Quantum Integrable Systems and Special Functions.I

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Abstract

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] as a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972]. For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2} q$, the wave functions of such systems are related to polynomials in l variables (l is a rank of root system) and they are a generalization of Gegenbauer polynomials and Jack polynomials [Ja 1970]. In [Pe 1998a], it was proved that the series for the product of two such polynomials is a κ-deformation of the Clebsch–Gordan series. This yields recurrence relations for these polynomials, in particular, for generalized zonal polynomials on symmetric spaces.

The present paper follows papers [Pe 1998a], [PRZ 1998]. In last of them, the recurrence relations were used to compute the explicit expressions for A_2 type polynomials, i.e., for the wave functions of the three-body Calogero–Sutherland system.

As it was shown by Ragoucy, Zaugg and the author of this paper (see Appendix 2), the similar results are also valid in A_2 case for the more general two-parameter deformation ((q, t)-deformation) introduced by Macdonald [Ma 1988].

1 Introduction

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] (see also [OP 1978] and [OP 1983]) as a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972]. Such systems depend on one real parameter κ (for root systems of the type A_n, D_n and E_6, E_7, E_8), on two parameters (for the type B_n, C_n, F_4 and G_2) and on three parameters for the type BC_n. These parameters are related to the coupling constants of the quantum system.

For the potential $v(q) = \kappa(\kappa-1)\ \sin^{-2} q$ and special values of parameter κ, the wave functions correspond to the characters of the compact simple Lie groups ($\kappa = 1$) [We 1925/26] or to zonal spherical functions on symmetric spaces ($\kappa = 1/2, 2, 4$) [Ha 1958], [He 1978]. At arbitrary values of κ, they provide an interpolation between these objects.

This class has many remarkable properties. Here we mention only one: the wave functions of such systems are a natural generalization of special functions (hypergeometric functions) to the case of several variables. The history of this problem and some results may be found in [OP 1983]. In [Pe 1998a], it was shown that the product of two wave functions is a finite linear combination of analogous functions, namely, of functions that appear in the corresponding

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Clebsch–Gordan series. In other words, this deformation (κ-deformation) does not change the Clebsch–Gordan series. For rank 1, we obtain the well-known cases of the Legendre, Gegenbauer and Jacobi polynomials, and the limiting cases of the Laguerre and Hermite polynomials (see for example [Vi 1968]). Some other cases were also considered in [He 1955], [Ja 1970], [Ko 1974], [Ja 1975], [Vr 1976], [Se 1977], [KS 1978], [Ma 1982], [Pr 1984], [Vr 1984], [HO 1987], [He 1987], [Op 1988a], [Op 1988b], [Ma 1988], [La 1989], [Op 1989], [St 1989], [KS 1995] and [CP 1997]. In [PRZ 1998], κ-deformed Clebsch–Gordan series was used in order to obtain the explicit expressions for the generalized Gegenbauer polynomials [2] of type A_2 what gives the explicit solution of the three-body Calogero–Sutherland model. For special values of $\kappa = 1/2, 2, 4$, these formulae give the explicit expressions for zonal polynomials of type A_2.

In Appendix 2, are presented analogous results obtained by E. Ragoucy, Ph. Zaugg and the author for two-parameter family of polynomials of type A_2 introduced by Ruijsenaars [Ru 1987] and Macdonald [Ma 1988].

2 General description

The systems under consideration are described by the Hamiltonian (for more details, see [OP 1983])

$$H = \frac{1}{2} p^2 + U(q), \qquad p^2 = (p, p) = \sum_{j=1}^{l} p_j^2, \tag{2.1}$$

where $p = (p_1, ..., p_l)$, $p_j = -i\,\partial/\partial q_j$, is a momentum operator, and $q = (q_1, ..., q_l)$ is a coordinate vector in the l-dimensional vector space $V \sim \mathbb{R}^l$ with standard scalar product $(\cdot\,,\,\cdot)$. The potential $U(q)$ is constructed by means of a certain system of vectors $R^+ = \{\alpha\}$ in V (the so-called *root systems*):

$$U = \sum_{\alpha \in R^+} g_\alpha^2\, v(q_\alpha), \qquad q_\alpha = (\alpha, q), \qquad g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1), \qquad g_\alpha = g_\beta, \quad \text{if } (\alpha, \alpha) = (\beta, \beta). \tag{2.2}$$

Such systems are completely integrable for potentials of five types (see [OP 1983] for A_l; [HO 1987], [He 1987], [Op 1988a], [Op 1988b] and [Op 1989] for a general case). They are a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972] for which $\{\alpha\} = \{e_i - e_j\}$, $\{e_j\}$ being a standard basis in V.

In this paper, we consider in details only the case of A_2 with potential $v(q) = \sin^{-2} q$.

3 Root systems

We give here only basic definitions. For more details, see [Hu 1972], [Ma 1988] and [OV 1990].

[2] In some papers, the name Jack polynomials [Ja 1970] is used. However, the Jack polynomials are a very special case of polynomials under consideration. So, we prefer to use the name generalized Gegenbauer polynomials for the general case and Jack polynomials for the special case.

Let V be a l-dimensional real vector space with a standard scalar product $(\cdot\,,\,\cdot)$, $(\alpha,\beta) = \sum \alpha_j\beta_j$, and let s_α be a reflection in the hyperplane through origin orthogonal to the vector α,

$$s_\alpha\, q = q - (q, \alpha^\vee)\, \alpha, \qquad \alpha^\vee = \frac{2}{(\alpha,\alpha)}\, \alpha\,. \tag{3.1}$$

Consider a finite set of non-zero vectors $R = \{\alpha\}$ generating V and satisfying the following conditions:

1. For any $\alpha \in R$, the reflection s_α conserves $R : s_\alpha R = R$.

2. For all $\alpha, \beta \in R$, we have $(\alpha^\vee, \beta) \in \mathbb{Z}$.

The set $\{s_\alpha\}$ generates the finite group $W(R)$ which is called the Weyl group of R. The root system R is called *reduced* one if only $\pm\alpha$ are vectors being collinear to α in R.

Let us choose the hyperplane which does not contain any root. This hyperplane divide the root system on two subsets, $R = R^+ \bigcup R^-$, where R^+ is called the set of positive roots. In R^+, there is the basis $\{\alpha_1, \ldots, \alpha_l\}$ such that $\alpha = \sum_j n_j\alpha_j$, $n_j \geq 0$, for any $\alpha \in R^+$. It is called the set of simple roots. The root system R is called *irreducible* if it cannot decompose onto two non-empty subsets R_1 and R_2 being orthogonal to each other.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots, let R^+ be the set of positive roots, and $\{\lambda_j\}$ be a dual basis or the basis of fundamental weights: $(\lambda_j, \alpha_k) = \delta_{jk}$.

Let Q be the root lattice, and Q^+ be the cone of positive roots

$$Q = \left\{ \beta\colon \beta = \sum_{j=1}^{l} m_j\alpha_j, \quad m_j \in \mathbb{Z} \right\}; \qquad Q^+ = \left\{ \gamma\colon \gamma = \sum_{j=1}^{l} n_j\alpha_j, \quad n_j \in \mathbb{N} \right\}. \tag{3.2}$$

Let P be a weight lattice, and P^+ be a cone of dominant weights:

$$P = \left\{ \lambda\colon \lambda = \sum_{j=1}^{l} m_j\lambda_j, \quad m_j \in \mathbb{Z} \right\}; \qquad P^+ = \left\{ \mu\colon \mu = \sum_{j=1}^{l} n_j\lambda_j, \quad n_j \in \mathbb{N} \right\}. \tag{3.3}$$

Following [Vr 1976] and [Ma 1988], we define a partial order on P as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$, or $(\lambda,\, \lambda_j) \geq (\mu,\, \lambda_j)$ for all $j = 1, \ldots, l$. The set of linear combinations over $\mathbb{R}$ of functions $f_\lambda(q) = \exp\{2i(\lambda, q)\}$, $\lambda \in P$, $q \in V$, may be considered as the group algebra A over $\mathbb{R}$ of the free Abelian group P. For any $\lambda \in P$, we denote the corresponding element of A as $e^\lambda \sim f_\lambda(q)$. So, $e^\lambda\, e^\mu = e^{\lambda+\mu}$, $(e^\lambda)^{-1} = e^{-\lambda}$, and the identity element of A as $e^0 = 1$. Then e^λ, $\lambda \in P$, form an $\mathbb{R}$-basis of A.

