

Twisting of monoidal structures

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Abstract

This article is devoted to the investigation of the deformation (twisting) of monoidal structures, such as the associativity constraint of the monoidal category and the monoidal structure of monoidal functor. The sets of twistings have a (non-abelian) cohomological nature. Using this fact the maps from the sets of twistings to some cohomology groups (Hochschild cohomology of K-theory) are constructed. The examples of monoidal categories of bimodules over some algebra, modules and comodules over bialgebra are examined. We specially concentrate on the case of free tensor category.

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1 introduction

According to Tannaka-Krein theory [4] a group-like object (group, Lie algebra etc.) is in some sense equivalent to its category of representation with natural tensor product operation plus forgetful functor to vector spaces. Accurately speaking to extract the group (Lie algebra) from its representations category we must regard some extra-structures (monoidal structures) on the category and the functor (section 2).

Through this point of view the deformation (quantization) of the group (Lie algebra) is the deformation of its representation category and forgetful functor. In many cases (semisimple groups and Lie algebras) the deformation does not change the category, tensor product on it and the forgetful functor. So the deformation consists in a change of the monoidal structures of the category and the functor.

In the present paper we shall deal with only this kind of deformations.

It appears that this deformations are presented by a twistings of monoidal structures by the automorphisms of some monoidal functors (section 3).

The set of twistings can be identified with the non-abelian cohomology sets of the cosimplicial complex naturally attached to the monoidal functor (section 4).

These non-abelian cohomologies can be abelianized by means of infinitesimal methods (section 5) or algebraic K-theory (section 6).

The examples of monoidal categories of bimodules (section 7), modules and comodules over bialgebra (section 8) and free tensor category (sections 9,10) are examined.

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2 endomorphisms of functors

Fix a commutative ring k . All categories and functors will be k -linear. It means that all Hom -sets are k -modules and the functors induces k -linear maps between Hom -sets.

For an associative k -algebra R we will denote by $R - Mod$ the category of left R -modules and by $R - mod$ the subcategory in $R - Mod$ of finitely generated modules. For example if k is a field then $k - Mod$ is the category of all vector spaces over k and $k - mod$ is its subcategory of finite-dimensional spaces.

Another definition of k -linearity of the category \mathcal{A} consists in assigning of the

which is associative respectively tensor product in $k - Mod$. In this case k -linearity of the functor F between k -linear categories means that F preserves the action of $k - Mod$.

It is easy to see that the second version of k -linearity implies the first and in the case of categories which have infinite (finite) products. The first version implies the existence of the action of the category, which is non-canonically equivalent to $k - Mod$ ($k - mod$).

Here we shall use the second definition of k -linearity.

Suppose that a faithful (injective on the morphisms) [16] k -linear functor $F : \mathcal{A} \rightarrow k - Mod$ have a left (right) k -linear adjoint functor $G : k - Mod \rightarrow \mathcal{A}$. It means that morphisms of functors

$$\alpha : id_{k-Mod} \rightarrow FG, \quad \beta : GF \rightarrow id_{\mathcal{A}},$$

are given and the compositions of endomorphisms

$$\begin{aligned} F &\xrightarrow{F(\alpha)} FGF \xrightarrow{\beta F} F \\ G &\xrightarrow{\alpha G} GFG \xrightarrow{G(\beta)} G \end{aligned}$$

are identical (the definition of the right adjoint can be obtained from this by replacing F by G).

The k -linearity of the functor G involves that

$$G(V) = E \otimes V, \quad V \in k - Mod$$

for some object E from \mathcal{A} . We will call this object a *generator* (*cogenerator* in the case when G is a right adjoint to F). The (co)generator E is called *projective* (*injective*) if the morphism β (α) of the corresponding adjunction is surjective (*injective*).

Proposition 1 *Let $H : \mathcal{A} \rightarrow \mathcal{B}$ be a right (left) exact functor between abelian categories and E is a projective generator (injective cogenerator). Then the homomorphism of algebras*

$$End(H) \rightarrow C_{End_{\mathcal{B}}(H(E))}(H(End_{\mathcal{A}}(E))),$$

which sends the endomorphism γ to its specialization γ_E , is an isomorphism .

Proof:

Suppose that E is a projective generator (the proof in the case of injective cogenerator can be obtained by the inversion of arrows).

For any object A from \mathcal{A} consider the commutative diagram

$$\begin{array}{ccc} H(A) & \xleftarrow{H(\beta_A)} & H(E) \otimes F(A) \\ \gamma_A \downarrow & & \downarrow \gamma_E \\ H(A) & \xleftarrow{H(\beta_A)} & H(E) \otimes F(A) \end{array}$$

The surjectivity of $H(\beta)$ implies that the homomorphism

$$\text{End}(H) \rightarrow \text{End}_{\mathcal{B}}(E)$$

is injective.

Conversely, consider the left exact sequence of functors

$$0 \longleftarrow H \xleftarrow{H(\beta)} H(E) \otimes F \longleftarrow H(E) \otimes FK$$

where K is a kernel of β . The element r of the centralizer $C_{\text{End}_{\mathcal{B}}(H(E))}(H(\text{End}_{\mathcal{A}}(E)))$ defines an endomorphisms of two right functors of this sequence and hence an endomorphism of H . It is obvious that the specialization of this endomorphism coincides with r .

3 monoidal categories

The (k -linear) *tensor product* $\mathcal{G} \boxtimes \mathcal{H}$ of two (k -linear) categories \mathcal{G} and \mathcal{H} is a category, whose objects are the pairs of objects of \mathcal{G} and \mathcal{H}

$$(X, Y) \quad X \in \mathcal{G}, Y \in \mathcal{H}$$

and morphisms are tensor product of morphisms from \mathcal{G} and \mathcal{H}

$$\text{Hom}_{\mathcal{G} \boxtimes \mathcal{H}}((X_1, Y_1), (X_2, Y_2)) = \text{Hom}_{\mathcal{G}}(X_1, X_2) \otimes_k \text{Hom}_{\mathcal{H}}(Y_1, Y_2)$$

The *tensor product* of functors $F : \mathcal{G}_1 \rightarrow \mathcal{H}_1$ and $G : \mathcal{G}_2 \rightarrow \mathcal{H}_2$ is a functor $F \boxtimes G : \mathcal{G}_1 \boxtimes \mathcal{G}_2 \rightarrow \mathcal{H}_1 \boxtimes \mathcal{H}_2$, which sends the pair (X, Y) to $(F(X), G(Y))$. The *monoidal category* is a category \mathcal{G} with a bifunctor

$$\otimes : \mathcal{G} \boxtimes \mathcal{G} \longrightarrow \mathcal{G} \quad (X, Y) \mapsto X \otimes Y$$

which called *tensor (or monoidal) product*. This functor must be equipped with a functorial collection of isomorphisms (so-called *associativity constraint*)

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z \quad \text{for any } X, Y, Z \in \mathcal{G}$$

which satisfies to the following *pentagon axiom*:
the diagram

$$\begin{array}{ccc} X \otimes (Y \otimes (Z \otimes W)) & \xrightarrow{\varphi_{X,Y,Z \otimes W}} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{\varphi_{X \otimes Y, Z, W}} & ((X \otimes Y) \otimes Z) \otimes W \\ I_X \otimes \varphi_{Y, Z, W} \downarrow & & & & \uparrow \varphi_{X, Y, Z} \otimes I_W \\ X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{\varphi_{X, Y \otimes Z, W}} & & & (X \otimes (Y \otimes Z)) \otimes W \end{array}$$

is commutative for any objects $X, Y, Z, W \in \mathcal{G}$.

Consider two tensor products of objects X_1, \dots, X_n from \mathcal{G} with an arbitrary arrangement of the brackets. The coherence theorem [16] states that the pentagon axiom implies the existence of a unique isomorphism between them, which is the composition of the associativity constraints.

An object 1 together with the functorial isomorphisms

$$\rho_X : X \otimes 1 \rightarrow X \quad \lambda_X : 1 \otimes X \rightarrow X$$

in a monoidal category \mathcal{G} is called a *unit* if $\lambda_1 = \rho_1$ and the diagrams

$$\begin{array}{ccc} 1 \otimes (X \otimes Y) & \xrightarrow{\lambda_{X \otimes Y}} & X \otimes Y \\ & \searrow \varphi_{1, X, Y} & \nearrow \lambda_X \otimes I_Y \\ & (1 \otimes X) \otimes Y & \end{array}$$

$$\begin{array}{ccc} X \otimes (1 \otimes Y) & \xrightarrow{\rho_X \otimes I_Y} & X \otimes Y \\ & \searrow \varphi_{X, 1, Y} & \nearrow I_X \otimes \lambda_Y \\ & (X \otimes 1) \otimes Y & \end{array}$$

$$\begin{array}{ccc} X \otimes (Y \otimes 1) & \xrightarrow{\rho_{X \otimes Y}} & X \otimes Y \\ & \searrow \varphi_{X, Y, 1} & \nearrow I_X \otimes \rho_Y \\ & (X \otimes Y) \otimes 1 & \end{array}$$

are commutative for any $X, Y \in \mathcal{G}$. It is easy to see that the unit object is unique up to isomorphism.

A *monoidal functor* between monoidal categories \mathcal{G} and \mathcal{H} is a functor $F : \mathcal{G} \rightarrow \mathcal{H}$, which is equipped with the functorial collection of isomorphisms (the so-called *monoidal structure*)

$$\gamma_{X, Y} : F(X \otimes Y) \rightarrow F(X) \otimes F(Y) \quad \text{for any } X, Y \in \mathcal{G}$$

for which the following diagram is commutative for any objects $X, Y, Z \in \mathcal{G}$

$$\begin{array}{ccccc} F(X \otimes (Y \otimes Z)) & \xrightarrow{c_{X, Y \otimes Z}} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{I \otimes c_{Y, Z}} & F(X) \otimes (F(Y) \otimes F(Z)) \\ F(\varphi_{X, Y, Z}) \downarrow & & & & \psi_{F(X), F(Y), F(Z)} \downarrow \\ F((X \otimes Y) \otimes Z) & \xrightarrow{c_{X \otimes Y, Z}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{c_{X, Y} \otimes I} & (F(X) \otimes F(Y)) \otimes F(Z) \end{array}$$

A morphism $f : F \rightarrow G$ of monoidal functors F and G is called *monoidal* if the diagram

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{c_{X,Y}} & F(X) \otimes F(Y) \\ J_{X \otimes Y} \downarrow & & J_X \otimes J_Y \downarrow \\ G(X \otimes Y) & \xrightarrow{d_{X,Y}} & G(X) \otimes G(Y) \end{array}$$

is commutative for any $X, Y \in \mathcal{G}$.

Denote by $Aut^{\otimes}(F)$ the group of monoidal automorphisms of the monoidal functor F .

Monoidal categories \mathcal{G} and \mathcal{H} are *equivalent* (as monoidal categories) if there are mutually inverse monoidal functors $F : \mathcal{G} \rightarrow \mathcal{H}$ and $G : \mathcal{H} \rightarrow \mathcal{G}$ with monoidal isomorphisms $F \circ G \simeq id_{\mathcal{H}}$ and $G \circ F \simeq id_{\mathcal{G}}$. It is easy to see that monoidal categories are equivalent if and only if there is a monoidal functor between them which is an equivalence of categories.

Let us denote by $Aut^{\otimes}(\mathcal{G})$ the group of monoidal autoequivalences of the monoidal category \mathcal{G} and by $Aut(\mathcal{G}, \otimes)$ the group of autoequivalences of the category \mathcal{G} which preserves the tensor product \otimes .

