Donaldson-Thomas invariants

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1 Stability conditions

Here I'll introduce a refined version of Bridgeland's stability condition on a triangulated category (see [Br]). It can be called a *compact non-commutative algebraic variety endowed with a polarization*. Here is the data:

- a triangulated $\mathbf{k}$-linear category $\mathcal{C}$ where $\mathbf{k}$ is a base field,
- a homomorphism $K_0(\mathcal{C}) \to \Lambda$ where $\Lambda \simeq \mathbb{Z}^r$ is a free abelian group of finite rank,
- an additive map $Z : \Lambda \to \mathbb{C}$,
- a collection C^{ss} of (isomorphism classes of) non-zero objects in $\mathcal{C}$ called the semistable ones, such that $Z(\mathcal{E}) \neq 0$ for any $\mathcal{E} \in \mathcal{C}^{ss}$,
- a choice $\log Z(\mathcal{E}) \in \mathbb{C}$ of the logarithm of $Z(\mathcal{E})\ \forall\, \mathcal{E} \in \mathcal{C}^{ss}$.

Also we assume that it makes sense to speak about families of objects of $\mathcal{C}$ paramertized by a scheme over $\mathbf{k}$. A typical example of such category is $D^b(CohX)$, the bounded derived category of the category of coherent sheaves on a smooth compact algebraic variety $X/\mathbf{k}$. Lattice Λ can be thought as the image of $K_0(\mathcal{C})$ in $H^*(X)$ under the map given by the Chern character.

More generally, for a non-necessary compact smooth variety X endowed with a closed compact subset $X_0 \subset X$ the corresponding category consists of complexes of sheaves with cohomology supported at X_0. Another example is the homotopy category of finite complexes of free A-modules with finite-dimensional cohomology where A is a finitely generated associative algebra of finite cohomological dimension.

For $\mathcal{E} \in \mathcal{C}^{ss}$ we denote by $Arg(\mathcal{E}) \in \mathbb{R}$ the imaginary part of $\log Z(\mathcal{E})$.

The above data should satisfy the following axioms:

- $\forall\, \mathcal{E} \in \mathcal{C}^{ss}$ and $\forall\, n \in \mathbb{Z}$ we have $\mathcal{E}[n] \in \mathcal{C}^{ss}$ and $\log Z(\mathcal{E}[n]) = \log Z(\mathcal{E}) + \pi i n$,
- $\forall\, \mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}^{ss}$ with $Arg(\mathcal{E}_1) > Arg(\mathcal{E}_2)$ we have $\mathrm{Hom}\,(\mathcal{E}_1, \mathcal{E}_2) = 0$,

- for any object $\mathcal{E} \in \mathcal{C}$ there exists $n \geq 0$ and a chain of morphisms $0 = \mathcal{E}_0 \to \mathcal{E}_1 \to \dots \to \mathcal{E}_n = \mathcal{E}$ (an analog of a filtration) such that the corresponding "quotients" $F_i := Cone(\mathcal{E}_i \to \mathcal{E}_{i+1})$ are semistable and $Arg(\mathcal{F}_0) > Arg(\mathcal{F}_1) > \dots > Arg(\mathcal{F}_{n-1})$,
- $\forall \lambda \in \Lambda \in \{0\}$ the "moduli stack" $\mathcal{M}^{ss}_\lambda$ of semistable objects in class λ is an Artin stack of finite type,
- pick a norm $\| \cdot \|$ on $\Lambda \otimes \mathbb{R}$, then $\exists\, C > 0$ such that $\forall \mathcal{E} \in \mathcal{C}^{ss}$ one has $|Z(\mathcal{E})| > C \parallel \mathcal{E} \parallel$.

The last condition implies that the set $\{Z(\mathcal{E}) \in \mathbb{C} \,|\, \mathcal{E} \in \mathcal{C}^{ss}\}$ is a discrete subset of $\mathbb{C}$ with at most polynomially growing density at infinity. Also it implies that the stability condition is locally finite in the sense of Bridgeland.

Any stability condition gives a bounded t-structure on $\mathcal{C}$ with the corresponding heart consisting of semistable objects $\mathcal{E}$ with $Arg(\mathcal{E}) \in (0, \pi]$, and their extensions. The case of classical Mumford stability with respect to an ample line bundle is not an example of Bridgeland stability, it is rather a limiting degenerate case of it.

For given $\mathcal{C}$ and Λ denote by $Stab(\mathcal{C})$ the set of stability conditions $(Z, \mathcal{C}^{ss}, (\log Z(\mathcal{E}))_{\mathcal{E} \in \mathcal{C}^{ss}})$ (we skip Λ from the notation). This space can be endowed with certain non-trivial Hausdorff topology.

