Asymptotic Solutions to Fuchsian Equations in Several Variables

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Abstract

The aim of this paper is to construct asymptotic solutions to multidimensional Fuchsian equations near points of their degeneracy. Such construction is based on the theory of resurgent functions of several complex variables worked out by the authors in the paper [1]. This theory allows to construct the explicit resurgent solutions to Fuchsian equations and also to investigate the evolution equations (Cauchy problems) with operators of Fuchsian type in their right-hand parts.

1 Introduction

In this paper we construct asymptotic solutions to equations of Fuchsian type in several variables. By equations of Fuchsian type we mean equations of the form

$$\hat{H}u \stackrel{\text{def}}{=} H\left(x, x\frac{\partial}{\partial x}\right)u = 0 \tag{1}$$

where $x = (x^1, ..., x^n)$ is a point in the Cartesian space $\mathbf{R}^n$,

$$x\frac{\partial}{\partial x} = \left(x^1\frac{\partial}{\partial x^1}, ..., x^n\frac{\partial}{\partial x^n}\right)$$

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and the function $H(x,p)$ is a polynomial with respect to the variable p. Such operators were studied earlier from different point of wiev by M. Kashiwara and T. Kawai [2], V. Maslov [3] R. Melrose [4], B.-W. Schulze [5] B. Ziemian [6], [7].

Evidently, this equation is degenerated on the union of coordinate planes $\{x^i = 0\}$ and, hence, one can expect that the solutions will have singularities on this union. Our goal is to construct asymptotic solutions to such an equation at points of its singularities.

We remark that the set of singularities, that is, the union of hyperplanes $\{x^i = 0\}$ can be stratified in such a way that the strata are coordinate planes of different dimension. Renumerating the coordinates, if necessary, one can write down the equation of each stratum in the form

$$A_k = \left\{x^1 = 0, ..., x^k = 0\right\} \tag{2}$$

where k is codimension of the strata A_k.

If we are intended to construct the asymptotic solution (by smoothness) to equation (1) in a neighbourhood of a point of stratum (2) then we can state that the group of variables $(x^1, ..., x^k)$ plays quite a different role than the group $(x^{k+1}, ..., x^n)$. Indeed, the variables of the first group are the variables transversal to the singularity manifolds $\{x^i = 0, i = 1, ..., k\}$ which pass through the considered point and, hence, these variables are parameters of the asymptotic expansion under construction. At the same time, the variables of the second group are not small near the considered point and can be considered simply as parameters. To express this difference in the explicit form we shall slightly change the notation denoting the variables of the first group by $x = (x^1, ..., x^n)$ and the variables of the second one by $y = (y^1, ..., y^k)$. Using this notation we can rewrite equation (1) in the form

$$\hat{H}u = H\left(x, y, x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u = 0. \tag{3}$$

Further, from technical reason it is convenient to consider the operator $\hat{H}$ included in the equation (3) as a differential operator of the form

$$H\left(x, x\frac{\partial}{\partial x}\right)$$

whose coefficients lie in the space of differential operators in variables y lying in the Cartesian space $\mathbf{R}^n$ or, more generally, on some smooth manifold. It is also convenient to complexify the problem with respect to the variables x.

For constructing asymptotic solutions to the equation (3) we use the theory of resurgent functions of several independent variables worked out by the authors (see [1]). We also remark that the one-dimensional case of such construction was considered in the paper [8] by B.-W. Schulze, B. Sternin, and V. Shatalov.

The outline of the paper is as follows. In Section 2 we construct asymptotic expansions of the resurgent type for solutions to equation (3). In Section 3 we consider the corresponding evolution equations. Finally, in Section 4 we present two concrete examples of the introduced technique.

2 Statement of the problem

Let us proceed with exact definitions. Consider a Fuchsian equation of the form

$$\hat{H}u \stackrel{\text{def}}{=} H\left(x, x\frac{\partial}{\partial x}\right)u(x) = f(x) \tag{4}$$

where $\hat{H}$ is a differential operator of the form

$$\hat{H} = \sum_{|\alpha|\leq m} a_\alpha(x)\left(x\frac{\partial}{\partial x}\right)^\alpha \tag{5}$$

with analytical coefficients $a_\alpha(x)$. Here $x = (x^1, ..., x^n)$ is a point of the Cartesian complex space $\mathbf{C}^n$,

$$x\frac{\partial}{\partial x} = \left(x^1\frac{\partial}{\partial x^1}, ..., x^n\frac{\partial}{\partial x^n}\right)$$

and we construct asymptotical solutions to equation (4) with respect to smoothness in a neighbourhood of the origin. The coefficients $a_\alpha(x)$ of operator (5) can be operator-valued functions of x with values in the space of differential operators in $\mathbf{C}^k_y$, $y = (y^1, ..., y^k)$ or, more generally, in the space of differential operators on a smooth manifold Y; in the last case we shall denote by y local coordinates on Y.

