ROTA–BAXTER ALGEBRAS AND NEW COMBINATORIAL IDENTITIES

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Abstract. The word problem for an arbitrary associative Rota–Baxter algebra is solved. This leads to a noncommutative generalization of the classical Spitzer identities. Links to other combinatorial aspects are indicated.

Keywords: Rota–Baxter relation; free algebras; word problem; quasi-symmetric functions; noncommutative symmetric functions; Hopf algebra; Dynkin idempotent; pre-Lie relation; Bohnenblust–Spitzer identities; Bogoliubov recursion

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1. Introduction and definitions

Nearly forty years ago, a class of combinatorial formulas for random variables were recast by Rota as identities in the theory of Baxter maps [3]. The key result was the solution of the word problem, for associative *commutative* algebras endowed with such maps [28, 29]; see also [6]. This showed the equivalence of the combinatorics of fluctuations with that of classical symmetric functions. Since then, operators of the Baxter type kept showing up in all sorts of applications, and lately in the Connes–Kreimer Hopf algebraic approach to renormalization [8, 12, 15]. However, in many instances the algebra in question is not commutative. We revisit and solve the word problem, and find the corresponding identities, in the general case. Roughly speaking, we are led to replace symmetric functions of commuting variables by quasi-symmetric functions of non-commuting ones. Sequences of 'noncommutative Spitzer identities' ensue; they are related with Lam's approach [20, 21] to the Dyson and Magnus expansions for ordinary differential equations. We arrive at those formulae by use of the Dynkin idempotent of free Lie algebra theory.

Definition 1.1. *Let $\mathbb{K}$ be a field of characteristic zero. Let A be a $\mathbb{K}$-algebra, not necessarily associative nor commutative nor unital. An operator $R \in \mathrm{End}(A)$ satisfying the relation*

$$Ra\,Rb = R(Ra\,b + aRb) + \theta R(ab), \quad \text{for all} \quad a, b \in A, \tag{1}$$

is said Rota–Baxter of weight $\theta \in \mathbb{K}$. The pair (A, R) is a weight θ Rota–Baxter algebra (RBA).

The Rota–Baxter identity (1) prompts the definition of a new product $a *_R b := Ra\,b + aRb + \theta ab$, $a, b \in A$.

Proposition 1.1. *The linear space underlying A equipped with the product $*_R$ is again a* RBA *of the same weight with the same Rota–Baxter map. We denote it by (A_R, R). If A is associative, so is A_R.*

We call $*_R$ the *Rota–Baxter double product*. Clearly R becomes an algebra map from A_R to A. Note that $\tilde{R} := -\theta \mathrm{id}_A - R$ is Rota–Baxter as well, and $*_{\tilde{R}} = -*_R$. One may think of Rota–Baxter operators as generalized integrals. Indeed, relation (1) for the weight $\theta = 0$ corresponds to the integration-by-parts identity for the Riemann integral; the reader will have no difficulty in checking duality of (1) with the 'skewderivation' rule

$$\delta(ab) = \delta a\,b + a\delta b + \theta\delta a\delta b.$$

For instance, the finite difference operator of step $-\theta$, given by $\delta f(x) := \theta^{-1}(f(x-\theta) - f(x))$, is a skewderivation. The summation operator $Zf(x) := \sum_{n\geq 1} \theta f(x + \theta n)$ is Rota–Baxter of weight θ, and we find $\delta Z = \mathrm{id} = Z\delta$ on suitable classes of functions. Scaling $R \to \theta^{-1}R$ reduces the study of RBAs of nonvanishing weight to the case $\theta = 1$. For notational simplicity we proceed with this one, returning to general weight when convenient. We assume henceforth we are dealing with associative RBAs; non-associative ones will arise later in an ancillary role.

2. Main result

We extend to our noncommutative setting Rota's notion of *standard* RBA. Let $X = (x_1, \ldots, x_n, \ldots)$ be a countably infinite, ordered set of variables and $T(X)$ the tensor algebra over X. The elements of X are called noncommutative polynomials (over X). Consider the pair $(\mathcal{A}, \rho)$, where $\mathcal{A}$ is the algebra of countable sequences $\Upsilon \equiv (y_1, \ldots, y_n, \ldots)$ of elements $y_i \in T(X)$ with pointwise addition and product, and ρ given by

$$\rho\Upsilon = (0, y_1, y_1 + y_2, y_1 + y_2 + y_3, \ldots).$$

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By abuse of notation we regard X itself as an element of $\mathcal{A}$. The component y_p of Υ is denoted Υ_p.

Lemma 2.1. *The algebra $\mathcal{A}$ together with $\rho \in \mathrm{End}(\mathcal{A})$ defines a weight $\theta = 1$ Rota–Baxter algebra structure.*

This is a straightforward verification. We remark that ρ is left-invertible.

