Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2011 (43)

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MPIM 11-43

A Generalized Vertex Operator Algebra for Heisenberg Intertwiners

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July 20, 2011

Abstract

We consider the extension of the Heisenberg vertex operator algebra by all its irreducible modules. We give an elementary construction for the intertwining vertex operators and show that they satisfy a complex parametrized generalized vertex operator algebra. We illustrate some of our results with the example of integral lattice vertex operator superalgebras.

*Supported by a Science Foundation Ireland Research Frontiers Grant, and by Max-Planck-Institut für Mathematik, Bonn

1 Introduction

The concept of a Vertex Operator Algebra (VOA) was introduced in [1] and [2] and is essentially a rigorous algebraic approach to chiral conformal field theory in physics. This paper is devoted to one the most basic examples, namely the rank l Heisenberg VOA M^l i.e. the chiral CFT consisting of l free bosons. An important application is the construction of a VOA V containing a Heisenberg subVOA M^l where we decompose V into irreducible M^l-modules or extend V by V-modules or twisted V-modules related to the Heisenberg structure. Thus in [3] it is demonstrated how to extend a simple VOA V by g-twisted V-modules for automorphisms g generated by Heisenberg vectors [4]. It is also shown in [3] that intertwining vertex operators constructed on the larger space form a Generalized VOA [5]. In this paper we consider the extension of the Heisenberg VOA M^l to the space, $\mathcal{M}$, given by the union of all irreducible M^l-modules. We give an elementary construction for the intertwining vertex operators on $\mathcal{M}$ and show that these operators satisfy a complex parametrized Generalized VOA of a more general type than that defined in [5]. We illustrate some of our results with the example of lattice VOSA V_L for even or odd integral lattice L and consider the extension of V_L by twisted sectors for automorphisms generated by Heisenberg elements.

We begin in Section 2 with definitions and some properties of VOA modules and twisted modules. We consider the creative intertwiner vertex operators operators for a VOA V-module W i.e. operators whose modes map V onto W. In particular, we describe a version of Dong's Lemma and Goddard's uniqueness theorem for such operators. In Section 3 we consider the rank l Heisenberg VOA M^l with irreducible module M_α for $\alpha \in \mathbb{C}^l$. We give an elementary construction of the creative intertwiner (very similar in structure to vertex operators for a lattice VOA). In Section 4 we construct a $\mathbb{C}$-parametrized Generalized VOA on $\mathcal{M} = \cup_{\alpha\in\mathbb{C}^l} M_\alpha$, the union of all irreducible M^l-modules. We also discuss skew-symmetry and show that there exists a unique invertible invariant symmetric bilinear form on $\mathcal{M}$. Finally in Section 5 we consider the example of the lattice VOSA V_L for an even or odd integral lattice L. Using the Generalized VOA structure we construct the g-twisted V_L-module for an automorphism g generated by a Heisenberg vector and show that this isomorphic to Li's construction [4]. Finally, we conclude with a generalization of one of the main results of ref. [3] for V_L.

2 Creative Intertwiners for a Vertex Operator Algebra

We begin with a brief review of aspects of Vertex Operator Algebras and their modules, see refs. [1], [2], [6], [7], [8], [9] for more details. In particular, we are interested in describing properties of creative intertwiners which can proved by a suitable modification of standard results in VOA theory.

We define the standard formal series

$$\delta\left(\frac{x}{y}\right) = \sum_{n\in\mathbb{Z}} x^n y^{-n}, \tag{1}$$

$$(x+y)^\kappa = \sum_{m\geq 0} \binom{\kappa}{m} x^{\kappa-m} y^m, \tag{2}$$

for any formal variables x, y, κ where $\binom{\kappa}{m} = \frac{\kappa(\kappa-1)\ldots(\kappa-m+1)}{m!}$.

Definition 2.1 *A Vertex Operator Superalgebra (VOSA) is determined by a quadruple $(V, Y, \mathbf{1}, \omega)$ as follows: V is a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with parity $p(u) = 0$ or 1 for $u \in V_{\bar{0}}$ or $V_{\bar{1}}$ respectively. V also has a $\frac{1}{2}\mathbb{Z}$-grading with $V = \bigoplus_{r\in\frac{1}{2}\mathbb{Z}} V_r$ with $\dim V_r < \infty$. $\mathbf{1} \in V_0$ is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below.*

Y is a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ for formal variable z so that for any vector $u \in V$ we have a vertex operator

$$Y(u, z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1}. \tag{3}$$

The linear operators (modes) $u(n) : V \to V$ satisfy creativity

$$Y(u, z)\mathbf{1} = u + O(z) \tag{4}$$

and lower truncation

$$u(n)v = 0, \tag{5}$$

for each $u, v \in V$ and $n \gg 0$. For the conformal vector ω

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}, \tag{6}$$

where $L(n)$ satisfies the Virasoro algebra for some central charge c

$$[\,L(m), L(n)\,] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m,-n}. \tag{7}$$

Each vertex operator satisfies the translation property

$$Y(L(-1)u,z) = \partial_z Y(u,z). \tag{8}$$

The Virasoro operator $L(0)$ provides the $\frac{1}{2}\mathbb{Z}$-grading with $L(0)u = ru$ for $u \in V_r$ and with $r \in \mathbb{Z} + \frac{1}{2}p(u)$. Finally, the vertex operators satisfy the Jacobi identity

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - (-1)^{p(u,v)}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y\left(Y(u,z_0)v,z_2\right), \end{aligned} \tag{9}$$

where $p(u,v) = p(u)p(v)$.

Remark 2.2 *$(V, Y, \mathbf{1}, \omega)$ is called a Vertex Operator Algebra (VOA) when $V_{\bar{1}} = 0$.*

Amongst other properties, these axioms imply locality, associativity, commutativity and skew-symmetry:

$$(z_1-z_2)^m Y(u,z_1)Y(v,z_2) = (-1)^{p(u,v)}(z_1-z_2)^m Y(v,z_2)Y(u,z_1), \tag{10}$$

$$(z_0+z_2)^n Y(u,z_0+z_2)Y(v,z_2)w = (z_0+z_2)^n Y(Y(u,z_0)v,z_2)w, \tag{11}$$

$$u(k)Y(v,z) - (-1)^{p(u,v)}Y(v,z)u(k) = \sum_{j\geq 0}\binom{k}{j}Y(u(j)v,z)z^{k-j}, \tag{12}$$

$$Y(u,z)v = (-1)^{p(u,v)}e^{zL(-1)}Y(v,-z)u, \tag{13}$$

for $u, v, w \in V$ and integers $m, n \gg 0$ [6], [7], [9].

We define the notion of a V-module [6], [8], [10].

Definition 2.3 *A V-module for a VOSA V is a pair (W, Y_W) where W is a $\mathbb{C}$-graded vector space $W = \bigoplus_{r\in\mathbb{C}} W_r$ with $\dim W_r < \infty$ and where $W_{r+n} = 0$*

for all r and $n \ll 0$. Y_W is a linear map $Y_W : V \to \mathrm{End}(W)[[z, z^{-1}]]$ defining a module vertex operator

$$Y_W(u, z) = \sum_{n\in\mathbb{Z}} u_W(n) z^{-n-1}, \tag{14}$$

for each $u \in V$ with modes $u_W : W \to W$. For the vacuum vector $Y_W(\mathbf{1}, z) = \mathrm{Id}_W$ and for the conformal vector

$$Y_W(\omega, z) = \sum_{n\in\mathbb{Z}} L_W(n) z^{-n-2}, \tag{15}$$

where $L_W(0)w = rw$ for $w \in W_r$. The module vertex operators satisfy the Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(u, z_1)Y_W(v, z_2) - (-1)^{p(u,v)}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(v, z_2)Y_W(u, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W\left(Y(u, z_0)v, z_2\right). \end{aligned} \tag{16}$$

A V-module W is irreducible if no proper, nonzero subspace is invariant under all $u_W(n)$. The above axioms imply that $L_W(n)$ of (15) satisfies the Virasoro algebra (7) for the same central charge c and that the translation property holds:

$$Y_W(L(-1)u, z) \;=\; \partial_z Y_W(u, z). \tag{17}$$

We next define the notion of a twisted V-module. Let g be a V-automorphism g i.e. a linear map preserving $\mathbf{1}$ and ω such that

$$gY(v, z)g^{-1} = Y(gv, z),$$

for all $v \in V$. We assume that V can be decomposed into g-eigenspaces

$$V = \oplus_{\rho\in\mathbb{C}} V^\rho,$$

where V^ρ denotes the eigenspace of g with eigenvalue $e^{2\pi i\rho}$.