In this paper we present the approach to the construction of resurgent solutions to such equations based on the theory of resurgent functions of several independent variables worked out by the authors in the paper [1]. This approach gives *explicit formulas* for the asymptotic solutions under consideration.

Under the above asumptions equation (4) can be written down in the form

$$\hat{H}u \stackrel{\text{def}}{=} H\left(x, y, x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u(x,y) = f(x,y) \tag{6}$$

where

$$H\left(x, y, x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \sum_{|\alpha|\leq m}\sum_{|\gamma|\leq m_\alpha} a^\gamma_\alpha(x,y)\left(x\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial y}\right)^\gamma.$$

Evidently, the singularities of solutions to equations (4) or (6) can have singularities on the union of the coordinate hyperplanes

$$x^i = 0, \quad i = 1, 2, ..., n.$$

As we had already mentioned, the construction of asymptotic solutions to equations (4) or (6) will be carried out with the help of the theory of multidimensional resurgent functions worked out by the authors in the paper [1]. In order to apply this theory to equations (4) or (6) we perform the change of variables

$$x^i = e^{\tau^i}, \quad i = 1, ...n. \tag{7}$$

and expand the coefficients $a_\alpha(x)$ into the Taylor's series in variable x:

$$a_\alpha(x) = \sum_{|\beta|\geq 0} a_{\alpha\beta}\, x^\beta = \sum_{|\beta|\geq 0} a_{\alpha\beta}\, e^{\tau\beta},$$

where

$$\tau\beta = \tau^1\beta_1 + \ldots + \tau^n\beta_n$$

and $a_{\alpha\beta}$ are differential operators in the space $\mathbf{C}^n$. Then the considered equation becomes

$$\sum_{|\alpha|\leq m} \sum_{|\beta|\geq 0} a_{\alpha\beta}\, e^{\tau\beta} \left(\frac{\partial}{\partial\tau}\right)^\alpha u(\tau) = f(e^\tau) \stackrel{\text{def}}{=} g(\tau). \tag{8}$$

This equation can be investigated with the help of the theory of resurgent functions of several variables presented in the paper [1].

3 Construction of the resurgent solutions

Now we are able to apply the resurgent functions theory to the construction of asymptotic solutions to equation (8). We recall [1] that a resurgent function of variables x is a function of the form

$$u(\tau) = l(U,\Omega) \stackrel{\text{def}}{=} \sum_j \int_{\Gamma_j} e^{-s} U(s,\tau)\, ds \tag{9}$$

where $U(s,\tau)$ is an analytic homogeneous hyperfunction (see [9]) of the variables (s,τ) and each Γ_j is a special contour surrounding some singular point $s_j = s_j(\tau)$ of the hyperfunction U. It is not needed that each singular point of the function U is surrounded by some contour Γ_j; the set of singular points included into expression (9) is called *a support* of the resurgent function u (see [1]). The support of the resurgent function u will be denoted by

$$\Omega = \Omega(\tau);$$

we emphasize that the support of the resurgent function can depend on τ. The contours Γ_j are shown on Figure 1.

The following affirmation takes place.

Theorem 1 *The commutation formulas between the operator l given by* (9) *and the differentiation operators have the form*

$$\frac{\partial}{\partial\tau^j} \circ l = l \circ \left[\left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial\tau^j}\right],$$

$$\frac{\partial}{\partial y^j} \circ l = l \circ \frac{\partial}{\partial y^j}.$$

