THE GABRIEL-ROITER MEASURE FOR RADICAL-SQUARE ZERO ALGEBRAS

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Abstract. Let Λ be a radical-square zero algebra over an algebraically closed field k with radical $\mathfrak{r}$ and $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}$ be the associated hereditary algebra. There is a well-known functor $F : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ which induces a stable equivalence. We show that the functor F keeps Gabriel-Roiter measures and Gabriel-Roiter factors. Thus one may study the Gabriel-Roiter measure for Λ using F and known facts for hereditary algebras. In particular, we study the middle terms of the almost split sequences ending at Gabriel-Roiter factor Λ-modules, and the relation between the preprojective partition for Λ and the take-off Λ-modules, when Λ is of s-tame type (see section 2.3.2 for the definition).

1. Introduction

Throughout the paper, we assume that k is an algebraically closed field. By algebras, we mean finite dimensional basic connected ones over k. We denote by $Q(\Lambda)$ the ordinary quiver (with relations) of an algebra Λ. We also denote by $\mathrm{mod}\,\Lambda$ the category of finite dimensional left Λ-modules and by $\mathrm{ind}\,\Lambda$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of indecomposable Λ-modules. We refer to [1] for non-explained notions and preliminaries.

The Gabriel-Roiter measure was introduced in [8] (first by Gabriel under the name 'Roiter measure'). We first recall some definitions.

Let $\mathbb{N}_1=\{1, 2, \cdots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets of $\mathbb{N}_1$. A total order on $\mathcal{P}(\mathbb{N}_1)$ can be defined as follows: If I,J are two different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I\backslash J) \cup (J\backslash I)$ belongs to J.

For a given algebra Λ and a Λ-module M, we denote by $|M|$ the length of M. Let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \cdots, |M_t|\}$ where $M_1 \subset M_2 \subset \cdots \subset$

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M_t is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure** (briefly **GR measure**) of M. If M is an indecomposable Λ-module, we call an inclusion $X \subset M$ with X indecomposable a **Gabriel-Roiter inclusion** (briefly **GR inclusion**) provided $\mu(M) = \mu(X) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(X)$. In this case, we call X a **Gabriel-Roiter submodule** of M and the corresponding factor M/X a **Gabriel-Roiter factor module**. A Gabriel-Roiter factor module is always indecomposable.

The Gabriel-Roiter measure has been studied in [8],[9],[3],[4]. As an interesting application, it was showed that Gabriel-Roiter submodules can be used to build orthogonal exceptional pairs for indecomposable modules over representation directed algebras [3],[10].

Let M be an indecomposable non-projective Λ-module over an algebra Λ and $0 \to \tau M \to X \to M \to 0$ be an almost split sequence. We denote by $\alpha(M)$ the number of the indecomposable summands of X. We say M has indecomposable middle term if $\alpha(M) = 1$. The middle term of Gabriel-Roiter factor modules over hereditary algebras of representation finite types and tame types were studied in [3],[4], respectively. In particular, we have

Theorem 1.1. *Let $\Lambda = kQ$ be a hereditary algebra and M be a GR factor Λ-module.*
(1) If Λ is of representation finite type, or tame type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$, and M is not injective, then $\alpha(M) = 1$.
(2) If Λ is of tame type $\tilde{\mathbb{A}}_n$ or $\tilde{\mathbb{D}}_n$ and $\alpha(M) > 1$, then M is preinjective with $\underline{\dim}\, M < \delta$, the minimal dimension vector corresponding to imaginary root. Thus, almost all but finitely many GR factor modules have indecomposable middle terms.

By using Gabriel-Roiter measure, Ringel obtained a partition of the module category of an artin algebra Λ of representation infinite type [10]: The module category consists of take-off part, central part and landing part. Moreover, It was showed that all indecomposable modules lying in the landing part are preinjective modules in the sense of Auslander and Smalø [2]. This naturally leads us to compare these two different kinds of partitions, obtained by Gabriel-Roiter measure approach and by Auslander and Smalø's method, of the module category. In [6], the following theorem in this direction was proved.

Theorem 1.2. *Let $\Lambda = kQ$ be a tame hereditary algebra. Then every indecomposable preprojective module lies in the take-off part.*

This paper is devoted to a study of the Gabriel-Roiter measure for radical-square zero k-algebras. Let Λ be such an algebra and $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}$ where $\mathfrak{r}$ is the radical of Λ. Then Γ is hereditary and there is a well-known functor $F : \operatorname{mod}\Lambda \rightarrow \operatorname{mod}\Gamma$ which induces a stable equivalence. We will see that

(1) The functor F leaves the Gabriel-Roiter measures invariant, that is, $\mu(X) = \mu(F(X))$ for each indecomposable Λ-module X (**Theorem A**).

(2) F sends GR factor modules to GR factor modules, that is, if X is a GR factor module over Λ, then $F(X)$ is a GR factor module over Γ (Proposition 4.4).

(3) F sends indecomposable preprojective Λ-modules to indecomposable preprojective Γ-modules (Proposition 5.5).

Therefore, some results concerning Gabriel-Roiter measure for radical-square zero algebras can be obtained using F and the facts that they are known for hereditary algebras. In particular, comparing with Theorem 1.1,1.2, we show the following theorems.