Theorem 2.1. *The Rota–Baxter subalgebra $(\mathcal{R}, \rho)$ of $\mathcal{A}$ generated by X is free on one generator in the category of* $\mathbb{K}$-RBAs.

In detail, our assertions are the following.

- $X \in \mathcal{R}$.
- The product in $\mathcal{R}$ is associative.
- ρ is a Rota–Baxter operator.
- Let (A, R) be any associative RBA and $a \in A$. There is a unique algebra map $h : \mathcal{R} \to A$ with $h(X) = a$ and such that $R \circ h = h \circ \rho$.

The pair $(\mathcal{R}, \rho)$ is what we call the *standard* RBA. The point of course is that the theorem allows us to prove the validity for any RBA A of an identity involving one element of A and R, by proving it for X in $(\mathcal{R}, \rho)$.

Only the last assertion above asks for proof. We sketch the main argument, following Rota and Smith [29] insofar as possible; a full-blooded proof will appear in [17]. The adaptation to the noncommutative setting requires some care. The lexicographical ordering $<_L$ for noncommutative monomials over X is useful; for any noncommutative polynomial P we write $\mathrm{Sup}\, P$ for the highest monomial in P for $<_L$ and extend the lexicographical ordering of noncommutative monomials to a partial ordering on $T(X)$. Namely, we write $P <_L P'$ whenever $\mathrm{Sup}\, P <_L \mathrm{Sup}\, P'$. Note that, for P, P' homogeneous noncommutative polynomials and z, t in $T(X)$, we have $P <_L P' \Rightarrow Pz <_L P'z$ and $z <_L t \Rightarrow Pz <_L Pt$. Henceforth we just employ the generic R for the Rota–Baxter map on the standard RBA; this should not lead to confusion.

Proof. (Main steps.) Let us call End-algebra any associative algebra W provided with a distinguished endomorphism T_W, so that an End-algebra morphism f from W to W' satisfies $f \circ T_W = T_{W'} \circ f$. Write $\mathcal{L}$ for the free End-algebra on one generator Z. The elements of $\mathcal{L}$ are linear combinations of all symbols obtained from Z by iterative applications of the endomorphism T and of the associative product; they look like $ZT^2(TZT^3Z)$, and so on. We call these symbols $\mathcal{L}$-monomials. A RBA A is an End-algebra together with relation (1) on $T_A \equiv R$. Denote by $\mathcal{F}$ the free RBA on one generator Y. Between the three algebras $\mathcal{L}$, $\mathcal{F}$, and $\mathcal{R}$ there are the following maps: unique End-algebra maps F, U from $\mathcal{L}$ to $\mathcal{F}$, respectively $\mathcal{R}$, sending Z to Y, respectively X; and a unique onto Rota–Baxter map h' sending Y to X. Moreover $U = h' \circ F$.

We have to show the existence of an inverse for h' in the RBA category. Clearly $\ker F \subseteq \ker U$. We need only prove that $\ker U \subseteq \ker F$.

Any $l \in \mathcal{L}$ can be written uniquely as a linear combination of $\mathcal{L}$-monomials. We write $\mathrm{Max}\, l$ for the maximal number of T's occurring in the monomials, so that, say, $\mathrm{Max}(ZT^2(ZTZ) + Z^3T^2Z\ Z) = 3$. We call α, a $\mathcal{L}$-monomial, *elementary* iff it can be written as either Z^i, $i \geq 0$ or as a product $Z^{i_1}Tb_1\ Z^{i_2} \cdots Tb_k\ Z^{i_{k+1}}$, where the b_is are elementary, and $i_2, \ldots, i_k$ are strictly positive integers, while i_1 and i_{k+1} may be equal to zero; this definition makes sense by induction on $\mathrm{Max}\, \alpha$. It turns out that every element l of $\mathcal{L}$ can be written as the sum of a linear combination of elementary monomials with an element r_l such that $F(r_l) = 0$. This is due to the fact that, up to the addition of suitable elements in $\ker F$, products like $TcTd$ can be iteratively cancelled from the expression of l using the Rota–Baxter relation (1).

We claim that for p large enough and $l \neq l'$, with l, l' elementary monomials, we have $\mathrm{Sup}\, U(l)_p \neq \mathrm{Sup}\, U(l')_p$, from which the required $\ker U \subseteq \ker F$ follows. Our assertion can be verified by induction on $\mathrm{Max}\, l$, using that U is an End-algebra map. □

Corollary 2.1. *The images of the elementary monomials of $\mathcal{L}$ in $\mathcal{R}$ form a linear basis of the free RBA on one generator.*

3. Two pertinent Hopf algebras

Inductively define in a general RBA (A, R),

$$Ra^{[n+1]} = R\big(Ra^{[n]}a\big) \quad \text{and} \quad Ra^{\{n+1\}} = R\big(aRa^{\{n\}}\big).$$

with the convention that $Ra^{[1]} = Ra = Ra^{\{1\}}$ and $Ra^{[0]} = 1 = Ra^{\{0\}}$, with the unit adjoined if need be. These iterated compositions with R appear in the context of Spitzer-like formulas; of course there is no difference between $Ra^{[n]}$ and $Ra^{\{n\}}$ in the commutative context.