The quotient $\psi \circ \varphi^{-1}$ of two associativity constraints ψ and φ of the functor \otimes is an isomorphism of the functor $\otimes \circ (\otimes \boxtimes id) = id^{\otimes 3}$. Thus we have a map

$$Q : MS(\mathcal{G}) \times MS(\mathcal{G}) \longrightarrow Aut(id^{\otimes 3})$$

from the direct square of the set $MS(\mathcal{G}) = MS(\mathcal{G}, \otimes)$ of all monoidal structures of the category \mathcal{G} with the tensor product \otimes (that is the set of all associativity constraints of the tensor product \otimes) to the group $Aut(id^{\otimes 3})$ of automorphisms of the functor $id^{\otimes 3}$.

Analogously, the quotient $d \circ c^{-1}$ of two monoidal structures c and d of functor F is an isomorphism of the functor $\otimes \circ (F \boxtimes F) = F^{\otimes 2}$. Thus there is a map

$$Q : MS(F) \times MS(F) \longrightarrow Aut(F^{\otimes 2})$$

from the direct square of the set $MS(F)$ of all monoidal structures of the functor F to the group $Aut(F^{\otimes 2})$.

It immediately follows from the definitions that these maps satisfy the conditions

$$Q(x, x) = 1 \quad \text{and} \quad Q(x, y)Q(y, z) = Q(x, z)$$

The map

$$Q(\varphi, ?) : MS(\mathcal{G}) \longrightarrow Aut(id^{\otimes 3})$$

is injective for any associativity constraint φ of the tensor product \otimes of the category \mathcal{G} . The image of this map consists of those isomorphisms $\alpha \in Aut(id^{\otimes 3})$

for which the diagram

$$\begin{array}{ccccc}
 X \otimes (Y \otimes (Z \otimes W)) & \xrightarrow{\varphi \circ \alpha} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{\varphi \circ \alpha} & ((X \otimes Y) \otimes Z) \otimes W \\
 \downarrow I \otimes \varphi \circ \alpha & & & & \uparrow \varphi \circ \alpha \otimes I \\
 X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{\varphi \circ \alpha} & & \xrightarrow{\varphi \circ \alpha} & (X \otimes (Y \otimes Z)) \otimes W
 \end{array}$$

is commutative for any objects $X, Y, Z, W \in \mathcal{G}$.

For such α we call the monoidal category $\mathcal{G}(\alpha) = (\mathcal{G}, \varphi)(\alpha) = (\mathcal{G}, \alpha\varphi)$ a *twisted form* (twisting by α) of the monoidal category (\mathcal{G}, φ) .

Similarly, the map

$$Q(c, ?) : MS(F) \rightarrow Aut(F^{\otimes 2})$$

is injective for any monoidal structure c of the functor F and its image consists of those isomorphisms $\alpha \in Aut(F^{\otimes 2})$ for which the diagram

$$\begin{array}{ccccc}
 F(X \otimes (Y \otimes Z)) & \xrightarrow{c \circ \alpha} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{I \otimes c \circ \alpha} & F(X) \otimes (F(Y) \otimes F(Z)) \\
 \downarrow \varphi & & & & \downarrow \psi \\
 F((X \otimes Y) \otimes Z) & \xrightarrow{c \circ \alpha} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{c \circ \alpha \otimes I} & (F(X) \otimes F(Y)) \otimes F(Z)
 \end{array}$$

is commutative for any objects $X, Y, Z \in \mathcal{G}$.

The pair $(F, \alpha c) = (F, c)(\alpha) = F(\alpha)$ will be called a *twisted form* of the monoidal functor (F, c) .

The notion of the isomorphism of monoidal functors defines an equivalence relation " \sim " on the set $MS(F)$

$$c \sim d \Leftrightarrow (F, c) \simeq (F, d) \quad c, d \in MS(F).$$

Denote by $Ms(F)$ the factor of $MS(F)$ by the relation \sim .

The inclusion $Q(c, ?)$ induces an equivalence relation on its image (and even on the whole group $Aut(F^{\otimes 2})$):

$\alpha \sim \beta$ iff there is an automorphism f of the functor F such that the diagram

$$\begin{array}{ccc}
 F(X \otimes Y) & \xrightarrow{c \circ \alpha} & F(X) \otimes F(Y) \\
 \downarrow f & & \downarrow I \otimes f \\
 F(X \otimes Y) & \xrightarrow{c \circ \beta} & F(X) \otimes F(Y)
 \end{array}$$

is commutative for any objects $X, Y \in \mathcal{G}$. It is easy to see that this relation on the group $Aut(F^{\otimes 2})$ comes from the action of the group $Aut(F)$.

In the case of the identity functor the set $Ms(id_{\mathcal{G}})$ has a group structure and coincides with the kernel of the natural map

$$Aut^{\otimes}(\mathcal{G}) \rightarrow Aut(\mathcal{G}, \otimes).$$

is commutative for any objects $X, Y \in \mathcal{G}$. It is easy to see that this relation on the group $Aut(F^{\otimes 2})$ comes from the action of the group $Aut(F)$.

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$$Aut^{\otimes}(\mathcal{G}) \rightarrow Aut(\mathcal{G}, \otimes).$$

In a similar way we can define the factor $Ms(\mathcal{G})$ by the equivalence relation " \approx " on the set $MS(\mathcal{G})$, which corresponds to the equivalence of monoidal categories. Since the definition of the relation " \approx " uses all autoequivalences of the category it is often difficult to verify it. It is useful to consider a more weak relation " \sim " which corresponds to the equivalence by means of identity functor [21].

Namely, $\varphi \sim \psi$ iff the identity functor $id_{\mathcal{G}}$ can be equipped with a structure of monoidal functor between $(\mathcal{G}, \otimes, \varphi)$ and $(\mathcal{G}, \otimes, \psi)$.

The factor $MS(\mathcal{G})/\sim$ will be denoted by $\overline{MS}(\mathcal{G})$.

The corresponding relation on the image of inclusion $Q(\varphi, ?)$ (on the group $Aut(id_{\mathcal{G}}^{\otimes 3})$) has the form:

$\alpha \sim \beta$ iff there is an automorphism c of the functor $F^{\otimes 2}$ such that the diagram

$$\begin{array}{ccccc} F(X \otimes (Y \otimes Z)) & \xrightarrow{c} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{I \otimes c} & F(X) \otimes (F(Y) \otimes F(Z)) \\ \varphi \circ \alpha \downarrow & & & & \varphi \circ \beta \downarrow \\ F((X \otimes Y) \otimes Z) & \xrightarrow{c} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{c \otimes I} & (F(X) \otimes F(Y)) \otimes F(Z) \end{array}$$

is commutative for any objects $X, Y, Z \in \mathcal{G}$.

It is evident that this relation on the group $Aut(id_{\mathcal{G}}^{\otimes 3})$ comes from the action of $Aut(id_{\mathcal{G}}^{\otimes 2})$ on it.

The group $Aut(\mathcal{G}, \otimes)$ acts on the set $Ms(\mathcal{G})$ in the following way:

$$F(\varphi)_{X,Y,Z} = F(\varphi_{F^{-1}(X), F^{-1}(Y), F^{-1}(Z)})$$

where $\varphi \in Ms(\mathcal{G})$ and $F \in Aut(\mathcal{G}, \otimes)$. The factor of this action coincides with the set $\overline{MS}(\mathcal{G})$.

4 endomorphisms of monoidal functor

The collection of the endomorphisms algebras of tensor powers of a monoidal functor F can be equipped with the structure of cosimplicial complex $E(F)$. = $End(F^{\otimes *})$.

The image of the coface map

$$\partial_{n+1}^i : End(F^{\otimes n}) \rightarrow End(F^{\otimes n+1}) \quad i = 0, \dots, n+1$$

here ϕ_i is the unique isomorphism between $F(\dots(X_1 \otimes \dots) \otimes X_{n+1}) = F^{\otimes n+1}(X_1, \dots, X_{n+1})$ and $F(\dots(X_1 \otimes \dots) \otimes (X_i \otimes X_{i+1})) \otimes \dots \otimes X_{n+1}$. The specialization of the image of the coboundary map

$$\sigma_{n-1}^i : \text{End}(F^{\otimes n}) \rightarrow \text{End}(F^{\otimes n+1}) \quad i = 0, \dots, n-1$$

is

$$\sigma_{n-1}^i(\alpha)_{X_1, \dots, X_{n-1}} = \alpha_{X_1, \dots, X_i, 1, X_{i+1}, \dots, X_{n-1}}.$$

We may also define the zero component of this complex as the endomorphism algebra of the unit object of the category \mathcal{H} which can be regarded as the endomorphism algebra of the functor

$$F^{\otimes 0} : k\text{-Mod} \rightarrow \mathcal{H}, \quad F^{\otimes 0}(V) = V \otimes 1$$

The coface maps

$$\partial_1^i : \text{End}_{\mathcal{H}}(1) \rightarrow \text{End}(F^{\otimes 1}), \quad i = 0, 1$$

has the form

$$\partial_1^0(a) = \rho(I_1 \otimes a)\rho^{-1}, \quad \partial_1^1(a) = \lambda(I_1 \otimes a)\lambda^{-1};$$

here ρ and λ are the structural isomorphisms of the unit object 1.

The invertible elements of this complex of algebras form a cosimplicial complex of the (generally non-commutative) groups $A_*(F) = \text{Aut}(F^{\otimes *})$. Nevertheless, for small n the n -th cohomology of the complex $A(F)_*$ can be defined.

Let us consider the cosimplicial complex E_* of algebras as a functor [9]

$$E : \delta \rightarrow k\text{-Alg}$$

from the category δ of finite ordered sets with nondecreasing maps to the category $k\text{-Alg}$ of k -algebras. In particular, E_n is the value of the functor E on the ordered set of $n+1$ elements $[n] = \{0, 1, \dots, n\}$ and

$$\partial_n^i = E(\partial_n^i), \quad \sigma_n^i = E(\sigma_n^i),$$

where

∂_n^i is the increasing injection which does not take the value $i \in [n]$,

σ_n^i is the nondecreasing surjection which takes twice the value $i \in [n]$.

We say that the *linking coefficient* of two nondecreasing maps $\tau, \pi : [l] \rightarrow [m]$ is equal to n iff there are decompositions into the nonintersecting unions

$$\text{Im}(\tau) = A_1 \cup \dots \cup A_s, \quad A_1 < \dots < A_s$$

$$\text{Im}(\pi) = B_1 \cup \dots \cup B_t, \quad B_1 < \dots < B_t \quad s+t = n+1$$

$$\text{and } A_1 \leq B_1 \leq A_2 \leq B_2 \dots$$

In particular the linking coefficient of ∂_n^i and ∂_n^{i+1} is equal to

$$1, \text{ if } n = 1; \quad 2, \text{ if } n = 2; \quad 2n - 3, \text{ if } n \geq 3$$

if $i > j - 1$ then the linking coefficient of ∂_n^i and ∂_n^j is equal to

$$1, \text{ if } n = 2; \quad 2, \text{ if } n = 3; \quad 2n - 5, \text{ if } n \geq 4.$$

The functor $E : \delta \rightarrow k - Alg$ (the cosimplicial complex of algebras E_*) will be called *n-commutative* if

$$E(\tau)(a) \text{ commutes with } E(\pi)(b) \quad \text{for every } a, b \in E(l),$$

for any nondecreasing maps $\tau, \pi : [l] \rightarrow [m]$ whose linking coefficient is less or equal to n , if $n < 3$ and $2n - 1$, if $n \geq 3$.

