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GROUP ACTIONS ON SPHERES WITH RANK ONE ISOTROPY

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Abstract. Let G be a rank two finite group, and let $\mathcal{H}$ denote the family of all rank one p-subgroups of G, for which $\operatorname{rank}_p(G) = 2$. We show that a rank two finite group G which satisfies certain group-theoretic conditions admits a finite G-CW-complex $X \simeq S^n$ with isotropy in $\mathcal{H}$, whose fixed sets are homotopy spheres.

1. Introduction

Let G be a finite group. The unit spheres $S(V)$ in finite-dimensional orthogonal representations of G provide the basic examples of smooth G-actions on spheres. Moreover, character theory reveals intricate relations between the dimensions of the fixed sets $S(V)^H$, for subgroups $H \leq G$, and the structure of the isotopy subgroups $\{G_x \,|\, x \in S(V)\}$. We are interested in understanding how these basic invariants are constrained for smooth *non-linear* finite group actions on spheres.

We say that G has *rank k* if it contains a subgroup isomorphic to $(\mathbf{Z}/p)^k$, for some prime p, but no subgroup $(\mathbf{Z}/p)^{k+1}$, for any prime p. In this paper, we use chain complex methods to study the following problem, as a step towards smooth actions.

Question. For which finite groups G, does there exist a finite G-CW-complex $X \simeq S^n$ with all isotropy subgroups of rank one ?

The isotropy assumption implies that G must have rank ≤ 2, by P. A. Smith theory (see Corollary 7.3). Since every rank one finite group can act freely on a finite complex homotopy equivalent to a sphere (Swan [13]), we will restrict to groups of rank two.

There is another group theoretical *necessary* condition related to fusion properties of the Sylow subgroups. This condition involves the rank two finite group $\mathrm{Qd}(p)$ which is the group defined as the semidirect product

$$\mathrm{Qd}(p) = (\mathbf{Z}/p \times \mathbf{Z}/p) \rtimes SL_2(p)$$

with the obvious action of $SL_2(p)$ on $\mathbf{Z}/p \times \mathbf{Z}/p$. In his thesis, Ünlü [16, Theorem 3.3] showed that $\mathrm{Qd}(p)$ does not act on a finite CW-complex $X \simeq S^n$ with rank 1 isotropy. This means that any rank two finite group which includes $\mathrm{Qd}(p)$ as a subgroup cannot admit such actions.

More generally, we say $\mathrm{Qd}(p)$ is *p'-involved in G* if there exists a subgroup $K \leq G$, of order prime to p, such that $N_G(K)/K$ contains a subgroup isomorphic to $\mathrm{Qd}(p)$. The

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argument given by Ünlü in [16, Theorem 3.3] can be extended easily to obtain the stronger necessary condition (see Proposition 6.3):

($\ast$). *Suppose that there exists a finite G-CW-complex $X \simeq S^n$ with rank 1 isotropy. Then* $\mathrm{Qd}(p)$ *is not p'-involved in G, for any odd prime p.*

In the other direction, finite groups which do not p'-involve $\mathrm{Qd}(p)$ have some interesting complex representations. Jackson [9, Theorem 47] proved that, in this case, each Sylow p-subgroup of G has a *p-effective character* which respects fusion in G. We use these characters to reduce the isotropy from p-subgroups to rank one p-subgroups.

Let $\mathcal{F}$ be a family of subgroups of G closed under conjugation and taking subgroups. For constructing group actions on CW-complexes with isotropy in the family $\mathcal{F}$, a good algebraic approach is to consider projective chain complexes over the orbit category relative to the family $\mathcal{F}$ (see [6]).

Let $\mathcal{S}_G$ denote the set of primes p such that $\mathrm{rank}_p(G) = 2$. Let $\mathcal{H}_p$ denote the family of all rank one p-subgroups $H \leq G$, for $p \in \mathcal{S}_G$, and let $\mathcal{H} = \bigcup \{H \in \mathcal{H}_p \,|\, p \in \mathcal{S}_G\}$. Our main result is the following:

Theorem A. *Let G be a rank two finite group satisfying the following two conditions:*

(i) *G does not p'-involve* $\mathrm{Qd}(p)$ *for any odd prime $p \in \mathcal{S}_G$;*

(ii) *if $1 \neq H \in \mathcal{H}_p$, then* $\mathrm{rank}_q(N_G(H)/H) \leq 1$ *for every prime $q \neq p$.*

Then there exists a finite G-CW-complex X with isotropy in $\mathcal{H}$, such that X^H is a homotopy sphere for each $H \in \mathcal{H}$.

Our construction produces new *non-linear* G-CW-complex examples, for certain groups G which do not admit any orthogonal representations V with rank one isotropy on the unit sphere $S(V)$ (see Example 7.4). In Section 7, we give the motivation for condition (ii) on the q-rank of the normalizer quotients. It is used in a crucial way (at the algebraic level) in the construction of our actions, but it is not, in general, a necessary condition for the existence of a finite G-CW-complex $X \simeq S^n$ with rank 1 isotropy (see Example 7.5). Determining the full list of necessary conditions is still an open problem.

Theorem A is an extension of our earlier joint work with Semra Pamuk [6] where we have shown that the first non-linear example, the permutation group $G = S_5$ of order 120, admits a finite G-CW-complex $X \simeq S^n$ with rank one isotropy. Theorem A implies this earlier result since for $G = S_5$, the set $\mathcal{S}_G$ includes only the prime 2 and the second condition above holds since all Sylow p-subgroups of S_5 for odd primes are cyclic. More generally, we have:

Corollary B. *Let p be a fixed prime and G be a finite group such that* $\mathrm{rank}_p(G) = 2$, *and* $\mathrm{rank}_q(G) = 1$ *for every prime $q \neq p$. If G does not p'-involve* $\mathrm{Qd}(p)$ *when $p > 2$, then there exists a finite G-CW-complex $X \simeq S^n$ with rank one p-group isotropy.*

We will obtain Theorem A from a more general technical result, Theorem 6.1, which accepts as input a suitable collection of $\mathcal{F}_p$-representations (see Definition 4.1), and produces a finite G-CW complex. Theorem 6.1 is used to construct the action in Example 7.5 for $G = A_7$ with rank one p-group isotropy. In principle, it could be used to construct other interesting non-linear examples for finite groups with specified p-group isotropy.

Here is a brief outline of the paper. We denote the orbit category relative to a family $\mathcal{F}$ by $\varGamma_G = \mathrm{Or}_{\mathcal{F}} G$, and construct projective chain complexes over $R\varGamma_G$ for various p-local coefficient rings $R = \mathbf{Z}_{(p)}$. To prove Theorem 6.1, we first introduce *algebraic homotopy representations* (see Definition 2.6), as chain complexes over $R\varGamma_G$ satisfying algebraic versions of the conditions found in tom Dieck's geometric *homotopy representations* (see [15, II.10.1], [4], [5], and Remark 3.3). In Section 2 we show that the conditions in Definition 2.6 are necessary conditions for a chain complex over $R\varGamma_G$ to be homotopy equivalent to a chain complex of a geometric homotopy representation (see Proposition 2.7). Then in Section 3 we prove conversely that algebraic homotopy representations are realizable by geometric homotopy representations.

In Section 4, we construct p-local chain complexes where the isotropy subgroups are p-groups. In Section 5, we add homology to these local models so that these modified local complexes $\mathbf{C}^{(p)}$ all have exactly the same dimension function. Results established in [6] are used to glue these algebraic complexes together over $\mathbf{Z}\varGamma_G$, and then to eliminate a finiteness obstruction. In Section 6 we combine these ingredients to give a complete proof for Theorem 6.1 and Theorem A. We end the paper with a discussion about the necessity of the conditions in Theorem A. This discussion and the examples of nonlinear actions for the groups $G = A_6$ and A_7 can be found in Section 7.

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2. Algebraic homotopy representations

Let G be a finite group and $\mathcal{F}$ be a family of subgroups of G which is closed under conjugations and taking subgroups. The orbit category $\mathrm{Or}_{\mathcal{F}} G$ is defined as the category whose objects are orbits of type G/K, with $K \in \mathcal{F}$, and where the morphisms from G/K to G/L are given by G-maps:

$$\mathrm{Mor}_{\mathrm{Or}_{\mathcal{F}} G}(G/K, G/L) = \mathrm{Map}_G(G/K, G/L).$$

The category $\varGamma_G = \mathrm{Or}_{\mathcal{F}} G$ is a small category, and we can consider the module category over $\varGamma_G$. Let R be a commutative ring with unity. A (right) $R\varGamma_G$-module M is a contravariant functor from $\varGamma_G$ to the category of R-modules. We denote the R-module $M(G/K)$ simply by $M(K)$ and write $M(f)\colon M(L) \to M(K)$ for a G-map $f\colon G/K \to G/L$.

The category of $R\varGamma_G$-modules is an abelian category, so the usual concepts of homological algebra, such as kernel, direct sum, exactness, projective module, etc., exist for $R\varGamma_G$-modules. Note that an exact sequence of $R\varGamma_G$-modules $0 \to A \to B \to C \to 0$ is exact if and only if

$$0 \to A(K) \to B(K) \to C(K) \to 0$$

is an exact sequence of R-modules for every $K \in \mathcal{F}$. For an $R\varGamma_G$-module M the R-module $M(K)$ can also be considered as an $RW_G(K)$-module in an obvious way where $W_G(K) = N_G(K)/K$. We will follow the convention in [11] and consider $M(K)$ as a right $RW_G(K)$-module. In particular, we will consider the sequence above as an exact sequence of right $RW_G(K)$-modules. The further details about the properties of modules over the orbit category, such as the definitions of free and projective modules, can be found in [6] (see also Lück [11, §9,§17] and tom Dieck [15, §10-11]).

In this section we consider chain complexes $\mathbf{C}$ of $R\varGamma_G$-modules. When we say a chain complex we always mean a non-negative complex, i.e., $\mathbf{C}_i = 0$ for $i < 0$. We call a chain complex $\mathbf{C}$ *projective* (resp. *free*) if for all $i \geq 0$, the modules $\mathbf{C}_i$ are projective (resp. free). We say that a chain complex $\mathbf{C}$ is *finite* if $\mathbf{C}_i = 0$ for $i > n$, and the chain modules $\mathbf{C}_i$ are all finitely-generated $R\varGamma_G$-modules.

Given a G-CW-complex X, associated to it, there is a chain complex of $R\varGamma_G$-modules

$$\mathbf{C}(X^{?}; R): \quad \cdots \to R[X_n{}^{?}] \xrightarrow{\partial_n} R[X_{n-1}{}^{?}] \to \cdots \xrightarrow{\partial_1} R[X_0{}^{?}] \to 0$$

where X_i denotes the set of i-dimensional cells in X and $R[X_i{}^{?}]$ is the $R\varGamma_G$-module defined by $R[X_i{}^{?}](H) = R[X_i^H]$. We denote the homology of this complex by $H_*(X^{?}; R)$. If the family $\mathcal{F}$ includes the isotropy subgroups of X, then the complex $\mathbf{C}(X^{?}; R)$ is a chain complex of free $R\varGamma_G$-modules.

Given a finite dimensional G-CW-complex X, there is a *dimension function*

$$\operatorname{Dim} X \colon \mathscr{S}(G) \to \mathbf{Z},$$

given by $(\operatorname{Dim} X)(H) = \dim X^H$ for all $H \in \mathscr{S}(G)$ where $\mathscr{S}(G)$ denote the set of all subgroups of G. In a similar way, we define the following.