Theorem 1 *(Bridgeland) The forgetting map $Stab(\mathcal{C}) \to \mathbb{C}^r \simeq \mathrm{Hom}(\Lambda, \mathbb{C})$, $(Z, \mathcal{C}^{ss}, \dots) \mapsto Z$, is a local homeomorphism.*

Hence, $Stab(\mathcal{C})$ is a complex manifold, not necessarily connected. Under our assumptions one can show also that the group $Aut(\mathcal{C})$ acts properly discontinuously on $Stab(\mathcal{C})$. On the quotient orbifold $Stab(\mathcal{C})/Aut(\mathcal{C})$ there is a natural non-holomorphic action of $GL_+(2, \mathbb{R})$ arising from linear transformations of $\mathbb{R}^2 \simeq \mathbb{C}$ preserving the standard orientation. A similar geometric structure appears on the moduli spaces of holomorphic Abelian differentials, see e.g. [Z] for a recent review.

2 Donaldson-Thomas invariants

Let us assume that $\mathcal{C}$ is a CY3 category. i.e. it is endowed with a functorial pairing $\mathrm{Hom}(\mathcal{E}, \mathcal{F})^* \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{E}[3])$. For example, $\mathcal{C} = D^b(Coh(X))$ where X is a smooth compact 3-dimensional variety with trivialized canonical bundle. Deformation theory of any object $\mathcal{E} \in \mathcal{C}$ is governed by certain homotopy Lie algebra whose cohomology is $\oplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{E}, \mathcal{E}[n])$. If $\mathcal{E}$ is semistable then the amplitude of this algebra is in $[0, 1, 2, 3]$. In the case $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = k \cdot id_{\mathcal{E}}$ (such object is called a *Schur object*), one can modify the deformation complex of $\mathcal{E}$ and get a new one with the amplitude in $[1, 2]$, which lead to the possibility to define a virtual fundamental class. In the case when the moduli stack $\mathcal{M}^{ss}_\lambda$ is compact Hausdorff and consists of Schur objects, the virtual dimension is zero, and the class is just an integer $DT(\lambda)$. It is the virtual number of points in $\mathcal{M}^{ss}_\lambda$ and is called the Donaldson-Thomas invariant (see [DT]). For $\mathcal{C} = D^b(Coh(X))$

this happens when $\lambda = (1, 0, ?, ?) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X)$ and the stability condition is close to the Mumford stability associated with some polarization of X. Semistable objects under the consideration are torsion-free sheaves which are in fact the ideal sheaves of subschemes $S \subset X$ with $\dim S \leq 1$ (fat points and curves), and the moduli stack is the corresponding Hilbert scheme. In [MNOP] a remarkable conjecture was made relating the Donaldson-Thomas invariants $DT(1, 0, ?, ?)$ depending on degrees in $H^4(X) = H_2(X), H^6(X) = \mathbb{Z}$ with the Gromov-Witten invariants of X counting curves of all degrees and genera in X. In particular, in the case of fat points one get an identity (see [BF] for the proof):

$$\sum_{n \geq 0} DT(1, 0, 0, -n)\, q^n = [(1+q)(1-q^2)^2(1+q^3)^3 \ldots]^{-\chi(X)}, \qquad (1)$$

$$\Longrightarrow DT(1, 0, 0, -1) = -\chi, \;\; DT(1, 0, 0, -2) = \frac{\chi^2 + 5\chi}{2}, \ldots, \;\; \chi = \chi(X)$$

The work of K. Behrend (see [Be]) gives a way to define Donaldson-Thomas invariants for not necessarily compact loci in stacks $\mathcal{M}^{ss}_\lambda$ consisting of Schur objects. At the moment it is not clear how to extend Behrend's definition to non-Schur objects. Nevertheless, we hope that one can do it and define for a CY3 category $\mathcal{C}$ and a stability condition $(Z, \mathcal{C}^{ss}, \dots)$ where Z is "generic" (i.e. $\forall\, \lambda_1, \lambda_2 \in \Lambda$ with $\mathbb{R} \cdot Z(\lambda_1) = \mathbb{R} \cdot Z(\lambda_2) \subset \mathbb{C}$ one has $\mathbb{Q} \cdot \lambda_1 = \mathbb{Q} \cdot \lambda_2$), certain even function $DT : \Lambda \setminus \{0\} \to \mathbb{Z}$. It should be supported on classes of *indecomposable* semistable objects. Moreover, function DT should change "nicely" if we move Z in $\mathbb{C}^r$. D. Joyce proposed in [J] a hypothetical complicated rule describing the behaviour of function DT for abelian categories, i.e. in the special case when the t-structure does not change and $\mathcal{C}$ is the derived category of its heart. In the following section I'll describe a different proposal (by Y. Soibelman and myself), in the general triangulated case, which is (presumably) compatible with Joyce's.