For the cosimplicial complex of groups A_* let us denote by $z^n(A_*) \subset A_n$ the set of solutions of the equation

$$\prod_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \partial_{n+1}^{2i}(a) = \prod_{i=\lfloor \frac{n}{2} \rfloor}^0 \partial_{n+1}^{2i+1}(a)$$

and by " \sim " the binary relation on the group A_n :

$$a \sim b \Leftrightarrow \exists c \in A_{n-1} : a \left(\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} \partial_n^{2i}(c) \right) = \left(\prod_{i=\lfloor \frac{n-1}{2} \rfloor}^0 \partial_n^{2i+1}(c) \right) b$$

Proposition 2 *If A_* is a n-commutative cosimplicial complex of algebras, then*

1. A_m is a commutative group for $m < n$
2. $z_m(A_*)$ is a subgroup of A_m for $m \leq n$
3. " \sim " is an equivalence relation on the group A_m (induced by the action of A_m which preserves $z_m(A_*)$) for $m \leq n + 1$

Proof:

1. For $m < n$ and for any $a, b \in A_m$ the commutator $[\partial_{m+1}^i(a), \partial_{m+1}^{i+1}(b)]$ is equal to zero. So

$$[a, b] = \sigma_m^i([\partial_{m+1}^i(a), \partial_{m+1}^{i+1}(b)]) = 0$$

2,3. For $m \leq n$ and for any $a, b \in A_m$

$$\prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(a) \prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(b) = \prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(ab)$$

and

$$\prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(a) \prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(b) = \prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(ab)$$

2,3. For $m \leq n$ and for any $a, b \in A_m$

$$\prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(a) \prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(b) = \prod_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \partial_{m+1}^{2i}(ab)$$

and

$$\prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(a) \prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(b) = \prod_{i=\lfloor \frac{m}{2} \rfloor}^0 \partial_{m+1}^{2i+1}(ab)$$

It immediately follows from the definition of the coface maps that the complex of automorphisms $A_*(F)$ of a monoidal functor F is 1-commutative.

The group

$$z^1(F) = z^1(A_*(F)) = \{\alpha \in Aut(F), \partial_2^1(\alpha) = \partial_2^0(\alpha)\partial_2^2(\alpha)\}$$

is the same as the group of monoidal automorphisms $Aut^{\otimes}(F)$ of the functor F .

The set

$$z^2(F) = \{\alpha \in Aut(F^{\otimes 2}), \partial_3^0(\alpha)\partial_3^2(\alpha) = \partial_3^3(\alpha)\partial_3^1(\alpha)\}$$

coincides with the image of the map $Q(c, ?) : MS(F) \rightarrow Aut(F^{\otimes 2})$ and the equivalence relation \sim on $z^2(F)$ coincides with the relation on the image of $Q(c, ?)$ which corresponds to the isomorphism of monoidal functors. So the second cohomology of monoidal functor is isomorphic to the set of monoidal structures

$$h^2(F) = z^2(F) / \sim = Ms(F).$$

It follows from the functoriality of the automorphisms that the complex of automorphisms $A_*(id_{\mathcal{G}})$ of the identity functor id is 2-commutative.

The group structure on $h^2(\mathcal{G}) = h^2(id_{\mathcal{G}})$ corresponds to the composition of monoidal functors.

The set

$$z^3(\mathcal{G}) = z^3(id_{\mathcal{G}}) = \{\alpha \in Aut(id^{\otimes 3}), \partial_4^0(\alpha)\partial_4^2(\alpha)\partial_4^4(\alpha) = \partial_4^3(\alpha)\partial_4^1(\alpha)\}$$

is nothing else then the image of the map $Q(c, ?) : MS(\mathcal{G}) \rightarrow Aut(id^{\otimes 3})$.

The equivalence relation \sim on $z^3(\mathcal{G})$ coincides with the relation on the image of $Q(c, ?)$ which corresponds to the equivalence of monoidal categories by means of the identity functor. So the third cohomology of the monoidal category (of the identity functor) is isomorphic to the set of monoidal structures

$$h^3(\mathcal{G}) = z^3(\mathcal{G}) / \sim = Ms(\mathcal{G}).$$

5 tangent cohomology

Let K be a k -algebra and \mathcal{G} be a k -linear category.

The subcategory of K -modules \mathcal{G}_K in \mathcal{G} is a category of pairs (X, α) where X is an object of \mathcal{G} and $\alpha : K \rightarrow \text{End}_{\mathcal{G}}(X)$ is a homomorphism of k -algebras. The morphism from (X, α) to (Y, β) in \mathcal{G}_K is a morphism f from X to Y in \mathcal{G} which preserves the K -module structure ($f\alpha(c) = \beta(c)f$ for any $c \in K$).

The forgetful functor

$$\mathcal{G}_K \rightarrow \mathcal{G}, \quad (X, \alpha) \mapsto X$$

has the right adjoint

$$\mathcal{G} \rightarrow \mathcal{G}_K, \quad X \mapsto (K \otimes_k X, i \otimes I_X)$$

if the category \mathcal{G} is sufficiently large (e.g. we can multiply any object by the k -module K). Anyway we have no problem if K is a finitely generated projective k -module.

A functor $F : \mathcal{G} \rightarrow \mathcal{H}$ induces the functor

$$F_K : \mathcal{G}_K \rightarrow \mathcal{H}_K, \quad F(X, \alpha) = (F(X), F(\alpha))$$

such that the diagram of functors

$$\begin{array}{ccc} \mathcal{G}_K & \xrightarrow{F_K} & \mathcal{H}_K \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{F} & \mathcal{H} \end{array}$$

is commutative.

Proposition 3 *Let $F : \mathcal{G} \rightarrow \mathcal{H}$ is a right exact monoidal functor between abelian monoidal categories with right biexact tensor products.*

Then for any k -algebra K the homomorphism of cosimplicial complexes

$$Z(K) \otimes_k E^*(F) \rightarrow E^*(F_K), \quad c \otimes \gamma_{(X, \alpha)} = \alpha(c)\gamma$$

is an isomorphism (where $Z(K)$ is the center of the algebra K).

Proof: It is easy to see that the statement of the proposition follows from the following fact:

for the right exact functor $F : \mathcal{G} \rightarrow \mathcal{H}$ the homomorphism of algebras

$$Z(K) \otimes_k \text{End}(F) \rightarrow \text{End}(F_K), \quad c \otimes \gamma_{(X, \alpha)} = \alpha(c)\gamma$$

is an isomorphism.

The proof of this fact is analogous to the proof of the proposition (1).

Proposition 4 *The tangent space of the functor $h^n(F)$ of cohomologies of monoidal functor F at the unit point (the class of identity endomorphism) coincides with the cohomology $H^n(F)$ of the cochain complex associated with the cosimplicial complex $E^*(F)$ of abelian groups (the tangent cohomology of F).*

Proof

Since $E^*(F_{A_2}) = A_2 \otimes_k E^*(F)$, any element of the kernel of the map $A^n(F_{A_2}) \rightarrow A^n(F)$ has the form $1 + \epsilon\alpha$ for some $\alpha \in E^n(F)$.

The direct checking shows that the cocycle condition for $1 + \epsilon\alpha$ is the equation $d_n(\alpha) = 0$, where $d_n = \sum_{i=0}^{n+1} (-1)^i \partial_{n+1}^i$ is a cochain differential. The cocycles $1 + \epsilon\alpha$ and $1 + \epsilon\beta$ is equivalent if and only if α and β differs by the coboundary.

For example, the first cocycle module $Z^1(F)$ of the tangent complex coincides with the module of *differentiations* of the monoidal functor F

$$Diff(F) = \{l \in End(F), l_X \otimes_Y = l_X \otimes l_Y + l_X \otimes l_Y\}$$

which is a Lie algebra respectively the commutator in the algebra $End(F)$. The first tangent cohomology $H^1(F)$ is a factoralgebra of this Lie algebra.

The tensor product of endomorphisms of the functor F

$$End(F^{\otimes n}) \otimes End(F^{\otimes m}) \rightarrow End(F^{\otimes n+m})$$

defines on $E^*(F)$ a structure of cosimplicial algebra

$$\partial_{n+m+1}^i(\alpha \otimes \beta) = \begin{cases} \partial_{n+1}^i(\alpha) \otimes \beta, & i \leq n \\ \alpha \otimes \partial_{m+1}^{i-n}(\beta), & i > n \end{cases}$$

Hence the associated cochain complex $(E^*(F), d)$ is a differential graded algebra

$$d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^n \alpha \otimes d(\beta).$$

In particular, the tangent cohomologies $H^*(F)$ form a graded algebra.

Let us also note that the multiplication is skew-symmetric on the first component $H^1(F)$, because

$$\alpha \otimes \beta + \beta \otimes \alpha = d(-\alpha\beta), \quad \text{for } \alpha, \beta \in Z^1(F).$$

The *tangent cone* in the tangent space at the point $x \in h(k)$ to the functor $h : k - Alg \rightarrow Sets$ is the image of the map

$$ker(h(A_\infty) \rightarrow h(k)) \rightarrow ker(h(A_2) \rightarrow h(k)),$$

induced by the homomorphism of algebras

$$A_\infty = k[[\epsilon]] \rightarrow k[\epsilon | \epsilon^2 = 0] = A_2, \quad \epsilon \mapsto \epsilon.$$

In many cases it is sufficient (and more convenient) to consider the *first approximation of the tangent cone* which is the image of the map

$$\ker(h(A_3) \rightarrow h(k)) \rightarrow \ker(h(A_2) \rightarrow h(k)),$$

induced by the homomorphism of algebras

$$A_3 = k[\epsilon | \epsilon^3 = 0] \rightarrow k[\epsilon | \epsilon^2 = 0] = A_2, \quad \epsilon \mapsto \epsilon.$$

6 K-theory

In [19] Quillen associated to an abelian (exact) category \mathcal{A} a topological space $BQ\mathcal{A}$ such that exact functors between categories define the continuous maps between the corresponding spaces and isomorphisms of functors define the homotopies between the corresponding maps. In other words, Quillen's space is a 2-functor from the 2-category of abelian (exact) categories (with isomorphisms of functors as 2-morphisms) to the 2-category of topological spaces.

Waldhausen [27] proved that the 2-functor $K = \Omega BQ$ is permutable (in some sense) with the product. Namely, he constructed the continuous map $K(\mathcal{A}) \wedge K(\mathcal{B}) \rightarrow K(\mathcal{C})$ for any biexact functor $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$.

The homotopy groups $K_*(\mathcal{A}) = \pi_*(\mathcal{A})$ of the Waldhausen space $K(\mathcal{A})$ are called *algebraic K-theory* of the category \mathcal{A} .

Now we give the definition of Hochschild cohomology of topological ring spaces. The topological space K with a continuous map $\mu : K \wedge K \rightarrow K$ is called a *ring space*. A ring space (K, μ) is *associative* if there is a homotopy between $\mu(I \wedge \mu)$ and $\mu(\mu \wedge I)$.

A *bimodule* over an associative ring space (K, μ) is a space M together with the continuous maps

$$\nu : M \wedge K \rightarrow M, \quad v : K \wedge M \rightarrow M$$

and the homotopies

$$\nu(I \wedge \mu) \rightarrow \nu(\nu \wedge I), \quad v(I \wedge \nu) \rightarrow v(\nu \wedge I), \quad v(I \wedge v) \rightarrow v(\mu \wedge I)$$

For example the space of maps to the ring space is a bimodule over this ring space.

Denote by $[X, Y]$ the set of homotopy classes of continuous maps from X to Y . The *Hochschild complex* of a ring space K with coefficients in a bimodule space M is a semicosimplicial complex of sets

$$C_*(K, M), \quad C_n(K, M) = [K^{\wedge n}, M]$$

with the coface maps $\partial_n^i : C_{n-1}(K, M) \rightarrow C_n(K, M)$ defined as follows

$$\partial_n^i(f) = \begin{cases} \nu(I \wedge f), & i = 0 \\ f(I \wedge \dots \wedge \mu \wedge \dots \wedge I), & 1 \leq i \leq n \\ \nu(f \wedge id), & i = n + 1 \end{cases}$$

If M is a loop space then $C_*(K, M)$ is a complex of groups and these groups are abelian if M is a double loop space. In the second case the cohomology of the cochain complex associated with $C_*(K, M)$ will be called *Hochschild cohomology* ($HH^*(K, M)$) of ring space K with coefficient in the bimodule M .