Definition 2.1. The *dimension function* of a finite dimensional chain complex $\mathbf{C}$ over $R\varGamma_G$ is defined as the function $\operatorname{Dim} \mathbf{C} \colon \mathscr{S}(G) \to \mathbf{Z}$ given by

$$(\operatorname{Dim} \mathbf{C})(H) = \dim \mathbf{C}(H)$$

for all $H \in \mathcal{F}$, where the dimension of a chain complex of R-modules is defined as the largest integer d such $C_d \neq 0$. If $\mathbf{C}(H)$ is the zero complex or if H is a subgroup such that $H \notin \mathcal{F}$, then we define $(\operatorname{Dim} \mathbf{C})(H) = -1$.

Remark 2.2. Recall that a function $\underline{n} \colon \mathscr{S}(G) \to \mathbf{Z}$ is called a *super class function* if it is constant on conjugacy classes of G. We say that a super class function $\underline{n} : \mathcal{S}(G) \to \mathbf{Z}$ is *defined on* $\mathcal{F}$, if $\underline{n}(H) = -1$ for all subgroups $H \notin \mathcal{F}$. For such a function, we sometimes use the notation $\underline{n} : \mathcal{F} \to \mathbf{Z}$ instead of $\underline{n} : \mathscr{S}(G) \to \mathbf{Z}$.

In this sense, the dimension function $\operatorname{Dim} \mathbf{C}$ of a chain complex $\mathbf{C}$ over $R\varGamma_G$ is a super class function defined on $\mathcal{F}$. In a similar way, we can define the *homological dimension function* of a chain complex $\mathbf{C}$ of $R\varGamma_G$-modules as the function $\operatorname{HomDim} \mathbf{C} \colon \mathcal{F} \to \mathbf{Z}$ where for each $H \in \mathcal{F}$, the integer

$$(\operatorname{HomDim} \mathbf{C})(H) = \operatorname{hdim} \mathbf{C}(H)$$

is defined as the homological dimension of the complex $\mathbf{C}(H)$.

Let us write $(H) \leq (K)$ whenever $H^g \leq K$ for some $g \in G$. Here (H) denotes the set of subgroups conjugate to H in G. The notation $(H) < (K)$ means that $(H) \leq (K)$ but $(H) \neq (K)$.

Definition 2.3. We call a function $\underline{n} \colon \mathscr{S}(G) \to \mathbf{Z}$ *monotone* if it satisfies the property that $\underline{n}(K) \leqslant \underline{n}(H)$ whenever $(H) \leq (K)$. We say that a monotone function $\underline{n}$ is *strictly monotone* if $\underline{n}(K) < \underline{n}(H)$, whenever $(H) < (K)$. □

We have the following:

Lemma 2.4. *The dimension function of a projective chain complex of $R\Gamma_G$-modules is a monotone function.*

Proof. By the decomposition theorem for projective $R\Gamma_G$-modules [15, Chap. I, Theorem 11.18], every projective $R\Gamma_G$-module P is of the form $P \cong \oplus_H E_H P_H$ where P_H is a projective $N_G(H)/H$-module. If $\underline{n}(K) = n$, then $\mathbf{C}_n$ must have a summand $E_H P_H$ with $(K) \leq (H)$. But then we will have $\mathbf{C}_n(L) \neq 0$ for every $(L) \leq (K)$. Also if $H \notin \mathcal{F}$, then by the closure of $\mathcal{F}$ under taking subgroups, there can not be any $K \in \mathcal{F}$ with $(H) < (K)$. So we have $(\operatorname{Dim} \mathbf{C})(K) = -1$ for all $K \leq G$ with $(H) \leq (K)$. □

We are particularly interested in chain complexes which have the homology of a sphere when evaluated at every $K \in \mathcal{F}$. To specify the restriction maps in dimension zero, we will consider chain complexes which are augmented, i.e., chain complexes $\mathbf{C}$ together with a map $\varepsilon\colon \mathbf{C}_0 \to \underline{R}$ such that $\varepsilon \circ \partial_1 = 0$ where $\underline{R}$ denotes the constant functor. We sometimes consider ε as a chain map by considering $\underline{R}$ as a chain complex over $R\Gamma_G$ which is concentrated at zero. By the *reduced homology* of an augmented complex $\varepsilon\colon \mathbf{C} \to \underline{R}$, we always mean the homology of the chain complex

$$\widetilde{\mathbf{C}} = \{\cdots \to C_n \xrightarrow{\partial_n} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \underline{R} \to 0\}$$

where $\underline{R}$ is considered to be at dimension -1.

Definition 2.5. Let $\underline{n}$ be a super class function defined on $\mathcal{F}$, and let $\mathbf{C}$ be a chain complex over $R\Gamma_G$.

(i) We say that $\mathbf{C}$ an *R-homology $\underline{n}$-sphere* if $\mathbf{C}$ is an augmented complex with the reduced homology of $\mathbf{C}(K)$ is the same as the reduced homology of an $\underline{n}(K)$-sphere (with coefficients in R) for all $K \in \mathcal{F}$.

(ii) We say that $\mathbf{C}$ is *oriented* if the $W_G(K)$-action on the homology of $\mathbf{C}(K)$ is trivial for all $K \in \mathcal{F}$.

In transformation group theory, a G-CW-complex X is called a *homotopy representation* if it has the property that X^H is homotopy equivalent to the sphere $S^{n(H)}$ where $n(H) = \dim X^H$ for every $H \leq G$ (see tom Dieck [15, Section II.10]). Note that we do not assume that the dimension function is strictly monotone as in Definition II.10.1 in [15].

In [15, II.10], there is a list of properties that are satisfied by homotopy representations. We will use algebraic versions of these properties to define an analogous notion for chain complexes.

Definition 2.6. Let $\mathbf{C}$ be a finite projective chain complex over $R\Gamma_G$, which is an R-homology $\underline{n}$-sphere. We say $\mathbf{C}$ is an *algebraic homotopy representation* (over R) if

(i) The function $\underline{n}$ is a monotone function.

(ii) If $H, K \in \mathcal{F}$ are such that $n = \underline{n}(K) = \underline{n}(H)$, then for every G-map $f\colon G/H \to G/K$ the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an R-homology isomorphism.

(iii) Suppose $H, K, L \in \mathcal{F}$ are such that $H \leq K, L$ and let $M = \langle K, L\rangle$ be the subgroup of G generated by K and L. If $n = \underline{n}(H) = \underline{n}(K) = \underline{n}(L) > -1$, then $M \in \mathcal{F}$ and $n = \underline{n}(M)$.

In the remainder of this section we will assume that R is a principal ideal domain. The main examples for us are $R = \mathbf{Z}_{(p)}$ or $R = \mathbf{Z}$.

Proposition 2.7. *Let* $\mathbf{C}$ *be a finite projective chain complex over* $R\varGamma_G$*, which is an* R*-homology* $\underline{n}$*-sphere. If the equality* $\underline{n} = \operatorname{Dim}\mathbf{C}$ *holds, then* $\mathbf{C}$ *is an algebraic homotopy representation.*

Under condition (iii) of Definition 2.6, the isotropy family $\mathcal{F}$ has an important maximality property.

Corollary 2.8. *Let* $\mathbf{C}$ *be a projective chain complex of* $R\varGamma_G$*-modules, If condition* (iii) *holds, then the set of subgroups* $\mathcal{F}_H = \{K \in \mathcal{F} \,|\, (H) \leq (K),\ \underline{n}(K) = \underline{n}(H) > -1\}$ *has a unique maximal element, up to conjugation.*

Before we prove Proposition 2.7, we make some observations and give some definitions for projective chain complexes.

Lemma 2.9. *Let* $\mathbf{C}$ *be a projective chain complex over* $R\varGamma_G$*. Then, for every* G*-map* $f\colon G/H \to G/K$*, the induced map* $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ *is an injective map with an* R*-torsion free cokernel.*

Proof. It is enough to show that if P a projective $R\varGamma_G$-module, then for every G-map $f\colon G/H \to G/K$, the induced map $P(f)\colon P(K) \to P(H)$ is an injective map with a torsion free cokernel. Since every projective module is a direct summand of a free module, it is enough to prove this for a free module $P = R[X^?]$. Let $f\colon G/H \to G/K$ be the G-map defined by $f(H) = gK$. Then the induced map $P(f)\colon R[X^K] \to R[X^H]$ is the linearization of the map $X^K \to X^H$ given by $x \mapsto gx$. Since this map is one-to-one, we can conclude that $P(f)$ is injective with torsion free cokernel. □

When $H \leq K$ and $f\colon G/H \to G/K$ is the G-map defined by $f(H) = K$, then we denote the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ by r_H^K and call it the *restriction* map. When H and K are conjugate, i.e., $K = H^g$ for some $g \in G$, then the map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ induced by the G-map $f\colon G/H \to G/K$ defined by $f(H) = gK$ is called the *conjugation* map and usually denoted by c_K^g. Note that every G-map can be written as a composition of two G-maps of the above two types, so every induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ can be written as a composition of restriction and conjugation maps.

Since conjugation maps have inverses, they are always isomorphisms. So, the condition (ii) of Definition 2.6 is actually a statement only about restriction maps. To study the restriction maps more closely, we consider the image of $r_H^K\colon \mathbf{C}(K) \to \mathbf{C}(H)$ for a pair $H \leq K$ and denote it by $\mathbf{C}_H^K$. Note that $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ as a chain complex of R-modules. Also note that if $\mathbf{C}$ is a projective chain complex, then $\mathbf{C}_H^K$ is isomorphic to $\mathbf{C}(K)$, as a chain complex of R-modules, by Lemma 2.9.

Lemma 2.10. *Let* $\mathbf{C}$ *be a projective chain complex over* $R\varGamma_G$*. Suppose that* $K, L \in \mathcal{F}$ *such that* $H \leq K$ *and* $H \leq L$*, and let* $M = \langle K, L\rangle$ *be the subgroup generated by* K *and* L*. If* $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$ *then* $M \in \mathcal{F}$ *and we have* $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$*.*

Proof. As before it is enough to prove this for a free $R\varGamma_G$-module $P = R[X^?]$ where X is a G-set whose isotropy subgroups lie in $\mathcal{F}$. Note that the restriction maps r_H^K and r_H^L are linearizations of the maps $X^K \to X^H$ and $X^L \to X^H$, respectively, which are defined by inclusion of subsets. Then it is clear that the intersection of images of r_H^K and r_H^L would be $R[X^K \cap X^L]$ considered as an R-submodule of $R[X^H]$. There is a well known equality $X^K \cap X^L = X^M$ for fixed point sets. Therefore, if $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$, then we must have $X^M \neq \emptyset$. This implies that $M \in \mathfrak{F}$ and that $\operatorname{im} r_H^K \cap \operatorname{im} r_H^L = \operatorname{im} r_H^M$. □

Now, we are ready to prove Proposition 2.7.

Proof of Proposition 2.7. The first condition in Definition 2.6 follows from Lemma 2.4. For (ii) and (iii), we use the arguments similar to the arguments given in II.10.12 and II.10.13 in [15].

To prove (ii), let $f\colon G/H \to G/K$ be a G-map. By Lemma 2.9, the induced map $\mathbf{C}(f) : \mathbf{C}(K) \to \mathbf{C}(H)$ is injective with torsion free cokernel. Let $\mathbf{D}$ denote the cokernel of $\mathbf{C}(f)$. Then we have a short exact sequence of R-modules

$$0 \to \mathbf{C}(K) \to \mathbf{C}(H) \to \mathbf{D} \to 0$$

where both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have dimension n. Now consider the long exact *reduced* homology sequence (with coefficients in R) associated to this short exact sequence:

$$\cdots \to 0 \to H_{n+1}(\mathbf{D}) \to H_n(\mathbf{C}(K)) \xrightarrow{f^*} H_n(\mathbf{C}(H)) \to H_n(\mathbf{D}) \to \cdots$$

Note that $\mathbf{D}$ has dimension less than or equal to n, so $H_{n+1}(\mathbf{D}) = 0$ and $H_n(\mathbf{D})$ is torsion free. Since $H_n(\mathbf{C}(K)) = H_n(\mathbf{C}(H)) = R$, we obtain that f^* is an isomorphism. Since both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have no other reduced homology, we conclude that $\mathbf{C}(f)$ induces an R-homology isomorphism between associated augmented complexes. Since the induced map $\underline{R}(f) : \underline{R}(K) \to \underline{R}(H)$ is the identity map $\mathrm{id} : R \to R$, the chain map $\mathbf{C}(f)\colon \mathbf{C}(L) \to \mathbf{C}(K)$ is an R-homology isomorphism.