Definition 2.4 *A g-twisted V-module for a VOSA V is a pair (W^g, Y_g) where W^g is a $\mathbb{C}$-graded vector space $W^g = \bigoplus_{r\in\mathbb{C}} W^g_r$ with* $\dim W_r < \infty$

and where $W_{r+n} = 0$ *for all* r *and* $n \ll 0$. Y_g *is a linear map* $Y_g : V \to \mathrm{End}\, W^g\{z\}$, *the vector space of* $\mathrm{End}\, W^g$*-valued formal series in* z *with arbitrary complex powers of* z. *Then for* $v \in V^\rho$

$$Y_g(v, z) = \sum_{n \in \rho + \mathbb{Z}} v_g(n) z^{-n-1},$$

with $v_g(\rho + l)w = 0$ *for* $w \in W^g$ *and* $l \in \mathbb{Z}$ *sufficiently large. For the vacuum vector* $Y_g(\mathbf{1}, z) = \mathrm{Id}_{W^g}$ *and for the conformal vector*

$$Y_g(\omega, z) = \sum_{n \in \mathbb{Z}} L_g(n) z^{-n-2}, \tag{18}$$

where $L_g(0)w = rw$ *for* $w \in W^g$. *The* g*-twisted vertex operators satisfy the twisted Jacobi identity:*

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_g(u, z_1) Y_g(v, z_2) - (-1)^{p(u,v)} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_g(v, z_2) Y_g(u, z_1) \\ = z_2^{-1}\left(\frac{z_1 - z_0}{-z_2}\right)^{-\rho} \delta\left(\frac{z_1 - z_0}{-z_2}\right) Y_g(Y(u, z_0)v, z_2), \end{aligned} \tag{19}$$

for $u \in V^\rho$.

This definition is an extension of the standard one for g of finite order where $\rho \in \mathbb{Q}$ [11], [3] or for g unitary where $\rho \in \mathbb{R}$ [12]. These axioms imply that $L_g(n)$ of (18) satisfies the Virasoro algebra (7) for the same central charge c and that the translation property holds:

$$Y_g(L(-1)u, z) \;\; = \;\; \partial_z Y_g(u, z). \tag{20}$$

We lastly restrict ourselves to a VOA V with a V-module (W, Y_W) and introduce the notion of creative intertwining vertex operators that satisfy an intertwining algebra of type $\binom{W}{W\ V}$ in the terminology of ref. [6].

Definition 2.5 *A creative intertwining vertex operator* $\mathcal{Y}$ *for a VOA* V*-module* (W, Y_W) *is defined by a linear map* $\mathcal{Y} : W \to \mathrm{Lin}(V, W)[[z, z^{-1}]]$ *with*

$$\mathcal{Y}(w, z) = \sum_{n \in \mathbb{Z}} w(n) z^{-n-1}, \tag{21}$$

for each $w \in W$ with modes $w(n) : V \to W$. The intertwining vertex operator satisfies creativity

$$\mathcal{Y}(w, z)\mathbf{1} = w + O(z), \tag{22}$$

for each $w \in W$ and lower truncation

$$w(n)v = 0, \tag{23}$$

for each $v \in V$, $w \in W$ and $n \gg 0$. The intertwining vertex operators satisfy the Jacobi identity:

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(u, z_1)\mathcal{Y}(w, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(w, z_2)Y(u, z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}\left(Y_W(u, z_0)w, z_2\right), \end{aligned} \tag{24}$$

for all $u \in V$ and $w \in W$.

These axioms imply that the intertwining and module vertex operators satisfy the following form of translation, locality, associativity, commutativity and skew-symmetry:

$$\begin{aligned} \mathcal{Y}(L_W(-1)w, z) &= \partial_z \mathcal{Y}(w, z), & (25)\\ (z_1 - z_2)^m Y_W(u, z_1)\mathcal{Y}(w, z_2) &= (z_1 - z_2)^m \mathcal{Y}(w, z_2)Y(u, z_1), & (26)\\ (z_0 + z_2)^n Y_W(u, z_0 + z_2)\mathcal{Y}(w, z_2)v &= (z_0 + z_2)^n \mathcal{Y}(Y_W(u, z_0)w, z_2)v, & (27)\\ u_W(k)\mathcal{Y}(w, z) - \mathcal{Y}(w, z)u(k) &= \sum_{j \geq 0} \binom{k}{j} \mathcal{Y}(u_W(j)w, z)z^{k-j}, & (28)\\ \mathcal{Y}(w, z)v &= e^{zL_W(-1)}Y_W(v, -z)w, & (29) \end{aligned}$$

for $u, v \in V$, $w \in W$ and integers $m, n \gg 0$.

We may obtain creative intertwining versions of Dong's Lemma and Goddard's uniqueness theorem for VOAs. The proofs are omitted since they are easily modified versions of standard VOA methods e.g. [7]. Define an intertwining normal ordering for $u \in V$ and $w \in W$

$$: Y_W(u, z)\mathcal{Y}(w, z) := Y_W(u, z)_-\mathcal{Y}(w, z) + \mathcal{Y}(w, z)Y(u, z)_+, \tag{30}$$

where

$$Y(u,z)_- = \sum_{n\geq 0} u(n)z^{-n-1}, \qquad Y(u,z)_+ = \sum_{n<0} u(n)z^{-n-1}, \tag{31}$$

$$Y_W(u,z)_- = \sum_{n\geq 0} u_W(n)z^{-n-1}, \quad Y_W(u,z)_+ = \sum_{n<0} u_W(n)z^{-n-1}. \tag{32}$$

Theorem 2.6 (Dong's Lemma for Intertwiners) *For $u \in V$ and $w \in W$ then $: Y_W(u,z)\mathcal{Y}(w,z) :$ is local in the sense of (26) i.e. for each $v \in V$*

$$\begin{aligned} &(z_1 - z_2)^m Y_W(v,z_1) : Y_W(u,z_2)\mathcal{Y}(w,z_2) : \\ &= (z_1 - z_2)^m : Y_W(u,z_2)\mathcal{Y}(w,z_2) : Y(v,z_1), \end{aligned} \tag{33}$$

for $m \gg 0$.

Theorem 2.7 (Goddard's Uniqueness Theorem for Intertwiners) *Let $\mathcal{W}(z) \in \mathrm{Lin}(V,W)[[z,z^{-1}]]$ be local in the sense of (26) i.e. for each $u \in V$*

$$(z_1 - z_2)^m Y_W(u,z_1)\mathcal{W}(z_2) = (z_1 - z_2)^m \mathcal{W}(z_2)Y(u,z_1), \tag{34}$$

for $m \gg 0$. Suppose that for some $w \in W$

$$\mathcal{W}(z)\mathbf{1} = e^{zL_W(-1)}w, \tag{35}$$

then $\mathcal{W}(z) = \mathcal{Y}(w,z)$.