A ring (bimodule) structure on a space induces a graded ring (bimodule) structure on its homotopy groups.

The natural map $C_*(K, M)$ to the Hochschild complex $C_*(\pi_*(K), \pi_*(M))$ of the ring $\pi_*(K)$ with coefficients in the bimodule $\pi_*(M)$ induces the homomorphism of Hochschild cohomology

$$HH^*(K, M) \longrightarrow HH_*(\pi_*(K), \pi_*(M)).$$

Proposition 5 *Let $F : \mathcal{G} \rightarrow \mathcal{H}$ be an exact monoidal functor between abelian monoidal categories with biexact tensor products. Then there is a map of semi-cosimplicial complexes of groups*

$$A_*(F) = \text{Aut}(F^{\otimes *}) \longrightarrow C_*(K(\mathcal{G}), \Omega K(\mathcal{H}))$$

which defines the maps

$$h^i(F) \longrightarrow HH^i(K_*(\mathcal{G}), K_{*+1}(\mathcal{H}))$$

Proof:

The map $\text{Aut}(F^{\otimes *}) \longrightarrow C_*(K(\mathcal{G}), \Omega K(\mathcal{H}))$ sends the automorphism $\alpha \in \text{Aut}(F^{\otimes n})$ to the class of the corresponding autohomotopy of the map $K(F^{\otimes n})$.

The zero component of the previous map

$$h^i(F) \longrightarrow HH^i(K_0(\mathcal{G}), K_1(\mathcal{H}))$$

admits a more explicit description.

For an automorphism $\alpha \in \text{Aut}(F^{\otimes n})$ the class $[\alpha_{X_1, \dots, X_n}] \in K_1(\mathcal{H})$ depends only of the classes $[X_1], \dots, [X_n] \in K_0(\mathcal{G})$. Indeed, the exact sequence

$$0 \rightarrow Y_i \rightarrow X_i \rightarrow Z_i \rightarrow 0$$

can be extended to the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 \otimes \dots \otimes Y_i \otimes \dots \otimes X_n & \longrightarrow & X_1 \otimes \dots \otimes X_i \otimes \dots \otimes X_n & \longrightarrow & X_1 \otimes \dots \otimes Z_i \otimes \dots \otimes X_n \longrightarrow \\ & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ 0 & \longrightarrow & X_1 \otimes \dots \otimes Y_i \otimes \dots \otimes X_n & \longrightarrow & X_1 \otimes \dots \otimes X_i \otimes \dots \otimes X_n & \longrightarrow & X_1 \otimes \dots \otimes Z_i \otimes \dots \otimes X_n \longrightarrow \end{array}$$

which is commutative by the functoriality of α . Hence

$$[\alpha_{X_1, \dots, X_i, \dots, X_n}] = [\alpha_{X_1, \dots, Y_i, \dots, X_n}] + [\alpha_{X_1, \dots, Z_i, \dots, X_n}]$$

which means that $[\alpha_{X_1, \dots, X_n}]$ depends only of the classes $[X_1], \dots, [X_n]$ and can be regarded as an element $\bar{\alpha}$ of $\text{Hom}(K_0(\mathcal{G})^{\otimes n}, K_1(\mathcal{H}))$. Now the direct checking shows that the maps

$$\text{Aut}(F^{\otimes n}) \longrightarrow \text{Hom}(K_0(\mathcal{G})^{\otimes n}, K_1(\mathcal{H}))$$

commutes with the coface operators:

$$\begin{aligned} \overline{\partial_{n+1}^i(\alpha)}([X_1], \dots, [X_{n+1}]) &= [\partial_{n+1}^i(\alpha)_{X_1, \dots, X_{n+1}}] = \\ \left\{ \begin{array}{l} i = 0, \quad [I_{X_1} \otimes \alpha_{X_2, \dots, X_{n+1}}] = [X_1] \bar{\alpha}([X_2], \dots, [X_{n+1}]) \\ 1 \leq i \leq n, \quad [\alpha_{X_1, \dots, X_i} \otimes \alpha_{X_{i+1}, \dots, X_{n+1}}] = \bar{\alpha}([X_1], \dots, [X_i], [X_{i+1}], \dots, [X_{n+1}]) \\ i = n+1, \quad [\alpha_{X_1, \dots, X_n} \otimes I_{X_{n+1}}] = \bar{\alpha}([X_1], \dots, [X_n], [X_{n+1}]) \end{array} \right\} \\ &= \partial_{n+1}^i(\bar{\alpha})([X_1], \dots, [X_{n+1}]) \end{aligned}$$

Thus we have a map

$$h^i(F) \longrightarrow HH^i(K_0(\mathcal{G}), K_1(\mathcal{H}))$$

which asserts to the automorphism α the class $[\bar{\alpha}]$.

7 bimodules

A *bimodule* over an associative algebra R is a k -module M with the left and right R -module structures

$$\begin{aligned} R \otimes M &\rightarrow M & r \otimes m &\mapsto rm \\ M \otimes R &\rightarrow M & m \otimes r &\mapsto mr \end{aligned}$$

(left and right k -module structures coincides) such that

$$r(ms) = (rm)s \quad \text{for any } r, s \in R, m \in M.$$

A homomorphism ($\text{Hom}_{R-R}(M, N)$) from a bimodule M to a bimodule N is a k -linear map which preserves both left and right R -module structures.

For example, the endomorphism algebra $\text{End}_{R-R}(R \otimes_k R)$ of the bimodule $R \otimes_k R$ is isomorphic to the algebra $R^{op} \otimes R$ (R^{op} is the algebra with opposite multiplication). Any element $x \in R^{op} \otimes R$ defines the homomorphism f_x (for decomposable $x = r \otimes s$ f_x sends $p \otimes q$ from $R \otimes_k R$ to $pr \otimes sq$). Conversely the value of an endomorphism f of the bimodule $R \otimes_k R$ on the element $1 \otimes 1$ lies in $R \otimes_k R$.

The category $R\text{-Mod-}R$ of bimodules over R is an monoidal category with respect to tensor product \otimes_R of bimodules and trivial associativity constraint.

Proposition 6 *The complex of endomorphisms*

$$E_*(R - Mod - R) = End(id_{R-Mod-R}^{\otimes *})$$

of the identity functor of the category of bimodules $R - Mod - R$ is isomorphic to the complex $Z(R^{\otimes *+1})$ whose coface maps

$$\partial_n^i : Z(R^{\otimes *n}) \longrightarrow Z(R^{\otimes *n+1})$$

are induced by the homomorphisms of algebras

$$\partial_n^i : R^{\otimes *n} \longrightarrow R^{\otimes *n+1}$$

$$\partial_n^i(r_1 \otimes \dots \otimes r_n) = r_1 \otimes \dots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \dots \otimes r_n.$$

The isomorphism is realized by two mutually inverse maps:

$$End(id_{R-Mod-R}^{\otimes n}) \longrightarrow Z(R^{\otimes n+1})$$

which sends the endomorphism to its specialization on the objects $R \otimes_k R, \dots, R \otimes_k R$, and

$$Z(R^{\otimes n+1}) \longrightarrow End(id_{R-Mod-R}^{\otimes n})$$

which associates to an element $r = \sum_i r_0^i \otimes \dots \otimes r_n^i$ the endomorphism $\alpha(r)$ whose specialization on the objects $M_1, \dots, M_n \in R - Mod - R$ is

$$\alpha(r)_{M_1, \dots, M_n}(m_1 \otimes \dots \otimes m_n) = \sum_i r_0^i m_1 r_1^i \otimes \dots \otimes m_n r_n^i.$$

Proof:

The bimodule $R \otimes_k R$ is a projective generator in the category $R - Mod - R$. Indeed the functor

$$k - Mod \longrightarrow R - Mod - R \quad V \mapsto R \otimes_k V \otimes_k R$$

is a right adjoint of the forgetful functor $R - Mod - R \longrightarrow k - Mod$. The adjunction is given by the morphisms

$$R \otimes_k M \otimes_k R \rightarrow M \quad r \otimes m \otimes s \mapsto rms,$$

$$V \rightarrow R \otimes_k V \otimes_k R \quad v \mapsto 1 \otimes v \otimes 1.$$

Thus the endomorphism algebra $End(id_{R-Mod-R}^{\otimes n})$ is isomorphic to the centralizer of the subalgebra $End_{R-R}(R \otimes_k R)^{\otimes n}$ in the algebra $End_{R-R}(R^{\otimes *n+1})$. It is easy to see that the endomorphism f from $End_{R-R}(R^{\otimes *n+1})$ which commutes with the subalgebra $End_{R-R}(R \otimes_k R)^{\otimes n}$ is defined by its value $f(1 \otimes \dots \otimes 1) \in R^{\otimes *n+1}$. Indeed, since f commutes with $f_{r \otimes s}$, we have

$$f(r_1 \otimes \dots \otimes r_{n+1}) = f((r_1 \otimes 1) \otimes_R \dots \otimes_R (r_n \otimes r_{n+1})) =$$

$$f((f_{r_1 \otimes 1} \otimes_R \dots \otimes_R f_{r_n \otimes r_{n+1}})((1 \otimes 1) \otimes_R \dots \otimes_R (1 \otimes 1))) = \\ (f_{r_1 \otimes 1} \otimes_R \dots \otimes_R f_{r_n \otimes r_{n+1}})(f(1 \otimes \dots \otimes 1)).$$

Finally, R-polylinearity implies that $f(1 \otimes \dots \otimes 1)$ lies in $Z(R^{\otimes n+1})$.

Since the algebras $End(id_{R-Mod-R}^{\otimes n})$ are commutative, the cohomology $h^n(R-Mod-R) = h^n(id_{R-Mod-R})$ is defined for any n and coincides with the cohomology of the cochain complex associated to the cosimplicial complex of groups of invertible elements of the algebras $Z(R^{\otimes n+1})$. In the case of commutative algebra R this complex is called the *Amitsur complex* of the algebra R and its cohomology ($HA^*(R/k)$) the *Amitsur cohomology* [20]. In particular $h^1(R-Mod-R)$ is isomorphic to the relative Picard group $Pic(R/k)$ and $h^2(R-Mod-R)$ to the relative Brauer group $Br(R/k)$.

Using the restriction $h^n(R-Mod-R) \rightarrow h^n(R-mod-R)$ and the homomorphism from the cohomology of the identity functor to the Hochschild cohomology of the K-theory we can define the following maps

$$HA^n(R/k) \rightarrow HH^n(K_0(R-mod-R), K_1(R-mod-R)),$$

for the commutative algebra R .