Coming back to the standard RBA $(\mathcal{R}, R)$, notice that:

$$R(y_1, y_2, y_3, \dots)^{[2]} = R\big(R(y_1, y_2, y_3, \dots)\,(y_1, y_2, y_3, \dots)\big) = (0, 0, y_1y_2, y_1y_2 + y_1y_3 + y_2y_3, \dots).$$

This begins to give the game away. In general, the $(n+1)$-th entry of $R(y_1, y_2, y_3, \dots)^{[k]}$ is the elementary 'symmetric' function of degree k, restricted to the first n variables, the $(n+2)$-th entry is given by the same, restricted to $n+1$ variables, and so on. The quotes on 'symmetric' remind us here that the y_i do not commute. The pertinent notion here is Hivert's quasi-symmetric functions over a set of noncommuting variables [4, 22]. Denote as usual by $[n]$ the set of integers between 1 and n. Let f be a surjective map from $[n]$ to $[k]$. Then the quasi-symmetric function M_f over X associated to f is by definition $M_f\,X = \sum_\phi x_{\phi^{-1}\circ f(1)} \cdots x_{\phi^{-1}\circ f(n)}$, where ϕ runs over the set of increasing bijections between subsets of $\mathbb{N}$ of cardinality k and $[k]$. Let us represent f as the sequence of its values, $f = f(1), \dots, f(n)$, in the notation M_f. We also denote by M_f^l the image of M_f under the map sending x_i to 0 for $i > l$ and to itself otherwise. For example,

$$M_{1,3,3,2}\,X = x_1x_3x_3x_2 + x_1x_4x_4x_2 + x_1x_4x_4x_3 + x_2x_4x_4x_3 + \dots \quad \text{and} \quad M^3_{1,3,3,2}\,X = x_1x_3x_3x_2.$$

The linear span $\mathrm{NCQSym}(X)$ of the M_f is a subalgebra of the completion of the algebra of noncommutative polynomials over X which is naturally related to the Coxeter complex of type A_n and the corresponding Solomon–Tits and twisted descent algebras [25]. Finally, write $[n]$ for the identity map on $[n]$ and ω_n for the endofunction of $[n]$ reversing the ordering, so that $M_{\omega_n}\,X = M_{n,n-1,\dots,1}\,X$. Then we have

$$RX^{[n]} = (0, M^1_{[n]}\,X, M^2_{[n]}\,X, \dots, M^l_{[n]}\,X, \dots), \quad n \geq 1,$$

where $M^l_{[n]}\,X$ is at the $(l+1)$-th position in the sequence. Similarly

$$RX^{\{n\}} = (0, M^1_{\omega_n}\,X, M^2_{\omega_n}\,X, \dots, M^l_{\omega_n}\,X, \dots), \quad n \geq 1.$$

Proposition 3.1. *The elements $RX^{[n]}$ generate freely a subalgebra of $\mathcal{A}$ (respectively generate freely a subalgebra of the double RBA $\mathcal{A}_R$).*

The proofs are omitted for the sake of brevity; the first uses the observation that, for l big enough, we find $\mathrm{Sup}(M^l_{[n_1]}\,X \cdots M^l_{[n_k]}\,X) > \mathrm{Sup}(M^l_{[m_1]}\,X \cdots M^l_{[m_j]}\,X)$ with $n_1 + \dots + n_k = m_1 + \dots + m_j$ iff the sequence $(n_1, \dots, n_k)$ is smaller than the sequence $(m_1, \dots, m_j)$ in the lexicographical ordering. The second is a bit more involved, but the reader should have no problems to provide the details.

The algebra NCQSym of quasi-symmetric functions in noncommuting variables is naturally endowed with a Hopf algebra structure [4]. On the elementary quasi-symmetric functions $M_{[n]}$, the coproduct Δ acts as on a sequence of divided powers: $\Delta\big(M_{[n]}\big) = \sum_{i=0}^n M_{[i]} \otimes M_{[n-i]}$. Thus the $M_{[n]}$ generate a free subalgebra of NCQSym naturally isomorphic as a Hopf algebra to the classical descent algebra, which is a convolution subalgebra of the endomorphism algebra of $T(X)$ [27] —or equivalently, to the algebra of noncommutative symmetric functions (NCSF) described in [18]. The same construction goes over to the free algebras over the $RX^{[n]}$ for the pointwise product as well as the Rota–Baxter double product $*_R$. The first one is naturally provided with a cocommutative Hopf algebra structure for which the $RX^{[n]}$ form a sequence of divided powers, that is:

