Deformation theory of Calabi-Yau threefolds and Certain invariants of singularities

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Let Z be a Calabi-Yau threefold with isolated rational Gorenstein singularities, that is, Z is a projective variety of dimension three with isolated rational Gorenstein singularities, with trivial dualizing sheaf, and finally with $H^1(Z, \mathcal{O}_Z) = 0$. In the previous works [Na 1, Na-St, Na 2], we have considered the following problems in the case where Z has only isolated rational hypersurface singularities or only terminal singularities:

(1) When is the Kuranishi space Def(Z) smooth ?

(2) When can we deform Z to a smooth Calabi-Yau threefold ?

By [Na 1] the problem (1) always has a positive answer, and by [Na-St, Na 2] we can give rather satisfactory, sufficient (or necessary) conditions for Z to be smoothed by a flat deformation.

In turn, when Z does not necessarily have only hypersurface singularities, the situations are rather complicated; in [Gr] Mark Gross has studied it. For example, Def(Z) is not necessarily reduced (cf. [Gr, Example(2.4)]). As for (2), if Z has a quotient singularity, then Z is not, even locally, smoothable by Sclessinger [Sch]. But it should be remarked that some sufficient conditions for smoothings are obtained in [Gr]. In this paper we shall prove the following:

Theorem *Let Z be a Calabi-Yau threefold with isolated rational Gorenstein singularities. Assume that*

(1) *Z is $\mathbf{Q}$-factorial*;

(2) *every singularity on Z is locally smoothable*, and

(3) *the semi-universal deformation space $Def(Z, x)$ of each singularity (Z, x) is smooth.*

Then Z is smoothable by a flat deformation.

Example Let Z be a $\mathbf{Q}$-factorial Calabi-Yau threefold which admits only isolated rational Gorenstein codimension 3 points. Then Z is smoothable. In fact, any such point is a Pfaffian subscheme by [B-E]. By [K-L], it is smoothable. On the other hand, the semi-universal deformation space of a normal, Gorenstein codimension 3 point is smooth by [W].

Our method is, in principle, the same as [Na-St]. Let (X, x) be the germ of an isolated rational singularity. Let $\pi : (Y, E) \to (X, x)$ be a good resolution of (X, x), that is, E is a divisor with simple normal crossing. Define $\mu(X, x)$ to be the dimension of the cokernel of the map $(2\pi i)^{-1} d\log : H^1(Y, \mathcal{O}_Y^*) \otimes_{\mathbf{Z}} \mathbf{C} \to H^1(Y, \Omega_Y^1)$. By [Na 1, §5], $\mu(X, x)$ is independent of the choice of the resolution. We also define $\sigma(X, x)$ to be the rank of the finitely generated Abelian group $Weil(X, x)/Cart(X, x)$, where $Weil(X, x)$ (resp. $Cart(X, x)$) be the group of Weil (resp. Cartier) divisors of (X, x). The following is a key result.

Theorem 1. *Let (X, x) be an isolated rational Gorenstein singularity of dimension three. Assume that $\mu(X, x) = \sigma(X, x) = 0$. Then (X, x) is rigid.*

Proof Put $U = X \backslash x$. Identifying U with $\pi^{-1}(U)$, we have the map $\tau : H^1(U, \Omega_U^2) \to H_E^2(Y, \Omega_Y^2(logE)(-E))$ as a coboundary map of the sequence of local cohomology. By the vanishing theorem of Guillén, Navarro Aznar, Puerta and Steenbrink (cf. [St 2]), $H^2(Y, \Omega_Y^2(logE)(-E)) = 0$. Hence, τ is a surjection. On the other hand, by [Na-St, Proposition (2.1)], $\mu(X, x) = dim_{\mathbf{C}} H^1(Y, \Omega_Y^1(logE)(-E))$. Consider the map $d : H^1(Y, \Omega_Y^1(logE)(-E)) \to H^1(Y, \Omega_Y^2(logE)(-E))$. This map is a surjection by the proof of Theorem (1.1) from [Na-St]. By the assumption, $\mu(X, x) = 0$;and hence $H^1(Y, \Omega_Y^2(logE)(-E)) = 0$. From this it follows that τ is also an injection. In particular, we have $h^1(U, \Omega_U^2) = h_E^2(Y, \Omega_Y^2(logE)(-E))$. Consider the exact sequence

$$0 = H^1(Y, \Omega_Y^1(logE)(-E)) \to H^1(Y, \Omega_Y^1(logE)) \to H^1(E, \Omega_Y^1(logE) \otimes \mathcal{O}_E).$$

By duality, the middle term has the same dimension as $H_E^2(Y, \Omega_Y^2(logE)(-E))$, hence as $H^1(U, \Omega_U^2)$ by the above remark. Note that

$$H^1(E, \Omega_Y^1(logE) \otimes \mathcal{O}_E) = Gr_F^1 H_{\{x\}}^3(X, \mathbf{C}),$$

where F is the Hodge filtration of the mixed Hodge structure on $H_{\{x\}}^3(X)$ (cf. [St 1]). On the other hand, $h_{\{x\}}^3(X, \mathbf{C}) = \sigma(X, x)$ because (X, x) is an isolated rational singularity of dimension three (cf. the proof of [Na-St, Proposition (3.10)]). Thus, the third term in the exact sequence must vanish, and we have $H^1(U, \Omega_U^2) = 0$. Since (X, x) is an isolated Gorenstein singularity of dimension three, this implies that (X, x) is rigid by Schlessinger [Sch]. Q.E.D.