The Weyl group $W(R)$ acts on P and hence also on $A : s(e^\lambda) = e^{s\lambda}$ for $s \in W$ and $\lambda \in P$. Let A^W denotes the subalgebra of W-invariant elements of A. It is evident that each W-orbit in P contains only one point in P^+. Consequently, the monomial symmetric functions

$$m_\lambda = \sum_{\mu \in W\cdot\lambda} e^\mu, \qquad \lambda \in P^+ \tag{3.4}$$

form the $\mathbb{R}$-basis of A^W.

4 The Clebsch–Gordan series

Let us recall the main results of [Pe 1998a] and specialize them to the A_2 case with potential $v(q) = \sin^{-2} q$.

The Schrödinger equation for this quantum system has the form

$$H\,\Psi^\kappa = E(\kappa)\,\Psi^\kappa; \qquad H = -\,\Delta_2 + U(q_1, q_2, q_3), \qquad \Delta_2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j^2} \tag{4.1}$$

with potential

$$U(q_1, q_2, q_3) = \kappa(\kappa - 1)\left(\sin^{-2}(q_1 - q_2) + \sin^{-2}(q_2 - q_3) + \sin^{-2}(q_3 - q_1)\right). \tag{4.2}$$

The ground state wave function and its energy are

$$\Psi_0^\kappa(q) = \left(\prod_{j<k}^{3} \sin(q_j - q_k)\right)^\kappa, \qquad E_0(\kappa) = 8\,\kappa^2. \tag{4.3}$$

Substituting $\Psi_\lambda^\kappa = \Phi_\lambda^\kappa\,\Psi_0^\kappa$ in (4.1), we obtain

$$-\Delta^\kappa\,\Phi_\lambda^\kappa = \varepsilon_\lambda(\kappa)\,\Phi_\lambda^\kappa, \qquad \Delta^\kappa = \Delta_2 + \Delta_1^\kappa, \qquad \varepsilon_\lambda(\kappa) = E_\lambda(\kappa) - E_0(\kappa). \tag{4.4}$$

Here the operator Δ_1^κ takes the form

$$\Delta_1^\kappa = \kappa \sum_{j<k}^{3} \cot(q_j - q_k)\left(\frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_k}\right). \tag{4.5}$$

It is easy to see that the set of symmetric polynomials in variables $\exp(2iq_j)$ is invariant under the action of Δ^κ. Such polynomial m_λ is labelled by the $SU(3)$ highest weight $\lambda = m\lambda_1 + n\lambda_2$, with m, n being non-negative integers, and $\lambda_{1,2}$ being two fundamental weights. In general,

$$\Phi_\lambda^\kappa = \sum_{\mu \le \lambda} C_\lambda^\mu(\kappa)\,m_\mu, \quad \mu, \lambda \in P^+, \qquad m_\mu = \sum_{\nu \in W\cdot\mu} e^{2i(q,\nu)}, \tag{4.6}$$

where P^+ denotes the cone of dominant weights, W is the Weyl group, and $C_\lambda^\mu(\kappa)$ are some constants.

As it was shown in [Pe 1998a], the product of two wave functions is a finite sum of wave functions (a sort of the κ-deformed Clebsch–Gordan series):

$$\Phi_\mu^\kappa\,\Phi_\lambda^\kappa = \sum_{\nu \in D_\mu(\lambda)} C_{\mu\lambda}^\nu(\kappa)\,\Phi_\nu^\kappa. \tag{4.7}$$

In this equation, $D_\mu(\lambda) = (D_\mu + \lambda) \cap P^+$, where D_μ is a weight diagram of the representation with the highest weight μ.

Since Φ_μ^κ are symmetric functions of $\exp(2iq_j)$, it is convenient to use a new set of variables:

$$\begin{aligned} z_1 &= e^{2iq_1} + e^{2iq_2} + e^{2iq_3}, \\ z_2 &= e^{2i(q_1+q_2)} + e^{2i(q_2+q_3)} + e^{2i(q_3+q_1)}, \\ z_3 &= e^{2i(q_1+q_2+q_3)}. \end{aligned} \tag{4.8}$$

In the centre-of-mass frame ($\sum_i q_i = 0$), the wave functions depend only on two variables choosen as z_1 and z_2 (in this case, $z_3 = 1$). In these variables, up to a normalisation factor, we have

$$\Delta^\kappa = (z_1^2 - 3z_2)\,\partial_1^2 + (z_2^2 - 3z_1)\,\partial_2^2 + (z_1 z_2 - 9)\,\partial_1\partial_2 + (3\kappa+1)\,(z_1\partial_1 + z_2\partial_2), \tag{4.9}$$

where $\partial_i = \partial/\partial z_i$. Corresponding eigenvalues are

$$\varepsilon_{m,n}(\kappa) = m^2 + n^2 + mn + 3\,\kappa(m+n). \tag{4.10}$$

We shall use the normalization for polynomials Φ_λ^κ such that the coefficient at the highest monomial is equal to one. Denoting them by $P_{m,n}^\kappa$, we have

$$P_{m,n}^\kappa(z_1,z_2) = \sum_{p,q} C_{m,n}^{p,q}(\kappa)\, z_1^p\, z_2^q = z_1^m\, z_2^n + \text{lower terms}, \tag{4.11}$$

with $p+q \geq m+n$ and $p-q \equiv m-n \,(\mathrm{mod}\ 3)$. As it is easy to see, the first polynomials are

$$P_{0,0}^\kappa = 1, \qquad P_{1,0}^\kappa = z_1, \qquad P_{0,1}^\kappa = z_2. \tag{4.12}$$

Simple consequences of (4.7) for $P_\lambda^\kappa = P_{1,0}^\kappa$ or $P_{0,1}^\kappa$ are [Pe 1998]

$$\begin{aligned} z_1\, P_{m,n}^\kappa &= P_{m+1,n}^\kappa + a_{m,n}(\kappa)\, P_{m,n-1}^\kappa + c_m(\kappa)\, P_{m-1,n+1}^\kappa, \\ z_2\, P_{m,n}^\kappa &= P_{m,n+1}^\kappa + \tilde{a}_{m,n}(\kappa)\, P_{m-1,n}^\kappa + c_n(\kappa)\, P_{m+1,n-1}^\kappa, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} a_{m,n}(\kappa) &= \tilde{a}_{n,m}(\kappa) = c_n(\kappa)\, c_{m+n+\kappa}(\kappa), \\ c_m(\kappa) &= \frac{e(m)}{e(\kappa+m)}, \qquad e(m) = \frac{m}{m-1+\kappa}. \end{aligned} \tag{4.14}$$

Below we shall construct such polynomials using these recurrence relations.

5 A_1 case

Formulae of this section are well-known, see for example [Vi 1968]. However, the presentation of results being not standard is very useful for more complicated cases, for example, for A_2.

Recall that the unitary irreducible representation d of the Lie algebra $A_1 \sim su(2)$ is characterized by one integer non-negative number l, $d = d_l$, and the Clebsch–Gordan series has the well-known form:

$$d_k \otimes d_l = \oplus_{m=0}^{m_1} d_{|k-l|+2m}, \qquad m_1 = m_{\max} = \frac{1}{2}\,(k + l - |k - l|). \tag{5.1}$$

We begin with the simplest case

$$\begin{aligned} d_1 \otimes d_l &= d_{l-1} \oplus d_{l+1}, \quad l \geq 1, \\ d_1 \otimes d_0 &= d_1. \end{aligned}$$

Corresponding formulae for characters are

$$\begin{aligned} &\chi_1(\theta) \cdot \chi_l(\theta) = \chi_{l-1}(\theta) + \chi_{l+1}(\theta), \quad l \geq 1; \qquad \chi_l(\theta) = e^{il\theta} + e^{i(l-2)\theta} + \cdots + e^{-il\theta}, \\ &\chi_l(\theta) = P_l^1(z) = z^l + \cdots, \qquad z = 2\cos\theta. \end{aligned} \tag{5.2}$$

Hence for polynomials $P_l^1(z)$ we have the formula

$$z\, P_l^1(z) = P_{l+1}^1(z) + P_{l-1}^1(z), \qquad P_0^1 = 1, \quad P_1^1 = z. \tag{5.3}$$

This formula may be considered as a basic recurrence relation for these polynomials. From this, it follows the equation for the generating function

$$\begin{aligned} F^1(z,u) &= \sum_{l=0}^{\infty} P_l^1(z)\, u^l, \\ z\, F^1(z,u) &= u^{-1}\big(F^1(z,u) - 1\big) + u\, F^1(z). \end{aligned} \tag{5.4}$$

Hence

$$F^1(z,u) = \big(1 - zu + u^2\big)^{-1}. \tag{5.5}$$

Let us define now the κ-deformed generating function and corresponding system of polynomials

$$F^\kappa(z,u) = \big(1 - zu + u^2\big)^{-\kappa} = \sum_{l=0}^{\infty} \tilde{P}_l^\kappa(z)\, u^l. \tag{5.6}$$

This is a *basic formula.* We shall demonstrate below how to get all other formulae from it.