To prove (iii), observe that there is a Mayer-Vietoris type exact sequence associated to the pair of complexes $\mathbf{C}_H^K$ and $\mathbf{C}_H^L$ which gives an exact sequence of the form

$$0 \to H_n(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K) \oplus H_n(\mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K + \mathbf{C}_H^L) \to H_{n-1}(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to 0.$$

Here we again take the homology sequence as the reduced homology sequence.

Let $i^K\colon \mathbf{C}_H^K \to \mathbf{C}(H)$, $i_H^L\colon \mathbf{C}_H^L \to \mathbf{C}(H)$, and $j\colon \mathbf{C}_H^K + \mathbf{C}_H^L \to \mathbf{C}(H)$ denote the inclusion maps. We have zero on the left-most term since $\mathbf{C}_H^K + \mathbf{C}_H^L$ is an n-dimensional complex. To see the zero on the right-most term, note that by Lemma 2.9, $\mathbf{C}_H^K \cong \mathbf{C}(K)$ and $\mathbf{C}_H^L \cong \mathbf{C}(L)$ as chain complexes of R-modules, so they have the same homology. This gives that $H_i(\mathbf{C}_H^K) = H_i(\mathbf{C}_H^L) = 0$ for $i \leq n-1$.

Also note that by part (ii), the composition

$$H_n(\mathbf{C}(K)) \cong H_n(\mathbf{C}_H^K) \xrightarrow{i_*^K} H_n(\mathbf{C}_H^K + \mathbf{C}_H^L) \xrightarrow{j_*} H_n(\mathbf{C}(H))$$

is an isomorphism. So, j_* is surjective. Since $H_{n+1}(\mathbf{C}(H)/(\mathbf{C}_H^K + \mathbf{C}_H^L)) = 0$, we see that j_* is also injective. Therefore, j_* is an isomorphism. This implies that i_*^K is an isomorphism. Similarly one can show that $i_*^L\colon H_n(\mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K + \mathbf{C}_H^L)$ is also an isomorphism. Using these isomorphisms and looking at the exact sequence above, we conclude that $H_n(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \cong R$ and $H_i(\mathbf{C}_H^K \cap \mathbf{C}_H^L) = 0$ for $i \leq n-1$. So, $\mathbf{C}_H^K \cap \mathbf{C}_H^L$ is an R-homology n-sphere.

Since $n > -1$, this implies that $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$, and hence $M = \langle K, L\rangle \in \mathcal{F}$ by Lemma 2.10. Moreover, $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$. This proves that $\underline{n}(M) = n$ as desired. □

3. A Realization Theorem for Chain Complexes over $\mathbf{Z}\Gamma_G$

In [6], we proved the following realization theorem for free $\mathbf{Z}\Gamma_G$-module chain complexes, with respect to any family $\mathcal{F}$, which are $\mathbf{Z}$-homology $\underline{n}$-spheres satisfying certain extra conditions.

Theorem 3.1 ([6, Theorem 8.10], [12])**.** *Let $\mathbf{C}$ be a finite free chain complex of $\mathbf{Z}\Gamma_G$-modules which is a homology $\underline{n}$-sphere. Suppose that $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$. If $\mathbf{C}_i(H) = 0$ for all $i > \underline{n}(H) + 1$, and all $H \in \mathcal{F}$, then there is a finite G-CW-complex X such that $\mathbf{C}(X^?; \mathbf{Z})$ is chain homotopy equivalent to $\mathbf{C}$ as chain complexes of $\mathbf{Z}\Gamma_G$-modules.*

Note that a homology $\underline{n}$-sphere $\mathbf{C}$ with $\operatorname{Dim}\mathbf{C} = \underline{n}$, and $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, will automatically satisfy these conditions. So, it is interesting to ask under what conditions a projective chain complex $\mathbf{C}$ which is an $\mathbf{Z}$-homology $\underline{n}$-sphere is chain homotopy equivalent to one where the equality $\operatorname{Dim}\mathbf{C} = \underline{n}$ holds. It turns out that the conditions (i), (ii), and (iii) of Definition 2.6 are exactly what is needed. In other words, the additional input needed is that $\mathbf{C}$ should be an algebraic homotopy representation.

The main purpose of this section is prove the following theorem.

Theorem 3.2. *Let $\mathbf{C}$ be a finite free chain complex of $\mathbf{Z}\Gamma_G$-modules which is a homology $\underline{n}$-sphere. If $\mathbf{C}$ is an algebraic homotopy representation, then $\mathbf{C}$ is chain homotopy equivalent to a finite free chain complex $\mathbf{D}$ for which $\operatorname{Dim}\mathbf{D} = \underline{n}$. If, in addition, $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, then there is a finite G-CW-complex X such that $\mathbf{C}(X^?; \mathbf{Z})$ is chain homotopy equivalent to $\mathbf{C}$ as chain complexes of $\mathbf{Z}\Gamma_G$-modules.*

Remark 3.3. The construction actually produces a finite G-CW-complex X with the additional property that all the non-empty fixed sets X^H are simply-connected. Moreover, by construction, $W_G(H) = N_G(H)/H$ will act trivially on the homology of X^H. Therefore X will be an *oriented* geometric homotopy representation (in the sense of tom Dieck). From the perspective of Theorem A, since we don't specify any dimension function, a G-CW-complex X with all fixed sets X^H integral homology spheres will lead (by three-fold join) to a homotopy representation. The same necessary and sufficient conditions for existence apply.

The last sentence of Theorem 3.2 follows from Theorem 3.1. The first part says that under the conditions given in the theorem, the complex $\mathbf{C}$ is homotopy equivalent to a finite free chain complex $\mathbf{D}$ with $\operatorname{Dim}\mathbf{D}(H) = \operatorname{HomDim}\mathbf{D}(H) = \underline{n}(H)$ for every $H \in \mathcal{F}$.

In the remainder of this section we will again assume that R is a principal ideal domain. The main examples for us are $R = \mathbf{Z}_{(p)}$ or $R = \mathbf{Z}$, as before.

Definition 3.4. We say a chain complex $\mathbf{C}$ of $R\Gamma_G$-modules is *tight at* $H \in \mathcal{F}$ if

$$\operatorname{Dim}\mathbf{C}(H) = \operatorname{HomDim}\mathbf{C}(H).$$

We call a chain complex of $R\Gamma_G$-modules *tight* if it is tight at every $H \in \mathcal{F}$.

For possible future applications we will deduce Theorem 3.2 from the following algebraic result, together with Theorem 3.1.

Theorem 3.5. *Let* $\mathbf{C}$ *be a finite free chain complex of* $R\Gamma_G$*-modules which is a homology* $\underline{n}$*-sphere. If* $\mathbf{C}$ *is an algebraic homotopy representation over* R*, then* $\mathbf{C}$ *is chain homotopy equivalent to a finite free chain complex* $\mathbf{D}$ *which is tight.*

So to prove Theorem 3.5, we need to show that the complex $\mathbf{C}$ can be made *tight* at each $H \in \mathcal{F}$ by replacing it with a chain complex homotopic to $\mathbf{C}$.

Let H be a *maximal element* in $\mathcal{F}$. Consider the subcomplex $\mathbf{C}^{(H)}$ of $\mathbf{C}$ formed by free summands of $\mathbf{C}$ isomorphic to $R[G/H^{?}]$. The complex $\mathbf{C}^{(H)}$ is a complex of isotypic modules of type $R[G/H^{?}]$. Recall that free $R\Gamma_G$-module F is called *isotypic* of type G/H if it is isomorphic to a direct sum of copies of a free module $R[G/H^{?}]$, for some $H \in \mathcal{F}$. For extensions involving isotypic modules we have the following:

Lemma 3.6. *Let*

$$\mathcal{E}\colon 0 \to F \to F' \to M \to 0$$

be a short exact sequence of $R\Gamma_G$*-modules such that both* F *and* F' *are isotypic free modules of the same type* G/H*. If* $M(H)$ *is* R*-torsion free, then* $\mathcal{E}$ *splits and* M *is stably free.*

Proof. This is Lemma 8.6 of [6]. The assumption that R is a principal ideal domain ensures that finitely-generated R-torsion free modules are free. □

Note that $\mathbf{C}^{(H)}(H) = \mathbf{C}(H)$, since H is maximal in $\mathcal{F}$. This means that $\mathbf{C}^{(H)}$ is a finite free chain complex over $R\Gamma_G$ of the form

$$\mathbf{C}^{(H)}\colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0$$

which is a R-homology $\underline{n}(H)$-sphere, with $\underline{n}(H) \leq d$.

Lemma 3.7. *Let* $\mathbf{C}$ *be a finite free chain complex of* $R\Gamma_G$*-modules. Then* $\mathbf{C}$ *is chain homotopy equivalent to a finite free chain complex* $\mathbf{D}$ *which is tight at every maximal element* $H \in \mathcal{F}$.

Proof. We apply [6, Proposition 8.7] to the subcomplex $\mathbf{C}^{(H)}$, for each maximal element $H \in \mathcal{F}$. The key step is provided by Lemma 3.6. □

To make the complex $\mathbf{C}$ tight at every $H \in \mathcal{F}$ we use a downward induction, but the situation at an intermediate step is more complicated than the first step considered above.

Suppose that $H \in \mathcal{F}$ is such that $\mathbf{C}$ tight at every $K \in \mathcal{F}$ such that $(K) > (H)$. Let $\mathbf{C}^H$ denote the subcomplex of $\mathbf{C}$ with free summands of type $R[G/K^{?}]$ satisfying $(H) \leq (K)$. In a similar way, we can define the subcomplex $\mathbf{C}^{>H}$ of $\mathbf{C}$ whose free summands are of type $R[G/K^{?}]$ with $(H) < (K)$. The complex $\mathbf{C}^{>H}$ is a subcomplex of $\mathbf{C}^H$. Let us denote the quotient complex $\mathbf{C}^H/\mathbf{C}^{>H}$ by $\mathbf{C}^{(H)}$. As before the complex $\mathbf{C}^{(H)}$ is isotypic with isotropy type $R[G/H^{?}]$. We have a short exact sequence of chain complexes of free $R\Gamma_G$-modules

$$0 \to \mathbf{C}^{>H} \to \mathbf{C}^H \to \mathbf{C}^{(H)} \to 0.$$

By evaluating at H, we obtain an exact sequence of chain complexes

$$0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}^{H}(H) \to \mathbf{C}^{(H)}(H) \to 0$$

which is just the sequence

$$0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$$

defining the splitting functor S_H (see [11, Lemma 9.26]) Note that we also have a sequence

$$0 \to \mathbf{C}^{H} \to \mathbf{C} \to \mathbf{C}/\mathbf{C}^{H} \to 0.$$

If we can show that $\mathbf{C}^H$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at H, then by push-out of $\mathbf{D}'$ along the injective map $\mathbf{C}^H \to \mathbf{C}$, we can find a complex $\mathbf{D}$ homotopy equivalent to $\mathbf{C}$ which is tight at every $K \in \mathcal{F}$ with $(H) \leq (K)$. So it is enough to show that $\mathbf{C}^H$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at H.