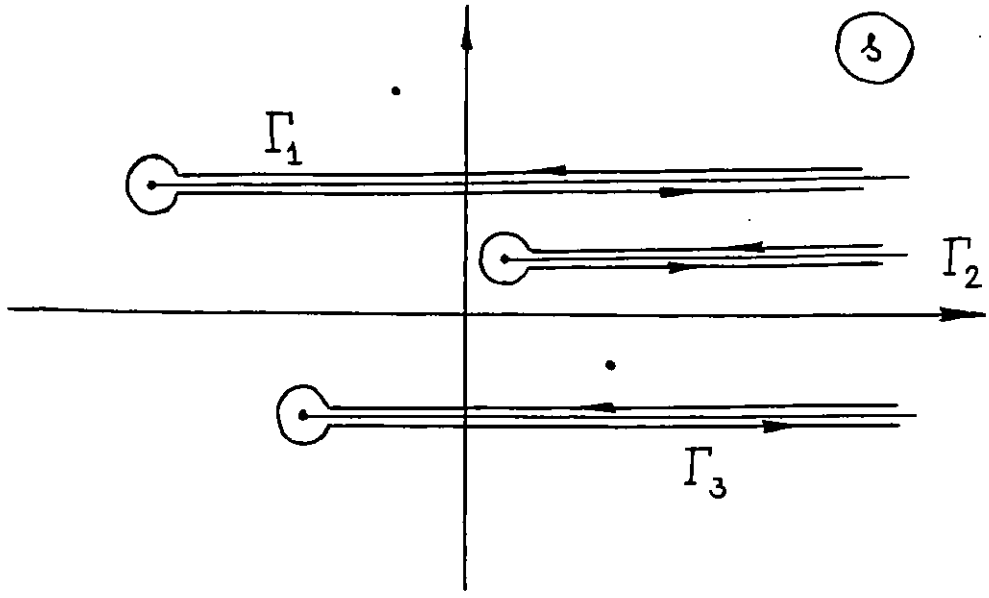

Figure 1:

Further, the formula

$$e^{\tau\beta} \circ l = l \circ T_{\tau\beta}$$

is valid. Here $T_{\tau\beta}$ is the shift operator along the axis s:

$$(T_{\tau\beta}U)(s,\tau) = U(s+\tau\beta,\tau).$$

Now we shall construct resurgent solutions to equation (8) provided that its right-hand part is a resurgent function. Applying the affirmation of the latter theorem to equation (8) we obtain the equation for the function $U(s,\tau)$:

$$\sum_{|\alpha|\le m} \sum_{|\beta|\ge 0} a_{\alpha\beta}\, T_{\tau\beta} \left[\left(\frac{\partial}{\partial s} \right)^{-1} \frac{\partial}{\partial \tau} \right]^{\alpha} U(s,\tau) = G(s,\tau) \tag{10}$$

where the function $G(s,\tau)$ corresponds to $g(\tau)$ under the action of the operator l. Equation (10) is considered as an equation in homogeneous hyperfunctions of the variable s.

We remark that, since we search for the asymptotic solutions to equation (5) in a neighbourhood of the origin, the variables

$$\tau^j = \ln\, x^j, \quad j = 1,\dots,n \tag{11}$$

vary in the region $Re\ \tau^j < 0$. Hence, we must construct asymptotic solutions with respect to smoothness to equation (10) only in this region. We recall that the coefficients $a_{\alpha\beta}$ of equation (10) are in general differential operators in variables $y \in \mathbf{C}^k$.

Note that if a point $s = S(\tau)$ is a singular point of the function $U(s,\tau)$ then the shifted function $T_{\tau\beta}[U(s,\tau)]$ has the singularity at the point

$$s = S(\tau) - \tau\beta$$

lying to the right from the point of the original singularity.

Suppose that for some value of τ the point $s = S_0(\tau)$ is the very left point of the support of the resurgent function u (such point will be referred as *the main singularity of the function* U). Then the supports of all terms of the left-hand side of (10) except for that corresponding to $\beta = 0$ lyes strictly to the right with respect to the main singularity.

This allows to use a recurrent procedure for constructing of a resurgent solution to the equation (10). Namely, we denote by $U^0(s,\tau)$ a solution 4o the equation

$$\sum_{|\alpha|\le m} a_{\alpha 0}\left[\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right]^{\alpha} U^0(s,\tau) = G(s,\tau), \tag{12}$$

which is the 'main part' of equation (10). Then we determine the subsequent functions $U^\beta(s,\tau)$ as solutions to equations

$$\begin{aligned}&\sum_{|\alpha|\le m} a_{\alpha 0}\left[\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right]^{\alpha} U^\beta(s,\tau)\\ &= -\sum_{|\alpha|\le m}\sum_{\beta'+\beta''=\beta} a_{\alpha\beta'}\,T_{\tau\beta'}\left[\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial \tau}\right]^{\alpha} U^{\beta''}(s,\tau)\end{aligned} \tag{13}$$

where the sum in the right do not contain the term with $\beta' = 0$. Ordering the set of functions

$$\left\{U^\beta(s,\tau),\ \beta \ge 0\right\} \tag{14}$$

in such a way that the product $\tau\beta$ decreases along this ordering, we see that system of equations (12), (13) determine a recurrent procedure for the set of functions (14) with one and the same principal part.

Now we denote by $u^\beta(\tau)$ the resurgent functions corresponding to the functions $U^\beta(s,\tau)$. Certainly, for doing so we must determine the supports of these resurgent functions. To begin with, we determine these supports in a neighbourhood of some fixed value of τ. The support of the function $U^0(s,\tau)$ can be chosen in arbitrary way but provided that it is contained in a sector of angle less than π bisected by the positive direction of the real axis in the complex plane s.The supports of the functions $u^\beta(\tau)$, $\beta \ge 0$ are chosen in such a way that these functions satisfy the equations

$$\sum_{|\alpha|\le m} a_{\alpha 0}\left(\frac{\partial}{\partial \tau}\right)^{\alpha} u^\beta(\tau) = -\sum_{|\alpha|\le m}\sum_{\beta'+\beta''=\beta} a_{\alpha\beta'}\,e^{\tau\beta'}\left(\frac{\partial}{\partial \tau}\right)^{\alpha} u^{\beta''}(\tau).$$

Evidently, this requirement uniquely determines the supports of $u^\beta(\tau)$.