First we have

$$\tilde{P}_l^\kappa(z) = \frac{(\kappa)_l}{l!}\, P_l^\kappa(z), \quad P_l^\kappa(z) = z^l - \frac{l(l-1)}{l-1+\kappa}\, z^{l-2} + \cdots, \tag{5.7}$$

where

$$(x)_n = x(x+1)\cdots(x+n-1). \tag{5.8}$$

Then

$$F_z^\kappa = \kappa u\, F^{\kappa+1}, \qquad F_{zz}^\kappa = \kappa(\kappa+1)u^2\, F^{\kappa+2},$$
$$F_u^\kappa = \kappa(z-2u)\, F^{\kappa+1}, \qquad F_{uu}^\kappa = \kappa(\kappa+1)\,(z-2u)^2\, F^{\kappa+2} - 2\kappa\, F^{\kappa+1}, \tag{5.9}$$

where $F_z = \partial F/\partial z$, $F_u = \partial F/\partial u$, and so on.

From this, it is easy to get the equation

$$\left(D_z^2 - 4\,\partial_z^2 + 2\kappa\, D_z\right) F^\kappa(z,u) = \left(D_u^2 + 2\kappa\, D_u\right) F^\kappa(z,u), \tag{5.10}$$

where $D_z = z\,\partial_z$, $D_u = u\,\partial_u$. Content-Length: 452 X-Lines: 11 Status: O

Substituting the expansion (5.6) for F^κ, we obtain the differential equation for $\tilde{P}_l^\kappa$,

$$\left((z^2-4)\,\partial_z^2 + (2\kappa+1) z\,\partial_z\right) \tilde{P}_l^\kappa = l(l+2\kappa)\,\tilde{P}_l^\kappa. \tag{5.11}$$

Let us rewrite now (5.9) in the form

$$\begin{aligned} &\left(1 - zu + u^2\right) F_u^\kappa = \kappa(z-2u)\, F^\kappa, \\ &F_u^\kappa - z\,(D_u + \kappa)\, F^\kappa + u\,(D_u + 2\kappa)\, F^\kappa = 0. \end{aligned} \tag{5.12}$$

From this, it follows the recurrence relation

$$(n+1)\,\tilde{P}_{n+1}^\kappa - z(n+\kappa)\,\tilde{P}_n^\kappa + (n-1+2\kappa)\,\tilde{P}_{n-1}^\kappa = 0. \tag{5.13}$$

At $\kappa = 1$, it is equivalent to the basic recurrence relation (5.3), and for polynomilas $P_n^\kappa(z)$ we have

$$z\,P_n^\kappa(z) = P_{n+1}^\kappa(z) + a_n(\kappa)\,P_{n-1}^\kappa(z), \qquad a_n(\kappa) = \frac{n(n-1+2\kappa)}{(n-1+\kappa)(n+\kappa)}, \qquad P_0^\kappa = 1, \quad P_1^\kappa = z. \tag{5.14}$$

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So, we may interprete (5.13), (5.14) as a special deformation of the Clebsch–Gordan series for the Lie algbera $A_1 \sim su(2)$.

From (5.9), also it follows

$$F_z^\kappa - u(D_z + \kappa)\,F^\kappa + u^2\,F_z^\kappa = 0. \tag{5.15}$$

Therefore, we have

$$\partial_z\,\tilde{P}_{n+1}^\kappa - (z\partial_z + \kappa)\,\tilde{P}_n^\kappa + \partial_z\,\tilde{P}_{n-1}^\kappa = 0. \tag{5.16}$$

Combining the expressions for F_u^κ and F_z^κ, we obtain

$$uF_u^\kappa = (z-2u)\,F_z^\kappa. \tag{5.17}$$

Hence

$$n\,\tilde{P}_n^\kappa - z\,\partial_z\tilde{P}_n^\kappa + 2\,\partial_z\tilde{P}_{n-1}^\kappa = 0. \tag{5.18}$$

Using (5.9), we get

$$\left(1 - \frac{1}{2}zu\right) F_z^\kappa = \frac{1}{2}u(u\,F_u^\kappa + 2\kappa\,F^\kappa). \tag{5.19}$$

From this, it follows that

$$\partial_z \tilde{P}^\kappa_{n+1} - \frac{1}{2}\, z\, \partial_z \tilde{P}^\kappa_n = \frac{1}{2}\,(n+2\kappa)\, \tilde{P}^\kappa_n. \tag{5.20}$$

We also have

$$\left(4 - z^2\right) \partial_z F^\kappa = 4\,\kappa u\, F^\kappa - z(z-2u)u\, F^\kappa_u = u(4\kappa + 2\,D_u)\, F^\kappa - zD_u\, F^\kappa. \tag{5.21}$$

Hence

$$\left(4 - z^2\right) \partial_z \tilde{P}^\kappa_n = 2\,(2\kappa + n - 1)\, \tilde{P}^\kappa_{n-1} - zn\, \tilde{P}^\kappa_n, \tag{5.22}$$

or

$$\left(\left(4 - z^2\right) \frac{d}{dz} + nz\right) \tilde{P}^\kappa_n = 2\,(2\kappa + n - 1)\, \tilde{P}^\kappa_{n-1}. \tag{5.23}$$

In similar way, we obtain

$$\left(\left(4 - z^2\right) \frac{d}{dz} - (n+1)z\right) \tilde{P}^\kappa_n = -\,(n + 2 - 2\kappa)\, \tilde{P}^\kappa_{n+1}, \tag{5.24}$$

and

$$\frac{1}{2}\,(2n+1)\left(4 - z^2\right) \partial_z \tilde{P}^\kappa_n = (n+1)(n+2\kappa-1)\, \tilde{P}^\kappa_{n-1} - n(n+2-2\kappa)\, \tilde{P}^\kappa_{n+1}. \tag{5.25}$$

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Finally let us give the important Rodrigues formula:

$$\tilde{P}^\kappa_n(z) = \frac{1}{2^n\, n!}\, \frac{1}{\left(z^2-4\right)^{\kappa-(1/2)}}\, \frac{d^n}{dz^n}\left(z^2-4\right)^{n+\kappa-(1/2)}. \tag{5.26}$$

Note also that

$$\frac{d}{dz}\, \tilde{P}^\kappa_n(z) = \kappa\, \tilde{P}^{\kappa+1}_{n-1}(z) \tag{5.27}$$

and that from the identity

$$F^\kappa = F^{\kappa_1} \cdot F^{\kappa_2}, \qquad \kappa = \kappa_1 + \kappa_2$$

it follows the formula

$$\tilde{P}^\kappa_n(z) = \sum_{n_1+n_2=n} \tilde{P}^{\kappa_1}_{n_1}(z)\, \tilde{P}^{\kappa_2}_{n_2}(z). \tag{5.28}$$

6 A_2 case

Now we proceed to the case of $A_2 \sim su(3)$. In this case, the representation d of A_2 is characterized by two non-negative numbers $d = d_{mn}$. We have two fundamental representations

$$d_{10}, \qquad d_{01}; \qquad (\dim d_{10} = \dim d_{01} = 3),$$

and also representations

$$d_{n0}, \qquad d_{0n}; \qquad \left(\dim d_{n0} = \dim d_{0n} = \frac{1}{2}\,(n+1)(n+2)\right),$$

$$d_{n1}\left(\dim d_{n1} = (n+1)(n+3)\right), \qquad d_{nn}\left(\dim d_{nn} = (n+1)^3\right).$$

We start with the Clebsch–Gordan series

$$\begin{array}{l} d_{10}\otimes d_{n+1,0} = d_{n+2,0}\oplus d_{n,1}, \\ d_{01}\otimes d_{n0} = d_{n,1}\oplus d_{n-1,0}. \end{array} \tag{6.1}$$

Excluding $d_{n,1}$, we obtain

$$d_{n-1,0}\ominus (d_{01}\otimes d_{n0})\oplus (d_{10}\otimes d_{n+1,0})\ominus d_{n+2,0} = 0,$$

or

$$\chi_{n-1,0} - z_2\,\chi_{n,0} + z_1\,\chi_{n+1,0} - \chi_{n+2,0} = 0, \tag{6.2}$$

where are introduced the notations:

$$\begin{array}{l} z_1 = \chi_{10} = e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}, \\ z_2 = \chi_{01} = e^{-i\theta_1} + e^{-i\theta_2} + e^{-i\theta_3}. \end{array} \tag{6.3}$$

From this, we obtain the expression for the generating function

$$\begin{aligned} F_0^1(z_1,z_2;u) &= \sum_{n=0}^{\infty} \chi_{n0}(z_1,z_2)\,u^n, \\ F_0^1(z_1,z_2;u) &= \left(1 - z_1 u + z_2\,u^2 - u^3\right)^{-1}. \end{aligned} \tag{6.4}$$

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Let us define now the κ-deformed functions $\tilde{P}^{\kappa}_{n,0}(z_1,z_2)$ by the formula

$$F^{\kappa}(z_1,z_2;\,u) = \left(1 - z_1\,u + z_2\,u^2 - u^3\right)^{-\kappa} = \sum_{n=0}^{\infty} \tilde{P}^{\kappa}_{n,0}(z_1,z_2)\,u^n. \tag{6.5}$$

Analogously to A_1 case, we may obtain all other formulae from this one.

