 K_{n+1}(D, \mathbb{Z}/l^s) \longrightarrow K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l) \longrightarrow K_n(D, \hat{\mathbb{Z}}_l) \longrightarrow 0,$$

we have $\varprojlim^1 K_{n+1}(D, \mathbb{Z}/l^s) = 0$. Hence

$$K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l) \simeq K_n(D, \hat{\mathbb{Z}}_l) \tag{X}$$

proving (b).

Now, for any exact category $\mathcal{C}$, we have $\varprojlim K_n(\mathcal{C}, \mathbb{Z}/l^s) \simeq K_n(\mathcal{C}, \hat{\mathbb{Z}}_l)$ for all $n \geq 2$ (see [10, lemma 8.2.2] or [7]). So, we have

$$\varprojlim_s K_n^{\mathrm{pr}}(D, \mathbb{Z})/l^s \simeq K_n(D, \hat{\mathbb{Z}}_l). \tag{XI}$$

From (X) and (XI) we now have $\varprojlim K_n^{\mathrm{pr}}(D, \mathbb{Z})/l^s \simeq K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l)$ — proving (a). It is profinite because $K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l) = \varprojlim K_n(D, \mathbb{Z}/l^s)$, where $K_n(D, \mathbb{Z}/l^s)$ is a finite group.

(c),(d). Recall that $K_n(\gamma\mathcal{F}, B)$ is by definition the K_n of the (exact) category of vector bundles on ${}_\gamma\mathcal{F}$ equipped with left B-module structure. Recall also from theorem 3.1.5 that for all $n \geq 1$, $K_n({}_\gamma\mathcal{F}, B)_l$ is a finite group and hence has no non-zero divisible subgroups. Hence, (c) follows from [10, theorem 8.2.1] or [8] and (d) follows from [10, corollary 8.2.1] or [8].

(e). We saw in the proof of 3.1.5 that $K_n(D, \mathbb{Z}/l^s)$ is a finite group for all $n \geq 2$. Hence $\varprojlim^1 K_n(D, \mathbb{Z}/l^s) = 0$ for all $n \geq 2$. But by [10, theorem 8.2.2(ii)] or [8]

$$\varprojlim{}^1 K_{m+1}(D, \mathbb{Z}/l^s) \simeq \mathrm{div}\, K_m^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l).$$

Hence $\mathrm{div}\, K_n^{\mathrm{pr}}(D, \hat{\mathbb{Z}}_l) = 0$ as required for all $n \geq 1$, so $\mathrm{div}\, K_n^{\mathrm{pr}}((\gamma\mathcal{F}, B), \hat{\mathbb{Z}}_l) = 0$. □

4.3.5. *Remarks.* (a) Let V be a Brauer-Severi variety over a p-adic field F. By a similar proof to that of 4.2.4, we have

(i) $K_n^{\mathrm{pr}}(V, \hat{\mathbb{Z}}_l) \simeq K_n(V, \hat{\mathbb{Z}}_l)$ is an l-complete profinite Abelian group.
(ii) $K_n(V)/l^s \simeq K_n^{\mathrm{pr}}(V, \hat{\mathbb{Z}}_l)/l^s$ and $K_n(V)[l^s] \simeq K_n^{\mathrm{pr}}(V, \hat{\mathbb{Z}}_l)[l^s]$.
(iii) Kernel and cokernel of $K_n(V) \to K_n^{\mathrm{pr}}(V, \hat{\mathbb{Z}}_l)$ are uniquely l-divisible.
(iv) $\operatorname{div} K_n^{\mathrm{pr}}(V, \hat{\mathbb{Z}}_l) = 0$.

(b) Finally, if ${}_\gamma X$ is as in 3.2.3, we have similar results to those of 4.2.4 for $K_n^{\mathrm{pr}}(({}_\gamma X, B), \hat{\mathbb{Z}}_l)$, etc.

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