$$\Delta\big(RX^{[n]}\big) = \sum_{0 \leq m \leq n} RX^{[m]} \otimes RX^{[n-m]};$$

this is just the structure inherited from the Hopf algebra structure on NCQSym. We call this algebra the free noncommutative Spitzer (Hopf) algebra on one generator, or the *Spitzer algebra* for short, and write $\mathcal{S}$ for it. When dealing with the $*_R$ product, the right subalgebra to consider, as it will emerge soon, is the algebra freely generated by the $RX^{[n]}X$. We also make it a Hopf algebra by requiring the free generators to form a sequence of divided powers, that is

$$\Delta_*\big(RX^{[n]}X\big) = 1 \otimes RX^{[n]}X + \sum_{0 \leq m \leq n-1} RX^{[n-m-1]}X \otimes RX^{[m]}X + RX^{[n]}X \otimes 1.$$

Thus it is convenient to set $RX^{[-1]}X = 1$. We call this Hopf algebra the *double Spitzer algebra*, and write $\mathcal{C}$ for it. We shall need the antipode S for both Hopf algebras. For obtaining it, recourse to Atkinson's theorem [2] seems the simplest method. Recall that we assume $\theta = 1$.

Theorem 3.1. **(Atkinson)** Let (A, R) be an associative unital Rota–Baxter algebra. Fix $a \in A$ and let x and y be defined by $x = \sum_{n\in\mathbb{N}} t^n Ra^{[n]}$ and $y = \sum_{n\in\mathbb{N}} t^n \tilde{R}a^{\{n\}}$, that is, as the solutions of the equations

$$x = 1 + tR(x\,a) \quad \text{and} \quad y = 1 + t\tilde{R}(a\,y),$$

in $A[[t]]$. We have the following factorization

$$x\big(1+at\big)y = 1, \quad \text{so that} \quad 1+at = x^{-1}y^{-1}.$$

Corollary 3.1. *Let (A,R) be an associative unital Rota–Baxter algebra. Fix $a \in A$ and assume x and y to solve the equations in the foregoing theorem. The inverses x^{-1} and y^{-1} solve the equations*

$$x^{-1} = 1 - tR(a\,y) \quad \text{and} \quad y^{-1} = 1 - t\tilde{R}(x\,a),$$

in $A[[t]]$.

One checks that $xx^{-1} = x^{-1}x = 1$ by using the definitions and the Rota–Baxter property. Similarly for y^{-1}.

Corollary 3.2. *The action of the antipode S on the Spitzer algebra $\mathcal{S}$, is given by*

$$S\big(RX^{[n]}\big) = -R\Big(X\,\tilde{R}X^{\{n-1\}}\Big).$$

Indeed, the Spitzer bialgebra is naturally graded. The series $\sum_{n\in\mathbb{N}} RX^{[n]}$ is a group-like element in $\mathcal{S}$. The inverse series computes the action of the antipode on the terms of the series. The corollary follows, since

$$\Big(\sum_{n\in\mathbb{N}} RX^{[n]}\Big)^{-1} = 1 - R\Big(X\Big(\sum_{n\in\mathbb{N}} \tilde{R}X^{\{n\}}\Big)\Big).$$

Corollary 3.3. *The action of the antipode S on the double Spitzer algebra $\mathcal{C}$ is given by*

$$S\big(RX^{[n]}X\big) = -\Big(X\,\tilde{R}X^{\{n\}}\Big). \tag{2}$$

For the proof, one can observe that the operator R induces an isomorphism of free graded algebras between $\mathcal{C}$ and $\mathcal{S}$ (which is the identity on scalars). That is, for any sequence of integers $i_1,\ldots,i_k$, we have:

$$R\big(RX^{[i_1]}X *_R \cdots *_R RX^{[i_k]}X\big) = RX^{[i_1+1]}\cdots RX^{[i_k+1]}.$$

Hence (2).

Corollary 3.4. *The free $*_R$ subalgebras of $\mathcal{A}$ generated by the $RX^{[n]}X$ and the $X\tilde{R}X^{\{n\}}$ are canonically isomorphic. The antipode exchanges the two families of generators. In particular, the $X\tilde{R}X^{\{n\}}$ form also a sequence of divided powers in the double Spitzer algebra.*

4. Enter the Dynkin map

The classical Dynkin operator is defined on the tensor algebra $T(X) := \bigoplus_{n\geq 0} T_n(X)$ over X by the left-to-right iteration of the associated Lie bracket,

$$D(x_1 \ldots x_n) = [\cdots[[x_1,x_2],x_3]\cdots,x_n],$$

where $[x,y] := xy - yx$, with $D|_{T_0(X)} = 0$ and $D|_{T_1(X)} = \mathrm{id}_X$. The Dynkin operator can be shown to be a quasi-idempotent —that is, its action on an homogeneous element of degree n satisfies $D^2 = nD$. In fact, we have $D : T(X) \to \mathrm{Lie}(X)$ since $T(X)$ is canonically the enveloping algebra of the free Lie algebra $\mathrm{Lie}(X)$ over X, and the associated projector D/n sends $T_n(X)$ to the component of degree n of $\mathrm{Lie}(X)$, see [27]. Now, D can be rewritten in purely Hopf algebraic terms as $S \star N$, where N is the grading operator and $\star$ the convolution product in $\mathrm{End}(T(X))$. This definition generalizes to any graded connected cocommutative or commutative Hopf algebra [26]. One actually deals there with a more general phenomenon, namely the possibility to define an action of the classical descent algebra on any graded connected cocommutative or commutative Hopf algebra [24].