Differentiating F^{κ} on u, z_1 and z_2, we get

$$F^{\kappa}_u = \kappa\left(z_1 - 2z_2\,u + 3u^2\right)F^{\kappa+1},$$

$$F^{\kappa}_{z_1,z_1} = \kappa(\kappa+1)u^2\,F^{\kappa+2}, \qquad F^{\kappa}_{z_2,z_2} = \kappa(\kappa+1)u^4\,F^{\kappa+2},$$

$$F^{\kappa}_{z_1,z_2} = -\,\kappa(\kappa+1)u^3\,F^{\kappa+2}, \qquad u\,F^{\kappa}_u = \kappa\left(z_1 u - 2\,z_2 u^2 + 3\,u^3\right)F^{\kappa+1},$$

and

$$\begin{aligned} D_u^2\,F^{\kappa} &= \Big\{\kappa\left(z_1 u - 4\,z_2 u^2 + 9u^3\right)\left(1 - z_1 u + z_2 u^2 - u^3\right) \\ &\quad + \kappa(\kappa+1)\left(z_1 u - 2z_2 u^2 + 3u^3\right)^2\Big\}\,F^{\kappa+2}, \qquad D_u = u\partial_u, \end{aligned}$$

or

$$\begin{aligned}&\left(1 - z_1 u + z_2\, u^2 - u^3\right) F^\kappa_u = \kappa\left(z_1 - 2\, z_2 u + 3\, u^2\right) F^\kappa,\\[2mm]
&(n+3)\,\tilde P^\kappa_{n+3} - z_1(n+2)\,\tilde P^\kappa_{n+2} + z_2(n+1)\,\tilde P^\kappa_{n+1} - n\,\tilde P^\kappa_n\\
&= \kappa z_1\,\tilde P^\kappa_{n+2} - 2\,\kappa z_2\,\tilde P^\kappa_{n+1} + 3\,\kappa\,\tilde P^\kappa_n.\end{aligned} \tag{6.6}$$

So, we obtain the important recurrence formula

$$(n+3)\,\tilde P^\kappa_{n+3,0} = (n+2+\kappa) z_1\,\tilde P^\kappa_{n+2,0} - (n+1+2\kappa) z_2\,\tilde P^\kappa_{n+1,0} + (n+3\kappa)\,\tilde P_{n,0}. \tag{6.7}$$

Now let us differentiate F^κ on z_1. We have

$$F^\kappa_{z_1} = \kappa u\, F^{\kappa+1}. \tag{6.8}$$

Hence

$$\partial_{z_1} P^\kappa_{n,0} = n\, P^{\kappa+1}_{n-1,0}. \tag{6.9}$$

Analogously,

$$F^\kappa_{z_2} = -\,\kappa u^2\, F^{\kappa+1}. \tag{6.10}$$

Therefore

$$\partial_{z_2} P^\kappa_{n,0} = -\,\frac{n(n-1)}{\kappa+n-1}\, P^{\kappa+1}_{n-2,0}.$$

Finally we have the basic differential equation for $F^\kappa(z_1, z_2;\, u)$:

$$\begin{aligned}&\left(\left(D^2_{z_1} + D^2_{z_2} + D_{z_1} D_{z_2}\right) - 3\, z_2 \partial^2_{z_1} - 3\, z_1 \partial^2_{z_2} - 9\,\partial_{z_1}\partial_{z_2} + 3\,\kappa(D_{z_1} + D_{z_2})\right) F^\kappa\\
=\;&\left(D^2_u + 3\,\kappa D_u\right) F^\kappa(z_1, z_2;\, u);\end{aligned}$$

$$D_{z_1} = z_1\,\partial_{z_1}, \qquad D_{z_2} = z_2\,\partial z_2, \qquad D_u = u\,\partial_u.$$

Let us note that the normalization of polynomials $\tilde P^\kappa_{n,0}(z_1, z_2)$ follows from the expression (6.5) for the generating function. Namely,

$$\tilde P^\kappa_{n0}(z_1, z_2) = \frac{(\kappa)_n}{n!}\, z_1^n + \cdots = \frac{(\kappa)_n}{n!}\, P^\kappa_{n,0}(z_1, z_2), \tag{6.11}$$

where

$$(\kappa)_n = (\kappa)(\kappa+1)\cdots(\kappa+n-1).$$

The main property of this normalization is that $\tilde P^\kappa_{n,0}(z_1, z_2)$ has a polynomial dependence on the parameter κ.

Now we shall consider other Clebsch–Gordan series for $\kappa = 1$:

$$d_{1,0} \otimes d_{n+1,0} = d_{n+2,0} \oplus d_{n,1}.$$

According to [Pe 1998], the analogous formula is valid for an arbitrary value of κ, i.e.,

$$a_n z_1\,\tilde P^\kappa_{n+1,0} = b_n\,\tilde P^\kappa_{n+2,0} + c_n\,\tilde P^\kappa_{n,1}, \tag{6.12}$$

where coefficients a_n, b_n and c_n do not depend on z_1 and z_2 but may depend on κ. Comparing of coefficients at z_1^{n+2} yields

$$a_n = \kappa + n + 1, \qquad b_n = n + 2.$$

From this relation, we may determine the function $\tilde{P}_{n,1}$ up to the normalizing constant c_n:

$$c_n(\kappa)\, \tilde{P}^{\kappa}_{n,1} = (\kappa + n + 1) z_1\, \tilde{P}^{\kappa}_{n+1,0} - (n+2)\, \tilde{P}^{\kappa}_{n+2,0}. \tag{6.13}$$

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Let us calculate now the generating function for both left-hand and right-hand sides of this equation,

$$\begin{aligned} G^{\kappa} &= \sum_{n=0}^{\infty} c_n(\kappa)\, \tilde{P}^{\kappa}_{n,1}(z_1, z_2), \\ G^{\kappa} &= \frac{\kappa z_1}{u}\, (F_0^{\kappa} - 1) + \frac{z_1}{u}\, F_1^{\kappa} - \frac{1}{u^2}\, (F_1^{\kappa} - \kappa z_1 u), \end{aligned} \tag{6.14}$$

where

$$F_0^{\kappa} = \sum_{n=0}^{\infty} \tilde{P}^{\kappa}_{n,0}\, u^n; \qquad F_1^{\kappa} = \sum_{n=0}^{\infty} n\, \tilde{P}^{\kappa}_{n,0}\, u^n = D_u\, F_0^{\kappa},$$

and

$$G^{\kappa} = \frac{1}{u^2}\, (\kappa z_1 u - (1 - z_1 u) D_u)\, F_0^{\kappa}.$$

Finally,

$$G^{\kappa} = \kappa \left(2 z_2 - (z_1 z_2 + 3)\, u + 2\, z_1 u^2 \right) F_0^{\kappa+1}. \tag{6.15}$$

From this, it follows three-term recurrence relation

$$\tilde{P}^{\kappa}_{n,1} = \kappa \left(2\, z_2\, \tilde{P}^{\kappa+1}_{n,0} - (z_1 z_2 + 3)\, \tilde{P}^{\kappa+1}_{n-1,0} + 2\, z_1\, \tilde{P}^{\kappa+1}_{n-2,0} \right).$$

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Let us give also the explicit expression for $\tilde{P}^{\kappa}_{n,0}(z_1, z_2)$ and $\tilde{P}^{\kappa}_{n,1}(z_1, z_2)$ in other variables

$$\begin{aligned} z_1 &= e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} = x_1 + x_2 + x_3, \\ z_2 &= e^{-i\theta_1} + e^{-i\theta_2} + e^{-i\theta_3} = x_1 x_2 + x_2 x_3 + x_3 x_1. \end{aligned}$$