Theorem B. *Let Λ be a radical-square zero algebra and Z be a GR factor module. Then*
(1) *If Λ is of finite type and Z is not simple, then $\alpha(Z) = 1$.*
(2) *If Λ is of s-tame type, then all but finitely many GR factor modules have indecomposable middle terms. Moreover, if Λ is $\tilde{\mathbb{D}}_n$-free and Z is not simple, then $\alpha(Z) = 1$.*

Here s-tame means that the ordinary quiver $Q(\Gamma)$ of the associated hereditary algebra Γ of Λ contains a tame quiver but not any wild quiver; and $\tilde{\mathbb{D}}_n$-free means that $Q(\Gamma)$ does not contain $\tilde{\mathbb{D}}_n$ as a component.

We also want to compare the above mentioned partitions for radical square-zero algebras of s-tame type. An algebra is called a strong take-off algebra if all indecomposable preprojective modules are in take-off part. Using F and the above Theorem 1.2, one may obtain the following result:

Theorem C. *Let Λ be a radical-square zero algebra, and Γ, F be as before. Assume that $Q(\Gamma)$ is a connected tame quiver. Then Λ is a strong take-off algebra.*

Some preliminaries of Gabriel-Roiter measure, the preprojective partition and radical-square zero algebras will be recalled in section 2. Section 3 is devoted to a discussion of some new properties of the functor F. In particular, a proof of Theorem A can be found there. In section 4, the almost split sequences ending at

Gabriel-Roiter factor modules over radical-square zero algebras of finite type and s-tame type are studied and Theorem B will be proved in this section. The proof of Theorem C will be presented in section 5.

2. PRELIMINARIES

In this section, we recall some preliminaries concerning the preprojective partition, the Gabriel-Roiter measure and radical-square zero algebras.

2.1. **Preprojective partition.** We first recall the preprojective partitions for artin algebras. The main reference will be [2].

Let Λ be an artin algebra. A partition $\mathcal{P}^0, \mathcal{P}^1, \cdots, \mathcal{P}^m, \cdots, \mathcal{P}^\infty$ of $\operatorname{ind}\Lambda$ is called a preprojective partition of $\operatorname{ind}\Lambda$ if
(1) $\cup_{i=0}^{\infty}\mathcal{P}^i = \operatorname{ind}\Lambda$
(2) $\mathcal{P}^i \cap \mathcal{P}^j = \emptyset$ if $i \neq j$
(3) For all $m < \infty$, $\mathcal{P}^m$ is a finite minimal cover for $\operatorname{ind}\Lambda - \cup_{j<m}\mathcal{P}^j$.

The indecomposable modules in $\mathcal{P}^i$ for some $i < \infty$ are called preprojective. We always denote by $\mathcal{P}_\Lambda$ the full subcategory of $\operatorname{ind}\Lambda$ consisting of preprojective modules.

Proposition 2.1. *Under the above notions, one has*
(1) $X \in \mathcal{P}^0$ *if and only if* X *is a projective module.*
(2) *If* $X \in \mathcal{P}^m$, *then there exists an indecomposable module* $Y \in \mathcal{P}^j$ *for some* $j < m$ *and an irreducible morphism* $Y \rightarrow X$.

Dually, preinjective partition can be defined.

2.2. **The Gabriel-Roiter measure.** We begin to recall some basic properties and the partition obtained by Gabriel-Roiter measure approach.

2.2.1. *Properties of Gabriel-Roiter incusions.* We first collect some properties of Gabriel-Roiter incusions. For a proof of the proposition, we refer to [8],[3].

Proposition 2.2. *Let* Λ *be a finite dimensional algebra and* $l : X \subset Y$ *be a GR inclusion. Then*
(1) Y/X *is indecomposable.*
(2) *Every morphism to* Y/X *which is not an epimorphism factors through the canonical projection* $Y \rightarrow Y/X$.
(3) *Every irreducible map to* Y/X *is surjective.*

(4) *If all irreducible map from indecomposable modules to Y are monomorphisms, then the inclusion l is an irreducible map.*

2.2.2. *Partition obtained by Gabriel-Roiter measure approach.* A subset $I \in \mathcal{P}(\mathbb{N}_1)$ is said to be a Gabriel-Roiter measure for Λ if there exists an indecomposable Λ-module M with $\mu(M) = I$. A measure I is said to be of **finite type** if there are only finitely many isomorphism classes of indecomposable modules with measure I. The following partition was obtained in [8].

Theorem 2.3. *Let Λ be a representation infinite artin algebra. Then there are Gabriel-Roiter measures I_t, I^t for Λ such that*

$$I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$$

and such that any other measure J satisfies $I_t < J < I^t$ for all t. Moreover, all these measures I_t and I^t are of finite type.

The measures $I_t(I^t)$ are called **take-off (landing)** measures and any other measure is called a **central** measure. An indecomposable module M with GR measure I is called a take-off (resp. central, landing) module if I is a take-off (resp. central, landing) measure. In [8], the following proposition was showed:

Proposition 2.4. *Let Λ be a representation infinite artin algebra. Then all landing modules are preinjective.*

2.3. **Radical-square zero algebras.** The main reference of this part will be [1], section X.2. Assume that Λ is a radical-square zero k-algebra. We denote by $\mathfrak{r}$ the radical of Λ. Let

$$\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix},$$

then Γ is a hereditary algebra with radical square zero.