In order to determine the supports of the resurgent function $u^\beta(\tau)$ for all values of τ one must perform the procedure of analytic continuation of the constructed resurgent function along paths in the complex plane $\mathbf{C}_s$. This procedure can be performed in a way usual in the theory of resurgent functions with the help of the so-called *transition homomorphism* (see, for example [10]). We shall not describe here this construction in detail.

We remark, that if the functions $u^\beta(\tau)$ are determined as it was described above, the series

$$u(\tau) = \sum_{\beta \geq 0} u^\beta(\tau) \tag{15}$$

converge in the space of resurgent function since we consider the domain in the space $\mathbf{C}^n_\tau$ where $Re\ \tau\beta \leq 0$ and, hence, the supports of the terms of this series lye in the half-plane $Re\ s > N$ for any value of N if $|\beta|$ is sufficiently large. The function (15) is exactly the required resurgent solution to the equation (8).

Note that since we search for resurgent solution of the initial equation, we must solve equation (12) for the *microfunction* U^0(as well as the subsequent equations for microfunctions U^β) in the class of *infinitely-continuable* microfunctions. To investigate the existence of such solutions we use the $\partial/\partial s$-transformation of ramifying analytic functions (see [11]). Applying this transformation to equation (12) we obtain the following equation for the image $\tilde{U^0}\ (s,p)$ of the function $U^0(s,\tau)$ under this transformation:

$$\sum_{|\alpha| \leq m} a_{\alpha 0}\, p^\alpha\ \ \tilde{U^0}\ (s,p) = \tilde{G}\ (s,p). \tag{16}$$

The latter equation is a family of operator equations in the space of functions of the variables y with parameters $p \in \mathbf{C}_n$. Note that the latter equation must be solved in the space of microfunctions, that is, we must solve equation (16) modulo holomorphic functions of (s,p). Similar to the case of differential equations with constant (numerical) coefficients, the set of singularities of solution $\tilde{U^0}\ (s,p)$ is determined by the set of points $p \in \mathbf{C}_n$ such that the operator

$$\tilde{H}\ (p) = \sum_{|\alpha| \leq m} a_{\alpha 0}\, p^\alpha$$

is not invertible in the considered space of functions of variable y . We denote this set by

$$\text{char}\ \hat{H} = \left\{ p : \tilde{H}\ (p)\ \text{is not invertible} \right\} \tag{17}$$

and call it *the characteristic set* of the operator $\hat{H}$. We input the following requirement on the operator $\hat{H}$.

Condition 1 The set *char* $\hat{H}$ is an analytic set in the space $\mathbf{C}_n$.

Under this condition the set of singularities of a solution to equation (12) is the union of some set which is characteristic with respect to the operator $\hat{H}$ with the set of singularities of the right-hand part $G(s,\tau)$. Suppose that the main singularity $s = S_0(\tau)$ of solution is

not determined by some singularity of the function $G(s,\tau)$. Then the function $S_0(\tau)$ must be a solution of the Hamilton-Jacobi equation

$$\left\{p:\ p=\frac{\partial S_0}{\partial \tau}\right\} \subset \operatorname{char} \hat{H}\ .$$

Now we are able to prove the existence of infinitely-continuable solutions to equation (12).

Theorem 2 *Suppose that the operator $\hat{H}$ satisfies Condition* 1. *Then equation* (12) *is solvable in the space of resurgent functions.*

Proof. To construct a solution to equaion (12) we choose a submanifold which is not everywhere characteristic with respect to this equation. Then a solution to any Cauchy problem with resurgent Cauchy data on this manifold will be a resurgent solution to equation (12). The existence of infinitely-continuable solution for such a problem (under the condition that the Cauchy data are infinitely-continuable) can be proved with the help of an explicit formula for solutions which has the same form as in the case of constant (numerical) coefficients (see [11]). The proof of the fact that this formula determines an infinitely-continuable solution to the Cauchy problem is quite similar to that in the cited book and we leave it to the reader. This proves the theorem.