As it was shown in [Pr 1984],

$$\tilde{P}^{\kappa}_{n,0} = \sum_{m_1, m_2, m_3} C^{n,0}_{m_1,m_2,m_3}(\kappa)\, x_1^{m_1} x_2^{m_2} x_3^{m_3}, \qquad m_1 + m_2 + m_3 = n, \qquad x_1\, x_2\, x_3 = 1, \tag{6.16}$$

where quantities $C^{n,0}_{m_1,m_2,m_3}(\kappa)$ are κ-deformed three-nomial coefficients

$$C^{n,0}_{m_1,m_2,m_3}(\kappa) = \frac{n!}{m_1!\, m_2!\, m_3!}\, \frac{(\kappa)_{m_1} (\kappa)_{m_2} (\kappa)_{m_3}}{(\kappa)_n}\,. \tag{6.17}$$

The function $\tilde{P}^{\kappa}_{n,1}$ may be defined by the formula

$$\begin{aligned}\tilde{P}^{\kappa}_{n-2,1}(z_1,z_2) &= z_1\tilde{P}^{\kappa}_{n-1,0} - \tilde{P}_{n,0},\\ \tilde{P}^{\kappa}_{n-2,1}(z_1,z_2) &= \sum_{m_1,m_2,m_3} C^{n,1}_{m_1,m_2,m_3}(\kappa)\, x_1^{m_1}x_2^{m_2}x_3^{m_3}.\end{aligned} \tag{6.18}$$

We obtain

$$C^{n,1}_{m_1,m_2,m_3}(\kappa) = C^{n-1,0}_{m_1-1,m_2,m_3}(\kappa) + C^{n-1,0}_{m_1,m_2-1,m_3}(\kappa) + C^{n-1,0}_{m_1,m_2,m_3-1} - C^{n,0}_{m_1,m_2,m_3}(\kappa). \tag{6.19}$$

Substituting here the explicit expression for $C^{n,0}_{m_1,m_2,m_3}(\kappa)$, we get

$$C^{n,1}_{m_1,m_2,m_3}(\kappa) = C^{n,0}_{m_1,m_2,m_3}(\kappa)\, S^{n,1}_{m_1,m_2,m_3}, \tag{6.20}$$

where

$$S^{n,1}_{m_1,m_2,m_3} = \frac{\kappa-1+n}{n}\left(\frac{m_1}{\kappa-1+m_1}+\frac{m_2}{\kappa-1+m_2}+\frac{m_3}{\kappa-1+m_3}-\frac{n}{\kappa-1+n}\right). \tag{6.21}$$

At $\kappa=1$, the quantity $S^{n,1}_{m_1,m_2,m_3}(\kappa)$ is equal to 2. As $\kappa\to\infty$, it has an order κ^{-1} and is a symmetric function of m_1, m_2, m_3. So, it should have a form

$$S^{n,1}_{m_1,m_2,m_3} = \frac{\alpha(\kappa-1)^2+\beta(\kappa-1)+\gamma}{(\kappa-1+m_1)(\kappa-1+m_2)(\kappa-1+m_3)}\,. \tag{6.22}$$

Now let us follow [PRZ 1998] and construct the general polynomials in terms of the simplest polynomials (Jack polynomials) $P^{\kappa}_{m,0}$ and $P^{\kappa}_{0,n}$. We get

$$P^{\kappa}_{m,0}\,P^{\kappa}_{0,n} = \sum_{i=0}^{\min(m,n)} \gamma^i_{m,n}\,P^{\kappa}_{m-i,n-i}. \tag{6.23}$$

This is a consequence of equation (4.7), with the notable difference that the sum on the right-hand side is over a restricted domain (actually, it parallels exactly the $SU(3)$ Clebsch–Gordan decomposition).

To prove this, let us assume that (6.23) is valid up to (m,n). Then using (4.13) and (4.14), we get

$$P^{\kappa}_{m,0}\,P^{\kappa}_{0,n+1} = \sum_{i=0}^{\min(m,n+1)} \gamma^i_{m,n+1}\,P^{\kappa}_{m-i,n+1-i} + c_n\,\delta^i_{m,n+1}\,P^{\kappa}_{m+1-i,n-1-i}, \tag{6.24}$$

where we defined

$$\gamma^i_{m,n+1} = \gamma^i_{m,n} + \tilde{a}_{m-i+1,n-i+1}\,\gamma^{i-1}_{m,n} - c_n\,c_{m-i+1}\,\gamma^{i-1}_{m,n-1}, \tag{6.25}$$

$$\begin{aligned}\delta^i_{m,n+1} &= c_n^{-1}\,c_{n-i}\,\gamma^i_{m,n} - \gamma^i_{m,n-1} - a_{m-i+1,n-i}\,\gamma^{i-1}_{m,n-1}\\ &\quad + c_{n-1}\,c_{\kappa+n-1}\,\gamma^{i-1}_{m,n-2}.\end{aligned} \tag{6.26}$$

From the polynomial normalization, we already know that $\gamma^0_{m,n} = 1$. After a straigthforward computation, the solution to (6.25) proved to be equal

$$\gamma^i_{m,n} = \frac{e(2\kappa+m+n+1-i)_{-i}}{e(1)_i\, e(m+1)_{-i}\, e(n+1)_{-i}}\,, \tag{6.27}$$

where

$$e(m) = \frac{m}{m-1+\kappa}\,.$$

It implies that $\delta^i_{m,n+1} = 0$ in (6.26). Let us give also the more explicit expression [3]

$$\gamma^i_{m,n} = \frac{(\kappa)^i\,(m)_i\,(n)_i\,(3\kappa+m+n-1-i)_i}{i!\,(\kappa+m-1)_i\,(\kappa+n-1)_i\,(2\kappa+m+n-i)_i}\,, \tag{6.28}$$

where

$$\begin{aligned} (x)^i &= x(x+1)\cdots(x+i-1),\\ (x)_i &= x(x-1)\cdots(x-i+1). \end{aligned} \tag{6.29}$$

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The constructive aspect of this formula is in its inverted form.

Theorem 1. *The generalized Gegenbauer polynomials $P^\kappa_{m,n}$ of type A_2 are given by the formula*

$$P^\kappa_{m,n} = \sum_{i=0}^{\min(m,n)} \beta^i_{m,n}\, P^\kappa_{m-i,0}\, P^\kappa_{0,n-i}\,, \tag{6.30}$$

where the constants $\beta^i_{m,n}$ are

$$\beta^i_{m,n} = \frac{(-1)^i}{i!}\,\frac{3\kappa+m+n-2i}{3\kappa+m+n-i}\,\frac{(m)_i\,(n)_i\,(\kappa)_i\,(3\kappa+m+n-1)_i}{(\kappa+m-1)_i(\kappa+n-1)_i(2\kappa+m+n-1)_i}\,. \tag{6.31}$$

Note that $\beta^i_{m,n}$ are obtained by using of the relation

$$\beta^i_{m,n} = -\sum_{j=0}^{i-1} \beta^j_{m,n}\,\gamma^{i-j}_{m-j,n-j}\,. \tag{6.32}$$

From this theorem, we see that the construction of a general polynomial $P^\kappa_{m,n}$ is similar to the construction of $SU(3)$ representations from tensor products of two fundamental representations.
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Likewise, we can consider other types of decompositions, such as

$$P^\kappa_{m,0}\,P^\kappa_{n,0} = \sum_{i=0}^{\min(m,n)} \tilde\gamma^i_{m,n}\,P^\kappa_{m+n-2i,i}. \tag{6.33}$$

[3] Note that this expression for $\gamma^i_{m,n}$ may be obtained from the general Macdonald formula [Ma 1995], however, the way of proof given in [PRZ 1998] is more convenient here.