Let $\mathcal{C}$ be the category whose objects are triples (X, Y, s) with $X, Y \in \operatorname{mod} \Lambda/\mathfrak{r}$ and $s : \mathfrak{r} \otimes_{\Lambda/\mathfrak{r}} X \to Y$ is a morphism of $\Lambda/\mathfrak{r}$ modules, and whose morphisms from (X, Y, s) to (X', Y', s') are pairs of morphisms of $\Lambda/\mathfrak{r}$ modules (f, g) such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{r} \otimes X & \xrightarrow{s} & Y \\ \downarrow{\scriptstyle 1_{\mathfrak{r}} \otimes f} & & \downarrow{\scriptstyle g} \\ \mathfrak{r} \otimes X' & \xrightarrow{s'} & Y' \end{array}$$

It is well-known that $\operatorname{mod}\Gamma$ is equivalent to $\mathcal{C}$.

The functor $F : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ is defined as follows: It sends a Λ-module X to a triple $(X/\mathfrak{r}X, \mathfrak{r}X, s_X)$ where $s_X : \mathfrak{r} \otimes X/\mathfrak{r}X \to \mathfrak{r}X$ is the canonical map given by $s(a \otimes \bar{x}) = ax$, and sends a morphism $f : X \to Y$ to a pair $(\bar{f}, f|)$ of induced maps where $f|$ is the restriction $f| : \mathfrak{r}X \to \mathfrak{r}Y$, and $\bar{f} : X/\mathfrak{r}X \to Y/\mathfrak{r}Y$ satisfies $\bar{f}(\bar{x}) = \overline{f(x)}$.

2.3.1. *Properties of F.* The functor F has many nice properties which will play an important role in our discussions. The statements here are taken from [1], section X.2.

Proposition 2.5. *Let Λ, Γ and F be as above. Then*
(1) $|X| = |F(X)|$.
(2) *F is full.*
(3) *$X \cong X'$ if and only if $F(X) \cong F(X')$.*
(4) *X is indecomposable if and only if $F(X)$ is indecomposable.*
(5) *X is projective if and only if $F(X)$ is projective.*
(6) *The indecomposable Γ-modules not isomorphic to some $F(M)$ are the simple projective Γ-modules $(0, S, 0)$ where S is a simple projective Λ-module.*
(7) *$F : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$ is a stable equivalence.*

Proposition 2.6. *Let Λ, Γ and F be as before. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of Λ-modules such that A and C are indecomposable and A is not simple. Then this sequence is an almost split sequence if and only if $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is an almost split sequence of Γ-modules.*

2.3.2. *Quiver of Γ.* One can easily obtain the quiver $Q(\Gamma)$ of Γ from $Q(\Lambda)$ as follows: For each vertex $a \in Q(\Lambda)$, there are two vertices a and a' in $Q(\Gamma)$; and for each arrow $a \xrightarrow{\beta} b$ in $Q(\Lambda)$, there is an arrow $a \xrightarrow{\beta} b'$ in $Q(\Gamma)$. Note that each component of $Q(\Gamma)$ is a quiver with sink-source orientation.

It is well-known that algebras which are stably equivalent are of same representation type (finite or infinite). In particular, if Λ is a radical-square zero algebra of representation finite type, then the quiver $Q(\Gamma)$ of Γ is a disjoint union of Dynkin quivers. We say that Λ is of *s-tame type* if $Q(\Gamma)$ consists of tame quivers $(\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8})$ and Dynkin quivers as components and contains at least one tame quiver. If Λ is of s-tame type and Q is a quiver, we say Λ is Q-free if $Q(\Gamma)$ does not contain a component Q' such that the underlying diagrams of Q and Q' are the same.

3. More properties of the functor F

Let Λ be a radical-square zero algebra, $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}$ and F be the associated hereditary algebra and functor as before. In section 2.3.1, we have seen some nice properties of F. We will see in this section some new properties of F. We always assume in this section that Λ is a fixed radical-square zero algebra.

Lemma 3.1. *Assume that $X \subset Y$ is a GR inclusion of Λ-modules. Then the following are equivalent:*
(1) *X is simple.*
(2) *Y is a local module.*
(3) $\operatorname{top} Y$ *is simple.*
(4) *X is contained in $\mathfrak{r}Y$.*

Proof. It is easily seen that X is simple if and only if Y is a local module with Loewy length 2, thus (1) and (2) are equivalent since $\mathfrak{r}^2 = 0$.