Theorems 2.6 and 2.7 imply

Proposition 2.8 *For $u \in V$ and $w \in W$ we have*

$$: Y_W(u,z)\mathcal{Y}(w,z) := \mathcal{Y}(u_W(-1)w,z). \tag{36}$$

3 Heisenberg Intertwiners

In this Section we explicitly construct the creative intertwining operators for the irreducible modules of the rank l Heisenberg VOA M^l generated by l weight one Heisenberg vectors a^i, $i = 1,\ldots,l$ with modes obeying

$$[\, a^i(n), a^j(m)\,] = n\delta_{n,-m}\delta_{i,j}. \tag{37}$$

M^l is spanned by the Fock basis

$$a^{i_1}(-k_1)a^{i_2}(-k_2)\ldots a^{i_r}(-k_r)\mathbf{1},\ k_i > 0, \tag{38}$$

for Virasoro vector $\omega = \frac{1}{2}\sum_{i=1}^{l} a^i(-1)^2\mathbf{1}$ with central charge l.

The irreducible modules for M^l, denoted by $M_\alpha = M^l \otimes e^\alpha$ (with $M^l \cong M_0$) are indexed by a complex l-tuple $\alpha = \{\alpha^1, \ldots, \alpha^l\} \in \mathbb{C}^l$ with

$$\begin{aligned} a^i_{M_\alpha}(0)(u \otimes e^\alpha) &= \alpha^i(u \otimes e^\alpha), && (39)\\ a^i_{M_\alpha}(n)(u \otimes e^\alpha) &= (a^i(n)u) \otimes e^\alpha,\ n \neq 0, && (40)\end{aligned}$$

for $Y_{M_\alpha}(a^i, z) = \sum_{n\in\mathbb{Z}} a^i_{M_\alpha}(n)z^{-n-1}$ and $u \in M^l$. From now on we will employ the standard abbreviations of writing $Y(u, z)$ in place of $Y_{M_\alpha}(u, z)$, $u(n)$ in place of $u_{M_\alpha}(n)$ and u in place of $u \otimes e^0$.

We next construct the creative intertwiner for $\mathcal{Y}(u\otimes e^\alpha, z)$ for $u\otimes e^\alpha \in M_\alpha$ for all $\alpha \in \mathbb{C}^l$. Much of the discussion is similar to the standard construction of lattice vertex operators e.g. [2], [7]. We first introduce the standard operators q^i conjugate to $a^i(0)$

$$\left[\, a^i(n), q^j\, \right] = \delta_{n,0}\delta_{i,j}, \tag{41}$$

and identify

$$u \otimes e^\alpha = e^{\alpha\cdot q}(u \otimes e^0) = e^{\alpha\cdot q}u,$$

where $\alpha \cdot q = \sum_{i=1}^{l} \alpha^i q^i$ for $\alpha \in \mathbb{C}^l$. We also define

$$Y_\pm(\alpha, z) = \exp\left(\mp \sum_{n>0} \frac{\alpha(\pm\, n)}{n} z^{\mp n}\right), \tag{42}$$

where $\alpha(n) = \alpha \cdot a(n)$. These operators obey the following [4], [3]

Proposition 3.1 *For all $\alpha \in \mathbb{C}^l$ and $u \in M^l$ we find*

$$\begin{aligned} Y_+(\alpha, z_1)Y_-(\beta, z_2) &= \left(1 - \frac{z_2}{z_1}\right)^{\alpha\cdot\beta} Y_-(\beta, z_2)Y_+(\alpha, z_1), && (43)\\ Y(Y_+(\alpha, -z_1)u, z_1)Y_-(\alpha, z_2) &= Y_-(\alpha, z_2)Y(Y_+(\alpha, -z_1 + z_2)u, z_1), && (44)\\ Y_+(\alpha, z_1)Y(u, z_2) &= Y(Y_+(\alpha, z_1 - z_2)u, z_2)Y_+(\alpha, z_1), && (45)\\ Y(Y_-(\alpha, z_1)u, z_2)Y_+(\alpha, z_2) &= z_2^{-\alpha(0)}(z_2 + z_1)^{\alpha(0)}Y_-(-\alpha, z_2)Y_-(\alpha, z_1 + z_2)\\ &\quad .Y(u, z_2)Y_+(\alpha, z_2 + z_1). && (46)\end{aligned}$$

We also have

Proposition 3.2 *For all $\alpha \in \mathbb{C}^l$ and $u \in M^l$ we find*

$$\begin{aligned} e^{-\alpha\cdot q}e^{zL(-1)}e^{\alpha\cdot q}e^{-zL(-1)} &= Y_-(\alpha,z), &(47)\\ e^{-\alpha\cdot q}e^{zL(1)}e^{\alpha\cdot q}e^{-zL(1)} &= Y_+(\alpha,\frac{1}{z}), &(48)\\ e^{-\alpha\cdot q}Y(u,z)e^{\alpha\cdot q} &= Y(Y_+(\alpha,-z)u,z). &(49)\end{aligned}$$

Proof. From (41) it follows that $[\,L(-1),q^i\,] = a^i(-1)$. Hence we find

$$\begin{aligned} e^{-\alpha\cdot q}e^{zL(-1)}e^{\alpha\cdot q} &= e^{z(L(-1)+\alpha(-1))}\\ &= Y_-(\alpha,z)e^{zL(-1)}, \end{aligned}$$

from refs. [4], [3] giving (47). (48) follows similarly.

To prove (49) we first show that

$$\left[\,q^i, Y(u,z)\,\right] = Y\left(X_+^i(z)u,z\right), \tag{50}$$

where $X_+^i(z) = \sum_{n>0}\frac{a^i(n)}{n}(-z)^n$. Assume that (50) holds for every Fock vector v with m Heisenberg modes and consider $u = a^j(-k-1)v$ for $k \geq 0$. Then (41) gives

$$\begin{aligned} \left[\,q^i, Y(u,z)\,\right] &= \frac{1}{k!}\left[\,q^i, :\partial_z^k Y(a^j,z)Y(v,z):\right]\\ &= \frac{1}{k!}\left[\,q^i, \partial_z^k Y(a^j,z)_-\,\right]Y(v,z)\\ &\quad +\frac{1}{k!}:\partial_z^k Y(a^j,z)Y(X_+^i(z)v,z):\\ &= Y\left(X_+^i(z)a^j(-k-1)v,z\right). \end{aligned}$$

using $\left[\,q^i, \partial_z^k Y(a^j,z)_+\,\right] = 0$ and

$$\frac{1}{k!}\left[\,q^i, \partial_z^k Y(a^j,z)_-\,\right] = (-z)^{-k-1}\delta_{i,j} = \left[\,X_+^i(z), a^j(-k-1)\,\right].$$

Hence (50) holds by induction in m. The general result (49) follows on exponentiating and using $Y_+(\alpha,-z) = e^{-\alpha\cdot X_+(z)}$. □

We may now construct the creative intertwiner in much the same way as for a lattice vertex operator e.g. [7]:

Theorem 3.3 *The creative intertwiner for $u \otimes e^{\alpha} \in M_{\alpha}$ for any $\alpha \in \mathbb{C}^{l}$ is given by*

$$\mathcal{Y}(u \otimes e^{\alpha}, z) \quad = \quad e^{\alpha \cdot q} c_{\alpha} Y_{-}(\alpha, z) Y(u, z) Y_{+}(\alpha, z) z^{\alpha(0)}, \tag{51}$$

where $c_{\alpha} = \mathrm{Id}_{M^{l}}$.