Theorem 4.1. *Let H be an arbitrary graded connected cocommutative Hopf algebra over a field of characteristic zero. The Dynkin operator $D \equiv S \star N$ induces a bijection between the group $G(H)$ of group-like elements of H and the Lie algebra $\mathrm{Prim}(H)$ of primitive elements in H. The inverse morphism from $\mathrm{Prim}(H)$ to $G(H)$ is given by*

$$h = \sum_{n\in\mathbb{N}} h_n \longmapsto \Gamma(h) := \sum_{n\in\mathbb{N}} \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{h_{i_1}\cdots h_{i_k}}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}. \tag{3}$$

This corresponds to Theorem 4.1 in our earlier work [16], establishing the same formula for characters and infinitesimal characters of graded connected commutative Hopf algebras. The proof follows from the one in that reference by dualizing the notions and identities, and thus can be omitted. In the particular case where H is a free associative algebra over a set of graded generators $y_1,\ldots,y_n,\ldots$ and H is provided with the structure of a cocommutative Hopf algebra by requiring the y_i to be a sequence of divided powers, the images of the generators y_i

under the action of D forms a sequence of primitive elements of H that generate freely H as an associative algebra. This stems directly from our theorem. Two examples of such a situation are well known. In the classical descent algebra the abstract Dynkin operator sends the identity of $T(X)$ to the classical Dynkin operator. This was put to use in [27]. Second, if H is the NCSF Hopf algebra, then H is generated as a free associative algebra by the complete homogeneous NCSF, which form a sequence of divided powers, the corresponding primitive elements under the action of the Dynkin operator being known as the power sums NCSF of the first kind [18].

The same machinery can be used to rederive the already known formulas for commutative RBAs, and moreover prove new formulas in the noncommutative framework. Let us compute the action of D on the generators of $\mathcal{C}$; that will give the action on the generators of $\mathcal{S}$, too. Denote for this purpose by π_* the product on $\mathcal{C}$. Using $N(1) = 0$ and $N(X) = 1$, there follows $D(RX^{[0]}X) = (S \star N)(X) = \pi_* \circ (S \otimes N) \circ \Delta_*(X) = \pi_* \circ (S \otimes N)(X \otimes 1 + 1 \otimes X) = X$. We then find:

$$
\begin{aligned}
D\big(RX^{[n-1]}X\big) &= (S \star N)\big(RX^{[n-1]}X\big) = \pi_* \circ (S \otimes N)\Big(\sum_{0 \le p \le n} RX^{[p-1]}X \otimes RX^{[n-p-1]}X\Big) \\
&= \sum_{0 \le p \le n} S\big(RX^{[p-1]}X\big) *_R N\big(RX^{[n-p-1]}X\big) \\
&= \sum_{0 \le p \le n-1} \Big(S\big(RX^{[p-1]}X\big) *_R N\big(RX^{[n-p-1]}\big)X + S\big(RX^{[p-1]}X\big) *_R RX^{[n-p-1]}X\Big) \\
&= \sum_{0 \le p \le n-1} S\big(RX^{[p-1]}X\big) *_R N\big(RX^{[n-p-1]}\big)X - S\big(RX^{[n-1]}X\big) \\
&= \sum_{0 \le p \le n-1} R\Big(S\big(RX^{[p-1]}X\big) *_R N\big(RX^{[n-p-2]}X\big)\Big)X \\
&- \sum_{1 \le p \le n-1} S\big(RX^{[p-1]}X\big)\tilde{R}\Big(R\big(N(RX^{[n-p-2]}X)\big)X\Big) - S\big(RX^{[n-1]}X\big).
\end{aligned}
$$

In the fourth line we used vanishing of $(S \star \mathrm{id})(RX^{[n-1]}X)$, then the simple identity $a *_R (Rb\,c) = R(a *_R b)c - a\tilde{R}(Rb\,c)$; the rest should be clear. After further simple manipulations, using (2) it comes

$$
D\big(RX^{[n-1]}X\big) = R\big(D(RX^{[n-2]}X)\big)X + X\tilde{R}\big(D(RX^{[n-2]}X)\big). \tag{4}
$$

The calculation suggests the introduction of a new product.