Assume that X is contained in the radical $\mathfrak{r}Y$. Then $\mathfrak{r}^2 = 0$ implies that $\mathfrak{r}Y$ is semisimple. It follows that X is semisimple, thus is simple since X is indecomposable. Therefore, Y is a local module and $\operatorname{top} Y$ is simple. □

Lemma 3.2. *Assume that* $\operatorname{top} Y$ *is simple, then $\mu(Y) = \mu(F(Y))$.*

Proof. Let $P \xrightarrow{\pi} Y$ be a projective cover. Then $\bar{\pi} : P/\mathfrak{r}P \to Y/\mathfrak{r}Y$ is an isomorphism and $\pi| : \mathfrak{r}P \to \mathfrak{r}Y$ is an epimorphism. It follows that

$$F(\pi) = (\bar{\pi}, \pi|) : (P/\mathfrak{r}P, \mathfrak{r}P, s_P) \to (Y/\mathfrak{r}Y, \mathfrak{r}Y, s_Y)$$

is an epimorphism. On the other hand, $F(P)$ is indecomposable and projective since P is (Proposition 2.5). Therefore, $\operatorname{top} F(Y)$ is simple and $\mu(Y) = \{1, |Y|\} = \{1, |F(Y)|\} = \mu(F(Y))$. □

Lemma 3.3. *Let $X \xrightarrow{l} Y$ be an inclusion of indecomposable Λ-modules such that X is not simple and $X \nsubseteq \mathfrak{r}Y$. Then $F(l)$ is a monomorphism.*

Proof. Since $X \nsubseteq rY$, the composition of l and the canonical projection $Y \xrightarrow{\pi} Y/\mathfrak{r}Y$ is not zero and the kernel of the composition πl is $X \cap \mathfrak{r}Y$. It follows from $\mathfrak{r}(X \cap \mathfrak{r}Y) \subset \mathfrak{r}^2 Y = 0$ that $X \cap \mathfrak{r}Y$ is a semisimple submodule of X. Thus $X \cap \mathfrak{r}Y \subseteq \operatorname{soc} X \subseteq \mathfrak{r}X$. The second inclusion holds since X is not simple and X is indecomposable. Then $X \cap \mathfrak{r}Y = \mathfrak{r}X$ and the induced map $\bar{l} : X/\mathfrak{r}X \to Y/\mathfrak{r}Y$ is injective. It turns out that $F(l) = (\bar{l}, l|) : F(X) \to F(Y)$ is a monomorphism. □

Lemma 3.4. *Let $F(X) \xrightarrow{g} F(Y)$ be a monomorphism for indecomposable Λ-modules X, Y. Then there exist a monomorphism $X \xrightarrow{f} Y$ such that $F(f) = g$.*

Proof. Since the functor F is full, there is a morphism $X \xrightarrow{f} Y$ such that $F(f) = g = (g_1, g_2)$. By the definition of F, we get a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{r}X & \longrightarrow & X & \longrightarrow & X/\mathfrak{r}X & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle f|=g_2} & & \big\downarrow {\scriptstyle f} & & \big\downarrow {\scriptstyle \bar{f}=g_1} & & \\
0 & \longrightarrow & \mathfrak{r}Y & \longrightarrow & Y & \longrightarrow & Y/\mathfrak{r}Y & \longrightarrow & 0
\end{array}$$

Therefore, f is a monomorphism since g_1, g_2 are both injective. □

Now we can prove the main theorem of this section.

Theorem A. *Let Y be an indecomposable Λ-module. Then $\mu(Y) = \mu(F(Y))$.*

Proof. We may assume that $\operatorname{top} Y$ is not simple (Lemma 3.2). For an inductive step, we assume that for any proper indecomposable submodule X of Y, $\mu(X) = \mu(F(X))$. We want to show $\mu(Y) = \mu(F(Y))$.

Let X be a GR submodule of Y. Since $\operatorname{top} Y$ is not simple, X is not contained in $\mathfrak{r}Y$ and the inclusion $X \subset Y$ induces an inclusion $F(X) \subset F(Y)$ (Lemma 3.3). We claim that $F(X)$ is a GR submodule of $F(M)$. Let T' be a GR submodule of $F(Y)$. Then T' is not simple since $\operatorname{top} F(Y)$ is not simple (otherwise $\operatorname{top} Y$ is simple). Note that the indecomposable Γ-modules which are not of the forms $F(M)$ for some Λ-module M are simple projective modules. Thus there is an indecomposable Λ-module T such that $F(T) = T'$. The inclusion $F(T) = T' \subset F(Y)$ induces an inclusion $T \subset Y$ (Lemma 3.4). By induction we have

$$\mu(F(T)) = \mu(T) \leq \mu(X) = \mu(F(X)) \leq \mu(F(T))$$

since X is a GR submodule of Y and $F(T) = T'$ is a GR submodule of $F(Y)$. It follows that $\mu(F(T)) = \mu(F(X))$ and $F(X)$ is a GR submodule of $F(Y)$. Therefore, the following equalities hold

$$\mu(F(Y)) = \mu(F(X)) \cup \{|F(Y)|\} = \mu(X) \cup \{|Y|\} = \mu(Y).$$

This finishes the proof. □

Remark. This proposition gives examples that different algebras have the same set of GR measures. Since F induces a stable equivalence which sends indecomposable modules to indecomposable modules, the following equality counting the isomorphism classes of indecomposable modules with the same GR measure holds for each GR measure I distinct from $\{1\}$.

$$|\{X \in \operatorname{ind}\Lambda : \mu(X) = I\}/\mathrm{iso}| = |\{Y \in \operatorname{ind}\Gamma : \mu(Y) = I\}/\mathrm{iso}|.$$

We now give an easy application of the theorem.