To conclude this section, we present the form of constructed asymptotic solutions in the case when this solution has *simple singularities*. We recall that the resurgent function (9) has simple singularities if the corresponding function $U(s,\tau)$ can be represented in a neighbourhood of its singular points in the form

$$U(s,\tau)=\frac{a_0(\tau)}{s-S(\tau)}+\ln(s-S(\tau))\sum_{j=0}^{\infty}\frac{(s-S(\tau))^j}{j!}a_{j+1}(\tau) \tag{18}$$

where $s=S(\tau)$ is an equation of the singularity set of U and the series in the right converge in a neighbourhood of $s=S(\tau)$. From the homogeneity properties of the function U it follows that the function $S(\tau)$ is a homogeneous function of the variables τ of degree 1 and the functions $a_{j+1}(\tau)$ are homogeneous functions of order $-(j+1)$. It is known that if the function $u(\tau)$ is a resurgent function with simple singularities then the point $s=S(\tau)$ of singularity corresponds to the term

$$e^{-S(\tau)}\sum_{j=0}^{\infty}a_j(\tau) \tag{19}$$

of the asymptotic expansion of this function for large values of $|\tau|$. Performing the change of variables (11) we come to the asymptotic expansion of the initial function $u(x)$ which is the sum of the following terms

$$e^{-S(\ln x)}\sum_{j=0}^{\infty}a_j(\ln x)$$

where S and a_j are homogeneous functions of degree 1 and $-j$ correspondingly.

4 Evolution equations.

In this section we consider the Cauchy problem

$$\begin{cases} \frac{\partial^m u}{\partial t^m} = \hat{H}\, u, \\ u|_{t=0} = u_0(x), ..., \left.\frac{\partial^{m-1} u}{\partial t^{m-1}}\right|_{t=0} = u_{m-1}(x), \end{cases} \tag{20}$$

where the operator $\hat{H}$ is an operator of the type (5). As above, using exponential change of variables (7) and expanding the coefficients of the operator $\hat{H}$ into Taylor series in x, we reduce problem (20) to the form

$$\begin{cases} \frac{\partial^m u}{\partial t^m} = \sum\limits_{|\alpha|\le m} \sum\limits_{|\beta|\ge 0} a_{\alpha\beta}\, e^{\tau\beta} \left(\frac{\partial}{\partial \tau}\right)^{\alpha} u, \\ u|_{t=0} = u_0(\tau), ..., \left.\frac{\partial^{m-1} u}{\partial t^{m-1}}\right|_{t=0} = u_{m-1}(\tau). \end{cases} \tag{21}$$

Remark 1 We recall that $a_{\alpha\beta}$ in the latter equation are supposed to be differential operators in variable $y \in \mathbf{C}^k$. More generally, we can assume that these operators contain differentiations with respect to t of order not more than $m-1$. In any case orders of the operators $a_{\alpha\beta}$ are supposed to be less or equal to $m - |\alpha|$.

Similar to the previous section we search for a solution to the problem (21) in the form of resurgent function (see equation (9)):

$$u(t,\tau) = \sum_j \int\limits_{\Gamma_j} e^{-s} U(s,t,\tau)\, ds,$$

where $U(s,t,\tau)$ is an infinitely-continuable analytic function in s. The corresponding Cauchy problem for the function $U(s,t,\tau)$ has the form

$$\begin{cases} \frac{\partial^m U}{\partial t^m} = \sum\limits_{|\alpha|\le m} \sum\limits_{|\beta|\ge 0} a_{\alpha\beta}\, T_{\tau\beta} \left[\left(\frac{\partial}{\partial s}\right)^{-1} \left(\frac{\partial}{\partial \tau}\right)\right]^{\alpha} U, \\ U|_{t=0} = U_0(s,\tau), ..., \left.\frac{\partial^{m-1} U}{\partial t^{m-1}}\right|_{t=0} = U_{m-1}(s,\tau). \end{cases} \tag{22}$$

We shall construct a solution to the problem (22) with the help of a recurrent procedure. Namely, we define the function $U^0(s,t,\tau)$ as a solution of the following Cauchy problem

$$\begin{cases} \frac{\partial^m U^0}{\partial t^m} = \sum\limits_{|\alpha|\le m} a_{\alpha 0} \left[\left(\frac{\partial}{\partial s}\right)^{-1} \left(\frac{\partial}{\partial \tau}\right)\right]^{\alpha} U^0, \\ U^0|_{t=0} = U_0(s,\tau), ..., \left.\frac{\partial^{m-1} U^0}{\partial t^{m-1}}\right|_{t=0} = U_{m-1}(s,\tau). \end{cases} \tag{23}$$