Lemma 3.8. *Let $\mathbf{C}$ be a finite free chain complex of $R\Gamma_G$-modules which is tight at every $K \in \mathcal{F}$ with $(H) < (K)$. Suppose that $n = \operatorname{hdim} \mathbf{C}(H) \geq \dim \mathbf{C}(K)$ for all $(H) < (K)$ and that $H_{n+1}(S_H\mathbf{C}) = 0$. Then $\mathbf{C}^H$ is homotopy equivalent to a finite free chain complex $\mathbf{D}'$ which is tight at every $K \in \mathcal{F}$ with $(H) \leq (K)$.*

Proof. We first observe that $\mathbf{C}^{>H}$ has dimension $\leq n$, since $\mathbf{C}^{>H}(K) = \mathbf{C}(K)$ for $(H) < (K)$, and $\dim \mathbf{C}(K) \leq n$. Let $d = \dim \mathbf{C}(H)$. If $d = n$, then we are done, so assume that $d > n$. Then $\dim \mathbf{C}^{(H)} = d$, and $\mathbf{C}^{(H)}$ is a complex of the form

$$\mathbf{C}^{(H)} \colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

We claim that the first map in the above chain complex is injective. Note that since $\mathbf{C}^{(H)}$ is isotypic of type (H), it is enough to show that this map is injective when it is calculated at H. In other words we claim that $H_d(\mathbf{C}^{(H)}(H)) = H_d(S_H\mathbf{C}) = 0$ when $d > n$. To show this consider the short exact sequence $0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$. Since the complex $\mathbf{C}^{>H}$ has dimension $\leq n$, the corresponding long exact sequence gives that $H_d(S_H\mathbf{C}) \cong H_d(\mathbf{C}(H)) = 0$ when $d > n+1$. If $d = n+1$, then this is true by the assumption in the lemma. Now we apply [6, Proposition 8.7] to $\mathbf{C}^{(H)}$ to obtain a tight complex $\mathbf{D}'' \simeq \mathbf{C}^{(H)}$, and then let $\mathbf{D}' \simeq \mathbf{C}^H$ denote the pullback of $\mathbf{D}''$ along the surjection $\mathbf{C}^H \to \mathbf{C}^{(H)}$. □

To complete the proof of Theorem 3.5, we need to show that if $\mathbf{C}$ is a finite free chain complex of $R\Gamma_G$-modules which is an R-homology $\underline{n}$-sphere, satisfying the conditions (i), (ii), and (iii) in Definition 2.6, then the conditions in Lemma 3.8 hold at an intermediate step of the downward induction.

Note that the condition about the dimensions follows from the condition (i) since when $\underline{n}$ is monotone, we have

$$\operatorname{HomDim} \mathbf{C}(H) = \underline{n}(H) \geq \underline{n}(K) = \operatorname{HomDim} \mathbf{C}(K) = \operatorname{Dim}(K)$$

for all $K \in \mathcal{F}$ with $(H) < (K)$. The other condition in the lemma, that $H_{n+1}(S_H\mathbf{C}) = 0$, follows from the conditions (ii) and (iii) but it is more involved to show this (see Corollary 3.11 below).

In the rest of the section, we assume that $\mathbf{C}$ is a finite projective chain complex of $R\Gamma_G$-modules, which is an R-homology $\underline{n}$-sphere, and satisfies the conditions (i), (ii), and (iii) in Definition 2.6. Assume also that $\mathbf{C}$ is tight for all $K \in \mathcal{F}$ with $(H) < (K)$ for some fixed subgroup $H \in \mathcal{F}$. Let $\mathcal{K}_H$ denote the set of all subgroups $K \in \mathcal{F}$ such that $H < K$ and $\underline{n}(H) = \underline{n}(K)$. By condition (iii) of Definition 2.6, this collection has a unique maximal element M. Let $\mathbf{C}_H^K$ denote the image of the restriction map

$$r_H^K \colon \mathbf{C}(K) \to \mathbf{C}(H),$$

for every $K \in \mathcal{F}$ with $H \leq K$. Note that $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ and by Lemma 2.9, it is isomorphic to $\mathbf{C}(K)$. Moreover, if $K \in \mathcal{K}_H$, then by condition (ii), the subcomplex $\mathbf{C}_H^K$ is an R-homology n-sphere and the map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{C}_H^K)$$

induced by the inclusion map $\mathbf{C}_H^M \hookrightarrow \mathbf{C}_H^K$ is an isomorphism. More generally, the following also holds.

Lemma 3.9. *Let $\mathbf{C}$ and $H \in \mathcal{F}$ be as above, and let $K_1, \dots, K_m$ be a set of subgroups in $\mathcal{K}_H$. Then the subcomplex $\sum_{i=1}^m \mathbf{C}_H^{K_i}$ is an R-homology n-sphere and the map*

$$H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \tag{3.10}$$

induced by the inclusion maps is an isomorphism.

Proof. The case $m = 1$ follows from the remarks above. For $m > 1$, we have the following Mayer-Vietoris type long exact sequence

$$0 \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_m) \to H_{n-1}(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to$$

where $\mathbf{D}_j = \sum_{i=1}^j \mathbf{C}_H^{K_i}$ for $j = m-1, m$. By the inductive assumption, we know that $\mathbf{D}_{m-1}$ is an R-homology n-sphere and the map $H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1})$ is an isomorphism.

Note that

$$\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m} = (\sum_{i=1}^{m-1} \mathbf{C}_H^{K_i}) \cap \mathbf{C}_H^{K_m} = \sum_{i=1}^{m-1} (\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m}) = \sum_{i=1}^{m-1} \mathbf{C}_H^{\langle K_i, K_m \rangle}$$

where the last equality follows from Lemma 2.10. We can apply Lemma 2.10 here because $\mathbf{C}_H^M \subseteq \mathbf{C}_H^K$ for all $K \in \mathcal{K}_H$ gives that $\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m} \neq 0$ for every $i = 1, \dots, m-1$. Note that we also obtain $\langle K_i, K_m \rangle \in \mathcal{K}_H$ for all i. Applying our inductive assumption again to these subgroups, we obtain that $\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}$ is an R-homology n-sphere and that the induced map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m})$$

is an isomorphism. This gives that $H_i(\mathbf{D}_m) = 0$ for $i \leq n-1$. We also obtain a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(\mathbf{C}_H^M) & \longrightarrow & H_n(\mathbf{C}_H^M) \oplus H_n(\mathbf{C}_H^M) & \longrightarrow & H_n(\mathbf{C}_H^M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) & \longrightarrow & H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) & \longrightarrow & H_n(\mathbf{D}_m) & \longrightarrow & 0 \end{array}$$

Since all the vertical maps except the map φ are known to be isomorphisms, we obtain that φ is also an isomorphism by the five lemma. This completes the proof. □

Corollary 3.11. *Let* $\mathbf{C}$ *and* $H \in \mathcal{F}$ *are as above. Then* $H_{n+1}(S_H\mathbf{C}) = 0$.

Proof. Let $\mathcal{K}_H = \{K_1, \dots, K_m\}$. By condition (ii), we know that the composition

$$H_n(\mathbf{C}(M)) \xrightarrow{\cong} H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. However, we have just proved that the middle map is an isomorphism, and that all the modules involved in the composition are isomorphic to R. Therefore, the map induced by inclusion

$$H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. Note that if $(H) \leq (K)$ and $\underline{n}(K) < n$, for some $K \in \mathcal{F}$, then $\dim \mathbf{C}(K) < n$. This means that

$$H_n(\mathbf{C}^{>H}(H)) \cong H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \cong H_n(\mathbf{C}(H)).$$

From the exact sequence $0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$, and the fact that $\operatorname{HomDim} \mathbf{C}(H) = n$, we conclude that $H_{n+1}(S_H\mathbf{C}) = 0$, as required. □

4. Construction of the preliminary local models

Our main technical tool is provided by Theorem 6.1, which gives a method for constructing finite G-CW-complexes, with isotropy in a given family. This theorem will be proved by applying the realization statement of Theorem 3.2. To construct a suitable finite free chain complex $\mathbf{C}$ over $\mathbf{Z}\varGamma_G$, we work one prime at a time to construct local models $\mathbf{C}^{(p)}$, and then apply the glueing method for chain complexes developed in [6, Theorem 6.7].

The main input of Theorem 6.1 is a compatible collection of unitary representations for the p-subgroups of G. We give the precise definition in a more general setting.

Definition 4.1. Let $\mathcal{F}$ be a family of subgroups of G. We say that $\mathbf{V}(\mathcal{F})$ is an $\mathcal{F}$-*representation* for G, if $\mathbf{V}(\mathcal{F}) = \{V_H \in \operatorname{Rep}(H, U(n)) \mid H \in \mathcal{F}\}$ is a compatible collection of (non-zero) unitary H-representations. The collection is *compatible* if $f^*(V_K) \cong V_H$ for every G-map $f\colon G/H \to G/K$.

For any finite G-CW-complex X, we let $\mathrm{Iso}(X) = \{K \le G \mid X^K \neq \emptyset\}$ denote the *isotropy family* of the G-action on X. This suggests the following notation:

Definition 4.2. Let $\mathbf{V}(\mathcal{F})$ be an $\mathcal{F}$-representation for G. We let

$$\mathrm{Iso}(\mathbf{V}(\mathcal{F})) = \{L \le H \mid S(V_H)^L \neq \emptyset, \text{ for some } V_H \in \mathbf{V}(\mathcal{F})\}$$

denote the *isotropy family* of $\mathbf{V}(\mathcal{F})$. We note that $\mathrm{Iso}(\mathbf{V}(\mathcal{F}))$ is a sub-family of $\mathcal{F}$.

Example 4.3. Our first example for these definitions will be a compatible collection of representations for the family $\mathcal{F}_p$ of all p-subgroups, with p a fixed prime dividing the order of G. In this case, an $\mathcal{F}_p$-representation $\mathbf{V}(\mathcal{F}_p)$ can be constructed from a suitable representation $V_p \in \mathrm{Rep}(P, U(n))$, where P denotes a p-Sylow subgroup of G. The representations V_H can be constructed for all $H \in \mathcal{F}_p$, by extending V_p to conjugate p-Sylow subgroups and by restriction to subgroups. To ensure a compatible collection $\{V_H\}$, we assume that V_p *respects fusion* in G, meaning that $\chi_p(gxg^{-1}) = \chi_p(x)$ for the corresponding character χ_p, whenever $gxg^{-1} \in P$ for some $g \in G$ and $x \in P$.

We will now specify an isotropy family $\mathcal{J}$ that will be used in the rest of the paper.

Definition 4.4. Let $\{\mathbf{V}(\mathcal{F}_p) \mid p \in \mathcal{S}_G\}$ be a collection of $\mathcal{F}_p$-representations, for a set $\mathcal{S}_G$ of primes dividing the order of G. Let $\mathcal{J}_p = \mathrm{Iso}(\mathbf{V}(\mathcal{F}_p))$ and $\mathcal{J} = \bigcup\{\mathcal{J}_p \mid p \in \mathcal{S}_G\}$ denote the isotropy families.

We note that $\mathcal{J}$ contains no isotropy subgroups of composite order, since each $\mathcal{J}_p$ is a family of p-subgroups. Let $\Gamma_G = \mathrm{Or}_{\mathcal{J}}\, G$ and $\Gamma_G(p)$ denote the orbit category $\mathrm{Or}_{\mathcal{J}_p}\, G$ over the family $\mathcal{J}_p$. A chain complex $\mathbf{C}$ over $R\Gamma_G(p)$ can always be considered as a complex of $R\Gamma_G$-modules, by taking the values $\mathbf{C}(H)$ at subgroups $H \notin \mathcal{J}_p$ as zero complexes.