Corollary 3.5. *Let Λ be a representation finite radical-square zero algebra and Y be an indecomposable Λ-module. Then up to isomorphism, Y has at most 3 GR submodules.*

Proof. In [3], it was showed that each indecomposable module over a representation finite hereditary k-algebra has, up to isomorphism, at most 3 GR submodules. Let Γ and F be the associated hereditary algebra and functor. If $\operatorname{top} Y$ is not simple, then from the proof of Theorem A, we have that F sends a GR submodule of Y to a GR submodule of $F(Y)$. Therefore, $F(Y)$, thus Y, has at most 3 GR submodules up to isomorphism. If $\operatorname{top} Y$ is simple, then $\operatorname{top} F(Y)$ is simple and $\mu(F(Y)) = \{1, |F(Y)|\}$. Since up to isomorphism, $F(Y)$ has at most 3 GR submodules, we have $|F(Y)| \leq 4$ and thus $|Y| \leq 4$. Now it is easily seen that Y has at most 3 GR submodules up to isomorphism. □

4. Gabriel-Roiter factors over radical-square zero algebras

Let Λ be a radical-square zero algebra, $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}$ and F be the associated hereditary algebra and functor as before. Since F keeps the GR measures and GR inclusions except for the simple-top cases (this can be seen from the proof of Theorem A). Thus the problems concerning the GR measure for radical-square zero algebras can be reduced to those for hereditary ones. Thus when studying the almost split sequences ending at GR factor modules over a radical-square zero algebra Λ, one may start with the hereditary cases and then deals with the general cases using F.

4.1. **Hereditary cases.** The following lemma will paly an important role. For a proof we refer to [3].

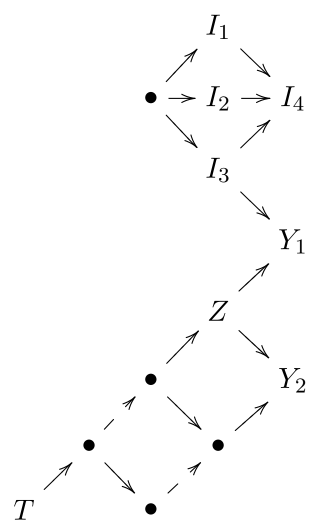
Lemma 4.1. *Let $\Lambda = k\mathbb{A}_n$ be a hereditary algebra and Z be a GR factor module. Then Z is a uniserial module.*

As a special case of Theorem 1.1, the following theorem can be obtained.

Theorem 4.2. *Let $\Lambda = kQ$ be a radical-square zero hereditary algebra with Q a Dynkin quiver or a tame quiver of type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$. Let $X \subset Y$ be a GR inclusion. If $Z = Y/X$ is not simple, then $\alpha(Z) = 1$.*

Proof. We have seen that in these cases, if Z is a GR factor module which is not injective, then $\alpha(Z) = 1$. Thus we need to show that if Z is an injective module such that $\alpha(Z) \geq 2$, then Z is simple. We show for D_n case, and the other cases are similar.

Assume for a contradiction that Z is injective and not simple such that $\alpha(Z) = 2$. Then there is an irreducible map $Z \to Y_1$. Since Λ is with radical square zero, i.e. Q is of sink-source orientation, thus there is another irreducible map $Z \to Y_2$ with $Y_1 \ncong Y_2$. Without loss of generality, we may assume that the full subquiver of the Auslander-Reiten quiver is of the following form where I_j is an injective module for each j and T is indecomposable such that $\alpha(T) = 1$:



Since Y_2 is not zero, then the composition of irreducible maps from T to Z is a monomorphism. It follows from Proposition 2.2(2) that there is a homomorphism from T to Y. Thus there is a path $T \to Y \to Z$ in the Auslander-Reiten quiver. In particular, it is easily seen that $\dim\mathrm{Hom}\,(Y, I_j) = 0$ for indecomposable injective module I_j. We may consider Y as a $k\mathbb{A}_m$-module where $\mathbb{A}_m$ is obtained by deleting the vertices of $\mathbb{D}_n$ corresponding to I_j for $1 \leq j \leq 4$. Thus Z is GR factor module over $k\mathbb{A}_m$. However, Z is clearly not a uniserial module since Y_1, Y_2 are different simple factor modules. This contradicts Lemma 4.1

The possibility of $\alpha(Z) = 3$ can be easily excluded. The proof is thus finished. □

Proposition 4.3. *Let $\Lambda = kQ$ be a radical-square zero hereditary algebra of type $\tilde{\mathbb{A}}_n$ (with $n \geq 1$ an odd number). The Gabriel-Roiter measures of indecomposable Λ-modules can be calculated as follows:*
(1) *Each indecomposable preprojective module has length $2m+1$ and GR measure $\{1, 3, 5, \cdots, 2m+1\}$.*
(2) *Each indecomposable exceptional regular module has length $2m$ and GR measure $\{1, 2, 4, 6, \cdots, 2m\}$. The GR measure of a homogeneous regular simple module H_1 is $\mu(H_1) = \{1, 3, 5, \cdots, n, n+1\}$. Thus each homogeneous module H_i of length $i(n+1)$ has GR measure $\{1, 3, 5, \cdots, n, n+1, 2(n+1), \cdots, i(n+1)\}$.*
(3) *The GR submodules of an indecomposable preinjective module are regular modules (exceptional regular modules if $n > 1$). Each indecomposable preinjective module has length $2m+1$ and GR measure $\{1, 2, 4, 6, \cdots, 2m, 2m+1\}$.*
(4) *The GR factor modules with decomposable middle term are exactly the injective simple modules.*