3 New wall-crossing formula

Assume that Λ is endowed with a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle : \Lambda \otimes \Lambda \to \mathbb{Z}$ such that $\forall\, \mathcal{E}, \mathcal{F} \in \mathcal{C}$

$$\langle \mathcal{E}, \mathcal{F} \rangle = \sum_{n \in \mathbb{Z}} (-1)^n rk \; \mathrm{Hom}(\mathcal{E}, \mathcal{F}[n])$$

Consider the Lie algebra over $\mathbb{Q}$ with basis $(e_\lambda)_{\lambda \in \Lambda}$ and the commutator given by the formula

$$[e_{\lambda_1}, e_{\lambda_2}] = (-1)^{\langle \lambda_1, \lambda_2 \rangle} \langle \lambda_1, \lambda_2 \rangle \; e_{\lambda_1 + \lambda_2}$$

This Lie algebra is isomorphic (non-canonically) to the algebra of Laurent polynomials on the algebraic torus $\mathrm{Hom}(\Lambda, \mathbb{G}_m)$, endowed with a translation-invariant Poisson bracket associated with the form $\langle \cdot, \cdot \rangle$.

Let $Z : \Lambda \to \mathbb{C}$ be an additive map, generic in the sense introduced above, and let $DT : \Lambda \to \mathbb{Z}$ be an even map supported on the set of $\lambda \in \Lambda$ such that

$|Z(\lambda)| > C \parallel \lambda \parallel$ for some constant $C > 0$ (here $\parallel \cdot \parallel$ is a norm on $\Lambda \otimes \mathbb{R}$). We associate with any angle $V \subset \mathbb{C}$ with center at zero (V is strictly less than $180°$) a group element given by an infinite product

$$A(V) := \overrightarrow{\prod_{\lambda \in Z^{-1}(V)}} \exp\left(DT(\lambda) \sum_{n=1}^{\infty} \frac{e_{n\lambda}}{n^2}\right) \qquad (2)$$

The product takes value in certain pro-nilpotent Lie group G_V. We will describe its Lie algebra. Let us consider the convex cone $U = U(V)$ in $\Lambda \otimes \mathbb{R}$ which is the convex hull of the set of points $v \in Z^{-1}(V)$ such that $|Z(v)| > C \parallel v \parallel$. The Lie algebra $Lie(G_V)$ is defined to be the infinite product $\prod_{\lambda \in \Lambda \cap U} \mathbb{Q} \cdot e_\lambda$.

The right arrow in the upperscript in (2) means that the product is taken in the *clockwise* order on the set of directions of rays $\mathbb{R}_+ \cdot Z(\lambda) \subset V \subset \mathbb{C}$.

Now we are able to formulate a rule describing the modification of function DT as we move additive map Z continuously. First of all, for any given $\lambda \in \Lambda$ the value $DT(\lambda)$ should jump only on a locally-finite collection of walls in $\mathbb{C}^r = \mathrm{Hom}(\Lambda, \mathbb{C})$. Let $(Z_t)_{t \in [0,1]}$ be a generic piece-wise smooth path in the complex vector space $\mathbb{C}^r$. For a countable set of values of t the map Z_t will be not generic. Our rule says (roughly) that $A(V)$ stays the same as long as no lattice point $\lambda \in \Lambda$ with $DT_t(\lambda) \neq 0$ crosses the boundary of $Z_t^{-1}(V)$. Of course such bad crossings could happen at infinitely many values of t, but for infinitesimally small intervals in the parameter space $[0, 1]$ (in the sense of non-standard analysis) we can avoid such crossings.

One can check that this rule is equivalent to the following. Consider the value t_0 in the parameter space for which the map Z_{t_0} is not generic. In this case we have either a non-zero vector $\lambda \in \Lambda \setminus \{0\}$ with $Z_{t_0}(\lambda) = 0$, or a rank two lattice $\Lambda' \simeq \mathbb{Z}^2$, $\Lambda' \subset \Lambda$ such that its image $Z_{t_0}(\Lambda')$ is contained in a real line $\mathbb{R} \cdot e^{i\alpha} \subset \mathbb{C}$. In the first case all the values of DT will not jump, in the second case only the values $DT(\lambda)$ for $\lambda \notin \Lambda'$ will not jump. The rule describing the change of values $DT(\lambda)$ for $\lambda \in \Lambda'$ is purely two-dimensional. We will describe it now.