In this section we construct a p-local chain complex $\mathbf{C}^{(p)}(0)$ over $R\Gamma_G(p)$, for $R = \mathbf{Z}_{(p)}$, which we call a *preliminary local model* (see Definition 4.10). From this construction we will obtain a dimension function $\underline{n}^{(p)} \colon \mathcal{J}_p \to \mathbf{Z}$. By taking joins we can assume that these dimension functions have common value at $H = 1$. In the next section, these preliminary local models will be "improved" at each prime p by adding homology as specified by the dimension functions $\underline{n}^{(q)} \colon \mathcal{J}_q \to \mathbf{Z}$, for all $q \in \mathcal{S}_G$ with $q \neq p$. The resulting complexes $\mathbf{C}^{(p)}$ over the orbit category $R\Gamma_G$ will all have the same dimension function

$$\underline{n} = \bigcup \{\underline{n}^{(p)} \mid p \in \mathcal{S}_G\} \colon \mathcal{J} \to \mathbf{Z},$$

and satisfy conditions needed for the glueing method.

The first step is based on a construction of Connolly and Prassidis [2, §2].

Theorem 4.5. *Let G be a finite group. Suppose that:*

(i) *B is a finite-dimensional G-CW-complex, with fixed sets B^H simply-connected for all $H \le G$;*
(ii) *$\mathbf{V}(\mathcal{F})$ is an $\mathcal{F}$-representation for G, with $\mathcal{F} = \mathrm{Iso}(B)$.*

Then there exists a finite-dimensional G-CW-complex E, with isotropy $\mathrm{Iso}(\mathbf{V}(\mathcal{F}))$, such that all fixed sets E^H are simply-connected.

Proof. The given G-CW-complex B and the given $\mathcal{F}$-representation $\mathbf{V}(\mathcal{F})$ can be combined with a construction of Connolly and Prassidis [2, §2] to produce an integer $k \geq 1$ and a G-fibration $\pi\colon E \to B$, such that the fibre

$$\pi^{-1}(\sigma) \simeq_H S(V_H^{\oplus k}),$$

for all cells $\sigma \subset B$ with stabilizer $G_\sigma = H$. The construction given in [2, §2] only works when all the representations V_H are free H-representations, but this construction later generalized by Klaus [10, Proposition 2.7] to an arbitrary family of unitary representations. A more detailed version of this construction can also be found in [17, Proposition 4.3] (see also Ilhan [7]). Here $V_H^{\oplus k}$ denotes the direct sum $V_H \oplus \cdots \oplus V_H$ of k copies of the unitary representation V_H, and $\simeq_H$ denotes "H-equivariant homotopy equivalence". The G-CW-complex E has the required properties. □

The basic input for our local models is the construction of finite-dimensional G-CW-complexes with good p-local properties.

Corollary 4.6. *Let G be a finite group, and let $\mathbf{V}(\mathcal{F}_p)$ be an $\mathcal{F}_p$-representation for G for some $p \in \mathcal{S}_G$. Then there exists a finite-dimensional G-CW-complex E, which is p-locally homotopy equivalent to a sphere, with isotropy $\mathcal{J}_p = \operatorname{Iso}(\mathbf{V}(\mathcal{F}_p))$. Moreover, all the fixed sets E^H are simply-connected.*

Proof. We recall a result of Jackowski, McClure and Oliver [8, Proposition 2.2]: there exists a simply-connected, finite dimensional G-CW-complex B which is $\mathbf{F}_p$-acyclic and has finitely many orbit types with isotropy in the family of p-subgroups $\mathcal{F}_p$ in G. The quoted result applies more generally to all compact Lie groups and produces a complex with p-toral isotropy (meaning a compact Lie group P whose identity component P_0 is a torus, and P/P_0 is a finite p-group). For G finite, the p-toral subgroups are just the p-subgroups. The property that all fixed sets B^H are simply-connected is established in the proof.

We now apply Theorem 4.5 to this G-CW-complex B and the given $\mathcal{F}_p$-representation $\mathbf{V}(\mathcal{F}_p)$. The resulting G-CW-complex E has the required properties. In particular, since B is $\mathbf{F}_p$-acyclic then for each p-subgroup H, the fixed point set B^H will be also $\mathbf{F}_p$-acyclic (and $B^H \neq \emptyset$). Hence E is a p-local sphere. □

We now let $R = \mathbf{Z}_{(p)}$, and consider the finite dimensional chain complex $\mathbf{C}(E^{?}; R)$ of free $R\varGamma_G(p)$-modules. By taking joins, we may assume that this complex has "homology gaps" of length $> l(\varGamma_G)$ as required for [6, Theorem 6.7], and that all the non-empty fixed sets E^H have $\underline{n}(H) \geq 3$ and trivial action of $W_G(H)$ on homology. Let $\underline{n}^{(p)}\colon \mathcal{J}_p \to \mathbf{Z}$ denote the dimension function $\operatorname{HomDim} \mathbf{C}(E^{?}; R)$.

The following result applies to chain complexes over $R\varGamma_G$ with respect to any family $\mathcal{F}$ of subgroups.

Lemma 4.7. *Let R be a noetherian ring and G be a finite group. Suppose that $\mathbf{C}$ is an n-dimensional chain complex of projective $R\varGamma_G$-modules with finitely generated homology groups. Then $\mathbf{C}$ is chain homotopy equivalent to a finitely-generated projective n-dimensional chain complex over $R\varGamma_G$.*

Proof. Note that the chain modules of $\mathbf{C}$ are not assumed to be finitely-generated, but $H_i(\mathbf{C}) = 0$ for $i > n$. We first apply Dold's "algebraic Postnikov system" technique [3, §7], to chain complexes over the orbit category (see [6, §6]).

According to this theory, given a positive projective chain complex $\mathbf{C}$, there is a sequence of positive projective chain complexes $\mathbf{C}(i)$ indexed by positive integers such that $f\colon \mathbf{C} \to \mathbf{C}(i)$ induces a homology isomorphism for dimensions $\leq i$. Moreover, there is a tower of maps

$$\begin{array}{ccccc}
 & & \mathbf{C}(i) & & \\
 & \nearrow & \downarrow & & \\
 & & \mathbf{C}(i-1) & \xrightarrow{\alpha_i} & \Sigma^{i+1}\mathbf{P}(H_i) \\
 & \nearrow & \vdots & & \\
\mathbf{C} & \longrightarrow & \mathbf{C}(1) & \xrightarrow{\alpha_2} & \Sigma^{3}\mathbf{P}(H_2) \\
 & \searrow & \downarrow & & \\
 & & \mathbf{C}(0) & \xrightarrow{\alpha_1} & \Sigma^{2}\mathbf{P}(H_1)
\end{array}$$

such that $\mathbf{C}(i) = \Sigma^{-1}\mathbf{C}(\alpha_i)$, where $\mathbf{C}(\alpha_i)$ denotes the algebraic mapping cone of α_i, and $\mathbf{P}(H_i)$ denotes a projective resolution of the homology module $H_i = H_i(\mathbf{C})$.

By assumption, since the homology modules H_i are finitely-generated and R is noetherian, we can choose the projective resolutions $\mathbf{P}(H_i)$ to be finitely-generated in each degree. Therefore, at each step the chain complex $\mathbf{C}(i)$ consists of finitely-generated projective $R\varGamma_G$-modules, and $\mathbf{C}(n) \simeq \mathbf{C}$ has homological dimension $\leq n$. Now, since $H^{n+1}(\mathbf{C}(n); M) = H^{n+1}(\mathbf{C}; M) = 0$, for any $R\varGamma_G$-module M, we conclude that $\mathbf{C}(n)$ is chain homotopy equivalent to an n-dimensional finitely-generated projective chain complex by [11, Prop. 11.10]. $\square$

Remark 4.8. This result generalizes [11, 11.31:ex. 2] or [14, Satz 9].

Lemma 4.9. *The chain complex $\mathbf{C}(E^{?}; R)$ is chain homotopy equivalent to an oriented R-homology $\underline{n}^{(p)}$-sphere $\mathbf{C}^{(p)}(0)$, which is an algebraic homotopy representation.*

Proof. The chain complex $\mathbf{C}(E^{?}; R)$ is finite dimensional and free over $R\varGamma_G$, but may not be finitely-generated. However, by the conclusion of Corollary 4.6, the homology groups $H_*(\mathbf{C}(E^{?}; R))$ are finitely generated since $\mathbf{C}(E^{?}; R)$ is an R-homology $\underline{n}$-sphere. The result now follows from Lemma 4.7, which produces a finite length projective chain complex $\mathbf{C}^{(p)}(0)$ of finitely-generated $R\varGamma_G(p)$-modules. Note that $\mathbf{C}(E^{?}; R)$ satisfies the conditions (i)-(iii) in Definition 2.6, so $\mathbf{C}^{(p)}(0)$ also satisfies these conditions (which are chain-homotopy invariant), hence $\mathbf{C}^{(p)}(0)$ is an algebraic homotopy representation. $\square$

Note that $\mathbf{C}^{(p)}(0)$ is an algebraic homotopy representation, meaning that it satisfies the condition (i), (ii), and (iii) in Definition 2.6, even though $\operatorname{Dim} \mathbf{C}^{(p)}$ may not be equal to $\underline{n}^{(p)} = \operatorname{HomDim} \mathbf{C}^{(p)}(0)$.

By taking joins, we may assume that there exists a common dimension $N = \underline{n}^{(p)}(1)$, at $H = 1$, for all all $p \in \mathcal{S}_G$. Moreover, we may assume that $N + 1$ is a multiple of any given integer m_G (to be chosen below). We now obtain the "global" dimension function

$$\underline{n} = \bigcup \{\underline{n}^{(p)} \mid p \in \mathcal{S}_G\} \colon \mathcal{J} \to \mathbf{Z},$$

where $\underline{n}^{(p)} = \operatorname{HomDim} \mathbf{C}^{(p)}(0)$, for all $p \in \mathcal{S}_G$, and $\underline{n}(1) = N$.

Definition 4.10 (*Preliminary local models*)**.** Let $\mathcal{S}_G = \{p \mid \operatorname{rank}_p G \geq 2\}$, and let m_G denote the least common multiple of the q-periods for G (as defined in [13, p. 267]), over all primes q for which $\operatorname{rank}_q G = 1$. We assume that $\underline{n}(1) + 1$ is a multiple of m_G.

(i) We will take the chain complex $\mathbf{C}^{(p)}(0)$ constructed in Lemma 4.9 for our preliminary model at each prime $p \in \mathcal{S}_G$.
(ii) If $\operatorname{rank}_q G = 1$, we take $\mathcal{J}_q = \{1\}$ and $\mathbf{C}^{(q)}(0)$ as the $R\Gamma_G$-chain complex $E_1\mathbf{P}$ where $\mathbf{P}$ is a periodic resolution of R as a RG-module with period $\underline{n}(1) + 1$ (for more details, see the proof of Theorem 5.2 below, or [6, Section 9B]).

This completes the construction of the preliminary local models at each prime dividing the order of G, for a given family of $\mathcal{F}_p$-representations. In the next section we will modify these preliminary models to get p-local chain complexes $\mathbf{C}^{(p)}$ over $R\Gamma_G$ which are R-homology $\underline{n}$-spheres for the dimension function $\underline{n}$ described above.