Definition 4.1. *Let (A, R) be an associative Rota–Baxter algebra. Consider the binary operation*

$$
a \bullet_R b := Ra\,b - bRa - ba = [Ra, b] - ba = Ra\,b + b\tilde{R}a, \tag{5}
$$

and the elements $c^{(1)}(a_1) := a_1$ and $c^{(n)}(a_1, \ldots, a_n) := \big(\cdots\big((a_1 \bullet_R a_2) \bullet_R a_3\big) \cdots \bullet_R a_{n-1}\big) \bullet_R a_n$ for $n > 1$. We further define $c^{(n)}(a)$ as the n-times iterated product $c^{(n)}(a, \ldots, a) = \big(\cdots\big((a \bullet_R a) \bullet a\big) \cdots \bullet_R a\big) \bullet_R a$. As well $C^{(n)}(a) := R\big(c^{(n)}(a)\big)$.

All these parenthesis are unavoidable, as the composition $\bullet_R$ is not associative. Nevertheless, it is (left) Vinberg or pre-Lie. Recall that a left pre-Lie algebra V is a vector space, together with a bilinear product $\bullet : V \otimes V \to V$, satisfying the left pre-Lie relation $(a \bullet b) \bullet c - a \bullet (b \bullet c) = (b \bullet a) \bullet c - b \bullet (a \bullet c)$, for $a, b, c \in V$. This is enough for the commutator $[a, b]_\bullet := a \bullet b - b \bullet a$ to satisfy the Jacobi identity. Hence the algebra of commutators of elements of V is a Lie algebra, justifying the nomenclature. The reader verifies that $a \bullet^R b := [a, Rb] - ba$ defines a right pre-Lie product. See [7] for more details on pre-Lie structures.

Lemma 4.1. *Let (A, R) be an associative Rota–Baxter algebra. The binary composition* (5) *defines a left pre-Lie structure on A, which we call the left Rota–Baxter pre-Lie product.*

The lemma follows by direct inspection. In conclusion, we have proved:

Theorem 4.2. *The action of the Dynkin operator D on the generators $RX^{[n]}$ of the Spitzer algebra (respectively on the generators $RX^{[n]}X$ of the double Spitzer algebra) is given by*

$$
D(RX^{[n]}) = C^{(n)}(X), \quad \textit{respectively by} \quad D(RX^{[n]}X) = c^{(n)}(X).
$$

Together with Theorem 4.1 this immediately implies

Theorem 4.3. *(First fundamental identity for noncommutative RBAs) We have the following identity in the Spitzer algebra $\mathcal{S}$*

$$RX^{[n]} = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{C^{(i_1)}(X)\cdots C^{(i_k)}(X)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}. \tag{6}$$

Proposition 4.1. *(Second fundamental identity for noncommutative RBAs) We have the following identity in the double Spitzer algebra $\mathcal{C}$*

$$RX^{[n-1]}X = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{c^{(i_1)}(X) *_R \cdots *_R c^{(i_k)}(X)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.$$

The Theorem and Proposition follow readily from our Theorem 4.1 by applying the inverse Dynkin map (3).

5. Generalized Bohnenblust–Spitzer identities

If (A,R) is a commutative Rota–Baxter algebra of weight θ with Rota–Baxter operator R, then on $A[[t]]$ the following identity by Spitzer holds [3, 31]:

$$\sum_{m\in\mathbb{N}} t^m Ra^{[m]} = \exp\big(\theta^{-1}R\log(1+\theta at)\big) = \exp\big(-\sum_{m>0}(-t^m/m)\,\theta^{m-1}Ra^m\big). \tag{7}$$

In the framework of the commutative standard RBA this becomes Waring's formula relating elementary and power symmetric functions [30, Chapter 4]. From (7) for $\theta=1$ it follows

$$n!\,Ra^{[n]} = \sum_{\sigma}(-1)^{n-k(\sigma)}Ra^{|\tau_1|}\,Ra^{|\tau_2|}\cdots Ra^{|\tau_{k(\sigma)}|}.$$

Here the sum is over all permutations σ of $[n]$ and $\sigma=\tau_1\tau_2\cdots\tau_{k(\sigma)}$ is the decomposition of σ into disjoint cycles [29]. We denote by $|\tau_i|$ the number of elements in τ_i. By polarization one obtains the classical formula [29]:

$$\sum_{\sigma} R\Big(R\big(\cdots(Ra_{\sigma(1)})a_{\sigma(2)}\cdots\big)a_{\sigma(n)}\Big) = \sum_{\pi\in\mathcal{P}_n}(-1)^{n-|\pi|}\prod_{\pi_i\in\pi}(m_i-1)!\;R\Big(\prod_{j\in\pi_i}a_j\Big). \tag{8}$$

Here π now runs through all unordered set partitions $\mathcal{P}_n$ of $[n]$; by $|\pi|$ we denote the number of blocks in π; and $m_i:=|\pi_i|$ is the size of the particular block π_i. Those are often called Bohnenblust–Spitzer formulas. Incidentally, they are very useful in proving identities for articulated graphs in renormalization theory [13].