Proof. We outline the calculations, for details we refer to [5].
1. Since all irreducible maps between indecomposable preprojective modules are injective, the GR submodules of an indecomposable preprojective module are given by irreducible maps (Proposition 2.2(4)).
2. Every exceptional regular simple module X is of length 2. This can be seen from the statements for string algebras and Auslander-Reiten sequences involving string modules with indecomposable middle terms. Thus $\mu(X) = \{1, 2\} > \mu(Y)$ for all preprojective module Y. Thus the exceptional sequence of irreducible monomorphisms $X = X_1 \to X_2 \to \cdots \to X_j$ gives rise to a GR filtration for each X_j.
3. If H_1 is a homogeneous regular simple module, then the GR submodule of H_1 is preprojective. Each indecomposable preprojective module with length smaller than $|H_1| = n+1$ is isomorphic to a submodule of H_1, thus $\mu(H_1) = \{1, 3, 5, \cdots, n, n+1\}$. It is clear that $\mu(H_1) > \mu(X)$ for every indecomposable preprojective module X.
4. Every GR submodule of a non-simple preinjective is regular.
5. A GR factor module Z with decomposable middle term is preinjective. Then the corresponding GR inclusion $X \subset Y$ with $Y/X = Z$ satisfies that either X is preprojective and Y is regular, or X is regular and Y is preinjective. Then from the list of the GR measures, it is easily seen that Z is a simple injective module. □

4.2. **General cases.** In this section, we start to deal with the general cases. Let Λ be a radical-square zero algebra, $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{r} & 0 \\ \mathfrak{r} & \Lambda/\mathfrak{r} \end{pmatrix}$ and F be as before. We will see that the functor F sends GR factor Λ-modules to GR factor Γ-modules. Using this fact and the consideration in 4.1 of hereditary cases, one may obtain a proof of Theorem B.

Proposition 4.4. *Let Λ be a radical-square zero algebra and Γ, F be as before. Let $l : X \subset Y$ be a GR inclusion of Λ-modules. Then the Γ-module $F(Y/X)$ is also a GR factor module.*

Proof. First assume that $\operatorname{top} Y$ is simple. Let $P \xrightarrow{f} Y$ be the projective cover of Y and $\pi : Y \to Y/X$ be the canonical projection. Then P is indecomposable since $\operatorname{top} Y$ is simple and the composition πf is a projective cover of Y/X. Thus $\pi(\mathfrak{r}Y) = \pi f(\mathfrak{r}P) = \mathfrak{r}(Y/X)$, and $\bar{\pi}(Y/\mathfrak{r}Y) = \overline{\pi f}(P/rP) = (Y/X)/\mathfrak{r}(Y/X)$. If follows that the induced maps $\pi|, \bar{\pi}$ are both epimorphism. That is, $F(\pi) : F(Y) \to F(Y/X)$ is an epimorphism. The $\ker F(\pi)$ is thus a simple Γ-module since $|\ker F(\pi)| = |F(Y)| - F(Y/X)| = |Y| - |Y/X| = |X| = 1$. Since $\operatorname{top} F(Y)$ is also simple, the inclusion $\ker F(\pi) \subset F(Y)$ is a GR inclusion and $F(Y/X)$ is a GR factor module.

Now assume that $\operatorname{top} Y$ is not simple. Then X is not contained in the radical $\mathfrak{r}Y$ and not simple. From the proof of Theorem A, one obtains that $F(X)$ is a GR submodule of $F(Y)$, and $\operatorname{coker} F(l) = (\operatorname{coker}(\bar{l}), \operatorname{coker}(l|), s)$ is indecomposable and not projective. We obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{r}X & \to & X & \to & X/\mathfrak{r}X & \to & 0 \\
 & & \downarrow l & & \downarrow l & & \downarrow \bar{l} & & \\
0 & \to & \mathfrak{r}Y & \to & Y & \to & Y/\mathfrak{r}Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{coker}(l|) & \to & \operatorname{coker} l = Z & \to & \operatorname{coker} \bar{l} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that $\operatorname{coker} \bar{l}$ is not zero since otherwise, the composition of the canonical maps $X \xrightarrow{l} Y \to Y/\mathfrak{r}Y$ is an epimorphism, and this implies $X + \mathfrak{r}Y = Y$ which is impossible.