Proof. Using Proposition 3.1 and skew-symmetry (29) we find that

$$\begin{aligned} \mathcal{Y}(u \otimes e^{\alpha}, z) v &= e^{zL(-1)} Y(v, -z)(u \otimes e^{\alpha}) \\ &= e^{zL(-1)} Y(v, -z) e^{\alpha \cdot q} u \\ &= e^{\alpha \cdot q} Y_{-}(\alpha, z) e^{zL(-1)} Y(Y_{+}(\alpha, z) v, -z) u \\ &= e^{\alpha \cdot q} Y_{-}(\alpha, z) Y(u, z) Y_{+}(\alpha, z) v, \end{aligned}$$

for all $u, v \in M^{l}$. This implies that

$$\mathcal{Y}(u \otimes e^{\alpha}, z) = e^{\alpha \cdot q} Y_{-}(\alpha, z) Y(u, z) Y_{+}(\alpha, z) b_{\alpha}(z), \tag{52}$$

where $b_{\alpha}(z) \in \mathrm{End}(M^{l})[[z, z^{-1}]]$ with

$$b_{0}(z) = \mathrm{Id}_{M^{l}}, \quad [\, a^{i}(n), b_{\alpha}(z)\,] = 0, \quad b_{\alpha}(z) v = v, \tag{53}$$

for all $v \in M^{l}$. Translation (25) and $L(-1)(\mathbf{1} \otimes e^{\alpha}) = \alpha \cdot a \otimes e^{\alpha}$ imply

$$\partial_{z} \mathcal{Y}(\mathbf{1} \otimes e^{\alpha}, z) \quad = \quad \mathcal{Y}(\alpha \cdot a \otimes e^{\alpha}, z).$$

Using (52) we find that $z \partial_{z} b_{\alpha}(z) = \alpha(0) b_{\alpha}(z)$ giving

$$b_{\alpha}(z) = c_{\alpha} z^{\alpha(0)}, \tag{54}$$

for z independent operator c_{α}. Applying (53) we conclude that $c_{\alpha} = \mathrm{Id}_{M^{l}}$ and hence the result follows. □

4 Generalized Vertex Operator Algebra for Heisenberg Intertwiners

4.1 A Generalized Vertex Operator Algebra

Let $\mathcal{M} = \cup_{\beta \in \mathbb{C}} M_{\beta}$ denote the union of all the irreducible modules for $M^{l} = M_{0}$. Using (40), the creative intertwining operator $\mathcal{Y}(u \otimes e^{\alpha}, z)$ has a natural extension to an intertwiner vertex operator in $\mathrm{Lin}(\mathcal{M},\mathcal{M})[[z,z^{-1}]]$ where now c_α acts on M_β as a scalar

$$c_\alpha u\otimes e^\beta=\epsilon(\alpha,\beta)u\otimes e^\beta,$$

for $\epsilon(\alpha,\beta)\in\mathbb{C}^\times$ and $\epsilon(\alpha,0)=\epsilon(0,\alpha)=1$. As for lattice VOAs (e.g. [2], [7]), we define a cocycle system over $\mathbb{C}^l$ as an additive group. Define

$$e^\alpha=e^{\alpha\cdot q}c_\alpha, \tag{55}$$

for all $\alpha\in\mathbb{C}^l$ so that

$$\mathcal{Y}(u\otimes e^\alpha,z)\ =\ e^\alpha Y_-(\alpha,z)Y(u,z)Y_+(\alpha,z)z^{\alpha(0)}. \tag{56}$$

We assume that the operators (55) satisfy an associative algebra for 2-cocycle $\epsilon(\alpha,\beta)$ such that

$$e^\alpha e^\beta=\epsilon(\alpha,\beta)e^{\alpha+\beta},\quad e^0=1. \tag{57}$$

Associated to the cocycle system is the commutator function $C(\alpha,\beta)$ with

$$e^\alpha e^\beta=C(\alpha,\beta)e^\beta e^\alpha,$$

where

$$C(\alpha,\beta)=C(\beta,\alpha)^{-1}=\frac{\epsilon(\alpha,\beta)}{\epsilon(\beta,\alpha)}. \tag{58}$$

Associativity implies $C(\alpha,\beta)$ is skewsymmetric and bilinear:

$$\begin{aligned}C(\alpha,\beta)&=C(\beta,\alpha)^{-1},\\ C(\alpha+\beta,\gamma)&=C(\alpha,\gamma)C(\beta,\gamma),\\ C(\alpha,\beta+\gamma)&=C(\alpha,\beta)C(\alpha,\gamma).\end{aligned} \tag{59}$$

Example 4.1 *Suppose $l=2m$ and let $\alpha=(\alpha^1,\alpha^2)$ and $\beta=(\beta^1,\beta^2)$ for $\alpha^i,\beta^j\in\mathbb{C}^m$. Then*

$$C(\alpha,\beta)=\zeta^{\alpha^1\cdot\beta^2-\alpha^2\cdot\beta^1},$$

for any $\zeta\in\mathbb{C}^\times$ satisfies (59).

Lemma 4.2 *The cocycle factors $\epsilon(\alpha,\beta)$ can be chosen such that $\epsilon(\alpha,-\alpha)=1$ for all $\alpha\in\mathbb{C}^l$.*

Proof. Apply associativity to $e^\alpha e^{-\alpha} e^\alpha$ to find

$$\epsilon(\alpha, -\alpha) = \epsilon(-\alpha, \alpha), \quad C(\alpha, -\alpha) = 1. \tag{60}$$

As for lattice VOAs (e.g. [7]), we may redefine e^α to be $\epsilon_\alpha e^\alpha$ for the same commutator function $C(\alpha, \beta)$ for any $\epsilon_\alpha \in \mathbb{C}^\times$ with $\epsilon_0 = 1$ where $\epsilon(\alpha, \beta)$ is redefined as $\epsilon_\alpha \epsilon_\beta \epsilon_{\alpha+\beta}^{-1} \epsilon(\alpha, \beta)$. Define an ordering on $\zeta \in \mathbb{C}$ with $\zeta > 0$ if $\Re(\zeta) > 0$ or if $\Re(\zeta) = 0$ and $\Im(\zeta) > 0$. Choose

$$\epsilon_\alpha = \begin{cases} \epsilon(\alpha, -\alpha)^{-1} & \text{if } \alpha^1 > 0 \text{ or if } \alpha^1 = \ldots = \alpha^{m-1} = 0 \text{ and } \alpha^m > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{61}$$

Hence $\epsilon(\alpha, -\alpha)$ is redefined as unity. $\square$

We also define the operator [4], [3]

$$\Delta(\alpha, z) = z^{\alpha(0)} Y_+(\alpha, -z). \tag{62}$$

Using (49) and the cocycle structure (57) and (58) we immediately find:

Lemma 4.3 *For all $\beta \in \mathbb{C}^l$ and $u \otimes e^\alpha \in M_\alpha$*

$$(e^\beta)^{-1} \mathcal{Y}(u \otimes e^\alpha, z) e^\beta = C(\alpha, \beta) \mathcal{Y}(\Delta(\beta, z)(u \otimes e^\alpha), z). \tag{63}$$

The operators (56) with the above cocycle structure satisfy a natural extension from rational to complex parameters of the notion of a Generalized VOA as described in Chapt. 9 of [5] and utilized in [3]. In this case, the operators (56) obey creativity and translation with Heisenberg Virasoro vector and satisfy a generalized Jacobi identity as follows

Theorem 4.4 *The vertex operators $\mathcal{Y}(u \otimes e^\alpha, z) \in \mathrm{Lin}(\mathcal{M}, \mathcal{M})[[z, z^{-1}]]$ satisfy the generalized Jacobi identity*

$$\begin{aligned} & z_0^{-1} \left(\frac{z_1 - z_2}{z_0} \right)^{-\alpha\cdot\beta} \delta\left(\frac{z_1 - z_2}{z_0} \right) \mathcal{Y}(u \otimes e^\alpha, z_1)\, \mathcal{Y}(v \otimes e^\beta, z_2) \\ & -C(\alpha, \beta) z_0^{-1} \left(\frac{z_2 - z_1}{z_0} \right)^{-\alpha\cdot\beta} \delta\left(\frac{z_2 - z_1}{-z_0} \right) \mathcal{Y}(v \otimes e^\beta, z_2)\, \mathcal{Y}(u \otimes e^\alpha, z_1) \\ & = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2} \right) \mathcal{Y}(\mathcal{Y}(u \otimes e^\alpha, z_0)(v \otimes e^\beta), z_2) \left(\frac{z_1 - z_0}{z_2} \right)^{\alpha(0)}, \end{aligned} \tag{64}$$

for all $u \otimes e^\alpha, v \otimes e^\beta \in \mathcal{M}$ with cocycle structure (57) *and* (58).