The generalization to *noncommutative* Bohnenblust–Spitzer formulas springs here from Theorem 4.3 and Proposition 4.1.

Theorem 5.1. *Let (A,R) be an associative Rota–Baxter algebra. For $a_i\in A$, $i=1,\ldots,n$, we have*

$$\sum_{\sigma} R\Big(R\big(\cdots(Ra_{\sigma(1)})a_{\sigma(2)}\cdots\big)a_{\sigma(n)}\Big) = \sum_{\sigma} R\Big(a_{\sigma(1)}\diamond_1 a_{\sigma(2)}\diamond_2\cdots\diamond_n a_{\sigma(n)}\Big), \quad \text{where} \tag{9}$$

$$a_{\sigma(i)}\diamond_i a_{\sigma(i+1)} = \begin{cases} a_{\sigma(i)} *_R a_{\sigma(i+1)}, & \max\,(\sigma(j)|j\le i)<\sigma(i+1)\\ a_{\sigma(i)}\bullet_R a_{\sigma(i+1)}, & \text{otherwise;}\end{cases}$$

furthermore consecutive $\bullet_R$ products should be performed from left to right, and always before the $*_R$ product.

The proof of this theorem involves some subtleties to be expounded elsewhere [17] in more detail. The reader might wish to ponder the first few cases. One readily finds

$$R\big(Ra_1\,a_2\big)+R\big(Ra_2\,a_1\big) = Ra_1\,Ra_2 + R(a_2\bullet_R a_1) = R\big(a_1 *_R a_2 + a_2\bullet_R a_1\big) = R\big(a_2 *_R a_1 + a_1\bullet_R a_2\big).$$

In quantum field theory, with R the Riemann integral, this formula gives the relation at second-order between the Dyson and Heisenberg representations for the scattering matrix. To check by direct calculation that

$$\begin{aligned}\sum_{\sigma\in S_3} R\big(R\big(Ra_{\sigma(1)}\,a_{\sigma(2)}\big)a_{\sigma(3)}\big) &= R(a_1 *_R a_2 *_R a_3) + R\big(a_1 *_R (a_3\bullet_R a_2)\big) + R\big(a_2 *_R (a_3\bullet_R a_1)\big)\\ &\quad + R\big((a_2\bullet_R a_1) *_R a_3\big) + R\big((a_3\bullet_R a_2)\bullet_R a_1\big) + R\big((a_3\bullet_R a_1)\bullet_R a_2\big)\\ &= Ra_1\,Ra_2\,Ra_3 + Ra_1\,R(a_3\bullet_R a_2) + Ra_2\,R(a_3\bullet_R a_1)\\ &\quad + R(a_2\bullet_R a_1)\,Ra_3 + R\big((a_3\bullet_R a_2)\bullet_R a_1\big) + R\big((a_3\bullet_R a_1)\bullet_R a_2\big)\end{aligned}$$

is already somewhat tedious. We give a practical rule for the decomposition in (9). Given any permutation σ of $[n]$, place a vertical bar to the left of σ_{i+1} iff it is bigger than all numbers to its left. For instance, for $n=3$ we obtain in the one-line notation the 'cut permutations' $(1|2|3),(21|3),(312),(1|32),(321),(2|31)$. The cuts indicate where the $*_R$ products, if any, should be located. Alternative rules could be devised, as (9) is symmetrical in its arguments —a non-obvious fact on the right hand side, demanding the pre-Lie properties for its verification. For the decomposition of $\sum_\sigma R\big(a_{\sigma(1)}R(a_{\sigma(2)}\cdots Ra_{\sigma(n)})\cdots\big)$ a rule is: place a vertical bar to the right of σ_i iff σ_i is smaller than all numbers to its right, and perform consecutive right pre-Lie products, $\bullet^R$, from right to left, and always before the $*_R$ product. For $n = 3$ the 'cut permutations' are then (1|2|3), (21|3), (31|2), (1|32), (321), (231); note the differences. In the commutative case $a \bullet_R b$ reduces to $-ab$ from any of the two previous forms, and we recover the classical Bohnenblust–Spitzer identities.