The proof is analogous to (6.23) (see footnote 3). The coefficients $\tilde{\gamma}^i_{m,n}$ are given by the formula

$$\tilde{\gamma}^i_{m,n} = \frac{(\kappa)^i\,(m)_i\,(n)_i\,(2\kappa+m+n-1-i)_i}{i!\,(\kappa+m-1)_i\,(\kappa+n-1)_i\,(\kappa+m+n-i)_i}. \tag{6.34}$$

Theorem 2. *There is another formula for polynomials $P^\kappa_{m,n}$ at $m \geq n$:*

$$\tilde{\gamma}^n_{m+n,n}\,P^\kappa_{m,n} = \sum_{i=0}^{n} \tilde{\beta}^i_{m,n}\,P^\kappa_{m+n+i,0}\,P^\kappa_{n-i,0}, \tag{6.35}$$

where

$$\tilde{\beta}^i_{m,n} = \frac{(-1)^i}{i!}\,(\kappa)_i\,\frac{(m+2i)}{m}\,\frac{(\kappa+m+n)^i}{(m+n+1)^i}\,\frac{(m)^i}{(\kappa+m+1)^i}\,\frac{(n)_i}{(\kappa+n-1)_i}. \tag{6.36}$$

This theorem follows directly from equation (6.33). The coefficients $\tilde{\beta}^i_{m,n}$ are found by using of the relation

$$\tilde{\beta}^i_{m,n} = -\left(\tilde{\gamma}^{n-i}_{m+n+i,n-i}\right)^{-1} \sum_{j=0}^{i-1} \tilde{\beta}^j_{m,n}\,\tilde{\gamma}^{n-i}_{m+n+j,n-j}. \tag{6.37}$$

As a by-product, let us specialize equation (6.30) to the case $\kappa = 1$, where $P^\kappa_{m,n}$ are nothing but the $SU(3)$ characters. We get

$$P^1_{m,n} = P^1_{m,0}\,P^1_{0,n} - P^1_{m-1,0}\,P^1_{0,n-1}. \tag{6.38}$$

From this, we easily deduce the generating function for $SU(3)$ characters (see e.g. [PS 1978])

$$G^1(u,v) = \sum_{m,n=0}^{\infty} u^m\,v^n\,P^1_{m,n} = \frac{1-uv}{(1-z_1\,u+z_2\,u^2-u^3)\,(1-z_2\,v+z_1\,v^2-v^3)}. \tag{6.39}$$

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Closing this section, let us underline that for κ=1/2, 1, 2 and 4, the obtained formulae yield the explicit expression of zonal polynomials for certain symmetric spaces listed below.

Table. *List of symmetric spaces with A_2 as a restricted root system*

Type of symmetric	X^- (non-compact)	X^+ (compact)	$\kappa = m/2$ (m is a multiplicity of roots)
$A\,\mathrm{I}$	$SL(3,\mathbb{R})/\mathbb{SO}(\nvDash)$	$SU(3)/SO(3)$	$1/2$
$A\,0$	$SL(3,\mathbb{C})/\mathbb{SU}(\nvDash)$	$SU(3)$	1
$A\,\mathrm{II}$	$SU_6^*/Sp(3)$	$SU(6)/Sp(3)$	2
$E\,\mathrm{IV}$	$E_6(-26)/F_4$	$E_6(-78)/F_4$	4

7 The integral representation for the case of $N = 3$

The integral representation for the case of $N = 2$ coincides with the integral representation for the Gegenbauer polynomials and is well known (see for example [Vi 1968]).

For the special values of κ ($\kappa = 1/2,\ 1\,,2\,,4$), wave functions are related to zonal spherical functions. The integral representation of these functions was obtained by Harish–Chandra by integrating on the some Lie group K [Ha 1958], [He 1978].

Following [CP 1997] let us consider now the A_2 case ($N = 3$) for $\kappa = 1/2$, $K = SO(3, \mathbb{R})$. The element $k \in K = SO(3, \mathbb{R})$ may be represented by three unit vectors orthogonal to each other

$$n,\, l,\, m; \qquad n^2 = l^2 = m^2 = 1, \qquad (n, l) = (l, m) = (m, n) = 0.$$

Here the integral representation for zonal spherical polynomials has the form

$$\Phi_{pq}(x) = \int_K [\Xi_1(x_j; n)]^p\, [\Xi_2(x_j; n, l)]^q\, d\mu(n, l), \qquad x_j = e^{iq_j}, \tag{7.1}$$

where

$$\Xi_1(x_j; n) = n_1^2\, x_1 + n_2^2\, x_2 + n_3^2\, x_3, \qquad \Xi_2(x_j; n, l) = \sum_{j<k} (n_j\, l_k - n_k\, l_j)^2\, x_j\, x_k,$$

and the integration is taken on the orthogonal group $K = SO(3, \mathbb{R})$, which is equivalent to the space of two unit orthogonal vectors n and l.

Noting that $m_k = \epsilon_{kij}\, n_i\, l_j$, we also have $x_1 x_2 = x_3^{-1}, \ldots$. Hence Φ_{pq} is given by (7.1) where

$$\Xi_2(x_j; n, l) = m_1^2\, x_1^{-1} + m_2\, x_2^{-1} + m_3\, x_3^{-1}. \tag{7.2}$$

For vectors n and m, the standard parametrization through the Euler angles φ, θ and ψ may be used:

$$\begin{aligned} n &= (\cos\varphi \sin\theta,\, \sin\varphi \sin\theta,\, \cos\theta), \quad m = \cos\psi \cdot a + \sin\psi \cdot b, \\ a &= (-\sin\varphi, \cos\varphi, 0), \quad b = (-\cos\varphi \cos\theta,\, -\sin\varphi \cos\theta,\, \sin\theta) \end{aligned} \tag{7.3}$$

with $d\mu(k) = d\mu(n, m) = A\, \sin\theta\, d\theta\, d\varphi\, d\psi$. Hence in the preceding expression we have a three-dimensional integral which may be calculated by using of the generating functions.

8 The generating function for the case of $N = 3$

Following [CP 1997] let us define the generating function by the formula

$$F(x_1, x_2, x_3; t_1, t_2) = \sum \Phi_{l_1, l_2}(x_1, x_2, x_3)\, t_1^{l_1}\, t_2^{l_2}. \tag{8.1}$$

Then we obtain the integral representation

$$F(x_1, x_2, x_3; t_1, t_2) = \int \left[\prod_{j=1}^{2} (1 - \Xi_j(x; n, \psi) t_j) \right]^{-1} d\mu(n)\, d\mu(\psi). \tag{8.2}$$

Let a and b be two unit orthogonal vectors in two-dimensional plane orthogonal to the vector n. Then on this plane an arbitrary unit vector m has the form $\cos\psi \cdot a + \sin\psi \cdot b$. Integrating first on $d\mu(\psi)$ we get:

$$F(x_1, x_2, x_3; t_1, t_2) = \int B^{-1}(n)\, C^{-1/2}(n)\, d\mu(n), \quad \int d\mu(n) = 1, \tag{8.3}$$

where

$$B = 1 - \left(n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3\right) t_1, \qquad C = \left(1 - x_2^{-1} t_2\right)\left(1 - x_3^{-1} t_2\right) n_1^2 + \cdots. \tag{8.4}$$

The crucial step for the further integration is the use of the formula

$$B^{-1}\, C^{-1/2} = \int_0^1 d\xi \left[B(1-\xi^2) + C\xi^2\right]^{-3/2}. \tag{8.5}$$

Using this, we obtain

$$F(x_1, x_2, x_3; t_1, t_2) = \int_0^1 d\xi \int [E(x_1, x_2, x_3; t_1, t_2, n, \xi)]^{-3/2}\, d\mu(n), \tag{8.6}$$

where

$$E(x_1, x_2, x_3; t_1, t_2, n, \xi) = \sum_j e_j(x_1, x_2, x_3; t_1, t_2, \xi)\, n_j^2. \tag{8.7}$$

We can now integrate on $d\mu(n)$. Finally we obtain the one-dimensional integral representation for the generating function

$$F(x_1, x_2, x_3; t_1, t_2) = \int_0^1 d\xi\, [H(x_1, x_2, x_3; t_1, t_2, \xi)]^{-1/2}, \tag{8.8}$$

where $H = h_1\, h_2\, h_3$, and functions $h_j(\xi; t_1, t_2)$ are given by formulae

$$h_j(\xi; t_1, t_2) = 1 - d_j(t_1, t_2)\left(1 - \xi^2\right), \qquad d_j(t_1, t_2) = \left(x_j t_1 + x_j^{-1} t_2 - t_1 t_2\right). \tag{8.9}$$