Moreover, $\operatorname{coker} \bar{l}$ is semisimple since $Y/\mathfrak{r}Y$ is. Thus $\operatorname{coker}(l|)$ contains rZ. It follows that $\operatorname{coker}(l|) = \mathfrak{r}Z$ and $\operatorname{coker} \bar{l} = Z/\mathfrak{r}Z$. Thus $F(Z) = (Z/\mathfrak{r}Z, \mathfrak{r}Z, s_Z) = (\operatorname{coker}(\bar{l}), \operatorname{coker}(l|), s)$ and $F(Y/X) \cong F(Y)/F(X)$ is a GR factor module. $\square$

After this preparation, we can formulate the following theorem:

Theorem B. *Let Λ be a radical-square zero algebra and Z be a GR factor module. Then*
(1) If Λ is of finite type and Z is not simple, then $\alpha(Z) = 1$.
(2) If Λ is of s-tame type, then all but finitely many GR factor modules have indecomposable middle terms. Moreover, if Λ is $\tilde{\mathbb{D}}_n$-free and Z is not simple. Then $\alpha(Z) = 1$.

Proof. (1) Let $0\to\tau Z\to X\to Z\to 0$ be an almost split sequence. Assume that τZ is simple, and $X = \oplus_{i=1}^m X_i$ is a decomposition of X into indecomposable summands. Then $|X_i| > |Z|$ holds since each irreducible map $X_i\to Z$ is surjective (Proposition 2.2). Then $1 = |\tau Z| = \sum_{i=1}^m |X_i| - |Z| \geq 1$ implies $m = 1$. If τZ is not simple, then Proposition 2.6 implies $0\to F(\tau Z)\to F(X)\to F(Z)\to 0$ is also an almost split sequence. Since $F(Z)$ is also a GR factor module (Proposition 4.4), from Theorem 4.2 and the fact that $F(Z)$ is not simple, one can conclude that the middle term $F(X)$ is indecomposable and therefore, X is indecomposable.

(2) We may use the same arguments as those in the proof of part (1) and Proposition 4.4. In addition, Proposition 4.3 should be used to deal with $\tilde{\mathbb{A}}_n$ case. □

Remark. An example that a non-simple GR factor module over $\Lambda = k\tilde{\mathbb{D}}_n$ with decomposable middle term is still missing. Thus, it is not clear if the condition $\tilde{\mathbb{D}}_n$-free can be omitted .

5. PREPROJECTIVE MODULES AND TAKE-OFF MODULES.

For a representation infinite artin algebra Λ, Ringel obtained in [8] a partition of $\operatorname{ind}\Lambda$, namely, take-off part, central part and landing part. On the other hand, there is already a preprojective partition (preinjective partition) obtained by Auslander and Smalø [2]. It is natural to compare these two different kinds of partitions. In [8], Ringel showed that all landing modules are preinjective modules.

It is interesting to study the dual case, that is, for a representation infinite algebra, if all indecomposable preprojective modules are take-off modules. However, it is not true in general. This leads to the following definition.

Definition 5.1. *A representation infinite algebra is called a take-off algebra if all indecomposable projective modules are take-off modules and is called a strong take-off algebra if all indecomposable preprojective modules are take-off modules.*

In [6], we have seen that all indecomposable preprojective module over tame hereditary algebras kQ are take-off modules, that is, algebras kQ with Q tame quivers are strong take-off algebras. We want to give some more examples of take-off algebras, namely, radical-square zero algebras of s-tame types.

Let Λ be a radical-square zero algebra, and Γ, F be as before. Then it is easily to get the following lemma since F does not change GR measures.

Lemma 5.2. *Let Λ be a radical-square zero algebra and Γ, F be as before. Let X be an indecomposable Λ-module. Then X is a take-off (resp. central, landing) Λ-module if and only if $F(X)$ is a take-off (resp. central, landing) Γ-module.*

Theorem C. *Let Λ be a radical-square zero algebra and Γ, F be as before. Assume that $Q(\Gamma)$ is a connected tame quiver. Then Λ is a strong take-off algebra.*

Proof. Let X be an indecomposable preprojective Λ-module. If $F(X)$ is a preprojective Γ-module, then $F(X)$ is a take-off module by Theorem 1.2. Therefore, by the above lemma, X is a take-off module.

Thus to finish the proof, it is sufficient to show that the functor F sends an indecomposable preprojective Λ-module to a preprojective Γ-module. For this we need some more considerations.

The following proposition in [7] gives, in some sense, a characterization of preprojective modules over radical-square zero algebras.

Proposition 5.3. *Let Σ be an algebra which is stably equivalent to a hereditary algebra and M be an indecomposable non-projective Σ-module. Then the following are equivalent:*
(1) *There is a projective module P and a chain of irreducible maps of indecomposable Σ-modules $P = C_0 \to C_1 \to \cdots \to C_n = M$.*
(2) *There are only a finite number of non-isomorphic indecomposable modules X such that $\underline{\mathrm{Hom}}(X, M) \neq 0$.*

Corollary 5.4. *Let Λ be a radical-square zero algebra and Γ, F be as before. Then the following are equivalent:*
(1) *There is a projective module P and a chain of irreducible maps of indecomposable Λ-modules $P = C_0 \to C_1 \to \cdots \to C_n = M$.*
(2) *There are only a finite number of non-isomorphic indecomposable Λ-modules X such that there exists $f : X \to M$ with $F(f) \neq 0$.*

Proof. Let $\mathcal{P}(X, M)$ be the subset of $\mathrm{Hom}\,(X, M)$ consisting of morphisms which factor through projective Λ-modules. Then $\mathcal{P}(X, M) = \mathrm{Hom}\,(X, rM)$ (see [1], section X.2) is the set of maps which are sent to zero by F. Thus $\underline{\mathrm{Hom}}\,(X, M) \neq 0$ if and only if there is a nonzero map $f \in \mathrm{Hom}\,(X, M)$ such that $F(f) \neq 0$. $\square$

Now we can finish the proof of Theorem C by showing the following proposition.