In the case of a Galois extension of fields K/k with the Galois group G [1] the previous homomorphism is identical:

$$H^n(G, R^*) = HA^n(R/k) \rightarrow HH^n(K_0(R-mod-R), K_1(R-mod-R)) = H^n(G, R^*).$$

Indeed, the category of bimodules $R-mod-R$ over Galois extension K/k is semisimple, its simple objects being parametrized by the elements of G , whence

$$K_*(R-mod-R) = K_*(K) \otimes_{\mathbb{Z}} \mathbb{Z}[G].$$

8 modules and comodules over bialgebra

A *bialgebra* [24] is an algebra H together with a homomorphisms of algebras

$$\Delta : H \rightarrow H \otimes H \quad (\text{coproduct})$$

$$\varepsilon : H \rightarrow k \quad (\text{counit})$$

for which

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta \quad (\text{coassociativity}),$$

$$(\varepsilon \otimes I)\Delta = (I \otimes \varepsilon)\Delta = I \quad (\text{axiom of counit}).$$

For example, the group algebra $k[G]$ of a group G is a bialgebra, where

$$\Delta : k[G] \rightarrow k[G] \otimes k[G] \quad \Delta(g) = g \otimes g,$$

$$\varepsilon : H \rightarrow k \quad (\text{counit})$$

for which

$$\begin{aligned} (\Delta \otimes I)\Delta &= (I \otimes \Delta)\Delta && (\text{coassociativity}), \\ (\varepsilon \otimes I)\Delta &= (I \otimes \varepsilon)\Delta = I && (\text{axiom of counit}). \end{aligned}$$

For example, the group algebra $k[G]$ of a group G is a bialgebra, where

$$\begin{aligned} \Delta : k[G] &\rightarrow k[G] \otimes k[G] && \Delta(g) = g \otimes g, \\ \varepsilon : k[G] &\rightarrow k && \varepsilon(g) = 1 \quad g \in G. \end{aligned}$$

Another example provided by the universal enveloping algebra $U[\mathfrak{g}]$ of the Lie algebra \mathfrak{g}

$$\begin{aligned} \Delta : U[\mathfrak{g}] &\rightarrow U[\mathfrak{g}] \otimes U[\mathfrak{g}] && \Delta(l) = l \otimes 1 + 1 \otimes l, \\ \varepsilon : U[\mathfrak{g}] &\rightarrow k && \varepsilon(l) = 0 \quad l \in \mathfrak{g}. \end{aligned}$$

The coproduct allows to define the structure of H -module on the tensor product $M \otimes_k N$ of two H -modules

$$h * (m \otimes n) = \Delta(h)(m \otimes n) \quad h \in H, m \in M, n \in N.$$

The coassociativity of coproduct implies that the standard associativity constraint for underlying k -modules

$$\varphi : L \otimes (M \otimes N) \rightarrow (L \otimes M) \otimes N \quad \varphi(l \otimes (m \otimes n)) = (l \otimes m) \otimes n$$

induces an associativity for this tensor product.

The counit defines the structure of H -module on the ground ring k which (by counit axiom) is a unit object respectively to the tensor product.

By another words the category of (left) H -modules $H\text{-Mod}$ is a monoidal category. It follows from the definition of tensor product in $H\text{-Mod}$ that the forgetful functor $F : H\text{-Mod} \rightarrow k\text{-Mod}$ is a monoidal functor with trivial monoidal structure.

Proposition 7 *1. The complex of endomorphisms $E_\bullet(F) = \text{End}(F^{\otimes \bullet})$ of forgetful functor of the category of modules $H\text{-Mod}$ over bialgebra H is isomorphic to the bar complex $H^{\otimes \bullet}$ of H (the coface maps*

$$\partial_n^i : H^{\otimes n-1} \rightarrow H^{\otimes n}$$

has the form

$$\partial_n^i(h_1 \otimes \dots \otimes h_n) = \begin{cases} 1 \otimes h_1 \otimes \dots \otimes h_n, & i = 0 \\ h_1 \otimes \dots \otimes \Delta(h_i) \otimes \dots \otimes h_n, & 1 \leq i \leq n \\ h_1 \otimes \dots \otimes h_n \otimes 1, & i = n + 1 \end{cases}$$

which associate to the element $x \in H^{\otimes n}$ the endomorphism of multiplying by x .
2. The complex of endomorphisms $E_*(H-Mod) = End(id_{H-Mod}^{\otimes *})$ of the identity functor of the category $H-Mod$ is isomorphic to the subcomplex of bar complex of H which consists of H -invariant elements (the subcomplex of centralizers $C_{H^{\otimes n}}(\Delta(H))$ of the images of diagonal embeddings).

Proof:

We will proof more general statement.

Let $f : F \rightarrow H$ be a homomorphism of bialgebras (it means that f is a homomorphism of algebras and $\Delta f = (f \otimes f)\Delta$). The restriction functor $f^* : H-Mod \rightarrow F-Mod$ is monoidal and the complex of its endomorphisms coincides with the subcomplex $C_{H^{\otimes *}}(\Delta(f(F)))$ in the bar complex of the bialgebra H . The proposition follows from this fact, because the forgetful functor from the category $H-Mod$ is the restriction functor, which corresponds to the unit embedding $k \rightarrow H$, and the identity functor is the restriction functor, which corresponds to the identity map $H \rightarrow H$.

The forgetful functor from the category $H-Mod$ have a right adjoint

$$k-Mod \longrightarrow H-Mod \quad V \mapsto H \otimes_k V$$

which endomorphism algebra coincides with an algebra of endomorphisms of H -module H and is isomorphic to H^{op} (the element $h \in H^{op}$ defines the endomorphism r_h of right multiplying by h).

Thus the endomorphism algebra $End((f^*)^{\otimes n})$ is isomorphic to the centralizer of $End_H(H)^{\otimes n}$ in the algebra $End_F(H^{\otimes n})$.

It is follows from the direct checking that an element f of this centralizer is defined by its value $f(1 \otimes \dots \otimes 1) \in H^{\otimes n}$. Indeed, since f commutes with right multiplications by elements of H we have

$$f(h_1 \otimes \dots \otimes h_n) = f((r_{h_1} \otimes \dots \otimes r_{h_n})(1 \otimes \dots \otimes 1)) = (r_{h_1} \otimes \dots \otimes r_{h_n})f(1 \otimes \dots \otimes 1)$$

The composition of endomorphisms from $C_{End_F(H^{\otimes n})}(End_H(H)^{\otimes n})$ corresponds to the multiplication of its values in $H^{\otimes n}$

$$fg(1 \otimes \dots \otimes 1) = f(g(1 \otimes \dots \otimes 1)) = f\left(\sum_i h_{i,1} \otimes \dots \otimes h_{i,n}\right) =$$

$$f(1 \otimes \dots \otimes 1)\left(\sum_i h_{i,1} \otimes \dots \otimes h_{i,n}\right) = f(1 \otimes \dots \otimes 1)g(1 \otimes \dots \otimes 1).$$

Finally, F -polylinearity of f means that $f(1 \otimes \dots \otimes 1)$ lies in the centralizer of the image of diagonal embedding $\Delta(F)$ in $H^{\otimes n}$.

The first cohomology $h^1(F) = z^1(F)$ of the forgetful functor $F : H-Mod \rightarrow k-Mod$ coincides with the group of invertible *group-like* elements of the bialgebra H [24]

$$G(H) = \{g \in H, \Delta(g) = g \otimes g\}.$$

The elements of $h^2(F)$ corresponds to the twistings of the coproduct in the Hopf algebra H [6]. Namely, the adjunction

$$\Delta^x(h) = x\Delta(h)x^{-1}, \quad h \in H$$

of the coproduct Δ by the element from

$$z^2(F) = \{x \in (H^{\otimes 2})^*, (1 \otimes x)(I \otimes \Delta)(x) = (x \otimes 1)(\Delta \otimes I)(x)\}$$

also satisfies to the coassociativity axiom.

If x also satisfies to the condition

$$(\varepsilon \otimes I)(x) = (I \otimes \varepsilon)(x) = 1$$

(which may be attained by the substitution of x by some equivalent to it), then the triple $(H, \Delta^x, \varepsilon)$ is a bialgebra (*twisted form* of bialgebra H).

It follows from the definitions that the complex of endomorphisms of the twisted form $F(x)$ of forgetful functor F coincides with the bar complex of the twisted form $(H, \Delta^x, \varepsilon)$ of bialgebra H .

The set $h^2(F)$ accepts also another description. Namely, $h^2(F)$ coincides with the set of Galois H -module coalgebras $Gal_{k-coalg}(H)$, which is isomorphic to H as H -modules [26].

The H -module coalgebra is a coalgebra L which coproduct is a homomorphism of H -modules. The H -module coalgebra L is *Galois* [2] if the map

$$H \otimes L \rightarrow L \otimes L, \quad h \otimes l \mapsto (h \otimes 1)\Delta(l)$$

is an isomorphism.

We have a map

$$h^2(F) \rightarrow Gal_{k-coalg}(H),$$

which sends the class of cocycle $x \in z^2(F)$ to the class of coalgebra (H, Δ_x) , where $\Delta_x(h) = \Delta(h)x^{-1}$, $h \in H$.

The first cohomology $h^1(H-Mod) = z^1(id_{H-Mod})$ of the identity functor coincides with the centre of the group of invertible group-like elements of the bialgebra H

$$Z(G(H)) = \{g \in Z(H)^*, \Delta(g) = g \otimes g\}.$$

The kernel of the map of punctured sets

$$h^2(H-Mod) \rightarrow h^2(F)$$

(the stabilizer of the action of the group $h^2(H-Mod)$ on the set $h^2(F)$) coincides with the kernel of natural homomorphism

$$Out_{bialg}(H) \rightarrow Out_{alg}(H)$$

from the group $Out_{bialg}(H)$ of outer automorphisms of bialgebra H (the factor of the group of automorphism of bialgebra H modulo conjugations by group-like elements of H) to the group $Out_{alg}(H)$ of outer automorphisms of the algebra H . Indeed, an element of second kernel is a conjugation by an element $h \in H^*$ such that $y = \Delta(h)(h \otimes h)^{-1}$ lies in the centralizer $C_{H \otimes 2}(\Delta(H))$. Hence, y represents the element of $h^2(H - Mod)$ which trivialises in $h^2(F)$.

For example, in the case of the group bialgebra $k[G]$ the first cohomology $h^1(F)$ of forgetful functor is isomorphic to the group G . The second cohomology $h^2(F)$ coincides with the set $Gal_k(G)$ of Galois G -extension of k . The kernel of the map

$$h^2(k[G] - Mod) \rightarrow h^2(F)$$

coincides (in the case of finite group G and the field k such that $char(k) \nmid |G|$) with the factorgroup of *locally-inner* automorphisms of G (an automorphisms which not moves the conjugacy classes) modulo inner.

The cohomology of forgetful and identity functors in the case of the universal enveloping algebra $U[\mathfrak{g}]$ of the Lie algebra (over the field k) are trivial. It follows from the absence of non-trivial invertible elements in $U[\mathfrak{g}]$, which can be deduced from Poincare-Birkhoff-Witt theorem. But non-trivial invertible elements appears if we extend the scalars (ground ring k). For example, the tangent cohomology of forgetful and identity functors are non-trivial in general. It was shown in [6] that the natural homomorphism

$$\Lambda^* \mathfrak{g} = \Lambda^* H^1(F) \rightarrow H^*(F),$$

induced by the multiplication in $H^*(F)$ is an isomorphism. The first approximation of the tangent cone in $H^2(F)$ is given by the equation

$$[[\alpha, \alpha]] = 0,$$

where $[[\ , \]]$ is a component of the Lie superalgebra structure on $\Lambda^* \mathfrak{g}$ [5]

$$\begin{aligned} & [[x_1 \wedge \dots \wedge x_s, y_1 \wedge \dots \wedge y_t]] = \\ & \sum_{i,j} (-1)^{i+j} [x_i, y_j] \wedge x_1 \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_s \wedge y_1 \wedge \dots \wedge \widehat{y}_j \wedge \dots \wedge y_t, \end{aligned}$$

where \widehat{z} means that z not accours in the product .

It follows from the result of [7] that the tangent cone in $H^2(F)$ coincides with its first approximation.

In particular, for any Cartan subalgebra \mathfrak{h} the space $\Lambda^2 \mathfrak{h}$ (the so-called infinitesimal *Sudbery family*) lies in the first approximation and in the tangent cone [8, 22].

It can be deduced from the description of $H^*(F)$ that the tangent cohomology of the identity functor $H^*(U[\mathfrak{g}] - Mod)$ coincides with \mathfrak{g} -invariant elements in $\Lambda^* \mathfrak{g}$.