6. Remarks and applications

1. Lemma 4.1 is related to more recondite properties of RBAs [9]. Let $(\mathcal{D}, *)$ be an associative algebra and assume that it is represented on itself, from the left and from the right, with commuting actions. We write $\succ$ and $\prec$ for the left and right actions, respectively. Assume moreover that we have $a * b = a \prec b + a \succ b$; then $\mathcal{D}$ is by definition a dendriform dialgebra. In detail, the dendriform properties are

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z);\ (x \succ y) \prec z = x \succ (y \prec z);\ (x \prec y + x \succ y) \succ z = x \succ (y \succ z). \tag{10}$$

Conversely, the latter relations are enough to ensure associativity of $(\mathcal{D}, *)$. We refer to [19] for information on dendriform dialgebras. Now, any dendriform dialgebra $\mathcal{D}$ gives rise to a pre-Lie algebra and, in two different ways, to the same Lie algebra. The pre-Lie algebra structure is given by $x \bullet y := x \succ y - y \prec x$. As observed already in [9], generalizing an observation made by Aguiar for the weight-zero case [1], the notion applies in particular to weight $\theta \neq 0$ RBAs, since the associative and pre-Lie products $*_R$ and $\bullet_R$, respectively, are composed from sums and differences of the binary operations $a \prec_R b := -a\tilde{R}b$ and $a \succ_R b := Ra\, b$, that satisfy equations (10) and define therefore a dendriform dialgebra structure on any associative Rota–Baxter algebra. In the case of the Rota–Baxter pre-Lie composition, we indeed see that $[a, b]_{\bullet_R} = [Ra, b] + [a, Rb] + \theta[a, b] = a *_R b - b *_R a =: [a, b]_{*_R}$. Since free dendriform algebra is embedded in free RBA [10], most of our formulae can also be interpreted as universal formulae for the former.

The proof of the following proposition is left as an exercise.

Proposition 6.1. *Let (A, R) be an associative Rota–Baxter algebra. The left pre-Lie algebra $(A, \bullet_R)$ with the left Rota–Baxter pre-Lie product is a Rota–Baxter pre-Lie algebra of the same weight, with Rota–Baxter map R.*

2. The formulae developed in this paper actually apply without restriction to any associative RBA, in particular to the solution of differential equations —to reestablish general weight in the pre-Lie product formulas amounts simply to replace in (5) the product ba by θba, thus the case $\theta = 0$ is included in our considerations. We actually drew inspiration for this paper from that subject: mainly from the path-breaking papers by Lam [20, 21] and recent work by two of us [5]. In fact, Theorem 4.3 yields the most efficient way to organize the terms coming from two standard methods to solve differential equations, the Dyson–Chen expansion and the Magnus series; the advantage of writing the Magnus series in this way has been recently recognized by the practitioners [23]. Lam did obtain our formulas for $Ra^{\{n\}}$ for the case $\theta = 0$; this arose from the need to prove deep theorems with strong physical roots on approximations to quantum chromodynamics. Part of the magic of the subject is how little needs to change when $\theta \neq 0$. In regard to the following remark, if we define the Magnus series coefficients K_n by $d/dt \log x(t) = \sum_{n>0}^\infty t^n K_n$, for $x = x(t) = \sum_{m\in\mathbb{N}} t^m Ra^{[m]}$. Then the relation between the $C^{(n)}$ and the K_n is precisely the one between power sums NCSF of the first and of the second kind [18].

3. It would be nice to be able to derive the new Bohnenblust–Spitzer identities at one stroke from an equation like the commutative Spitzer formula (7). On the one hand, one of us participated in an attempt in this direction a few years ago by [11], with the net result that in the noncommutative case $\sum_m t^m Ra^{[m]}$ is still a functional of $\log(1 + at)$, through a non-linear recursion (for which existence and unicity were proven) called, for want of a better name, the Baker–Campbell–Hausdorff recursion, e.g. see [14]. However, in practice work with this functional was painful. On the other hand, there is a direct link between that recursion and the Magnus series. Explicit expressions for the latter are known; and so in some sense the solution to the Baker–Campbell–Hausdorff recursion has been staring at us for a while. Concretely, consider the generating series $\psi(t) = \sum_{n>0}^\infty t^{-n} C^{(n)}(X)$. From (4) it follows that the generating series $x(t) := \sum_{m\in\mathbb{N}} t^m\, RX^{[m]}$ solves the initial value problem

$$dx(t)/dt = x(t)\,\psi(t); \qquad x(0) = 1.$$

The exponential form for the solution of this equation amounts precisely to the aforementioned relation between power sums NCSF of the first and of the second kind [18]. However, these formulas are rather clumsy. They will be revisited elsewhere [17].

4. Last, but not least, we comment on an important application. Use of general Spitzer-like identities for noncommutative Rota–Baxter algebras is bound to deepen the Connes–Kreimer algebraic understanding of renormalization in perturbative quantum field theory [8]. Bogoliubov's counterterm recursion has been examined

in the light of RBAs and Atkinson's theorem [11, 12, 15]. The above presented treatment for the equations in Theorem 3.1 applies, therefore pointing to a closed expression for the recursive process of renormalization. As shown in [16], the Dynkin operator is a key ingredient for the mathematical understanding of the combinatorial processes underlying the Bogoliubov recursion. One can envisage a complete solution of the latter with our kind of Lie algebraic tools; this will be dealt with in forthcoming work [17].

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