Example 4.11. In the proof of Theorem A we will be using the setting of Example 4.3. Suppose that G is a rank two finite group which does not p'-involve $\operatorname{Qd}(p)$, for any odd prime p. We let $\mathcal{S}_G$ be the set of primes p where $\operatorname{rank}_p G = 2$. Under this condition, a result of Jackson [9, Theorem 47] asserts that G admits a p-*effective* p-local character V_p. Here p-effective means that when V_p is restricted to an elementary abelian subgroup E of rank 2 then it has no trivial summand. This guarantees that the set of isotropy subgroups $\mathcal{J}_p = \operatorname{Iso}(S(V_p))$ consists of the rank one p-subgroups. In this setting, our preliminary local models arise from the following special case:

Corollary 4.12. *Let G be a finite rank two group with $\operatorname{rank}_p G = 2$. If G does not p'-involve $\operatorname{Qd}(p)$ when $p > 2$, then there exists a simply-connected, finite-dimensional G-CW-complex E with rank one p-group isotropy, which is p-locally homotopy equivalent to a sphere.*

5. Construction of the local models: adding homology

Let G be a finite group and let $\mathcal{S}_G = \{p \mid \operatorname{rank}_p G \geq 2\}$. We will use the notation $\mathcal{J}_p = \operatorname{Iso}(\mathbf{V}(\mathcal{F}_p))$, for $p \in \mathcal{S}_G$, as given in Definition 4.4. For $p \notin \mathcal{S}_G$ we have $\mathcal{J}_p = \{1\}$. We will continue to work over the orbit category $\Gamma_G = \operatorname{Or}_{\mathcal{J}} G$ where $\mathcal{J} = \bigcup \{\mathcal{J}_p \mid p \in \mathcal{S}_G\}$. For each prime p dividing the order of G, let $\mathbf{C}^{(p)}(0)$ denote the preliminary p-local model given in Definition 4.10, and denote the homological dimension function of $\mathbf{C}^{(p)}(0)$ by $\underline{n}^{(p)} \colon \mathcal{J}_p \to \mathbf{Z}$ for all primes dividing the order of G.

We now fix a prime q dividing the order of G, and let $R = \mathbf{Z}_{(q)}$. In Theorem 5.2, we will show how to add homology to the preliminary local model $\mathbf{C}^{(q)}(0)$, to obtain an algebraic homotopy representation with dimension function $\underline{n}^{(p)} \cup \underline{n}^{(q)}$ for any prime $p \in \mathcal{S}_G$ such that $p \neq q$. After finitely many such steps, we will obtain our local model $\mathbf{C}^{(q)}$ over $R\Gamma_G$ with dimension function $\operatorname{HomDim} \mathbf{C}^{(q)} = \underline{n}$.

Remark 5.1. In order to carry out the above construction, certain conditions on the dimension function must hold. In particular, if there exists a q-local model $\mathbf{C}^{(q)}$ with isotropy in $\mathcal{J}_p \cup \mathcal{J}_q$, where $p \in \mathcal{S}_G$, then for every p-subgroup $1 \neq H \in \mathcal{J}_p$, we can consider the $RN_G(H)/H$ complex $\mathbf{C}^{(q)}(H)$. This is a finite length chain complex of finitely generated modules which has the R-homology of an $\underline{n}(H)$-sphere.

Since $R = \mathbf{Z}_{(q)}$, if we take a q-subgroup $Q \in N_G(H)/H$ with $H \neq 1$, and restrict $\mathbf{C}^{(q)}(H)$ to Q, we obtain a finite dimensional *projective* RQ-complex (see [6, Lemma 3.6]). This forces Q to be a rank one subgroup. So, to be able to perform the above adding process, we must have the condition that for every p-group $1 \neq H$, the normalizer quotients $W_G(H) = N_G(H)/H$ have q-rank ≤ 1. Note that a similar argument also shows that for $q \notin \mathcal{S}_G$, we must have $\operatorname{rank}_q G \leq 1$. This explains the rank conditions that appear in Theorem 5.2 and Theorem 6.1.

We may also assume that $\underline{n}(H)+1$ is a multiple of the q-period of $W_G(H)$ and the gaps between non-zero homology dimensions are large enough: more precisely, for all $K, L \in \mathcal{J}$ with $\underline{n}(K) > \underline{n}(L)$, we have $\underline{n}(K) - \underline{n}(L) \geq l(\varGamma_G)$, where $l(\varGamma_G)$ denotes the length of the longest chain of maps in the category $\varGamma_G$. We can easily guarantee both of these conditions by taking joins of the preliminary local models we have constructed. The main result of this section is the following:

Theorem 5.2. *Let G be a finite group and let $R = \mathbf{Z}_{(q)}$. Suppose that $\mathbf{C}$ is an algebraic homotopy representation over R, such that*

(i) $\mathbf{C}$ *an (oriented) R-homology $\underline{n}^{(q)}$-sphere of projective $R\varGamma_G(q)$-modules;*
(ii) *If $1 \neq H \in \mathcal{J}_p$, then* $\operatorname{rank}_q(N_G(H)/H) \leq 1$, *for every prime $p \neq q$.*

Then there exists an algebraic homotopy representation $\mathbf{C}^{(q)}$ over R, which is an (oriented) R-homology $\underline{n}$-sphere over $R\varGamma_G$.

We will add the homology specified by the dimension function $\underline{n}^{(p)}$, at a prime $p \neq q$, by an inductive construction using the number of nonzero homology dimensions. The starting point of the induction is the given complex $\mathbf{C}$. Let $n_1 > n_2 > \cdots > n_s$ denote the set of dimensions $\underline{n}(H)$, over all $H \in \mathcal{J}_p$. Note that, since the dimension function $\underline{n}$ comes from a unitary representation, we have $n_s \geq 1$. Let us denote by $\mathcal{F}_i$, the collection of subgroups $H \in \mathcal{J}_p$ such that $\underline{n}(H) = n_i$.

Suppose that we have constructed a finite projective chain complex $\mathbf{C}$ over $R\varGamma_G$, satisfying the conditions (i)-(iii) of Definition 2.6, which has the property that $\operatorname{HomDim} \mathbf{C}(H) = \underline{n}(H)$ for all $H \in \mathcal{F}_{\leq k}$ where $\mathcal{F}_{\leq k} = \bigcup_{i \leq k} \mathcal{F}_i$. Our goal is to construct a new finite dimensional projective complex $\mathbf{D}$ which also satisfies the conditions (i)-(iii) of Definition 2.6, and has the property that $\operatorname{HomDim} \mathbf{D}(H) = \underline{n}(H)$ for all $H \in \mathcal{F}_i$ with $i \leq k+1$.

We will construct the complex $\mathbf{D}$ as an extension of $\mathbf{C}$ by a finite projective chain complex whose homology is isomorphic to the homology that we need to add. Note that since the constructed chain complex $\mathbf{D}$ must satisfy the conditions (i)-(iii), the homology we need to add should satisfy the condition that for every $H \leq K$ with $H, K \in \mathcal{F}_{k+1}$, the restriction map on the added homology module is an R-homology isomorphism.

Definition 5.3. Let J_i denote the $R\varGamma_G$-module which has the values $J_i(H) = R$ for all $H \in \mathcal{F}_i$, and otherwise $J_i(H) = 0$. The restriction maps $r_H^K \colon J_i(K) \to J_i(H)$ for every

$H, K \in \mathcal{F}_i$ such that $H \leq K$, and the conjugation maps $c^g : J_i(K) \to J_i({}^gK)$ for every $K \in \mathcal{F}$ and $g \in G$, are assumed to be the identity maps.

In this notation, the chain complex $\mathbf{D}$ must have homology isomorphic to J_i in dimension n_i for all $i \leq k+1$, and in dimension zero the homology of $\mathbf{D}$ should be isomorphic to $\underline{R}$ restricted to $\mathcal{F}_{k+1}$. It is in general a difficult problem to find chain projective complexes whose homology is given by a block of R-modules with prescribed restriction maps. But in our situation we will be able to do this using some special properties of the poset of subgroups in $\mathcal{F}_i$ coming from condition (iii) (see Corollary 2.8).

Lemma 5.4. *For $1 \leq i \leq s$, each poset $\mathcal{F}_i$ is a disjoint union of components where each component has a unique maximal subgroup up to conjugacy.*

Note that for every $K \in \mathcal{J}_p$, the Sylow q-subgroup of the normalizer quotient $W_G(K) = N_G(K)/K$ has q-rank equal to one, hence it is q-periodic.

The proof of Theorem 5.2. By our starting assumption, the q-period of $W_G(K)$ divides $\underline{n}(K)+1$. So by Swan [13], there exists a periodic projective resolution $\mathbf{P}$ with

$$0 \to R \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

over the group ring $RW_G(K)$ where $n = \underline{n}(K)$. Note that this statement includes the possibility that Sylow q-subgroup of $W_G(K)$ is trivial since in that case R would be projective as a $RW_G(K)$-module, and we can easily find a chain complex of the above form by adding a split projective chain complex.

Now suppose that $K \in \mathcal{J}_p$ is such that (K) is a maximal conjugacy class in $\mathcal{F}_{k+1}$. Consider the $R\varGamma_G$-complex $E_K\mathbf{P}$ where E_K denotes the extension functor defined in [6, Sect. 2C]. By definition

$$E_K(\mathbf{P})(H) = \mathbf{P} \otimes_{R[W_G(K)]} R[(G/K)^H]$$

for every $H \in \mathcal{F}$. We define the chain complex $E_{k+1}\mathbf{P}$ as the direct sum of the chain complexes $E_K\mathbf{P}$ over all representatives of isomorphism classes of maximal elements in $\mathcal{F}_{k+1}$. Let $\mathbf{N}$ denote the subcomplex of $E_{k+1}(\mathbf{P})$ obtained by restricting $E_K(\mathbf{P})$ to subgroups $H \in \mathcal{F}_{\leq k}$. Let $I_{k+1}\mathbf{P}$ denote the quotient complex $E_{k+1}(\mathbf{P})/\mathbf{N}$. We have the following:

Lemma 5.5. *The homology of $I_{k+1}\mathbf{P}$ is isomorphic to J_{k+1} at dimensions 0 and n_{k+1} and zero everywhere else.*

Proof. The homology of $I_{k+1}\mathbf{P}$ at $H \in \mathcal{F}_{k+1}$ is isomorphic to

$$\bigoplus \{R \otimes_{R[W_G(K)]} R[(G/K)^H] : (K) \text{ maximal in } \mathcal{F}_{k+1}\}$$

at dimensions 0 and n_{k+1} and zero everywhere else. Note that $(G/K)^H = \{gK \colon H^g \leq K\}$. If gK is such that $H^g \leq K$, then $H \leq {}^gK$. Now by condition (iii), we must have $\langle K, {}^gK\rangle \in \mathcal{F}_{k+1}$. But (K) was a maximal conjugacy class in $\mathcal{F}_{k+1}$, so we must have $K = {}^gK$, hence $g \in N_G(K)$. This gives $1 \otimes gK = 1 \otimes 1$ in $R \otimes_{R[W_G(K)]} R[(G/K)^H]$. Therefore

$$R \otimes_{R[W_G(K)]} R[(G/K)^H] \cong R$$

for every $H \in \mathcal{F}_{k+1}$. Also, by the same argument H can not be included in two different maximal subgroups in $\mathcal{F}_{k+1}$. So we have $I_{k+1}(\mathbf{P})(H) \cong R$ for all $H \in \mathcal{F}_{k+1}$. Since the restriction maps are given by the inclusion map of fixed point sets $(G/H)^U \hookrightarrow (G/H)^V$ for every $U, V \in \mathcal{F}_{k+1}$ with $V \leq U$, we can conclude that all restriction maps are identity maps. This completes the proof of the lemma. $\square$

The above lemma shows that the homology of $I_{k+1}\mathbf{P}$ is exactly the $R\Gamma_G$-module that we would like to add to the homology of $\mathbf{C}$. To construct $\mathbf{D}$ we use an idea similar to the idea used in [6, Section 9B]. Observe that for every $R\Gamma_G$-chain map $f\colon \mathbf{N} \to \mathbf{C}$, there is a push-out diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{N} & \longrightarrow & E_K\mathbf{P} & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{C}_f & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0\ . \end{array}$$

The homology of $\mathbf{N}$ is only nonzero at dimensions 0 and n_{k+1} and at these dimensions the homology is only nonzero at subgroups $H \in \mathcal{F}_{\leq k}$. At these subgroups the homology of $\mathbf{N}(H)$ is isomorphic to the direct sum of modules of the form $R \otimes_{RW_G(K)} R[(G/K)^H]$. Note that for every $H \in \mathcal{F}_{\leq k}$, there is an augmentation map

$$\varepsilon_H\colon R \otimes_{RW_G(K)} R[(G/K)^H] \to R$$

which takes $r \otimes gK$ to r for every $r \in R$. The collection of these maps gives a map of $R\Gamma_G$-modules denoted $\varepsilon_K\colon E_K R \to R$. Taking the sum over all conjugacy classes of maximal subgroups, we get a map $\sum_K \varepsilon_K\colon \oplus_K E_K R \to R$. Repeating the arguments given in [6, Section 9B], it is easy to see that if f is a chain map such that the induced map on zeroth homology $f_*\colon H_0(\mathbf{N}) \to H_0(\mathbf{C})$ is the same map as the sum of augmentation maps $\sum_K \varepsilon_K$, then the chain complex $\mathbf{C}_f$ will have the identity map as the restriction maps on zeroth homology. At dimension n_{k+1} we will have zero map since the homology of $\mathbf{C}$ is zero at dimension n_{k+1} by assumption.