There is another type of monoidal categories which can be associated to the bialgebras, namely categories of comodules.

A (right) *comodule* over a bialgebra H [24] is a k -module M with the homomorphism $\psi : M \rightarrow M \otimes H$ such that

$$(I \otimes \Delta)\psi = (\psi \otimes I)\psi \quad (\text{coassociativity of comodule structure}),$$

$$(I \otimes \varepsilon)\psi = I \quad (\text{counitality}).$$

For example, the ground ring k has the comodule structure

$$k \rightarrow k \otimes H = H, \quad c \mapsto c \otimes 1 = c1.$$

Another example provides by the bialgebra H with the coproduct Δ regarded as a comodule structure.

A *morphism of comodules* (M, ψ) and (N, ϕ) is a k -linear map $f : M \rightarrow N$ which preserves the comodule structure:

$$\phi f = (f \otimes I)\psi.$$

We will denote the set of morphisms of comodules by $Hom^H(M, N)$. For example, the endomorphism algebra $End^H(H)$ of comodule H is isomorphic to the $Hom_k(H, k)$ (*dual algebra* to H) with the multiplication (*convolution*)

$$p * q = (p \otimes q)\Delta, \quad p, q \in Hom_k(H, k).$$

Indeed, an endomorphism $f \in End^H(H)$ defines a map $p_f = \varepsilon f \in Hom_k(H, k)$.

Conversely, a linear map $p : H \rightarrow k$ defines an endomorphism $f_p = (p \otimes I)\Delta$.

The product $\mu : H \otimes H \rightarrow H$ of the bialgebra H allows to define a comodule structure on the tensor product (over k) of any two comodules (M, ψ) and (N, ϕ)

$$\gamma : M \otimes N \rightarrow M \otimes N \otimes H, \quad \gamma = (I_{M \otimes N} \otimes \mu)t_{H, N}(\psi \otimes \phi).$$

The associativity of the product μ implies that the (trivial) associativity constraint of underlying k -modules is an associativity for tensor product \otimes_k in the category of comodules $Comod - H$. It immediately follows from the definitions that the comodule k is a unit object of this category and the forgetful functor $F : Comod - H \rightarrow k - Mod$ is a monoidal functor with trivial monoidal structure.

Proposition 8 1. *The complex of endomorphisms $E_*(F) = End(F^{\otimes *})$ of forgetful functor of the category of comodules $Comod - H$ over bialgebra H is isomorphic to the cobar complex $Hom_k(H^{\otimes *}, k)$ of H (with coface maps*

$$\partial_n^i : Hom_k(H^{\otimes n-1}, k) \longrightarrow Hom_k(H^{\otimes n}, k)$$

which are induced by the homomorphisms $d_n^i : H^{\otimes n} \rightarrow H^{\otimes n-1}$

$$d_n^i(h_1 \otimes \dots \otimes h_n) = \begin{cases} \varepsilon(h_1) \otimes \dots \otimes h_n, & i = 0 \\ h_1 \otimes \dots \otimes h_i h_{i+1} \otimes \dots \otimes h_n, & 1 \leq i \leq n \\ h_1 \otimes \dots \otimes \varepsilon(h_n), & i = n + 1 \end{cases}$$

codegeneration maps are induced by the homomorphisms $s_n^i : H^{\otimes n} \rightarrow H^{\otimes n+1}$

$$s_n^i(h_1 \otimes \dots \otimes h_n) = h_1 \otimes \dots \otimes \Delta(h_i) \otimes \dots \otimes h_n$$

The isomorphism is realized by two mutually inverse maps:

$$\text{End}(F^{\otimes n}) \rightarrow \text{Hom}(H^{\otimes n}, k)$$

which sends the endomorphism to its specialization on the objects H, \dots, H , and

$$\text{Hom}(H^{\otimes n}, k) \rightarrow \text{End}(F^{\otimes n})$$

which associate to the element $x \in \text{Hom}(H^{\otimes n}, k)$ the endomorphism of convolution with x .

2. The complex of endomorphisms $E_*(\text{Comod} - H) = \text{End}(id_{\text{Comod} - H}^{\otimes *})$ of the identity functor of the category $\text{Comod} - H$ is isomorphic to the subcomplex of the cobar complex of H which consists of H -coinvariant elements (the subcomplex of centralizers $C_{\text{Hom}(H^{\otimes n}, k)}(\delta(\text{Hom}(H, k)))$ of the images of diagonal embeddings of $\text{Hom}(H, k)$).

Proof:

As in that case of modules over bialgebra we can proof more general statement. Let $f : H \rightarrow F$ is a homomorphism of bialgebras. The corestriction functor

$$f_* : H - \text{Mod} \rightarrow F - \text{Mod}, \quad f_*(M, \psi) = (M, (I \otimes f)\psi)$$

is a monoidal and the complex of its endomorphisms coincides with the subcomplex $C_{\text{Hom}(H^{\otimes *}, k)}(\delta(f(\text{Hom}(F, k))))$ in the cobar complex of the bialgebra H . The proposition follows from this fact, because the forgetful functor from the category $H - \text{Mod}$ is the corestriction functor which corresponds to the comit $H \rightarrow k$ and the identity functor is the corestriction functor which corresponds to the identity map $H \rightarrow H$.

The forgetful functor from the category $\text{Comod} - H$ have a left adjoint

$$k - \text{Mod} \rightarrow \text{Comod} - H \quad V \mapsto V \otimes_k H$$

which endomorphism algebra coincides with the algebra of endomorphisms of comodule H .

Thus the endomorphism algebra $\text{End}((f^*)^{\otimes n})$ is isomorphic to the centralizer of $\text{End}^H(H)^{\otimes n}$ in the algebra $\text{End}^F(H^{\otimes n})$.

It is follows from the direct checking that the element α of this centralizer is

defined by the map $p_\alpha = (\varepsilon \otimes \dots \otimes \varepsilon)f \in \text{Hom}(H^{\otimes n})$. It is easy to see that the composition of the endomorphisms α and β from $C_{\text{End}_F(H^{\otimes n})}(\text{End}_H(H)^{\otimes n})$ corresponds to the convolution of p_α and p_β . Finally, F -polylinearity of α means that p_α lies in the centralizer of the image of diagonal embedding $\delta(\text{Hom}(F, k))$ in $\text{Hom}(H^{\otimes n}, k)$.

For a *cocommutative* bialgebra H (cocommutativity means that $t\Delta = \Delta$, where t is a permutation of factors) the cobar complex is co-commutative. In particular, the cohomology $h^n(F)$ of forgetful functor $F: \text{Comod} - H \rightarrow K - \text{Mod}$ coincides with the cohomology of identity functor $h^n(\text{Comod} - H)$ and is well defined for any n .

This cohomology was considered by Sweedler [25]. He proved that in the case of group bialgebra of the group G the cohomology of forgetful functor is isomorphic to the group cohomology of G with coefficients in the (trivial G -module) of invertible elements of the group ring k

$$h^n(F_{\text{Comod} - k[G]}) \simeq H^n(G, k^*)$$

and in the case of universal enveloping algebra $U[\mathfrak{g}]$ of the Lie algebra \mathfrak{g} the cohomology of forgetful functor coincides with the cohomology of Lie bialgebra \mathfrak{g} with coefficients in the group ring k

$$h^n(F_{\text{Comod} - U[\mathfrak{g}]}) \simeq H^n(\mathfrak{g}, k).$$

Let us note that for the category of comodules over group algebra of the group G the homomorphism from the cohomology of the forgetful functor to the Hochschild cohomology of K -theory

$$H^n(G, k^*) = h^n(F_{\text{Comod} - k[G]}) \rightarrow HH^n(K_0(\text{Comod} - k[G]), K_0(k - \text{Mod})) = H^n(G, k^*)$$

is identical.

It follows from the fact that the category $\text{Comod} - k[G]$ (the category of G -graded vector spaces) is a semisimple and all its simple objects are invertible (and parametrized by G).

9 free tensor category. unitary R-matrices

Let now k is a field. The free k -linear (abelian) tensor category \mathcal{T}_k [4] is a cartesian product

$$\times_{n \geq 0} k[S_n] - \text{mod}$$

of the categories of finite-dimensional representations of the symmetric groups S_n .

For the field of characteristic zero \mathcal{T}_k is a semisimple category whose simple

of the categories of finite-dimensional representations of the symmetric groups S_n .

For the field of characteristic zero \mathcal{T}_k is a semisimple category whose simple objects correspond to the partitions. We will denote by $S^\mu X$ the simple object of \mathcal{T}_k labeled by the partition μ . Specifically, the object $1 = S^{(0)}X$ corresponding to the (trivial) partition of 0 is a unit object in \mathcal{T}_k and the object $X = S^{(1)}X$ which corresponds to the (trivial) partition of 1 is a tensor generator in \mathcal{T}_k (any object of \mathcal{T}_k is a direct summand of some sum of tensor powers of X). Note that

$$\text{Hom}_{\mathcal{T}}(X^{\otimes n}, X^{\otimes m}) = \begin{cases} k[S_n], & n = m \\ 0, & n \neq m \end{cases}$$

It was pointed by Yu.I. Manin [13] that a monoidal functor from the category \mathcal{T}_k to the category of k -vector spaces $k\text{-Mod}$ is nothing as a unitary solution of the *quantum Yang-Baxter equation* (*unitary quantum R-matrix*), i.e. an automorphism $R \in \text{Aut}(V^{\otimes 2})$ of tensor square of some vector space V such that

$$R_1 R_2 R_1 = R_2 R_1 R_2, \quad R^2 = 1,$$

where $R_1 = R \otimes I_V$ and $R_2 = I_V \otimes R$ are an automorphisms of tensor cube of V .

Indeed, the value $F(\tau)$ of the monoidal functor F on the automorphism $\tau \in \text{Aut}(X^{\otimes 2})$ (the generator of S_2) is a unitary quantum R -matrix on the vector space $V = F(X)$. The condition $R^2 = 1$ is obvious and quantum Yang-Baxter equation follows from Coxeter relation in S_3 . The pair $(F(X), F(\tau))$ is defined the functor F because the category \mathcal{T}_k is generated by the object X and the morphism τ .

It is natural to ask when the functors F_R and F_S , corresponding to the quantum R -matrices R , and S are isomorphic as functors (are twisted forms of each other as monoidal functors).

It is easy to see that two additive functors F and G between semisimple categories \mathcal{A} and \mathcal{B} are isomorphic if and only if they induces the same homomorphism between Grothendieck groups

$$K_0(F) = K_0(G) : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B}).$$

In the case of characteristic zero the Grothendieck ring of the free tensor category \mathcal{T}_k coincides with the free λ -ring (generating by one element $x = [X]$) and is isomorphic (as a ring) to the ring of polynomials of infinitely many variables [12]

$$K_0(\mathcal{T}_k) = \mathbb{Z}[x, \lambda^2 x, \lambda^3 x, \dots],$$

where $\lambda^n x = [A^n X]$ is a class of the simple object $A^n X$ corresponding to the non-trivial one-dimensional representation of S_n . The homomorphism f from $K_0(\mathcal{T}_k)$ to $\mathbb{Z} = K_0(k\text{-mod})$ is defined by its values $f(\lambda^n x)$ or by the formal series $H_f(t) = \sum_{n \geq 0} f(\lambda^n x) t^n$ (*Hilbert series* of f). This term comes from the fact

that $\Lambda^* X = \bigoplus_{n \geq 0} \Lambda^n X$ is a graded algebra in (appropriate extension of) the category \mathcal{T}_k (*exterior algebra of X*). For any monoidal functor $F : \mathcal{T}_k \rightarrow k\text{-mod}$ $F(\Lambda^* X)$ is a graded algebra and $H_{K_0(F)}(t)$ coincides with its Hilbert series.