Proof. The proof is similar to that of Theorem 3.5 of [3]. Using (49)

$$
\begin{aligned}
&\mathcal{Y}(u\otimes e^{\alpha}, z_1)\mathcal{Y}(v\otimes e^{\beta}, z_2)(w\otimes e^{\gamma})\\
&= z_1^{\alpha\cdot(\beta+\gamma)} z_2^{\beta\cdot\gamma} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y(Y_+(\beta, -z_1)u, z_1)\\
&\quad .Y_+(\alpha, z_1)Y_-(\beta, z_2)Y(v, z_2)Y_+(\beta, z_2)(w\otimes e^{\gamma})\\
&= z_1^{\alpha\cdot\gamma} z_2^{\beta\cdot\gamma}(z_1-z_2)^{\alpha\cdot\beta} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y(Y_+(\beta, -z_1)u, z_1)\\
&\quad .Y_-(\beta, z_2)Y_+(\alpha, z_1)Y(v, z_2)Y_+(\beta, z_2)(w\otimes e^{\gamma})\\
&= z_1^{\alpha\cdot\gamma} z_2^{\beta\cdot\gamma}(z_1-z_2)^{\alpha\cdot\beta} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y_-(\beta, z_2)Y(Y_+(\beta, -z_1+z_2)u, z_1)\\
&\quad .Y(Y_+(\alpha, z_1-z_2)v, z_2)Y_+(\alpha, z_1)Y_+(\beta, z_2)(w\otimes e^{\gamma}),
\end{aligned}
$$

using (43) and (44). Thus

$$
\begin{aligned}
&z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{-\alpha\cdot\beta}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}(u\otimes e^{\alpha}, z_1)\mathcal{Y}(v\otimes e^{\beta}, z_2)(w\otimes e^{\gamma})\\
&= z_0^{\alpha\cdot\beta} z_1^{\alpha\cdot\gamma} z_2^{\beta\cdot\gamma} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y_-(\beta, z_2)\\
&\quad .z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(Y_+(\beta, -z_0)u, z_1)Y(Y_+(\alpha, z_0)v, z_2)\\
&\quad .Y_+(\alpha, z_1)Y_+(\beta, z_2)(w\otimes e^{\gamma}),
\end{aligned}
$$

using $\delta\left(\frac{z_1-z_2}{z_0}\right)(z_1-z_2)^n = \delta\left(\frac{z_1-z_2}{z_0}\right)z_0^n$ for any integer n. Similarly, we find

$$
\begin{aligned}
&C(\alpha,\beta)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{-\alpha\cdot\beta}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}(v\otimes e^{\beta}, z_2)\mathcal{Y}(u\otimes e^{\alpha}, z_1)(w\otimes e^{\gamma})\\
&= z_0^{\alpha\cdot\beta} z_1^{\alpha\cdot\gamma} z_2^{\beta\cdot\gamma} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y_-(\beta, z_2)\\
&\quad .z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(Y_+(\alpha, z_0)v, z_2)Y(Y_+(\beta, -z_0)u, z_1)\\
&\quad .Y_+(\alpha, z_1)Y_+(\beta, z_2)(w\otimes e^{\gamma}).
\end{aligned}
$$

Hence on applying the Jacobi identity for the Heisenberg VOA, the left hand side of (64) applied to $w\otimes e^{\gamma}$ gives

$$
\begin{aligned}
&z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) z_0^{\alpha\cdot\beta} z_1^{\alpha\cdot\gamma} z_2^{\beta\cdot\gamma} e^{\alpha}e^{\beta} Y_-(\alpha, z_1)Y_-(\beta, z_2)\\
&.Y(B, z_2)Y_+(\alpha, z_1)Y_+(\beta, z_2)(w\otimes e^{\gamma}).
\end{aligned} \tag{65}
$$

for $B = Y(Y_+(\beta, -z_0)u, z_0)Y_+(\alpha, z_0)v$.

In a similar way

$$\begin{aligned}
&\mathcal{Y}\left(\mathcal{Y}(u\otimes e^{\alpha},z_0)(v\otimes e^{\beta}),z_2\right)(w\otimes e^{\gamma})\\
&= z_0^{\alpha\cdot\beta}\epsilon(\alpha,\beta)\mathcal{Y}\left(Y_-(\alpha,z_0)B\otimes e^{\alpha+\beta},z_2\right)(w\otimes e^{\gamma})\\
&= z_0^{\alpha\cdot\beta}z_2^{(\alpha+\beta)\cdot\gamma}e^{\alpha}e^{\beta}Y_-(\beta,z_2)Y_-(\alpha,z_2)Y\left(Y_-(\alpha,z_0)B,z_2\right)\\
&\quad .Y_+(\alpha,z_2)Y_+(\beta,z_2)(w\otimes e^{\gamma}).
\end{aligned}$$

Employing (46) the right hand side of (64) applied to $w\otimes e^{\gamma}$ therefore gives

$$\begin{aligned}
&z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\alpha\cdot\gamma}\delta\left(\frac{z_1-z_0}{z_2}\right)z_0^{\alpha\cdot\beta}\left(z_2+z_0\right)^{\alpha\cdot\gamma}z_2^{\beta\cdot\gamma}e^{\alpha}e^{\beta}\\
&.Y_-(\beta,z_2)Y_-(\alpha,z_0+z_2)Y(B,z_2)Y_+(\alpha,z_2+z_0)Y_+(\beta,z_2)(w\otimes e^{\gamma}).
\end{aligned}\tag{66}$$

Finally, using the identity

$$\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_1-z_0}{z_2}\right)^{\kappa}=\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\kappa},$$

for $\kappa\in\mathbb{C}$, we find that (65) and (66) are equal. Thus the theorem holds. □

4.2 Skew-Symmetry and an Invariant Form

In order to formulate a generalization of skew-symmetry (13) and (29) applicable to $\mathcal{Y}(u\otimes e^{\alpha},z)$ we firstly define for formal parameters z,κ

$$(-z)^{\kappa}=e^{i\pi N\kappa}z^{\kappa},\tag{67}$$

where N is an odd integer parameterizing the formal branch cut.

Lemma 4.5 *The operators $\mathcal{Y}(u\otimes e^{\alpha},z)$ satisfy the skew-symmetry property*

$$\mathcal{Y}(u\otimes e^{\alpha},z)(v\otimes e^{\beta})=e^{-i\pi N\alpha\cdot\beta}C(\alpha,\beta)e^{zL(-1)}\mathcal{Y}(v\otimes e^{\beta},-z)(u\otimes e^{\alpha}).\tag{68}$$

Proof. Using (29), (47) and (63) we have $\mathcal{Y}(u\otimes e^{\alpha},z)(v\otimes e^{\beta})$ is given by

$$\begin{aligned}
&\mathcal{Y}(u\otimes e^{\alpha},z)e^{\beta}v\\
&=C(\alpha,\beta)e^{\beta}\mathcal{Y}(\Delta(\beta,z)(u\otimes e^{\alpha}),z)v\\
&=C(\alpha,\beta)e^{\beta}e^{zL(-1)}Y(v,-z)\Delta(\beta,z)(u\otimes e^{\alpha})\\
&=C(\alpha,\beta)e^{zL(-1)}e^{\beta}Y_-(\beta,-z)Y(v,-z)Y_+(\beta,-z)z^{\alpha\cdot\beta}(u\otimes e^{\alpha})\\
&=e^{-i\pi N\alpha\cdot\beta}C(\alpha,\beta)e^{zL(-1)}\mathcal{Y}(v\otimes e^{\beta},-z)(u\otimes e^{\alpha}).\ \square
\end{aligned}$$