Then, for each multiindex $\beta \neq 0$ we determine the function $U^\beta(s,t,\tau)$ as a solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial^m U^\beta}{\partial t^m} = \sum\limits_{|\alpha|\le m} a_{\alpha 0} \left[\left(\dfrac{\partial}{\partial s}\right)^{-1}\left(\dfrac{\partial}{\partial \tau}\right)\right]^\alpha U^\beta \\ \quad + \sum\limits_{|\alpha|\le m}\sum\limits_{\beta'+\beta''=\beta} a_{\alpha\beta'}\, T_{\tau\beta'} \left[\left(\dfrac{\partial}{\partial s}\right)^{-1}\left(\dfrac{\partial}{\partial \tau}\right)\right]^\alpha U^{\beta''}, \\ U^\beta\big|_{t=0} = 0, \ldots, \dfrac{\partial^{m-1} U^\beta}{\partial t^{m-1}}\Big|_{t=0} = 0, \end{cases} \tag{24}$$

where the last sum do not contain the term with $\beta' = 0$. Since $\mathrm{Re}(\tau\beta) \le 0$, the second term in the right-hand part of the equation in (24) contains the functions $U^{\beta''}$ only with $\mathrm{Re}(\tau\beta'') > \mathrm{Re}\,(\tau\beta)$. Hence, if we order the set of functions $\left\{U^\beta, \beta \ge 0\right\}$ in such a way that $\mathrm{Re}(\tau\beta)$ do not increase, the set of Cauchy problems determines a recurrent procedure for determining these functions. Certainly, for problems (23), (24) were solvable in the class of infinitely continuable functions, one has to impose some requirements on the operator included into the right-hand part of problem (23); such requirements will be imposed below. However, if we assume that the recurrent system (23), (24) is solvable in the required functional class, then the series

$$u(t,\tau) = \sum_{\beta\ge 0} u^\beta(t,\tau) \tag{25}$$

converges in the space of resurgent functions (here $u^\beta(t,\tau)$ are resurgent functions corresponding to the functions $U^\beta(s,t,\tau)$) since the supports of terms $u^\beta(t,\tau)$ move to the left along the described ordering. The resurgent function (25) evidently is a resurgent solution to the problem (21).

Now let us formulate the condition under which equation (23) is solvable in classes of infinitely-continuable functions. To do this, we apply the $\partial/\partial s$-transformation [11] to the problem (23). Denoting by

$$\tilde{U}^\beta\,(s,t,p) = F_{\partial/\partial s}\left(U^\beta(s,t,\tau)\right) \tag{26}$$

we come to the following Cauchy problems

$$\begin{cases} \dfrac{\partial^m \tilde{U}^0}{\partial t^m} = \sum\limits_{|\alpha|\le m} a_{\alpha 0}\, p^\alpha\, \tilde{U}^0, \\ \tilde{U}^0\Big|_{t=0} = \tilde{U}_0\,(s,p), \ldots, \dfrac{\partial^{m-1}\tilde{U}^0}{\partial t^{m-1}}\Big|_{t=0} = \tilde{U}_{m-1}\,(s,p), \end{cases} \tag{27}$$

$$\begin{cases} \dfrac{\partial^m \tilde{U}^\beta}{\partial t^m} = \sum\limits_{|\alpha|\le m} a_{\alpha 0}\, p^\alpha \tilde{U}^\beta + \sum\limits_{|\alpha|\le m}\sum\limits_{\beta'+\beta''=\beta} a_{\alpha\beta'}\, T_{\tau\beta'}\, p^\alpha\, \tilde{U}^{\beta''}, \\ \tilde{U}^\beta\Big|_{t=0} = 0, \ldots, \dfrac{\partial^{m-1}\tilde{U}^\beta}{\partial t^{m-1}}\Big|_{t=0} = 0 \end{cases} \tag{28}$$

for the functions (26). Evidently, the solvability of the proble- (23) is equivalent to the solvability of the problem (27), so we must impose the following condition.

Condition 2 The resolvent operator for problem (27) exists for each value of $p \in \mathbf{C}_n$ and determines an analytical family of operators with parameter p.

Let us describe a situation in which Condition 2 will be valid. Suppose that the order of the operator[1] $a_{\alpha 0}$ equals to $m - |\alpha|$ and that the operator

$$\frac{\partial^m}{\partial t^m} - a_{00}$$

is strictly hyperbolic (see Remark 1 above). Then it is evident that Condition 2 is valid.

To conclude this section we shall investigate the singularities of the functions $U^\beta(s,t,\tau)$ provided that the Cauchy data $U_j(s,\tau)$ of problem (23) have simple singularities. This means that the functions $U_j(s,\tau)$, $j = 1,...,m-1$ can be represented in the form

$$U_j(s,\tau) = \frac{a_0(\tau)}{s - S(\tau)} + \ln(s - S(\tau)) \sum_{i=0}^{\infty} \frac{(s - S(\tau))^i}{i!} a_{i+1}(\tau) \tag{29}$$

near each point $s = S(\tau)$, where the series on the right of the latter equality converges. Then, as it follows from the *stationary phase formula* for $\partial/\partial s$-transformation (see [11], [12], [13]), the functions $\widetilde{U}_j\ (s,p)$ have the same form