Proposition 9 *Two R -matrices R and S (over the field k of characteristic zero) defines isomorphic monoidal functors*

$$F_R, F_S : \mathcal{T}_k \rightarrow k\text{-mod}$$

if and only if its Hilbert series $H_R(t) = H_{F_R}(t)$, $H_S(t) = H_{F_S}(t)$ coincides.

The functor $F : \mathcal{G} \rightarrow \mathcal{H}$ between monoidal categories is called *quasimonoidal* if it preserves the tensor product

$$F(X \otimes Y) \simeq F(X) \otimes F(Y), \quad \text{for any } X, Y \in \mathcal{G}.$$

For the quasimonoidal functor $F : \mathcal{T}_k \rightarrow k\text{-mod}$ the nonnegativity of values of $K_0(F)$ on the classes of simple objects $S^\mu X$ implies some conditions on the Hilbert series $H_{K_0(F)}(t)$.

Proposition 10 *Let k is an algebraically closed field of characteristic zero. The homomorphism $f : K_0(\mathcal{T}_k) \rightarrow \mathbb{Z}$ is a class $K_0(F)$ of some quasimonoidal functor $F : \mathcal{T}_k \rightarrow k\text{-mod}$ if and only if there are real positive a_i, b_j , $i = 1, \dots, n$, $j = 1, \dots, m$ such that*

$$H_f(t) = \frac{\prod_{i=0}^n (1 + a_i t)}{\prod_{i=0}^m (1 - b_i t)}.$$

Proof:

It is not hard to verify that the homomorphism $F : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$ between Grothendieck rings of semisimple k -linear monoidal categories with finite-dimensional spaces of morphisms is a class of some quasimonoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$ iff the values of f on the classes of simple objects of \mathcal{A} are non-negative (has non-negative coefficients in expressions via the classes of simple objects of \mathcal{B}).

The class $s^\mu x$ of the object $S^\mu X$ as an element of $K_0(\mathcal{T}_k)$ can be written as polynomial of $\lambda^n x$. Namely [12, 15]

$$s^\mu x = \det \left(\lambda^{\mu'_i - i + j} x \right)_{1 \leq i, j \leq n},$$

where $\mu' = (\mu'_1, \dots, \mu'_m)$ is a dual partition for partition μ ($\mu'_i = |\{j, \mu_j \geq i\}|$). Hence the non-negativity of values of the homomorphism $f : K_0(\mathcal{T}_k) \rightarrow \mathbb{Z}$ on $s^\mu x$ for all partitions μ means total positivity of the sequence $\{f(\lambda^n x)\}$ (by definition *total positivity* [10] of the sequence $\{a_n\}$ is non-negativity of all finite minors of the infinite Tepliz matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is known [10] that the generating function $\sum_{n \geq 0} a_n$ of the total positive sequence $\{a_n\}$ have the form

$$e^{ct} \frac{\prod_{i=0}^{\infty} (1 + a_i t)}{\prod_{i=0}^{\infty} (1 - b_i t)},$$

for some real positive a_i, b_j, c such that $\sum_i a_i + b_j < \infty$.

Hence the generating function is meromorphic in $\{t, |t| \leq 1\}$ and has finitely many zeros and poles in this circle. One theorem of Salem [23] implies that this function is rational.

The Hilbert series of R -matrix can be expressed in terms of some its symbolic invariants.

Proposition 11 [14] *Consider the series*

$$\Psi_R(t) = (\dim V)t + \sum_{n \geq 2} \text{tr}(R_1 \dots R_{n-1})t^n$$

for the R -matrix R . Then

$$H_R(t) = \exp\left(\int \frac{\Psi_R(-t)}{t} dt\right)$$

Proof:

The object $\Lambda^n X$ is a image of the idempotent

$$P_{\Lambda^n X} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma.$$

Hence the dimension of $F_R(\Lambda^n X)$ can be expressed in terms of the traces of some products of R -matrix R

$$\dim F_R(\Lambda^n X) = \text{Tr}(F_R(P_{\Lambda^n X})) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \text{Tr}(R_\sigma),$$

where $R_\sigma = F_R(\sigma)$. Since any permutation σ is similar to the product of independent cycles $\tau_1 \dots \tau_k$ (of length l_1, \dots, l_k respectively), then

$$\text{Tr}(R_\sigma) = \psi^{l_1} \dots \psi^{l_k}, \quad \text{where } \psi^l_R = \text{tr}(R_1 \dots R_{l-1}) \quad (\dim V, l = 1).$$

The number of permutations in S_n which has the cyclic structure corresponding to the partition $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}_n$ of the number n ($\sum i \mu_i = n$) equals $\frac{n!}{\prod_i \mu_i i^{\mu_i}}$. The sign of these partitions is $(-1)^{n+m}$, where $m = \sum \mu_i$. Hence

$$\text{Tr}(F_R(P_{\Lambda^n X})) = \sum_{\mu \in \mathcal{P}_n} (-1)^n \prod_i \left(\frac{-\psi^l_R}{i}\right)^{\mu_i} \frac{1}{\mu_i!}$$

and

$$H_R(t) = \sum_{n \geq 0} (-t)^n \sum_{\mu \in \mathcal{P}_n} \prod_i \left(\frac{-\psi_R^i}{i} \right)^{\mu_i} \frac{1}{\mu_i!}.$$

On the other side,

$$\begin{aligned} \exp \left(\int \frac{\Psi_R(-t)}{t} dt \right) &= \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{i \geq 0} (-t)^i \frac{\psi_R^i}{i} \right)^n = \\ &= \sum_{\mu \in \mathcal{P}_n} (-1)^n \prod_i \left(\frac{-\psi_R^i}{i} \right)^{\mu_i} \frac{1}{\mu_i!}. \end{aligned}$$

Example 1 If R is a unitary R -matrix, then $S = -R$ is also unitary R -matrix. Since

$$\text{tr}(S_1 \dots S_{n-1}) = (-1)^{n-1} \text{tr}(R_1 \dots R_{n-1}),$$

then

$$\Psi_S(t) = \dim V t - \sum_{n \geq 2} \text{tr}(R_1 \dots R_{n-1}) (-t)^n = -\Psi_R(-t)$$

and

$$H_S(t) = \exp \left(\int \frac{\Psi_S(-t)}{t} dt \right) = \exp \left(- \int \frac{\Psi_R(t)}{t} dt \right) = H_R(-t)^{-1}.$$

So the set of R -matrices with Hilbert series $H(t)$ is isomorphic to the set of R -matrices with Hilbert series $H(-t)^{-1}$.

Example 2 Let A be a commutative algebra, $a \in A \otimes A$ an element such that $at(a) = 1$ ($t : A \otimes A \rightarrow A \otimes A$ is a permutation of the factors) and M a A -module. Then

$$R = R(A, a, M) = L(a)t \in \text{Aut}(M \otimes M)$$

is a unitary R -matrix on the vector space M (here $L(a)$ is a operator of left multiplying by a). Indeed,

$$\begin{aligned} R_1 R_2 R_1 &= L(a_{12})t_1 L(a_{23})t_2 L(a_{12})t_1 = \\ &= L(a_{12}t_1(a_{23})t_1 t_2(a_{12}))t_1 t_2 t_1 = L(a_{12}a_{13}a_{23})t_1 t_2 t_1. \end{aligned}$$

On the other hand

$$R_2 R_1 R_2 = L(a_{23})t_2 L(a_{12})t_1 L(a_{23})t_2 =$$

$$L(a_{23}t_2(a_{12})t_2t_1(a_{23}))t_2t_1t_2 = L(a_{23}a_{13}a_{12})t_2t_1t_2.$$

Hence the quantum Yang-Baxter equation for R follows from commutativity of $A^{\otimes 3}$ and Coxeter relation for the permutation t . The unitarity of R follows from the condition $at(a) = 1$.

Define some invariant of elements $a \in A \otimes A$ for which $at(a) = 1$. Since algebra A is commutative, then the multiplication $\mu : A \otimes A \rightarrow A$ is a homomorphism of algebras and

$$1 = \mu(at(a)) = \mu(a)\mu(t(a)) = (\mu(a))^2.$$

So $\mu(a) = \pm 1$.

Now we can calculate the Hilbert series of R . Since

$$R_1 \dots R_{n-1} = L(a_{12})t_1 \dots L(a_{n-1n})t_{n-1} =$$

$$L(a_{12}t_1(a_{23}) \dots t_1 \dots t_{n-1}(a_{n-1n}))t_1 \dots t_{n-1} = L(a_{12} \dots a_{1n})t_1 \dots t_{n-1},$$

then

$$\text{tr}(R_1 \dots R_{n-1}) = \text{tr}(L(a_{12} \dots a_{1n})t_1 \dots t_{n-1}) =$$

$$\text{tr}(\mu(a_{12} \dots a_{1n})t_1 \dots t_{n-1}) = (\mu(a))^{n-1} \dim M.$$

Hence

$$\Psi_R(t) = \dim M \sum_{n \geq 1} (\mu(a))^{n-1} (-t)^n = \frac{\dim M t}{1 - \mu(a)t}$$

and

$$H_R(t) = \exp\left(\int \frac{\dim M}{1 - \mu(a)t} dt\right) = (1 + \mu(a)t)^{\mu(a) \dim M}.$$

10 free tensor category. Hecke algebras

This section is devoted to the investigation of the monoidal structures on the category \mathcal{T}_k .

We begin with the classification of (quasi)monoidal autoequivalences of the category \mathcal{T}_k .

Proposition 12 *Let f is a graded homomorphism $K_0(\mathcal{T}_k) \rightarrow K_0(\mathcal{T}_k)$ for which*

$$f(x) = x, \quad f(\lambda^2 x) = \lambda^2 x, \quad \text{and}$$

$$f(s^\mu x) = \sum n_{\mu, \nu} s^\nu x, \quad \text{where } n_{\mu, \nu} \geq 0.$$

Then f is identical.

Proof:

Let us prove the lemma by the induction on the degree.

Suppose that $f(s^\mu x) = s^\mu x$ for any partition $\mu \in \mathcal{P}_m$ where $m \leq n$.

To prove that $f(s^\mu x) = s^\mu x$ for $\mu \in \mathcal{P}_{n+1}$ it is sufficient to show that $f(\lambda^{n+1} x) = \lambda^{n+1} x$. It follows from the partial case of the Littlewood-Richardson formula

$$x \cdot s^\mu x = \sum_{\mu \subset \nu, |\nu \setminus \mu|=1} s^\nu x$$

that

$$f(\lambda^{n+1} x) + f(s^{(n,1)} x) = f(x \cdot \lambda^n x) = x \cdot \lambda^n x = \lambda^{n+1} x + s^{(n,1)} x.$$

The summand $f(s^{(n,1)} x)$ can not equal to zero, because in this case

$$s^2 x \cdot \lambda^n x = f(s^2 x) f(\lambda^n x) = f(s^2 x \cdot \lambda^n x) = f\left(\sum_{\nu \subset (n,1)} s^\nu x\right) = 0.$$

By another side the expression of $f(s^{(n,1)} x)$ can not contain $\lambda^{n+1} x$, because the right (and hence left) side of the equality

$$\begin{aligned} f(s^{(n,1)} x) + f(s^{(n-1,1,1)} x) + f(s^{(n-1,2)} x) &= f(x \cdot s^{(n-1,1)} x) = x \cdot s^{(n-1,1)} x = \\ &= s^{(n,1)} x + s^{(n-1,1,1)} x + s^{(n-1,2)} x \end{aligned}$$

not contain it.

Hence $f(s^{(n,1)} x) = s^{(n,1)} x$ and $f(\lambda^{n+1} x) = \lambda^{n+1} x$.