Proposition 2 *Let Z be a $\mathbf{Q}$-factorial Calabi-Yau threefold with isolated rational Gorenstein singularities. Let $\pi : Y \to Z$ be a resolution of Z. Let p_i $(1 \leq i \leq n)$ be the singular points on Z such that either $\mu(Z, p_i) > 0$ or $\sigma(Z, p_i) > 0$, and let E_i be the exeptional set over p_i. Let Z_i be mutually disjoint, contractible, Stein open neighborhoods of $p_i \in Z$. Set $Y_i = \pi^{-1}(Z_i)$. Consider the diagram*

$$Ext^1(\Omega_Z^1, \mathcal{O}_Z) \xrightarrow{\alpha} \bigoplus_{1 \leq i \leq n} H^0(Z_i, T_{Z_i}^1) \xleftarrow{\oplus_{1 \leq i \leq n} \beta_i} \bigoplus_{1 \leq i \leq n} H^1(Y_i, \Theta_{Y_i}).$$

Then there is an element $\eta \in Ext^1(\Omega_Z^1, \mathcal{O}_Z)$ such that $\alpha(\eta)_i \notin im(\beta_i)$ for all i.

Proof. Let $Sing(Z) = \{p_1, ..., p_n, p_{n+1}, ..., p_m\}$ and let $U = Z \backslash \{p_1, ..., p_m\}$. By [Sch] or [Na 1, §5], $Ext^1(\Omega_Z^1, \mathcal{O}_Z) \cong H^1(U, \Theta_U)$. On the other hand, $H^0(Z_i, T_{Z_i}^1) \cong H_{p_i}^2(Z, T_Z^0)$

by [Fr]. Thus, the map α is identified with the composition of the coboundary map $H^1(U,\Theta_U) \to \bigoplus_{1\le i\le m} H^2_{p_i}(Z,T^0_Z)$ of the exact sequence of local cohomology and the projection $\bigoplus_{1\le i\le m} H^2_{p_i}(Z,T^0_Z) \to \bigoplus_{1\le i\le n} H^2_{p_i}(Z,T^0_Z)$. Since $H^2_{p_i}(Z,T^0_Z) \cong H^2_{p_i}(Z,\pi_*\Omega^2_Y)$ and $\Theta_U \cong \Omega^2_U$, we have the following exact commutative diagram

$$\begin{array}{ccccc} H^1(U,\Omega^2_U) & \xrightarrow{\gamma} & H^2_E(Y,\Omega^2_Y) & \longrightarrow & H^2(Y,\Omega^2_Y) \\ \| & & \uparrow \bigoplus \phi_i & & \\ H^1(U,\Theta_U) & \xrightarrow{\alpha} & \bigoplus_{1\le i\le m} H^2_{p_i}(Z,T^0_Z) \cong H^2(Z,T^1_Z) & & \end{array}$$

Denote by ι_i the natural map $H^2_{E_i}(Y,\Omega^2_Y) \to H^2(Y,\Omega^2_Y)$. In the above diagram, ϕ_i is factorized as follows:

$$H^2_{p_i}(Z,T^1_Z) \xrightarrow{\phi_i'} H^2_{E_i}(Y,\Theta_Y) \to H^2_{E_i}(Y,\Omega^2_Y).$$

We shall prove that the map

$$\iota_i : H^2_{E_i}(Y,\Omega^2_Y) \to H^2(Y,\Omega^2_Y)$$

is not an injection for each $i \le n$. If this is proved, then we take a non-zero element $\zeta_i \in Ker(\iota_i)$ for each i. By the above diagram, there is an element $\eta \in H^1(U,\Theta_U)$ such that $\phi_i \circ \alpha(\eta) = \zeta_i \neq 0$. In particular, we have $\phi_i \circ \alpha(\eta) \neq 0$. We then see that $\alpha(\eta)_i \notin \text{image}(\beta_i)$ by the exact sequence

$$H^1(Y_i,\Theta_{Y_i}) \xrightarrow{\beta_i} H^2_{p_i}(Z,T^0_Z) \xrightarrow{\phi_i'} H^2_{E_i}(Y,\Theta_Y).$$

We shall finish the proof by showing the following claim.