From this it follows that if $z_1 = x_1 + x_2 + x_3$ and $z_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$, then

$$\begin{aligned} H \;=\;& a_0^3 - a_0^2\left[z_1\tau_1 + z_2\tau_2\right] + a_0\left[z_2\tau_1^2 + z_1\tau_2^2 + (z_1 z_2 - 3)\tau_1\tau_2\right] \\ & - \left[\tau_1^3 + \tau_2^3 + \tau_1\tau_2\left[(z_2^2 - 2z_1)\,\tau_1 + (z_1^2 - 2z_2)\,\tau_2\right]\right], \end{aligned} \tag{8.10}$$

where $a_0 = 1 + \left(1 - \xi^2\right) t_1 t_2$, $\tau_1 = \left(1 - \xi^2\right) t_1$, $\tau_2 = \left(1 - \xi^2\right) t_2$. Note that from (8.10) it follows that integral (8.8) is elliptic one and may be expressed in terms of standard elliptic integrals.
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Expanding $F(x_1, x_2, x_3; t_1, t_2)$ in power series of the variable t_2 we obtain

$$F(x_1, x_2, x_3; t_1, t_2) = \sum_{q=0}^{\infty} F_q(x_1, x_2, x_3; t_1)\, t_2^q. \tag{8.11}$$

We have

$$F_0(x_1, x_2, x_3; t) = \int_0^1 d\xi \, [H_0]^{-1/2}, \tag{8.12}$$

and

$$F_1 = \frac{1}{2} \int_0^1 d\xi \, H_1 \, [H_0]^{-3/2}, \tag{8.13}$$

where

$$\begin{aligned} H_0 &= 1 - z_1 \tau_1 + z_2 \tau_1^2 - \tau_1^3, \\ H_1 &= \left(1 - \xi^2\right) z_2 - \left[3\xi^2 + z_1 z_2 \left(1 - \xi^2\right)\right] \tau_1 + \left[2 z_1 \xi^2 + \left(1 - \xi^2\right) z_2^2\right] \tau_1^2 - z_2 \tau_1^3. \end{aligned}$$

From integral representation (8.8), many useful formulae may be obtained. Here we give only one of them, namely, as z_1, $z_2 \to \infty$,

$$\Phi_{m,n}(z_1, z_2) \approx A_{m,n} \, z_1^p \, z_2^q, \qquad A_{m,n} = \frac{(1/2)_m \, (1/2)_n}{(1)_m \, (1)_n} \, \frac{(1)_{m+n}}{(3/2)_{m+n}} \, . \tag{8.14}$$

One can also be shown [Pe 1998b] that for arbitrary κ the analogous integral representation is valid

$$F^\kappa(z_1, z_2; t_1, t_2) = \sum_{m,n=1}^{\infty} A_{m,n}(\kappa) \, P^\kappa_{m,n}(z_1, z_2) \, t_1^m \, t_2^n = \int_0^1 [H]^{-\kappa} d\mu^\kappa(\xi), \tag{8.15}$$

where H is given by (8.10), $P^\kappa_{m,n}(z_1, z_2) \sim z_1^m z_2^n$ as z_1, $z_2 \to \infty$, and

$$d\mu^\kappa(\xi) = [\xi(1 - \xi^2)]^{2\kappa - 1}, \quad A_{m,n}(\kappa) = \frac{1}{2} \, \frac{\Gamma(\kappa)\Gamma(2\kappa)}{\Gamma(3\kappa)} \, \frac{(\kappa)_m \, (\kappa)_n}{(1)_m \, (1)_n} \, \frac{(2\kappa)_{m+n}}{(3\kappa)_{m+n}}. \tag{8.16}$$

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Note that for the case under consideration another integral representation for $N = 3$ was obtained in [KS 1995] in terms of ${}_3F_2$ hypergeometric series.

Appendix 1. List of polynomials $P^\kappa_{m,n}$ with $m+n \le 4$

Following [PRZ 1998], we list here the polynomials $P^\kappa_{m,n}$ with $m+n \le 4$:

$$\begin{aligned}
P^\kappa_{2,0} &= z_1^2 - \frac{2}{\kappa+1}\, z_2, \\
P^\kappa_{1,1} &= z_1 z_2 - \frac{3}{2\kappa+1}, \\
P^\kappa_{3,0} &= z_1^3 - \frac{6}{\kappa+2}\, z_1 z_2 + \frac{6}{(\kappa+1)(\kappa+2)}, \\
P^\kappa_{2,1} &= z_1^2 z_2 - \frac{2}{\kappa+1}\, z_2^2 - \frac{3\kappa+1}{(\kappa+1)^2}\, z_1, \\
P^\kappa_{4,0} &= z_1^4 - \frac{12}{\kappa+3}\, z_1^2 z_2 + \frac{12}{(\kappa+2)(\kappa+3)}\, z_2^2 + \frac{24}{(\kappa+2)(\kappa+3)}\, z_1, \\
P^\kappa_{3,1} &= z_1^3 z_2 - \frac{6}{\kappa+2}\, z_1 z_2^2 - \frac{3(3\kappa+2)}{(\kappa+2)(2\kappa+3)}\, z_1^2 + \frac{30}{(\kappa+2)(2\kappa+3)}\, z_2, \\
P^\kappa_{2,2} &= z_1^2 z_2^2 - \frac{2}{\kappa+1}\,(z_1^3 + z_2^3) - \frac{12(\kappa-1)}{(\kappa+1)(2\kappa+3)}\, z_1 z_2 + \frac{9(\kappa-1)}{(\kappa+1)^2(2\kappa+3)}\,.
\end{aligned}$$

Appendix 2. Some formulae for Macdonald Polynomials for A_2 case (by A.M. Perelomov, E. Ragoucy and Ph. Zaugg)

Here we give only some necessary for us information. For other results and details, see [Ma 1995].

The Macdonald polynomials of type A_2 may be defined as polynomial eigenfunctions of the Macdonald difference equation

$$M^1\, P^{(q,t)}_{m,n}(x_1,x_2,x_3) = \lambda\, P^{(q,t)}_{m,n}(x_1,x_2,x_3), \qquad P_{m,n} = z_1^m z_2^n + \text{lower terms}, \tag{8.1}$$

$$z_1 = x_1+x_2+x_3, \qquad z_2 = x_1x_2+x_2x_3+x_3x_1, \qquad z_3 = x_1x_2x_3,$$

where

$$M^1 = \prod_{j\neq k} \frac{(tx_j - x_k)}{(x_j - x_k)}\, T_j, \qquad T_1\, f(x_1,x_2,x_3) = f(qx_1,x_2,x_3), \ldots. \tag{8.2}$$

The Macdonald polynomials satisfy the following recurrence relation

$$z_1\, P^{(q,t)}_{m,n}(z_1,z_2,z_3) = P^{(q,t)}_{m+1,n} + a_{m,n}(q,t)\, P^{(q,t)}_{m,n-1} + b_{m,n}(q,t)\, P^{(q,t)}_{m-1,n+1}, \tag{8.3}$$

where

$$\begin{aligned}
a_{mn} &= c_n\, \tilde{c}_{m+n}, \qquad b_{mn} = c_m, \\
c_m(q,t) &= \frac{(1-q^m)(1-t^2q^{m-1})}{(1-tq^m)(1-tq^{m-1})},
\end{aligned} \tag{8.4}$$

$$\tilde{c}_{m+n}(q,t) \;=\; \left(\frac{1-tq^{m+n}}{1-t^2q^{m+n}}\right)\left(\frac{1-t^3q^{m+n-1}}{1-t^2q^{m+n-1}}\right).$$

Note that

$$c_m(q,1) = c_m(q,q) = 1, \qquad \tilde{c}_{m+n}(q,1) = \tilde{c}_{m+n}(q,q) = 1. \tag{8.5}$$

Now let us consider the case $t = q^k$, k being an integer. In this case, the explicit expression for the generating function of polynomials $P_{n,0}^{(k)} \equiv P_{n,0}^{(q,t)}$ has the form

$$\begin{aligned} G^{(k)}(u) &= \prod_{j=0}^{k-1} F^1(q^j u), \\ G^{(k)}(u) &= \sum C_n^k \, P_{n,0}^{(k)} \, u^n, \\ C_n^k &= \frac{[k]_n}{[1]_n}, \qquad [x]_n = [x][x+1]\cdots[x+n-1], \quad [x] = \frac{1-q^x}{1-x}. \end{aligned} \tag{8.6}$$

From this we get a three-term recurrence relation being analogous to (6.7)

$$[n+1]\,\tilde{P}_{n+1}^k = [k+n]\,z_1\,\tilde{P}_n^k - [2k+n-1]\,z_2\,\tilde{P}_{n-1}^k + [3k+n-2]\,z_3\,\tilde{P}_{n-2}^k, \tag{8.7}$$

which follows from the recurrence relation

$$\left(1 - z_1 u + z_2 u^2 - z_3 u^3\right) G^{(k)}(u) = \left(1 - z_1 u q^k + z_2 u^2 q^{2k} - z_3 u^3 q^{3k}\right) G(qu). \tag{8.8}$$

