Further Results on Supernumary Polylogarithmic Ladders

L. Lewin

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/91- 58

Further Results on Supernumary Polylogarithmic Ladders

Abstract

With the help of the PARI computer program a number of matters left unresolved from previous work have now been settled. In particular, ladders of the base ψ (see [1]) have been extended to the sixth order, and involve a new index, 60, found by the PARI program. The base from $(p, q) = (11, 7)$ has an additional index 20, and this combines with earlier results to produce a valid ladder. The apparent "barren" feature of certain equations is now explained in terms of a need to work with a sufficient number of results. It is confirmed that the equation with $(p, q) = (5, 3)$ does not indeed possess any supernumary results.

A complete investigation of the Salem number from an equation of the fourth degree is given: it possesses results to the $8th$ order. An introduction is given to similar, ongoing, studies for the smallest Salem number, which conjecturally extends to the $16th$ order.

Some ladder results for combined bases are found, with one such formula deducible from a three-variable dilogarithmic functional equation.

1. Introduction

The concept of supernumary ladders was first introduced in [1] in relation to the base equation $u^p + u^q = 1$, which, for general integral values of p and q, possesses 8 generic cyclotomic equations, leading to six dilogarithmic ladders and finishing with a single formula at the fifth order. For the many details the reader is referred to the original paper. However, there are particular values of p and q for which additional cyclotomic equations can be found, and which, in some cases, at least, lead to valid ladders that are not part of the generic set; it is these that are referred to as "supernumary". A number of features were left unresolved in the reference paper. They have now been further studied and are reported on here, together with additional results on Salem numbers and on combined bases.

2. Ladders for the Base ψ

2.1 This base satisfies the equation

$$\psi^4 + \psi^3 = 1 \tag{1}$$

and in [1] two supernumary cyclotomic equations of indices $N = 30$ and 42 were found. The former gave valid ladders as far as the $4th$ order and the latter to the second only. The reason that it went no further has now been traced to a programming error, and when this was rectified it, too, extended to the $4th$ order: and when combined with the other supernumary result of index 30, the two between them were found to extend to the fifth order. This is not enough, however, to yield anything at $n = 6$; a further result is needed. Using a method due to Zagier [2]a further cyclotomic equation, of index 60, has been located, and this, in conjunction with the others, extended to give a unique result at the sixth order. This augments the number of such bases that give valid ladders beyond $n = 5$ (the limit of Kummer's equations) from 3 to 5, since the additional base ϕ, satisfying $\phi^4 + \phi = 1$, gives closely related formulas. And, as reported later, some Salem numbers do so too.

2.2 From (5.14) of [1], define

$$L_n(42,\psi) = li\,(n,42) - li\,(n,21) - li\,(n,14) + li\,(n,7) - 2\,li\,(n,6) + 2\,li\,(n,2) - li\,(n,1) \tag{2}$$

with the shorthand notation

$$li\,(n,N) = Li_n\left(\psi^N\right)/N^{n-1} \tag{3}$$

Then, from (5.16), $L_n^{(2)}(42,\psi) = 0$ when $n = 2$, where

$$L_n^{(2)}(42,\psi) = 42\,L_n(42,\psi) - 37\,L_n(2,\psi) - 8\,\zeta\,(2)\,l\,(n-2) \tag{4}$$

with

$$l\,(n) = \log^n(\psi)/n! \tag{5}$$

At $n = 3$ the combination

$$L_n^{(3)}(42,\psi) = 14\,L_n^{(2)}(42,\psi) - 70\,L_n^{(2)}(7,\psi) + 109\,L_n^{(2)}(3,\psi) \tag{6}$$

is found numerically (to 50 decimal accuracy) to equal $9\,\zeta\,(3)$. A combination, free of $\zeta\,(3)$, of the same nature as (5.39) to (5.42) of [1] (to which the reader is referred for notation) is