Claim *The map ι_i is not an injection for $i \le n$.*

Proof We only have to prove that the dual map $\iota_i^* : H^1(Y,\Omega^1_Y) \to H^1(Y_i,\Omega^1_{Y_i})$ is not surjective. First note that $H^1(Y,\Omega^1_Y) \cong H^1(Y,\mathcal{O}^*_Y)\otimes_{\mathbf{Z}}\mathbf{C}$ because $H^2(Y,\mathcal{O}_Y) = 0$. Thus, ι_i^* is factorized as follows:

$$H^1(Y,\mathcal{O}^*_Y)\bigotimes_{\mathbf{Z}}\mathbf{C} \to H^1(Y_i,\mathcal{O}^*_{Y_i})\bigotimes_{\mathbf{Z}}\mathbf{C} \to H^1(Y_i,\Omega^1_{Y_i}).$$

The second map is an injection by [Na 1, §2, CLAIM] because (Z,p_i) is a rational singularity. Since Z is $\mathbf{Q}$-factorial, the image of the first map is the $\mathbf{C}$-vector space generated by $[E_{i,j}]$'s, where $E_{i,j}$ are two dimensional irreducible components of E_i and $[E_{i,j}]$ denote the corresponding line bundles. If $\sigma(Z,p_i) > 0$, then the first map is not a surjection, and hence ι_i^* is not surjective. On the other hand, if $\mu(Z,p_i) > 0$, then the second map is not surjective by definition. In particular, ι_i^* is not surjective. Q.E.D.

Proof of Theorem Let Z be a Calabi-Yau threefold which satisfies the conditions (1), (2) and (3) of Theorem. By (2), for every singularity on Z, either μ or σ is positive. In fact, if both of them are zero, then the singularity is rigid by Theorem 1. On the other hand, it is smoothable by (3), which is a contradiction. By the result of Gross [Gr, Theorem(2.1)], the conditions (3) assures that $Def(Z)$ is smooth. We shall use the same notation as Proposition 2. Let $Def(Z_i)$ be the semi-universal deformation space of Z_i and let $\mathcal{Z}_i$ be the semi-universal family over $Def(Z_i)$. By definition, $Def(Z_i)$ is smooth, and a general point of $Def(Z_i)$ parametrizes a smooth point. By the same argument as Theorem (2.4) from [Na-St], $Def(Z_i)$ has a stratification into Zariski locally closed, smooth subsets S_i^k ($k \geq 0$) with the following properties:

1. $\mathrm{Def}(Z_i) = \coprod_{k\geq 0} S_i^k$;
2. S_i^0 is a non-empty Zariski open subset of $\mathrm{Def}(Z_i)$, and $\mathcal{Z}_i$ is smooth over S_i^0;
3. S_i^k are of pure codimension in $\mathrm{Def}(Z_i)$ for all $k \geq 0$, and $\mathrm{codim}_{\mathrm{Def}(Z_i)} S_i^k < \mathrm{codim}_{\mathrm{Def}(Z_i)} S_i^{k+1}$;
4. If $k > l$, then $\bar{S}_i^k \cap S_i^l = \emptyset$;
5. $\mathcal{Z}_i$ has a simultaneous resolution on each S_i^k, that is, there is a resolution $\mathcal{Z}_i^k$ of $\mathcal{Z}_i \times_{\mathrm{Def}(Z_i)} S_i^k$ such that $\mathcal{Z}_i^k$ is smooth over S_i^k.

The origin of $Def(Z_i)$ is contained in the minimal stratum S_i^k. By definition, the flat family $\mathcal{Z}_i \times_{Def(Z_i)} S_i^k \to S_i^k$ admits a simultaneous resolution. This simultaneous resolution induces a resolution $\pi : Y_i \to Z_i$. Since each π_i is an isomorphism over smooth points, these fits together into a global resolution $\pi : Y \to X$. We here apply Proposition (2.3). Let $g : \mathcal{Z} \to \Delta$ be a small deformation of Z determined by $\eta \in Ext^1(\Omega_Z^1, \mathcal{O}_Z)$. It determines a holomorphic map $\varphi_i : \Delta \to Def(Z_i)$ with $\varphi_i(0) = 0$ for each i. Then the image of φ_i is not contained in S_i^k. Moreover, if we take a general point $t \in \Delta \setminus 0$, then $\varphi_i(t) \in S_i^{k'}$ for some $k' < k$ by the property 4. of the stratification. $\mathcal{Z}_t$ is also a $\mathbf{Q}$-factorial Calabi-Yau threefold with isolated rational Gorenstein singularities which satisfies the conditions (1), (2) and (3) in Theorem. In fact, $\mathbf{Q}$-factoriality, in this case, is preserved by a small deformation by Kollár-Mori [K-M, 12.1.10]. The condition (3) is satisfied because $\mathcal{Z}_i \to Def(Z_i)$ is a versal family of singularities on a suitable open neighborhood of the origin in $Def(Z_i)$ (cf. [Gra, Tyu] or [Pou, Theorem]). (2) is clearly satisfied. Thus, we can continue the same process as above for $\mathcal{Z}_t$ by using $Def(Z_i)$. Finally, we reach a Calabi-Yau threefolds whose sinularities all satisfy $\mu = \sigma = 0$. But these singularities are locally smoothable. By Theorem 1, this implies that the resulting Calabi-Yau threefold is smooth. Q.E.D.

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