$$\widetilde{U}_j\ (s,p) = \frac{\widetilde{a}_0\ (p)}{s - \widetilde{S}\ (p)} + \ln(s - \widetilde{S}\ (p)) \sum_{i=0}^{\infty} \frac{(s - \widetilde{S}\ (p))^i}{i!}\ \widetilde{a}_{i+1}\ (p)$$

near singular points $s = \widetilde{S}\ (p)$ where the function $\widetilde{S}\ (p)$ is a Legendre transform of the function $S(\tau)$. Now the singularities of the functions $\widetilde{U^\beta}\ (s,t,p)$ can be computed in a usual way with the help of the Hamilton flow along the trajectories of the operator included in problems (27) and (28). Note that, opposite to the case considered in Section 3 the operator itself does not originate any singularities of solution; all singularities are originated by the singularities of the Cauchy data.

5 Examples

In this section we consider two examples of constructing of a resurgent solution to stationary equation and to a Cauchy problem correspondingly.

Example 1. Let us consider the equation

$$\left[\left(x^1\frac{\partial}{\partial x^1}\right)^2 + \left(x^2\frac{\partial}{\partial x^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right] u(x^1,x^2,y) = 0,$$

[1] We recall that $a_{\alpha\beta}$ are supposed to be differential operators in variables y.

where x^1 and x^2 belong to a neighbourhood of the origin in the space $\mathbf{C}^2$ and y belongs to the unit circle S^1. The corresponding characteristic set (17) for this equation is the union of sets

$$\operatorname{char}_n \hat{H} = \{p_1^2 + p_2^2 - n^2 = 0\}$$

over all natural values of n. This follows from the fact that the operator

$$\frac{\partial^2}{\partial y^2} + p_1^2 + p_2^2$$

on the unit circle is not invertible exactly for values of $p = (p_1, p_2)$ such that $p_1^2 + p_2^2 = n^2$. Performing, similar to the general case, change (7) of variables and passing to the "resurgent images" $U(s, \tau^1, \tau^2, y)$ in accordance to the formula (9), we come to the equation for U of the form

$$\left[\left(\frac{\partial}{\partial s}\right)^{-2}\left(\frac{\partial}{\partial \tau^1}\right)^2 + \left(\frac{\partial}{\partial s}\right)^{-2}\left(\frac{\partial}{\partial \tau^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right] U(s, \tau^1, \tau^2, y) = 0. \tag{30}$$

As it follows from the considerations of Section 3 the singularities of a solution to the latter equation must be posited in the set $s = S(\tau^1, \tau^2, y)$ where the function S must be a solution to one of the Hamilton-Jacobi equations

$$\left(\frac{\partial S}{\partial \tau^1}\right)^2 + \left(\frac{\partial S}{\partial \tau^2}\right)^2 = n^2 \tag{31}$$

for some nonnegative integer n. We denote a solution of this equation by $S_n(\tau^1, \tau^2)$.

Now we can construct an asymptotic solution to equation (30) with simple singularities. Such solution has the form

$$U(s, \tau^1, \tau^2, y) = U_+(s, \tau^1, \tau^2)\, e^{iny} + U_-(s, \tau^1, \tau^2)\, e^{-iny} \tag{32}$$

where the functions $U_\pm$ are sulutions to the equation

$$\left[\left(\frac{\partial}{\partial s}\right)^{-2}\left(\frac{\partial}{\partial \tau^1}\right)^2 + \left(\frac{\partial}{\partial s}\right)^{-2}\left(\frac{\partial}{\partial \tau^2}\right)^2 - n^2\right] U_\pm(s, \tau^1, \tau^2) = 0. \tag{33}$$

Such form of solution is due to the fact that the functions $\exp(\pm iny)$ are eigenfunctions of the operator $\partial^2/\partial y^2$ on the unit circle S^1. Solutions to equation (33) of the form (29) corresponding to action (31) can easily be constructed with the help of the Maslov's canonical operator on the complex manifold (see [11]). We present here only the result of the computation of the action itself. The computations similar to those in the paper [14] give us the expression for the functions S_n:

$$S_n = S_n^\pm(\tau^1, \tau^2) = \pm n\sqrt{(\tau^1)^2 + (\tau^2)^2}$$

or

$$S_n = S_n(\tau^1, \tau^2) = n\left(a\tau^1 + b\tau^2\right)$$

where a and b are subject to the relation $a^2 + b^2 = 1$.

Due to formula (19) the terms $U_\pm$ of asymptotic expansion (32) corresponding to the actions $S_n^\pm(\tau^1, \tau^2)$ have the form

$$e^{\pm n\sqrt{(\ln x^1)^2+(\ln x^2)^2}} \sum_{j=0}^{\infty} a_j(\ln x^1, \ln x^2)$$

where $a_j(\tau^1, \tau^2)$ are homogeneous in (τ^1, τ^2) of degree $-j$.