Denote by $k \in \mathbb{Z}$ the value of the form $\langle \cdot, \cdot \rangle$ on the basis of $\Lambda' \simeq \mathbb{Z}^2$. We assume that $k \neq 0$, otherwise there will be no jump in values of DT on Λ'. The group elements which we are interested in can be identified with products of the following formal symplectomorphisms (automorphisms of $\mathbb{Q}[[x, y]]$ preserving the symplectic form $(xy)^{-1} dx \wedge dy$):

$$T_{a,b} : (x, y) \mapsto \left(x \cdot (1 - (-1)^{ab} x^a y^b)^b, y \cdot (1 - (-1)^{ab} x^a y^b)^{-a}\right), a, b \geq 0, a+b \geq 1$$

Any exact symplectomorphism ϕ of $\mathbb{Q}[[x, y]]$ can be decomposed uniquely into the clockwise and an anticlockwise product:

$$\phi = \overrightarrow{\prod_{a,b}} T_{a,b}^{k c_{a,b}} = \overleftarrow{\prod_{a,b}} T_{a,b}^{k \widetilde{c}_{a,b}}$$

with certain exponents $c_{a,b}, \widetilde{c}_{a,b} \in \mathbb{Q}$. These exponents should be interpreted as DT invariants. The passage from the clockwise order (when the slope $a/b \in$

$[0,+\infty]\cap\mathbb{P}^1(\mathbb{Q})$ decreases) to the anticlockwise order (the slope inreases) gives the change of $DT_{|\Lambda'}$ as we cross the wall. The *integrality* of DT invariants is not obvious, it follows from

Conjecture 1 *If one decomposes the product $T_{1,0}^k\cdot T_{0,1}^k$ in the opposite order:*

$$T_{1,0}^k\cdot T_{0,1}^k=\prod_{a/b\ increases}T_{a,b}^{kd(a,b,k)} \tag{3}$$

then $d(a,b,k)\in\mathbb{Z}\quad\forall a,b,k$.

Here are decompositions for $k=1,2$:

$$T_{1,0}\cdot T_{0,1}=T_{0,1}\cdot T_{1,1}\cdot T_{1,0} \tag{4}$$

$$T_{1,0}^2\cdot T_{0,1}^2=T_{0,1}^2\cdot T_{1,2}^2\cdot T_{2,3}^2\cdot\dots\cdot T_{1,1}^4\cdot T_{2,2}^2\cdot\dots\cdot T_{3,2}^2\cdot T_{2,1}^2\cdot T_{1,0}^2 \tag{5}$$

For $k\geq 3$ or $k\leq -1$ the decomposition is not yet known completely. Computer experiments give a conjectural formula for the diagonal term with slope $a/b=1$. The corresponding symplectomorphism is given by the map

$$(x,y)\mapsto(x\cdot F_k(xy)^k,y\cdot F_k(xy)^{-k})$$

where the series $F_k=F_k(t)\in\mathbb{Z}[[t]]$ is an algebraic hypergeometric series given for $k\geq 3$ by formulas

$$\sum_{n=0}^{\infty}\binom{(k-1)^2n+k-1}{n}\frac{t^n}{(k-2)n+1}=\exp\left(\sum_{n=1}^{\infty}\binom{(k-1)^2n}{n}\frac{k}{(k-1)^2}\frac{t^n}{n}\right)$$

The example (4) with $k=1$ is compatible with the expected behavior of DT invariants when we have two spherical objects $\mathcal{E}_1,\mathcal{E}_2\in\mathcal{C}$ (sphericity means that $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_i[n])=H^*(S^3)$) such that there exists only one non-trivial extension

$$\mathrm{Hom}(E_1,E_2[1])=\mathbf{k},\ \ \mathrm{Hom}(E_1,E_2[n])=0\text{ for }n\neq 1$$

In this case on one side of the wall we have two semistable objects $\mathcal{E}_1,\mathcal{E}_2$, on the other side we have three semistable objects $\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_{1+2}$ where $\mathcal{E}_{1+2}$ is the extension of $\mathcal{E}_2$ by $\mathcal{E}_1$.

Let X be a 3-dimensional Calabi-Yau manifold, consider the whole subcategory of $D^b(CohX)$ generated by $\mathcal{O}_X$ and the ideal sheaves J_x of all closed points $x\in X$. Some of the putative DT invariants for certain stability condition on this category are exactly the original DT invariants for fat points, see (1). One can check that after crossing a wall one obtains a much simpler function, with the only non-trivial values on $\Lambda\simeq\mathbb{Z}\oplus\mathbb{Z}$ given by

$$DT(1,0)=DT(-1,0)=1,\quad DT(0,n)=-\chi(X),\ \forall\, n\neq 0$$

The value $DT(1,0)=1$ corresponds to the isolated spherical object $\mathcal{O}_X$, values $DT(0,n)=-\chi(X)$ correspond to the "counting" of indecomposable torsion sheaves on X.

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