Using the previous proposition we can describe the (quasi)monoidal autoequivalences of \mathcal{T}_k .

Proposition 13 *There is unique non-identical (quasi)monoidal autoequivalence F of the category \mathcal{T}_k which is defined by the setting*

$$F(X) = X, \quad F(\tau) = -\tau.$$

Proof:

Let G is a quasimonoidal autoequivalence of the category \mathcal{T}_k .

Firstly let us show that the automorphism $K_0(G)$ of the Grothendieck ring $K_0(\mathcal{T}_k)$ either identical or coincides with $K_0(F)$ which sends $s^\mu x$ to $s^{\mu'} x$ (where μ' is a dual partition to μ).

It is easy to see that the quasimonoidal autoequivalence must send the simple object to the simple. In particular, $G(X) = X$, since X is unique object of \mathcal{T}_k whose square is a sum of two simple objects. For $G(\Lambda^2 X)$ there are two possibilities $\Lambda^2 X$ and $S^2 X$. If $G(\Lambda^2 X) = S^2 X$ we may replace G by FG , so we can assume that $G(\Lambda^2 X) = \Lambda^2 X$. Hence $K_0(G)$ is identity by the previous proposition.

The case of monoidal autoequivalence is more easy. From the previous consideration follows that such autoequivalens sends the generator X to itself and the direct checking shows that there are only two solutions in $End(X^{\otimes 2})$ of the equations

$$t^2 = 1, \quad t_1 t_2 t_1 = t_2 t_1 t_2.$$

Namely, the standart $t = \tau$ and $t = -\tau$.

Now we will construct of the map

$$a : Ms(\mathcal{T}_k) \rightarrow k.$$

Let ψ some associativity constraint of the category \mathcal{T}_k . Consider two homomorphisms of algebras

$$f_1, f_2 : End_{\mathcal{T}}(X^{\otimes 2}) \rightarrow End_{\mathcal{T}}(X^{\otimes 3}), \quad f_1(g) = g \otimes I, \quad f_2(g) = \psi^{-1}(I \otimes g)\psi,$$

here $X^{\otimes 3}$ means $X \otimes (X \otimes X)$.

As an algebras

$$End_{\mathcal{T}}(X^{\otimes 2}) = End_{\mathcal{T}}(\Lambda^2 X) \oplus End_{\mathcal{T}}(S^2 X) \simeq k \oplus k$$

and

$$End_{\mathcal{T}}(X^{\otimes 3}) = End_{\mathcal{T}}(\Lambda^3 X) \oplus End_{\mathcal{T}}(S^{(2,1)} X) \oplus End_{\mathcal{T}}(S^3 X) \simeq k \oplus M_2(k) \oplus k,$$

here $(2, 1)$ is a partition of the number 3.

Since

$$\Lambda^{\otimes 2} X \otimes X \simeq \Lambda^{\otimes 3} X \oplus S^{(2,1)} X$$

the images $p_i = f_i(p)$ of the projector $p \in End_{\mathcal{T}}(X^{\otimes 2})$ on the $\Lambda^{\otimes 2} X$ have the decompositions $(1, P_i, 0) \in k \oplus M_2(k) \oplus k$. In addition the projectors $P_i \in M_2(k)$ has rank 1, hence

$$P_1 P_2 P_1 = a P_1, \quad P_2 P_1 P_2 = a P_2 \quad \text{for some } a = a(\psi) \in k.$$

Example 3 A Hecke algebra $H_n(q)$, $q \in k$ is an algebra with generators t_i , $i = 1, \dots, n-1$ and defining relations

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad i = 1, \dots, n-2,$$

$$t_i t_j = t_j t_i, \quad |i - j| > 2,$$

$$(t_i + 1)(t_i - q) = 0, \quad i = 1, \dots, n-1.$$

It is known [3, 17] that in the case when q is not a root of unity the algebras $H_n(q)$ are semisimple for any n and its irreducible representations are

parametrized by the partitions.

A free heckian category generated by one object $\mathcal{T}_{k,q}$ is a cartesian product

$$\mathcal{T}_{k,q} = \times_{n \geq 0} H_n(q) - mod,$$

(here $H_0(q) = H_1(q) = k$) with the tensor product, which is induced by the homomorphisms of algebras

$$H_n(q) \otimes H_m(q) \rightarrow H_{n+m}(q), \quad t_i \otimes 1 \mapsto t_i, \quad 1 \otimes t_i \mapsto t_{n+i}.$$

We will denote by $1, X$ the objects of $\mathcal{T}_{k,q}$ which corresponds to the unique one-dimensional representations of $H_0(q), H_1(q)$ and by $\Lambda_q^n X$ the objects corresponding to the one-dimensional representations of $H_n(q)$ which sends t_i to -1 . The term free heckian category is explained by the fact that the category $\mathcal{T}_{k,q}$ is generated (as a monoidal category) by the object X and the automorphism $t \in \text{End}_{\mathcal{T}_q}(X^{\otimes 2})$ such that $(t+1)(t-q) = 0$ and $t_1 t_2 t_1 = t_2 t_1 t_2$, where as usually $t_1 = t \otimes 1, t_2 = 1 \otimes t \in \text{End}_{\mathcal{T}_q}(X^{\otimes 3})$.

It is known that (for the case of characteristic zero) the Littlewood-Richardson coefficients of the category $\mathcal{T}_{k,q}$ coincides with the Littlewood-Richardson coefficients of the category \mathcal{T}_k . Hence the monoidal category $\mathcal{T}_{k,q}$ is the category \mathcal{T}_k with another associativity constraint φ_q .

Let us note [3, 17] that $\Lambda_q^n X$ is a image of the projector

$$p_{\Lambda^n X} = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} t_\sigma,$$

(here $l(\sigma) = m$ and $t_\sigma = t_{i_1} \dots t_{i_m}$, if $\sigma = \tau_{i_1} \dots \tau_{i_m}$ is an uncancelled decomposition of the permutation σ in the product of Coxeter generators). Using this fact it is not hard to verify that $a(\varphi_q) = \frac{q}{(q+1)^2}$.

Example 4 The *crystal* [11] is a set B with the maps

$$\tilde{f}_i, \tilde{e}_i : B \rightarrow B \cup \{0\},$$

where $i \in \mathbb{N}$, for which

$$\tilde{f}_i(u) = v \Leftrightarrow \tilde{e}_i(v) = u \quad \text{for any } u, v \in B,$$

and the functions

$$\phi_i(u) = \max\{k \geq 0, \tilde{f}_i^k(v) \neq 0\},$$

$$\varepsilon_i(u) = \max\{k \geq 0, \tilde{e}_i^k(v) \neq 0\}$$

has finite value for any $u \in B$.

The morphism f of crystals B_1 and B_2 is a map of the sets $f : B_1 \rightarrow B_2 \cup \{0\}$ such that

$$f \tilde{f}_i = \tilde{f}_i f, \quad f \tilde{e}_i = \tilde{e}_i f \quad \forall i.$$

The category of crystals will be denoted by *Crystals*.

A tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is the set $B_1 \times B_2$ with the maps

$$\begin{aligned}\tilde{f}_i(u \otimes v) &= \begin{cases} \tilde{f}_i(u) \otimes v, & \phi_i(u) > \varepsilon_i(v) \\ u \otimes \tilde{f}_i(v), & \phi_i(u) \leq \varepsilon_i(v) \end{cases} \\ \tilde{e}_i(u \otimes v) &= \begin{cases} \tilde{e}_i(u) \otimes v, & \phi_i(u) \leq \varepsilon_i(v) \\ u \otimes \tilde{e}_i(v), & \phi_i(u) < \varepsilon_i(v) \end{cases}\end{aligned}$$

It is easy to see that the tensor product of two crystals is also a crystal and that the category *Crystals* is monoidal with identical associativity constraint.

The element $b \in B$ is called *highest weight* if $\tilde{e}_i(b) = 0 \quad \forall i$. The set of highest weight elements of the crystal B will be denoted by B^h . It follows from definition of the tensor product that

$$(B_1 \otimes B_2)^h \subseteq B_1^h \otimes B_2.$$

Let us consider the crystal $X = \{x_i, i \in \mathbb{N}\}$ with the maps

$$\tilde{e}_i(x_j) = \delta_{i,j-1} x_{j-1} \quad \tilde{f}_i(x_j) = \delta_{i,j+1} x_{j+1}.$$

It is easy to verify that any connected component of $X^{\otimes n}$ contains only one highest weight element and two components B_1 and B_2 are isomorphic iff

$$\phi(b_1) = \phi(b_2) \quad \forall i, \text{ where } B_1^h = \{b_1\}, B_2^h = \{b_2\}.$$

It can be proved by the induction that the sequence $\{\phi(b)\}$, where $b \in (X^{\otimes n})^h$, satisfies to the condition $\sum_i i\phi(b) = n$.

In other words indecomposable objects of the monoidal subcategory \mathcal{T}_0 in *Crystals* generated by the object X are parametrized by the partitions. It follows from the results of [18] that the Littelwood-Richardson coefficients of tensor product in \mathcal{T}_0 coincides with the standart.

The k -linear envelope $Crystals_k$ of the category of crystals is a category with the same objects and whose morphisms are (finite) k -linear combinations of the morphisms of *Crystals*

$$Hom_{Crystals_k}(B_1, B_2) = \langle Hom_{Crystals}(B_1, B_2) \rangle_k.$$

The category $Crystals_k$ is a semisimple monoidal category. In particular, the k -linear envelope $\mathcal{T}_{k,0}$ of the subcategory \mathcal{T}_0 in *Crystals* is a category \mathcal{T}_k with another associativity constraint φ_0 .

It can be verified directly, using the identifications

$$\Lambda_0^n X = \{x_{i_1} \otimes \dots \otimes x_{i_n} \in X^{\otimes n}, i_1 < \dots < i_n\},$$

that $a(\varphi_0) = 0$.

Proposition 14 *Let k is a field of characteristic zero. The fibres of the map*

$$a : Ms(\mathcal{T}_k) \rightarrow k$$

over $k^ \setminus \{ \frac{1}{2(\cos(\alpha)+1)}, \alpha \in \mathbb{Q}^* \}$ consists of one point.*

Proof:

Let ψ is a associativity constraint for the category \mathcal{T}_k .

The direct checking shows that the endomorphism $t = q1 - (q+1)p \in \text{End}_{\mathcal{T}}(X^{\otimes 2})$, where $\frac{q}{(q+1)^2} = a$ and p is a projector over $\Lambda^2 X$, satisfies to the equations

$$(t+1)(t-q) = 0, \quad t_1 t_2 t_1 = t_2 t_1 t_2.$$

Indeed, the first equation follows from the condition $p^2 = p$ and the second from $p_1 p_2 p_1 = a p_1$, $p_2 p_1 p_2 = a p_2$.

Using the freedom property of the category $\mathcal{T}_{k,q}$ we can define the monoidal functor

$$F_q : \mathcal{T}_{k,q} \rightarrow (\mathcal{T}_k, \psi),$$

which sends the generator of the Hecke algebra $H_2(q)$ to the endomorphism t . This functor sends the objects X and $\Lambda_q^2 X$ of $\mathcal{T}_{k,q}$ to X and $\Lambda^2 X$ respectively.

If $a(\psi) = a \in k^* \setminus \{ \frac{1}{2(\cos(\alpha)+1)}, \alpha \in \mathbb{Q}^* \}$, then q is not a nontrivial root of unity and the Grothendieck ring $K_0(\mathcal{T}_{k,q})$ coincides with $K_0(\mathcal{T}_k)$. The functor F_q induces the homomorphism $K_0(\mathcal{T}_{k,q}) \rightarrow K_0(\mathcal{T}_k)$, which preserves x and $\lambda^2 x$. Hence it is an equivalence by the proposition 12.

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