Unfortunately, we can not take $\mathbf{D}$ as $\mathbf{C}_f$ since the complex $I_{k+1}\mathbf{P}$ is not projective in general, and neither is $\mathbf{N}$. We note that finding a chain map $\mathbf{N}$ satisfying the given condition is not a easy task without projectivity (compare [6, Section 9B], where this complex was a projective). So we first need to replace the sequence $0 \to \mathbf{N} \to E_{k+1}\mathbf{P} \to I_{k+1}\mathbf{P} \to 0$ with a sequence of projective chain complexes.

Lemma 5.6. *There is a diagram of chain complexes where all the complexes $\mathbf{P}', \mathbf{P}'', \mathbf{P}'''$ are finite projective chain complexes over $R\Gamma_G$ and all the vertical maps induce isomorphisms on homology:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{N} & \longrightarrow & E_{k+1}\mathbf{P} & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0\ . \end{array}$$

Proof. Since $E_K\mathbf{P}$ is a projective chain complex of length n, $E_{k+1}\mathbf{P}$ is a finite projective chain complex. So, by [11, Lemma 11.6], it is enough to show that $\mathbf{N}$ is weakly equivalent to a finite projective complex. For this first note that $\mathbf{N} = \bigoplus \mathbf{N}_K$ is a direct sum of chain complexes $\mathbf{N}_K$ where $\mathbf{N}_K$ is the restriction of $E_K\mathbf{P}$ to subgroups $H \in \mathcal{F}_{\leq k}$. So it is enough to show that $\mathbf{N}_K$ is weakly equivalent to a finite projective chain complex. To prove this, we will show that for each i, the $R\Gamma_G$-module $\mathbf{N}_i := (\mathbf{N}_K)_i$ has a finite projective resolution. The module $\mathbf{N}_i$ is nonzero only at subgroups $H \in \mathcal{F}_{\leq k}$ and at each such a subgroup, we have

$$\mathbf{N}_i(H) = (E_K\mathbf{P}_i)(H) = \mathbf{P}_i \otimes_{RW_G(K)} R[(G/K)^H].$$

So, as an $RW_G(H)$-module $\mathbf{N}_i(H)$ is a direct summand of $R[(G/K)^H]$ which is isomorphic to

$$\bigoplus \{R\big[W_G(H)/W_{{}^gK}(H)\big] : K\text{-conjugacy classes of subgroups } H^g \leq K\}$$

as an $RW_G(H)$-module. Since K is a p-group, these modules are projective over the ground ring R because R is q-local. So, for each $H \in \mathcal{F}_{\leq k}$, the $RW_G(H)$-module $\mathbf{N}_i(H)$ is projective. Now consider the map

$$\pi \colon \oplus_H E_H\mathbf{N}_i(H) \to \mathbf{N}_i$$

induced by maps adjoint to the identity maps at each H. We can take $\oplus_H E_H\mathbf{N}_i(H)$ as the first projective module of the resolution, and consider the kernel $\mathbf{Z}_0$ of $\pi \colon \oplus_H E_H\mathbf{N}_i(H) \to \mathbf{N}_i$. Note that $\mathbf{Z}_0$ has smaller length and it also have the property that at each L, the $W_G(L)$ modules $\mathbf{Z}_0(L)$ are projective. This follows from the fact that $R[(G/H)^L]$ is projective as a $W_G(L)$-module by the same argument we used above. Continuing this way, we can find a finite projective resolution for $\mathbf{N}_i$ of length $\leq l(\Gamma)$. □

Now it remains to show that there is a chain map $f \colon \mathbf{P}' \to \mathbf{C}$ such that the induced map on zeroth homology $f_* \colon H_0(\mathbf{P}') \cong H_0(\mathbf{N}) \to H_0(\mathbf{C})$ is given by the sum of augmentation maps ε_K over the conjugacy classes of maximal subgroups K in $\mathcal{F}_{k+1}$. Then the complex $\mathbf{D}$ will be defined as the push-out complex that fits into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0\,. \end{array}$$

Since both $\mathbf{C}$ and $\mathbf{P}'''$ are finite projective chain complexes, $\mathbf{D}$ will also be a finite projective complex.

To construct the chain map $f \colon \mathbf{P}' \to \mathbf{C}$, first note that the chain complex $\mathbf{C}$ has no homology below dimension n_k. By assumption on the gaps between nonzero homology dimensions, we can assume that $n_k \geq n_{k+1} + l(\Gamma_G) \geq l(\mathbf{P}')$. So, starting with the sum of augmentation maps $\sum_K \varepsilon_K$ at dimensions zero, we can obtain a chain map as follows:

$$\begin{array}{ccccccccccc} \longrightarrow & 0 & \longrightarrow & P'_m & \xrightarrow{\partial_m^{P'}} & \cdots & \longrightarrow & P'_0 & \longrightarrow & H_0(\mathbf{N}) & \longrightarrow 0 \\ & & & \downarrow f_m & & & & \downarrow f_0 & & \downarrow \sum_K \varepsilon_K & \\ \longrightarrow & C_{m+1} & \longrightarrow & C_m & \xrightarrow{\partial_m^{C}} & \cdots & \longrightarrow & C_0 & \longrightarrow & H_0(\mathbf{C}) & \longrightarrow 0 \end{array}$$

where $m = l(\mathbf{P}')$. This completes the proof of Theorem 5.2. □

6. The Proof of Theorem A

In this section we establish our main technique for constructing actions on homotopy spheres, based on a given $\mathcal{P}$-representation, where $\mathcal{P} = \bigcup\{\mathcal{F}_p \,|\, p \in \mathcal{S}_G\}$ denote the family of all p -subgroups of G, for the primes p in a given set $\mathcal{S}_G$ (see Definitions 4.1 and 4.4). Theorem A stated in the introduction will follow from this theorem almost immediately once we use the family of p-effective characters constructed by M. A. Jackson [9]. The main technical theorem is the following:

Theorem 6.1. *Let G be a finite group and let $\mathcal{S}_G = \{p \,|\, \operatorname{rank}_p G \geq 2\}$. Suppose that*

(i) *$\mathbf{V}(\mathcal{F}_p)$ is a $\mathcal{F}_p$-representation for G, with $\operatorname{Iso}(\mathbf{V}(\mathcal{F}_p)) = \mathcal{J}_p$, for each $p \in \mathcal{S}_G$;*
(ii) *If $p \in \mathcal{S}_G$ and $1 \neq H \in \mathcal{J}_p$, then we have $\operatorname{rank}_q(N_G(H)/H) \leq 1$ for every $q \neq p$.*

Then there exists a finite G-CW-complex $X \simeq S^n$, with isotropy in $\mathcal{J} = \bigcup\{\mathcal{J}_p \,|\, p \in \mathcal{S}_G\}$, which is a geometric homotopy representation for G.

Remark 6.2. The construction we give in the proof of Theorem 6.1 gives a simply-connected homotopy representation X, with $\dim X^H \geq 3$, for all $H \in \mathcal{J}$, whenever $X^H \neq \emptyset$. It also relates the dimension function of X to the linear dimension functions $\operatorname{Dim} S(V_H)$, for $V_H \in \bigcup\{\mathbf{V}(\mathcal{F}_p) \,|\, p \in \mathcal{S}_G\}$ in the following way: for every prime $p \in \mathcal{S}_G$, there exists an integer $k_p > 0$ such that for every $H \in \mathcal{F}_p$, the equality $\dim X^H = \dim S(V_H^{\oplus k_p})^H$ holds.

As we discussed in the previous section, the condition on the q-rank of $N_G(H)/H$ is a necessary condition for the existence of such actions. Recall that this condition is used in an essential way in the proof of Theorem 5.2.

The proof of Theorem 6.1. By the realization theorem (Theorem 3.1) proved in Section 3, we only need to construct a chain complex of $\mathbf{Z}\Gamma_G$-modules satisfying the conditions (i), (ii) and (iii) of Definition 2.6. If we apply Theorem 5.2 to the preliminary local model constructed in Section 4, we obtain a finite projective complex $\mathbf{C}^{(p)}$, over the orbit category $\mathbf{Z}_{(p)}\Gamma_G$ with respect to the family $\mathcal{J}$, for each prime p dividing the order of G. In addition, $\mathbf{C}^{(p)}$ is an oriented $\mathbf{Z}_{(p)}$-homology $\underline{n}$-sphere, with the same dimension function $\underline{n} = \operatorname{HomDim} \mathbf{C}^{(p)}(0)$ coming from the preliminary local models. By construction, the complex $\mathbf{C}^{(p)}$ satisfies the conditions (i), (ii) and (iii) of Definition 2.6 for $R = \mathbf{Z}_{(p)}$.

We may also assume that $\underline{n}(H) \geq 3$ for every $H \in \mathcal{J}$, and that the gaps between non-zero homology dimensions have the following property: for all $K, L \in \mathcal{J}$ with $\underline{n}(K) > \underline{n}(L)$, we have $\underline{n}(K) - \underline{n}(L) \geq l(\Gamma_G)$ where $l(\Gamma_G)$ denotes the length of the longest chain of maps in the category Γ_G.

To complete the proof of Theorem 6.1, we first need to glue these complexes $\mathbf{C}^{(p)}$ together to obtain an algebraic $\underline{n}$-sphere over $\mathbf{Z}\Gamma_G$. By [6, Theorem 6.7], there exists a finite projective chain complex $\mathbf{C}$ of $\mathbf{Z}\Gamma_G$-modules, which is a $\mathbf{Z}$-homology $\underline{n}$-sphere, such that $\mathbf{Z}_{(p)} \otimes \mathbf{C}$ is chain homotopy equivalent to the local model $\mathbf{C}^{(p)}$, for each prime p dividing the order of G. The complex $\mathbf{C}$ has a (possibly non-zero) finiteness obstruction (see Lueck [11, §10-11]), but this can be eliminated by joins (see [6, §7]).

After applying [6, Theorem 7.6], we may assume that $\mathbf{C}$ is a finite free chain complex of $\mathbf{Z}\Gamma_G$-modules which is a $\mathbf{Z}$-homology $\underline{n}$-sphere. Moreover, $\mathbf{C}$ is an algebraic homotopy representation: it satisfies the conditions (i), (ii) and (iii) of Definition 2.6 for $R = \mathbf{Z}$, since these conditions hold locally at each prime.

We have now established all the requirements for Theorem 3.2. For the family $\mathcal{F}$ used in its statement, we use $\mathcal{F} = \mathcal{J}$. For all $H \in \mathcal{F}$, we have the condition $\underline{n}(H) \geq 3$. Now Theorem 3.2 gives a finite G-CW-complex $X \simeq S^n$ with isotropy $\mathcal{J}$ such that X^H is an homotopy sphere for every $H \in \mathcal{J}$. $\square$

Now we are ready to prove Theorem A.