Asymptotic solutions to nonhomogeneous equation

$$\left[\left(x^1\frac{\partial}{\partial x^1}\right)^2 + \left(x^2\frac{\partial}{\partial x^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right] u(x^1, x^2, y) = f(x^1, x^2, y)$$

with resurgent right-hand part $f(x^1, x^2, y)$ can be investigated with the help of the Green's function of equation (33). The corresponding computations are similar to those in the paper [8].

Example 2. Let us consider a Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \left[\left(x^1\dfrac{\partial}{\partial x^1}\right)^2 + \left(x^2\dfrac{\partial}{\partial x^2}\right)^2 + \dfrac{\partial^2}{\partial y^2}\right] u, \\ u|_{t=0} = u_0, \ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = u_1. \end{cases}$$

Here, similar to the previous example, the variable y changes on the unit circle S^1 and we construct asymptotic solutions in a neighbourhood of the origin in the space of the variables $x = (x^1, x^2)$. We require also that the functions u_0 and u_1 are resurgent functions of the variables $\tau = (\tau^1, \tau^2)$ determined by change (7) of variables. This means that the functions

$$u_0\left(e^{\tau^1}, e^{\tau^2}\right), \ u_1\left(e^{\tau^1}, e^{\tau^2}\right)$$

can be represented in the form of the integral (9) with the corresponding functions $U_0(s, \tau, y)$ and $U_1(s, \tau, y)$ correspondingly. Then the Cauchy problem for the function $U = U(s, t, \tau, y)$ has the form

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} = \left[\left(\dfrac{\partial}{\partial s}\right)^{-2}\left(\dfrac{\partial}{\partial \tau^1}\right)^2 + \left(\dfrac{\partial}{\partial s}\right)^{-2}\left(\dfrac{\partial}{\partial \tau^2}\right)^2 + \dfrac{\partial^2}{\partial y^2}\right] U, \\ U|_{t=0} = U_0, \ \left.\dfrac{\partial U}{\partial t}\right|_{t=0} = U_1. \end{cases}$$

Passing in this Cauchy problem to the image $\widetilde{U}\ (s, t, p, y)$ of the function $U(s, t, \tau, y)$ under the action of the $\partial/\partial s$-transformation we come to the following family of Cauchy problems

with the parameters $p = (p_1, p_2)$:

$$\begin{cases} \dfrac{\partial^2 \tilde{U}}{\partial t^2} = \left[(p_1)^2 + (p_2)^2 + \dfrac{\partial^2}{\partial y^2} \right] \tilde{U}, \\ \tilde{U}\Big|_{t=0} = \tilde{U}_0, \ \dfrac{\partial \tilde{U}}{\partial t}\Big|_{t=0} = \tilde{U}_1 \end{cases} \tag{34}$$

where $\tilde{U}_0$ and $\tilde{U}_1$ are the images of the Cauchy data U_0 and U_1 of problem (34) under the action of $\partial/\partial s$-transformation.

Suppose now that the functions $\tilde{U}_j$, $j = 1, 2$ have simple singularities, that is that

$$\tilde{U}_j\,(s,p,y) = \frac{a_0(p,y)}{s - S(p,y)} + \ln(s - S(p,y)) \sum_{i=0}^{\infty} \frac{(s - S(p,y))^i}{i!} a_{i+1}(p,y).$$

Suppose, in addition, that the action $S(p,y)$ satisfies the condition

$$\frac{\partial S(p,y)}{\partial y} \neq 0.$$

Then, as it follows from the paper [11] the asymptotic solution to problem (34) with respect to smoothness has the form

$$\tilde{U}\,(s,t,p,y) = \tilde{U^+}\,(s,t,p,y) + \tilde{U^-}\,(s,t,p,y)$$

where

$$\tilde{U^{\pm}}_j\,(s,t,p,y) = \frac{a_0^{\pm}(t,p,y)}{s - S^{\pm}(t,p,y)} + \ln(s - S^{\pm}(t,p,y)) \sum_{i=0}^{\infty} \frac{(s - S^{\pm}(t,p,y))^i}{i!} a_{i+1}^{\pm}(t,p,y)$$

and the functions $S^{\pm}(t,p,y)$ are solutions of the following Cauchy problem for Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{\partial S^{\pm}(t,p,y)}{\partial t} = \pm \dfrac{\partial S^{\pm}(t,p,y)}{\partial y}, \\ S^{\pm}(t,p,y)|_{t=0} = S(p,y). \end{cases}$$

The explicit asymptotics for the solutions to problem (34) with respect to smoothness can be obtained with the help of the Laplace-Radon integral operators on complex manifolds; the theory of these operators is presented in the book [11].

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