We next introduce an invariant form $\langle\,,\rangle$ on $\mathcal{M}$ associated with the Möbius map [6], [13], [14], [15]

$$\left(\begin{array}{cc} 0 & \lambda \\ e^{i\pi N}\lambda^{-1} & 0 \end{array}\right) : z \mapsto \frac{\lambda^2}{e^{i\pi N} z}, \tag{69}$$

for $\lambda \neq 0$ and with N of (67). Usually for a VOA we have $e^{-i\pi N} = -1$ and one takes $\lambda = \pm\sqrt{-1}$ since only integral powers of formal parameters occur. In the present case we define the adjoint of the vertex operator $\mathcal{Y}(u \otimes e^\alpha, z)$ by

$$\mathcal{Y}^\dagger\left(u \otimes e^\alpha, z\right) = \mathcal{Y}\left(e^{-z\lambda^{-2}L(1)}\left(\frac{\lambda}{e^{i\pi N}z}\right)^{2L(0)}(u \otimes e^\alpha), \frac{\lambda^2}{e^{i\pi N}z}\right). \tag{70}$$

In particular, for a Heisenberg generating vector a^i we have

$$Y(a^i, z)^\dagger = \sum_n a^{i\dagger}(n) z^{-n-1}, \quad a^{i\dagger}(n) = (-1)^{n+1}\lambda^{2n} a^i(-n). \tag{71}$$

This implies

$$Y_\pm^\dagger(\alpha, z) = Y_\mp\left(\alpha, -\frac{\lambda^2}{z}\right). \tag{72}$$

We also note that $e^{-z\lambda^{-2}L(1)}\left(\frac{\lambda}{e^{i\pi N}z}\right)^{2L(0)}(u \otimes e^\alpha)$ is given by

$$\begin{aligned}
&\left(e^{-i\pi N}\frac{\lambda}{z}\right)^{\alpha^2} e^{-z\lambda^{-2}L(1)}\left(\left(-\frac{\lambda}{z}\right)^{2L(0)} u\right) \otimes e^\alpha \\
&= \left(e^{-i\pi N}\frac{\lambda}{z}\right)^{\alpha^2} e^{-z\lambda^{-2}L(1)} e^{\alpha\cdot q}\left(-\frac{\lambda}{z}\right)^{2L(0)} u \\
&= \left(e^{-i\pi N}\frac{\lambda}{z}\right)^{\alpha^2} e^{\alpha\cdot q} Y_+\left(\alpha, \frac{\lambda^2}{z}\right) e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)} u \\
&= \left(e^{-i\pi N}\frac{\lambda}{z}\right)^{\alpha^2}\left(Y_+\left(\alpha, \frac{\lambda^2}{z}\right) e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)} u\right) \otimes e^\alpha,
\end{aligned}$$

where $\alpha^2 = \alpha \cdot \alpha$ and using (48). Hence we find

$$\begin{aligned}
&\mathcal{Y}^\dagger(u \otimes e^\alpha, z)\ (w \otimes e^\gamma) \\
= \ & \left(e^{-i\pi N}\frac{\lambda}{z}\right)^{\alpha^2}\left(e^{-i\pi N}\frac{\lambda^2}{z}\right)^{\alpha\cdot\gamma} e^\alpha Y_-\left(\alpha, -\frac{\lambda^2}{z}\right) \\
&.Y\left(Y_+\left(\alpha, \frac{\lambda^2}{z}\right) e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)} u, -\frac{\lambda^2}{z}\right) \\
&.Y_+\left(\alpha, -\frac{\lambda^2}{z}\right)(w \otimes e^\gamma) \\
= \ & \lambda^{2\alpha\cdot(\alpha+\gamma)-\alpha^2}(e^{i\pi N}z)^{-\alpha\cdot(\alpha+\gamma)} Y_-\left(\alpha, -\frac{\lambda^2}{z}\right) \\
&.Y\left(e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)} u, -\frac{\lambda^2}{z}\right) \\
&.Y_+\left(\alpha, -\frac{\lambda^2}{z}\right) e^\alpha(w \otimes e^\gamma) \\
= \ & z^{-\alpha(0)} Y_-\left(\alpha, -\frac{\lambda^2}{z}\right) Y^\dagger(u, z) Y_+\left(\alpha, -\frac{\lambda^2}{z}\right) \\
&.e^{-i\pi N\alpha(0)}\lambda^{2\alpha(0)-\alpha^2} e^\alpha(w \otimes e^\gamma).
\end{aligned}$$

Thus using (71) and (72) we have

$$\mathcal{Y}^\dagger(u \otimes e^\alpha, z) = z^{\alpha(0)^\dagger} Y_+^\dagger(\alpha, z) Y^\dagger(u, z) Y_-^\dagger(\alpha, z) e^{\alpha\dagger}, \tag{73}$$

where we define

$$e^{\alpha\dagger} = e^{-i\pi N\alpha(0)}\lambda^{2\alpha(0)-\alpha^2} e^\alpha. \tag{74}$$

Definition 4.6 *A bilinear form $\langle\, , \rangle$ on $\mathcal{M}$ is said to be invariant if for all $u \otimes e^\alpha$, $v \otimes e^\beta$, $w \otimes e^\gamma \in \mathcal{M}$ we have*

$$\langle\, \mathcal{Y}(u\otimes e^\alpha, z)(v\otimes e^\beta), w\otimes e^\gamma \,\rangle = e^{-i\pi N\alpha\cdot\beta} C(\alpha, \beta)\langle\, v\otimes e^\beta,\ \mathcal{Y}^\dagger(u\otimes e^\alpha, z)(w\otimes e^\gamma)\,\rangle. \tag{75}$$

We choose the normalization $\langle\, \mathbf{1}, \mathbf{1} \,\rangle = 1$. For $\alpha = \beta = \gamma = 0$ (75) reverts to the standard definition[1] of an invariant form on the Heisenberg VOA M^l which is unique, symmetric and invertible [13]. In general, we have

[1]up to an additional λ dependence arising from definition for the adjoint in (71).

Proposition 4.7 *The bilinear form* $\langle\,,\rangle$ *on* $\mathcal{M}$ *is unique, symmetric and invertible with*

$$\langle\, v\otimes e^{\beta}, w\otimes e^{\gamma}\,\rangle = \epsilon(\beta,-\beta)\lambda^{-\beta^2}\delta_{\beta+\gamma,0}\langle\, v,w\,\rangle. \tag{76}$$

Remark 4.8 *From Lemma 4.2 we note that we may choose* $\epsilon(\beta,-\beta)=1$.