The proof of Theorem A. Let G be a rank 2 finite group and let $\mathcal{S}_G$ denote the set of primes with $\operatorname{rank}_p G = 2$. Since it is assumed that G does not p'-involve $\operatorname{Qd}(p)$ for any odd prime p, we can apply [9, Theorem 47] and obtain a p-effective representation V_p, for every prime $p \in \mathcal{S}_G$. We apply Theorem 6.1 to the $\mathcal{F}_p$-representations $\mathbf{V}(\mathcal{F}_p)$ given by this collection $\{V_p\}$ (see Example 4.3). Since V_p is p-effective means that all isotropy subgroups in $\mathcal{H}_p$ are rank one p-subgroups (see Example 4.11), the isotropy is contained in the family $\mathcal{H}$ of rank one p-subgroups of G, for all $p \in \mathcal{S}_G$. We therefore obtain a finite G-CW-complex $X \simeq S^n$, with rank 1 isotropy in $\mathcal{H}$, such that X^H is an homotopy sphere (possibly empty) for every $H \in \mathcal{H}$. $\square$

The proof of Corollary B follows easily from Theorem A since if $\operatorname{rank}_q(G) \leq 1$, then for every p-group H, we must have $\operatorname{rank}_q(N_G(H)/H) \leq 1$. So we can apply Theorem A to obtain Corollary B.

Note that the condition about $\operatorname{Qd}(p)$ being not p'-involved in G is a necessary condition for the existence of actions of rank 2 groups on finite CW-complexes $X \simeq S^n$ with rank one isotropy.

Proposition 6.3. *If G acts on a finite complex X homotopy equivalent to a sphere with rank one isotropy, then G cannot p'-involve* $\operatorname{Qd}(p)$ *for every odd prime p.*

Proof. Suppose that G has a normal p'-subgroup K such that $\operatorname{Qd}(p)$ is included as a subgroup in $N_G(K)/K$. Then a p-Sylow supbgroup of $\operatorname{Qd}(p)$ lifts to $P \leq N_G(K)$. Let $a, c \in P$ be elements of order p, where c is a central element and a is a non-central element such that aK and cK are conjugate to each other in $N_G(K)/K$ via an element in $\operatorname{Qd}(p)$. This means that there is an element $g \in N_G(K)$ such that the equation $g^{-1}ag = ck$ holds for some $k \in K$. Since k has order prime to p, by taking a suitable p'-power of both sides, we obtain that subgroups $\langle a \rangle$ and $\langle c \rangle$ are conjugate to each other in G.

The rest of the proof follows from the argument given in the proof [16, Theorem 3.3]. Since P action on X has rank 1 isotropy subgroups, we have $X^E = \emptyset$ for every rank two p-subgroup $E \leq P$. Therefore $X^C = \emptyset$ by Smith theory, since otherwise $P/\langle c \rangle \cong \mathbf{Z}/p \times \mathbf{Z}/p$ would act freely on $X^{\langle c \rangle}$ which is a *mod* p homology sphere. Now consider the subgroup $E = \langle a, c \rangle$. Since $\langle a \rangle$ and $\langle c \rangle$ are conjugate to each other in G, all cyclic subgroups of E are conjugate to each other. In particular, we have $X^H = \emptyset$ for every cyclic subgroup H in E. This is a contradiction, since E can not act freely on X. $\square$

7. Discussion and examples

We first discuss the rank conditions in the statement of Theorem A. Suppose that X is a finite G-CW-complex. Recall that $\mathrm{Iso}(X) = \{H \mid H \leq G_x \text{ for some } x \in X\}$ denotes the minimal family containing all the isotropy subgroups of the G-action on X. We call this the *isotropy family.* Note that $H \in \mathrm{Iso}(X)$ if and only if $X^H \neq \emptyset$. We say that X has *rank k isotropy* if $\mathrm{rank}\, G_x \leq k$ for all $x \in X$ and there exists a subgroup H with $\mathrm{rank}\, H = k$ and $X^H \neq \emptyset$. Let $\mathcal{P}$ denote the family of all prime-power order subgroups of G.

Lemma 7.1. *Let G be a finite group, and let X be a finite G-CW-complex with $X \simeq S^n$.*

(i) *If H is a maximal p-subgroup in* $\mathrm{Iso}(X)$*, then* $\mathrm{rank}_p(N_G(H)/H) \leq 1$.

(ii) *If $1 \neq H \in \mathrm{Iso}(X) \subseteq \mathcal{P}$ is a p-subgroup, and X^H is an integral homology sphere, then* $\mathrm{rank}_q(N_G(H)/H) \leq 1$*, for all primes $q \neq p$.*

Proof. This follows from two basic results of P. A. Smith theory [1, III.8.1]), which state (i) that the fixed set of a p-group action on a finite-dimensional mod p homology sphere is again a mod p homology sphere (or the empty set), and (ii) that $\mathbf{Z}/p \times \mathbf{Z}/p$ can not act freely on a finite G-CW-complex X with the mod p homology of a sphere.

For any prime p dividing the order of G, let $H \in \mathrm{Iso}(X)$ denote a maximal p-subgroup with $X^H \neq \emptyset$. For any $x \in X^H$, we have $H \leq G_x$ and if $g \cdot x = x$, for some $g \in N_G(H)$ of p-power order, it follows that the subgroup $\langle H, g\rangle \leq G_x$. Since H was a maximal p-subgroup in $\mathrm{Iso}(X)$, we conclude that $g \in H$. Therefore $N_G(H)/H$ acts freely on the fixed set X^H, which is a mod p homology sphere, and hence $\mathrm{rank}_p(N_G(H)/H) \leq 1$.

If $q \neq p$ and H is a p-subgroup in $\mathrm{Iso}(X) \subseteq \mathcal{P}$, then any q-subgroup Q of $N_G(H)/H$ must act freely on X^H (since $x \in X^H$ implies G_x is a p-group). Since X^H is assumed to be an integral homology sphere, Smith theory implies that $\mathrm{rank}_q(Q) \leq 1$. □

Example 7.2. If G is the extra-special p-group of order p^3, then the centre $Z(G) = \mathbf{Z}/p$ can not be a maximal isotropy subgroup in $\mathrm{Iso}(X)$. On the other hand, we know that G acts on a finite complex $X \simeq S^n$ with rank one isotropy: just take the linear sphere $S(\mathrm{Ind}_{Z(G)}^G W)$ for some nontrivial one-dimensional representation W of $Z(G)$. So we can not require that G acts on $X \simeq S^n$ with $\mathrm{Iso}(X)$ containing all rank one subgroups.

For any prime p, we can restrict the G-action on X to a p-subgroup of maximal rank.

Corollary 7.3. *If X is a finite G-CW-complex with $X \simeq S^n$ and rank k isotropy, then* $\mathrm{rank}_p G \leq k+1$*, for all primes p.*

These results help to explain the rank conditions in Theorem A. If we have rank one isotropy, then we must assume that G has rank two. However, condition (ii) on the q-rank of the normalizer quotient is only necessary for p-subgroups H, with $q \neq p$, for which $X^H \neq \emptyset$ is an integral homology sphere. In order to get a complete list of necessary conditions, we must have more precise control of the structure of the isotropy subgroups.

Now we discuss two applications of Theorem A and Theorem 6.1.

Example 7.4. *The alternating group $G = A_6$ admits a finite G-CW-complex $X \simeq S^n$, with rank one isotropy.* This follows from Theorem A once we verify that G satisfies the necessary conditions. Note that A_6 has order $2^3 \cdot 3^2 \cdot 5 = 360$ so it automatically satisfies the condition about $\mathrm{Qd}(p)$, since it can not include an extra-special p-group of order p^3 for an odd prime p. For the q-rank condition, note that $\mathcal{S}_G = \{2, 3\}$, so we need to check this condition only for primes $p = 2$ and 3. Here are some easily verified facts:

- A 2-Sylow subgroup $P \leq G$ is isomorphic to the dihedral group D_8, so all rank one 2-subgroups are cyclic, and $\mathcal{H}_2 = \{1, C_2, C_4\}$.
- $N_G(C_2) = P$, and $\mathrm{rank}_3(N_G(C_2)/C_2) = 0$.
- $N_G(C_4) = P$ and $\mathrm{rank}_3(N_G(C_4)/C_4) = 0$.

Now, let Q be a Sylow 3-subgroup in G. Then $Q \cong C_3 \times C_3$.

- Any subgroup of order 3 in G is conjugate to $C_3^A = \langle (123) \rangle$ or $C_3^B = \langle (123)(456) \rangle$.
- $|N_G(C_3^A)/C_3^A| = 6$ and $\mathrm{rank}_2(N_G(C_3^A)/C_3^A) = 1$.
- $|N_G(C_3^B)/C_3^B| = 6$ and $\mathrm{rank}_2(N_G(C_3^B)/C_3^B) = 1$

We conclude that condition (ii) of Theorem A holds for this group.

We now give an example which does not satisfy the q-rank conditions in Theorem A, but where we can apply Theorem 6.1 directly.

Example 7.5. *The alternating group $G = A_7$ admits a finite G-CW-complex $X \simeq S^n$, with rank one isotropy.* The order of G is $2^3 \cdot 3^2 \cdot 5 \cdot 7$, so this group also automatically satisfies the $\mathrm{Qd}(p)$ condition. Here is a summary of the main structural facts:

- The 3-Sylow subgroup $Q \leq G$ is isomorphic to $C_3 \times C_3$.
- Any subgroup of order 3 in G is conjugate to $C_3^A = \langle (123) \rangle$ or $C_3^B = \langle (123)(456) \rangle$.
- The Sylow 2-subgroup of $N_G(C_3^A)$ is isomorphic to D_8.
- $|N_G(C_3^A)/C_3^A| = 24$ and $\mathrm{rank}_2(N_G(C_3^A)/C_3^A) = 2$.
- $N_G(C_3^B) \cong (C_3 \times C_3) \rtimes C_2$ and $\mathrm{rank}_2(N_G(C_3^B)/C_3^B) = 1$.
- $|N_G(C_2)| = 24$, and $\mathrm{rank}_3(N_G(C_2)/C_2) = 1$.
- $N_G(C_4) \cong D_8$ and $\mathrm{rank}_3(N_G(C_4)/C_4) = 0$

We see that the rank condition in Theorem A fails for 3-subgroups, since there is a cyclic 3-subgroup $H = C_3^A$ with $\mathrm{rank}_2(N_G(H)/H) = 2$. On the other hand, by applying Theorem 6.1 directly, we can still find a finite G-CW-complex $X \simeq S^n$, with rank one isotropy in the family generated by $\{1, C_2, C_4, C_3^B\}$.

In this case, we have $\mathcal{S}_G = \{2, 3\}$. For $p = 2$, we can use the $\mathcal{F}_2$-representation V_2 from [9], since A_7 satisfies the rank condition for 2-subgroups. It remains to show that there exists an $\mathcal{F}_3$-representation of G with isotropy subgroups only type B cyclic 3-subgroups. But this is easily constructed by taking V_3 as the direct sum of augmented permutation modules $I(Q/K_1) \oplus I(Q/K_2)$ where $K_1 = \langle (123)(456) \rangle$ and $K_2 = \langle (123)(465) \rangle$. It is clear that this representation respects fusion, and has isotropy given only by the cyclic 3-subgroups of type B lying in $\mathcal{F}_3$.

Remark 7.6. When G is a finite group with a rank two elementary abelian Sylow q-subgroup Q, the representation

$$V_q = \bigoplus \{ I(Q/K_i) : 1 \leq i \leq s \}$$

over some family of rank 1 subgroups K_i, which is closed under G-conjugacy, will give a q-effective representation which respects fusion. But for more general Sylow q-subgroups, the above representation may fail to be q-effective. Note that for V_q to be q-effective one needs to have exactly one double coset in $E\backslash Q/K_i$ for every K_i and for every rank 2 elementary abelian subgroup E of Q. This fails, for example, if $Q = C_{p^2} \times C_{p^2}$ is a non-elementary rank two abelian group.

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