Proposition 5.5. *Let Λ be a radical-square zero algebra and Γ, F be as before. Let X be an indecomposable preprojective Λ-module. Then $F(X)$ is a preprojective Γ-module.*

Proof. We denote by $\mathcal{P}^0_\Lambda, \mathcal{P}^1_\Lambda, \cdots, \mathcal{P}^m_\Lambda, \cdots$ the preprojective partition of $\mathrm{ind}\,\Lambda$. In Proposition 2.1, we have the following characterization:

If an indecomposable Λ-module X is contained in $\mathcal{P}^m_\Lambda$, then there exists an indecomposable module $Y \in \mathcal{P}^i_\Lambda$ with $i < m$ and an irreducible map $Y \to X$.

Assume that $Y \in \mathcal{P}^m_\Lambda$ is an indecomposable preprojective Λ-module. We use induction on m. If $m = 0$, then Y is projective and thus $F(Y)$ is projective. Assume that for all $i < m$ and all $X \in \mathcal{P}^i_\Lambda$, $F(X)$ is a preprojective Γ-module. Then $Y \in \mathcal{P}^m_\Lambda$ implies that there is an indecomposable Λ-module $X \in \mathcal{P}^j_\Lambda$ with $j < m$ and an irreducible map $X \xrightarrow{f} Y$. By induction, $F(X)$ is a preprojective Γ-module. If $j \neq 0$, i.e. X is not projective, then $F(f)$ is an irreducible map, and thus $F(Y)$ is preprojective since Γ is hereditary.

If $j = 0$, then X and $F(X)$ are projective. Assume for a contradiction that $F(Y)$ is not preprojective. First note that $F(f) \neq 0$. Namely, let $g : Q \to Y$ be a projective cover. Then the irreducible map $f : X \to Y$ factors through g since g is an epimorphism, say $f = gh$ for some $h : X \to Q$. However, f is irreducible and Y is not projective imply that h is a split monomorphism and thus $Q = X \oplus Q' \xrightarrow{g=(f,g')} Y$ is a projective cover. In particular the induced map $\bar{f} : X/\mathfrak{r}X \to Y/\mathfrak{r}Y$ is a monomorphism which implies $F(f)$ is not zero. Then using the facts that $F(X)$ is projective, $F(Y)$ is not preprojective and the property of left minimal almost split morphisms, we obtain an infinite sequence of indecomposable Γ-modules X'_i with $F(X) = X'_0$ and non-zero homomorphisms $X'_i \xrightarrow{g_i} F(Y)$. Since F is full and X'_i is not simple projective Γ-module for any i, we obtain a sequence of Λ-modules X_i with $X_0 = X$ and morphisms $f_i : X_i \to Y$ such that $F(X_i) = X'_i$, $F(f_i) = g_i$. In particular, we obtain infinitely many non-isomorphic indecomposable Λ-modules X_i with non-zero maps f_i such that $F(f_i) = g_i \neq 0$. This contradicts Corollary 5.4 since there is an irreducible map $X \to Y$ with X projective. Thus $F(Y)$ is preprojective. $\square$

References

[1] M.Auslander; I.Reiten; S.O.Smalø, Representation Theory of Artin Algebras. Cambridge studies in advanced mathematics **36** (Cambridge University Press, Cambridge, 1995).

[2] M.Auslander; S.O.Smalø, Preprojective modules over Artin algebras. J. Algebra **66** (1980), 61–122.

[3] B.Chen, he Gabriel-Roiter measure for representation-finite hereidtary algebras. *J.Algebra.* **309**(2007), 292–317.

[4] B.Chen, The Auslander-Reiten sequences ending at Gabriel-Roiter factor modules over tame hereditary algebras. to appear in J.Algebra Appl.

[5] B.Chen, The Gabriel-Roiter measures for $\tilde{\mathbb{A}}_n$. submitted.

[6] B.Chen, Comparison of Auslander-Reiten theory and Gabriel-Roiter measure approach to the module categories of tame hereditary algebras. to appear in Comm. Alg.

[7] M.I.Platzeck, Representation thoery of algebras stably equivalent to an hereditary Artin algebra. Trans. Amer. Math. Soc. **238**(1978), 89–128.

[8] C.M.Ringel, The Gabriel-Roiter measure. *Bull. Sci. math.* **129**(2005),726-748.

[9] C.M.Ringel, Foundation of the representation theory of Artin algebras, Using the Gabriel-Roiter measure. Proceedings of the 36th Symposium on Ring Theory and Representation Theory. **2**, 1–19, Symp. Ring Theory Represent Theory Organ. Comm., Yamanashi, 2004.

[10] C.M.Ringel, The theorem of Bo Chen and Hall polynomials. *Nagoya Math. J.* **183**(2006),143–160.

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