Proof. Since $\langle\, a(0)(v\otimes e^{\beta}), w\otimes e^{\gamma}\,\rangle = -\langle\, v\otimes e^{\beta}, a(0)(w\otimes e^{\gamma})\,\rangle$ it follows that

$$\langle\, v\otimes e^{\beta}, w\otimes e^{\gamma}\,\rangle = \delta_{\beta+\gamma,0}\langle\, v\otimes e^{\beta}, w\otimes e^{-\beta}\,\rangle.$$

Applying (74) we obtain

$$\begin{aligned}\langle\, v\otimes e^{\beta}, w\otimes e^{-\beta}\,\rangle &= \langle\, e^{\beta}v, w\otimes e^{-\beta}\,\rangle\\ &= \langle\, v, e^{\beta\dagger}(w\otimes e^{-\beta})\,\rangle\\ &= \epsilon(\beta,-\beta)\lambda^{-\beta^2}\langle\, v,w\,\rangle.\end{aligned}$$

But $\epsilon(\beta,-\beta)=\epsilon(-\beta,\beta)$ from (60) and $\langle\, v,w\,\rangle$ is symmetric, unique and invertible so the result holds. □

Remark 4.9 *The necessity for the external factors on the right hand side of* (75) *is apparent when we consider*

$$\begin{aligned}\langle\, e^{\alpha}(v\otimes e^{\beta}), w\otimes e^{\gamma}\,\rangle &= \epsilon(\alpha,\beta)\langle\, v\otimes e^{\alpha+\beta}, w\otimes e^{\gamma}\,\rangle\\ &= \epsilon(\alpha,\beta)\epsilon(\alpha+\beta,-\alpha-\beta)\lambda^{-(\alpha+\beta)^2}\delta_{\alpha+\beta+\gamma,0}\langle\, v,w\,\rangle.\end{aligned}$$

On the other hand (75) *implies that this is also given by*

$$\begin{aligned}&e^{-i\pi N\alpha\cdot\beta}C(\alpha,\beta)\langle\, v\otimes e^{\beta}, e^{\alpha\dagger}(w\otimes e^{\gamma})\,\rangle\\ &= e^{-i\pi N\alpha\cdot\beta}C(\alpha,\beta)e^{-i\pi N\alpha\cdot(\alpha+\gamma)}\lambda^{\alpha^2+2\alpha\cdot\gamma}\epsilon(\alpha,\gamma)\langle\, v\otimes e^{\beta}, w\otimes e^{\alpha+\gamma}\,\rangle\\ &= C(\alpha,\beta)\epsilon(\alpha,-\alpha-\beta)\epsilon(\beta,-\beta)\lambda^{-(\alpha+\beta)^2}\delta_{\alpha+\beta+\gamma,0}\langle\, v,w\,\rangle.\end{aligned}$$

The equality of these expressions is equivalent to the identity $(e^{\alpha}e^{\beta})e^{-\alpha-\beta} = C(\alpha,\beta)e^{\beta}(e^{\alpha}e^{-\alpha-\beta})$.

5 Lattice Vertex Operator Superalgebras

In this section we apply Theorem 4.4 to the example of a lattice VOSA V_L for an integral Euclidean rank l lattice L. We construct the g-twisted module for a V_L automorphism g generated by a Heisenberg vector in terms of Heisenberg modules so that the twisted Jacobi identity (19) is satisfied as a consequence of Theorem 4.4. The relationship between this and Li's construction [4] for a g-twisted module is discussed We also consider a generalization from rational to complex parameterized twisted V_L modules VOSA of a related generalized VOA discussed in [3].

Let L be a Euclidean lattice of rank l and define $V_L = \oplus_{\mu\in L} M_\mu$ with standard cocycle commutator e.g. [7]

$$C(\mu_1, \mu_2) = (-1)^{\mu_1\cdot\mu_2+\mu_1^2\mu_2^2}, \tag{77}$$

for $\mu_1, \mu_2 \in L$. Define parity on V_L by $p(u\otimes e^\mu) = \mu^2 \mod 2$ for $u\otimes e^\mu \in V_L$. Then Theorem 4.4 implies that $(V_L, \mathcal{Y}, \mathbf{1}, \omega)$ is a VOA for L even and a VOSA for L odd with invertible invariant form (75) obeying

$$\langle\, \mathcal{Y}(u\otimes e^{\mu_1}, z)(v\otimes e^{\mu_2}), w\otimes e^{\mu_3}\,\rangle = (-1)^{\mu_1^2\mu_2^2}\langle\, v\otimes e^{\mu_2}, \mathcal{Y}^\dagger(u\otimes e^{\mu_1}), z)(w\otimes e_3^{\mu})\,\rangle.$$

Consider the automorphism

$$g = e^{-2\pi i\alpha(0)}, \tag{78}$$

generated by the Heisenberg vector $-\alpha\cdot a \in V_L$ for any $\alpha \in \mathbb{C}^l$. Clearly M_μ is a g eigenspace for eigenvalue $e^{-2\pi i\alpha\cdot\mu}$. Let $V_{L+\alpha} = \oplus_{\mu\in L} M_{\mu+\alpha}$ so that $e^\alpha : V_L \to V_{L+\alpha}$. We then find using Definition 2.4 that:

Proposition 5.1 *$(V_{L+\alpha}, \mathcal{Y})$ is a g-twisted V_L-module.*

Proof. Theorem 4.4 with commutator (77) implies that

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}(u\otimes e^{\mu_1}, z_1)\mathcal{Y}(v\otimes e^{\mu_2}, z_2)(w\otimes e^{\mu_3+\alpha})\\
&-(-1)^{\mu_1^2\mu_2^2}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}(v\otimes e^{\mu_2}, z_2)\mathcal{Y}(u\otimes e^{\mu_1}, z_1)(w\otimes e^{\mu_3+\alpha})\\
&= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\mu_1\cdot\alpha}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}(\mathcal{Y}(u\otimes e^{\mu_1}, z_0)(v\otimes e^{\mu_2}), z_2)(w\otimes e^{\mu_3+\alpha}),
\end{aligned} \tag{79}$$

for $u \otimes e^{\mu_1}, v \otimes e^{\mu_2} \in V_L$ and $w \otimes e^{\mu_3+\alpha} \in V_{L+\alpha}$. The result holds on comparison with (19) where $\rho = -\mu_1 \cdot \alpha$. □

Using Lemma 63 we immediately find that Proposition 5.1 is equivalent to Li's construction for the g-twisted module for V_L [4]:

Corollary 5.2 $(V_{L+\alpha}, \mathcal{Y}) \cong (V_L, Y_g)$ *as g-twisted V_L-modules where*

$$Y_g(u \otimes e^{\mu}, z) = \mathcal{Y}(\Delta(\alpha, z)(u \otimes e^{\mu}), z).$$

For the Heisenberg basis and Virasoro vector we obtain g-twisted modes

$$\begin{aligned} a_g^i(n) &= (e^{\alpha})^{-1} a^i(n) e^{\alpha} \\ &= a^i(n) + \alpha^i \delta_{n,0}, \qquad (80) \\ L_g(n) &= (e^{\alpha})^{-1} L(n) e^{\alpha} \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_g(n+m) \cdot a_g(-m) : \\ &= L(n) + \alpha(n) + \frac{1}{2}\alpha^2 \delta_{n,0}, \qquad (81) \end{aligned}$$

satisfying the original Heisenberg and central charge l Virasoro algebras. In particular, the grading is determined by $L_g(0) = L(0) + \alpha(0) + \frac{1}{2}\alpha^2$.

For each Heisenberg vector $\alpha \cdot a$ we may also construct a $\mathbb{C}$-graded VOSA $(V_L, \mathcal{Y}, \mathbf{1}, \omega_\alpha)$ with the original vector space and vertex operators but with a new 'shifted' conformal vector $\omega_\alpha = \omega - \alpha(-2)\mathbf{1}$ [16], [17]. The $\mathbb{C}$-grading is determined by $L_\alpha(0) = L(0) + \alpha(0)$ with twisted Virasoro modes

$$L_\alpha(n) = L(n) + (n+1)\alpha(n),$$

satisfying the Virasoro algebra with central charge $c_\alpha = l - 12\alpha^2$. Thus we find [17]

$$L_\alpha(0) - \frac{c_\alpha}{24} = L_g(0) - \frac{l}{24},$$

a fact that has been usefully exploited to relate shifted and twisted partition and n-point functions [17], [18].