From here, we obtain

$$\tilde{P}_0^\kappa = 1, \quad \tilde{P}_1^\kappa = [\kappa]\,z_1, \quad \tilde{P}_2^\kappa = \frac{[\kappa+1][\kappa]}{[1][2]}\,z_1^2 - \frac{[2\kappa]}{[2]}\,z_2, \;\ldots \quad \tilde{P}_{m,0} = \frac{[1]_m}{[\kappa]_m}\,P_{m,0}.$$

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Let us give also the formula for the generating function for the case of arbitrary t:

$$G^{(q,t)}(u) = \prod_{j=0}^{\infty} \frac{(1-q^j t u)}{(1-q^j u)} = \sum_{j=0}^{\infty} c_j(q,t)\,u^j. \tag{8.9}$$

Using the results from the book [Ma 1995], it is not difficult to get the formulae

$$P_{m,0}^{(q,t)}\,P_{0,n}^{(q,t)} = \sum_{i=0}^{\min(m,n)} \gamma_{m,n}^i\,P_{m-i,n-i}^{(q,t)}\,, \tag{8.10}$$

where

$$\gamma_{m,n}^i = \frac{(t;q)_i\,(q^m;q^{-1})_i\,(q^n;q^{-1})_i\,(q^{m+n-1-i}t^3;q^{-1})_i}{(q;q)_i\,(q^{m-1}t;q^{-1})_i\,(q^{n-1}t;q^{-1})_i\,(q^{m+n-i}t^2;q^{-1})_i}; \tag{8.11}$$

and

$$P_{m,0}^{(q,t)}\,P_{n,0}^{(q,t)} = \sum_{i=0}^{\min(m,n)} \tilde{\gamma}_{m,n}^i\,P_{m+n-2i,i}^{(q,t)}\,, \tag{8.12}$$

where

$$\tilde\gamma^i_{m,n} = \frac{(t;q)_i\,(q^m;q^{-1})_i\,(q^n;q^{-1})_i\,(q^{m+n-i-1}t^2;q^{-1})_i}{(q;q)_i(q^{m-1}t;q^{-1})_i(q^{n-1}t;q^{-1})_i(q^{m+n-i}t;q^{-1})_i}\,. \tag{8.13}$$

Inverting formulae (8.10) and (8.11) as in the [PRZ 1998], we come to

Theorem 1a. *The Macdonald polynomilas $P^{(q,t)}_{m,n}$ of type A_2 are given by the formula*

$$P^{(q,t)}_{m,n} = \sum_{i=0}^{\min(m,n)} \beta^i_{m,n}\,P^{(q,t)}_{m-i,0}\,P^{(q,t)}_{0,n-i}, \tag{8.14}$$

where constants have the form

$$\begin{aligned}\beta^i_{m,n} &= (-1)^i\,q^{i(i-1)/2}\,\frac{(t;q^{-1})_i}{(q;q)_i}\,\frac{1-q^{m+n-2i}\,t^3}{1-q^{m+n-i}\,t^3}\\ &\times\frac{(q^m;q^{-1})_i\,(q^n;q^{-1})_i\,(q^{m+n-1}t^3;q^{-1})_i}{(q^{m-1}t;q^{-1})_i\,(q^{n-1}t;q^{-1})_i\,(q^{m+n-1}t^2;q^{-1})_i}\,.\end{aligned}$$

In similar way, we get

Theorem 2a. *The Macdonald polynomials $P^{(q,t)}_{m,n}$ of type A_2 are given by the formula*

$$\tilde\gamma^n_{m+n,n}\,P^{(q,t)}_{m,n} = \sum_{i=0}^{n}\tilde\beta^i_{m,n}\,P^{(q,t)}_{m+n+i,0}\,P^{(q,t)}_{n-i,0}, \qquad m\ge n \tag{8.15}$$

where constants $\tilde\gamma^i_{m,n}$ are given by formula (8.13), *and*

$$\tilde\beta^i_{m,n} = (-1)^i\,q^{i(i-1)/2}\,\frac{(t;q^{-1})_i}{(q;q)_i}\,\frac{(1-q^{m+2i})}{(1-q^m)}\,\frac{(q^m;q)_i\,(q^n;q^{-1})_i\,(q^{m+n}t;q)_i}{(q^{m+n+1};q)_i\,(q^{m+1}t;q)_i\,(q^{n-1}t;q^{-1})_i}. \tag{8.16}$$

From Theorems 1a and 2a, many interesting identities may be obtained. Here we give one of them:

$$\begin{aligned}S^k_{n,l} &= \sum_{i=0}^{l}(-1)^i\,q^{i(i-1)/2}\,\frac{[3k+n-2i]}{[3k+n-i]}\left(\prod_{j=1}^{i}\frac{[k-j+1]}{[j]}\,\frac{[3k+n-j]}{[2k+n-j]}\right)\\ &\times\left(\prod_{j=0}^{l-i-1}\frac{[k+j]}{[j+1]}\,\frac{[3k+n-l-i-j-1]}{[2k+n-l-i-j]}\right) = 0,\end{aligned} \tag{8.17}$$

where $[n] = (1-q^n)/(1-q)$. Content-Length: 5667 X-Lines: 115 Status: O

The simplest version of this formula is

$$\sum_{i=0}^{l}(-1)^i\,q^{i(i-1)/2}\,\frac{[k+l-i-1]!}{[i]!\,[l-i]!\,[k-i]!} = 0. \tag{8.18}$$

We give below the **list of polynomilas** $P_{m,n}^{(q,t)}$ at $m+n\le 4$

$$
\begin{aligned}
P_{0,0}^{(q,t)} &= 1, \qquad P_{1,0}^{(q,t)} = z_1, \qquad P_{0,1}^{(q,t)} = z_2,\\
P_{2,0}^{(q,t)} &= z_1^2 - \frac{(1-q)(1+t)}{(1-qt)}\, z_2,\\
P_{1,1}^{(q,t)} &= z_1 z_2 - \frac{(1-q)(1+t+t^2)}{(1-qt^2)}\, z_3,\\
P_{3,0}^{(q,t)} &= z_1^3 - \frac{(1-q)(2+q+t+2qt)}{1-q^2t}\, z_1 z_2 + \frac{(1-q)^2(1+q)(1+t+t^2)}{(1-qt)(1-q^2t)}\, z_3,
\end{aligned}
$$

$$
\begin{aligned}
P_{2,1}^{(q,t)} &= z_1^2 z_2 - \frac{(1-q)(1+t)}{1-qt}\, z_2^2 - \frac{(1-q^2)(1-qt^3)}{(1-qt)^2(1+qt)}\, z_1 z_3,\\
P_{4,0}^{(q,t)} &= z_1^4 - \frac{(1-q)(3+2q+q^2+t+2qt+3q^2t)}{1-q^3t}\, z_1^2 z_2\\
&+ \frac{(1-q)^2(1+q+q^2)(1+t)(1+qt)}{(1-q^2t)(1-q^3t)}\, z_2^2\\
&+ \frac{(1-q)^2(1+q)(2+q+q^2+t+2qt+q^2t+t^2+qt^2+2q^2t^2)}{(1-q^2t)(1-q^3t)}\, z_1 z_3,
\end{aligned}
$$

$$
\begin{aligned}
P_{3,1}^{(q,t)} &= z_1^3 z_2 - \frac{(1-q)(2+q+t+2qt)}{1-q^2t}\, z_1 z_2^2 - \frac{(1-q^3)(1-q^2t^3)}{(1-q^2t)(1-q^3t^2)}\, z_1^2 z_3\\
&+ \frac{(1-q)^2}{(1-q^2t)(1-q^3t^2)}\\
&\times \big(2+2q+q^2+2t+4qt+3q^2t+q^3t+t^2+\\
&3qt^2+4q^2t^2+2q^3t^2+qt^3+2q^2t^3+2q^3t^3\big)\, z_2 z_3;
\end{aligned}
$$

$$
\begin{aligned}
P_{2,2}^{(q,t)} &= z_1^2 z_2^2 - \frac{(1-q)(1+t)}{1-qt}\,(z_2^3 + z_1^3 z_3)\\
&- \frac{(1-q)(3q+q^2-3t+2q^2t+q^3t-t^2-2qt^2+3q^3t^2-qt^3-3q^2t^3)}{(1-qt)(1-q^3t^2)}\, z_1 z_2 z_3\\
&+ \frac{(1-q)^2(1+q)(1+t+t^2)(q-t+q^2t-qt^2+q^3t^2-q^2t^3)}{(1-qt)(1-q^2t^2)(1-q^3t^2)}\, z_3^2.
\end{aligned}
$$

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