We conclude with a generalization of one of the main results of ref. [3] where a generalized VOA is constructed from an extension of a simple VOA V by Heisenberg generated g-twisted modules with rational parameters via Li's construction. Applying this approach to V_L the generalized vertex operators act on g-twisted modules $\{V_{L+\alpha}\}$ in our notation. Applying Definition 3.3 of ref. [3] to the VOSA V_L with $\psi_\alpha = e^{-\alpha\cdot q}$ we define the vertex operator

$$
\begin{aligned}
Y_\alpha(u\otimes e^{\mu_1+\alpha},z)(v\otimes e^{\mu_2+\beta}) &= \psi_{-\alpha-\beta}Y_-(\alpha,z)\mathcal{Y}(\psi_\alpha\Delta(\beta,z)(u\otimes e^{\mu_1+\alpha_1}),z) \\
&\quad .\Delta(\alpha,-z)\psi_\beta(v\otimes e^{\mu_2+\beta}), \qquad (82)
\end{aligned}
$$

for $\mu_1,\mu_2\in L$. Since $\Delta(\alpha,-z)=Y_+(\alpha,z)(-z)^{\alpha(0)}$ we must employ the formal branch parameterization of (67) to find

$$
\begin{aligned}
Y_\alpha(u\otimes e^{\alpha+\mu_1},z)(v\otimes e^{\mu_2+\beta}) &= e^{i\pi N\alpha\cdot\mu_2}\epsilon(\mu_1,\mu_2)Y_-(\mu_1+\alpha,z)Y(u,z) \\
&\quad .Y_+(\mu_1+\alpha,z)z^{(\mu_1+\alpha)a(0)}(v\otimes e^{\mu_2+\beta}) \\
&= \frac{e^{i\pi N\alpha\cdot\mu_2}\epsilon(\mu_1,\mu_2)}{\epsilon(\mu_1+\alpha,\mu_2+\beta)}\mathcal{Y}(u\otimes e^{\mu_1+\alpha},z)(v\otimes e^{\mu_2+\beta}).
\end{aligned}
$$

Substituting into Theorem 4.4 results in

Proposition 5.3 *The vertex operators $Y_\alpha(u\otimes e^{\alpha+\mu},z)$ satisfy the generalized Jacobi identity*

$$
\begin{aligned}
& z_0^{-1}\left(\tfrac{z_1-z_2}{z_0}\right)^{\eta_{12}}\delta\left(\tfrac{z_1-z_2}{z_0}\right)Y_{\alpha_1}(u\otimes e^{\mu_1+\alpha_1},z_1)Y_{\alpha_2}(v\otimes e^{\mu_2+\alpha_2},z_2)(w\otimes e^{\mu_3+\alpha_3}) \\
& -C_{12}z_0^{-1}\left(\tfrac{z_2-z_1}{z_0}\right)^{\eta_{12}}\delta\left(\tfrac{z_2-z_1}{-z_0}\right)Y_{\alpha_2}(v\otimes e^{\mu_2+\alpha_2},z_2)Y_{\alpha_1}(u\otimes e^{\mu_1+\alpha_1},z_1)(w\otimes e^{\mu_3+\alpha_3}) \\
& = z_2^{-1}\left(\tfrac{z_1-z_0}{z_2}\right)^{-\eta_{13}}\delta\left(\tfrac{z_1-z_0}{z_2}\right)Y_{\alpha_1+\alpha_2}(Y_{\alpha_1}(u\otimes e^{\mu_1+\alpha_1},z_0)(v\otimes e^{\mu_2+\alpha_2}),z_2)(w\otimes e^{\mu_3+\alpha_3})
\end{aligned} \qquad (83)
$$

where

$$
\begin{aligned}
\eta_{12} &= -\alpha_1\cdot\alpha_2-\mu_1\cdot\alpha_2-\mu_2\cdot\alpha_1, \qquad (84)\\
\eta_{13} &= -\alpha_1\cdot\alpha_3-\mu_1\cdot\alpha_3-\mu_3\cdot\alpha_1, \qquad (85)
\end{aligned}
$$

and with commutator

$$
C_{12}=e^{i\pi N(\alpha_1\cdot\mu_2-\alpha_2\cdot\mu_1)}(-1)^{\mu_1^2\mu_2^2}. \qquad (86)
$$

Choosing the branch[2] (67) with $N=1$ Theorem 5.3 extends Theorem 3.5 of ref. [3] from rational to complex parametrized twisted modules of a VOSA

[2] This branch choice is an unstated assumption in eqn. (3.30) in the proof of Theorem 3.5 of ref. [3]

V_L. Furthermore, it is clear that the commutator factor of $e^{i\pi N(\alpha_1\cdot\mu_2-\alpha_2\cdot\mu_1)}$ arises solely from the branch choice made in (82). In fact, we may modify Defn. 3.3 of [3] and (82) by replacing the $\Delta(\alpha,-z)$ operator by $Y_+(\alpha,z)z^{\alpha(0)}$ in order to define a new vertex operator

$$\begin{aligned}\widehat{Y}_\alpha(u\otimes e^{\mu_1+\alpha},z)(v\otimes e^{\mu_2+\beta}) &= \psi_{-\alpha-\beta}Y_-(\alpha,z)\mathcal{Y}(\psi_\alpha\Delta(\beta,z)(u\otimes e^{\mu_1+\alpha_1}),z)\\ &\quad .Y_+(\alpha,z)z^{\alpha(0)}\psi_\beta(v\otimes e^{\mu_2+\beta}). \qquad (87)\end{aligned}$$

These operators satisfy the generalized Jacobi identity (83) with the standard lattice parity commutator $C_{12}=(-1)^{\mu_1^2\mu_2^2}$.

References

[1] R. Borcherds, Vertex algebras, Kac-Moody algebras and the Monster, Proc.Natl.Acad.Sci. U.S.A. **83** (1986) 3068–3071.

[2] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[3] C. Dong, H. Li, G. Mason, Simple currents and extensions of vertex operator algebras, Comm.Math.Phys. **180** (1996) 671–707.

[4] H. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, Contemp.Math. **193** (1996) 203–236.

[5] C. Dong, J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Mathematics, **112**, Birkhauser, Boston, 1993.

[6] I. Frenkel, Y-Z. Huang, J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem.Amer.Math.Soc. **104** no. 494 (1993).

[7] V. Kac, Vertex Operator Algebras for Beginners, University Lecture Series, Vol. 10, AMS, Boston, 1998.

[8] J. Lepowsky, H. Li, Introduction to Vertex Operator Algebras and their Representations, Progr.Math. **227**, Birkhauser, Boston, 2004.

[9] G. Mason, M.P. Tuite, Vertex operators and modular forms, *A Window into Zeta and Modular Physics* eds. K. Kirsten and F. Williams, MSRI Publications **57** 183–278, Cambridge University Press, Cambridge, 2010.

[10] X. Xu, Introduction to Vertex Operator Superalgebras and their Modules, Kluwer Academic, 1998.

[11] C. Dong, Twisted modules for vertex operator algebras associated with even lattices, J.Alg. **165** (1993) 91–112.

[12] C. Dong, Z. Lin, G. Mason, On vertex operator algebras as sl_2-modules, Arasu, K. T. (ed.) et al., *Groups, Difference Sets, and the Monster*, Proceedings of a special research quarter, Columbus, OH, USA, Spring 1993, Walter de Gruyter, Berlin, 1996, Ohio State Univ.Math.Res.Inst.Publ. **4** (1996) 349–362.

[13] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, J.Pure.Appl.Alg. **96** (1994) 2790–297.

[14] N. Scheithauer, Vertex algebras, Lie algebras and superstrings, J.Alg. **200** (1998) 363–403.

[15] M.P. Tuite, A. Zuevsky, Genus two partition and correlation functions for fermionic vertex operator superalgebras I, arXiv:1007.5203, to appear in Commun.Math.Phys.

[16] A. Matsuo, K. Nagatomo, A note on free bosonic vertex algebras and its conformal vector, J.Alg. **212** (1999) 395–418.

[17] C. Dong, G. Mason, Shifted vertex operator algebras, Math.Proc.Camb.Philos.Soc. **141** (2006) 67–80.

[18] G. Mason, M.P Tuite, A. Zuevsky, Torus n-point functions for $\mathbb{C}$-graded vertex operator superalgebras and continuous fermion orbifolds, Commun.Math.Phys. **283** (2008) 305–342.