$$L_n^{(4)}(42,\psi) = 42\left\{14\,L_n^{(2)}(42,\psi) - 70\,L_n^{(2)}(7,\psi) - 36\,L_n^{(2)}(4,\psi) + 82\,L_n^{(2)}(3,\psi)\right\} \tag{7}$$

and this, by construction, is zero at $n = 3$. At $n = 4$ it is found that

$$L_n^{(5)}(42,\psi) = L_n^{(4)}(42,\psi) + 42\,L_n^{(4)}(10,\psi) - \frac{1960}{3}\zeta\,(4)\,l\,(n-4) \tag{8}$$

is zero. A combination, valid at $n = 5$, is found as

$$-91\,L_5^{(5)}(42,\psi) + 824\,L_5^{(5)}(30,\psi) + 3594\;L_5^{(5)}(18,\psi) = 6948\,\zeta\,(5) \tag{9}$$

This is as far as the results can be taken without formulas with a new index.

2.3 The following cyclotomic equation was found by Zagier's method, and is readily confirmed by algebraic manipulation of the base equation (1). (It is the only additional cyclotomic equation satisfied by ψ)

$$1 - \psi^{60} = \left(1-\psi^{20}\right)\left(1-\psi^{15}\right)^3\left(1-\psi^{12}\right)\left(1-\psi^{5}\right)^{-3}\left(1-\psi^{2}\right) \tag{10}$$

and leads to constructing the ladder

$$L_n(60,\psi) = li\,(n,60) - li\,(n,20) - 3\,li\,(n,15) - li\,(n,12) + 3\,li\,(n,5) - li\,(n,2) \tag{11}$$

Define

$$L_n^{(2)}(60,\psi) = 60\,L_n(60,\psi) + 13\,L_n(2,\psi) + 13\,\zeta\,(2)\,l\,(n-2) \tag{12}$$

Then it is found numerically that this is zero at $n = 2$. A combination free of $\zeta\,(3)$ and zero at $n = 3$ is found as

$$L_n^{(4)}(60,\psi) = 60\,\left\{20\,L_n^{(2)}(60,\psi) + 140\,L_n^{(2)}(7,\psi) + 88\,L_n^{(2)}(4,\psi) + 67\,L_n^{(2)}(3,\psi)\right\} \tag{13}$$

Define

$$L_n^{(5)}(60,\psi) = L_n^{(4)}(60,\psi) + 232\,L_n^{(4)}(10,\psi) + 1747\,\zeta\,(4)\,l\,(n-4) \tag{14}$$

This expression is zero at $n = 4$, and at $n = 5$ it is found that

$$156\,L_5^{(5)}(60,\psi) + 1104\,L_5^{(5)}(30,\psi) + 17892\,L_5^{(5)}(18,\psi) = 28747\,\zeta\,(5) \tag{15}$$

Along with the original (5.46) of [1], two expressions clear of $\zeta(5)$ can now be constructed:

$$\begin{aligned} L_n^{(6)}(42,\psi) =& 637\,L_n^{(5)}(42,\psi) - 5768\,L_n^{(5)}(30,\psi) + 41688\;L_n^{(5)}(24,\psi) \\ &- 9525\,L_n^{(5)}(18,\psi) \end{aligned} \tag{16}$$

and

$$\begin{aligned} L_n^{(6)}(60,\psi) =& 1456\,L_n^{(5)}(60,\psi) + 10304\,L_n^{(5)}(30,\psi) - 229976\,L_n^{(5)}(24,\psi) \\ &+ 80751\,L_n^{(5)}(18,\psi) \end{aligned} \tag{17}$$

Both of these are zero at $n = 5$. At $n = 6$ it is then found that

$$69390\,L_6^{(6)}(60,\psi) + 419310\,L_6^{(6)}(42,\psi) + 651115309\,\zeta\,(6) = 0 \tag{18}$$

It may be noticed that the coefficients of the leading terms in (16) and (17), namely 637 and 1456, are both simple multiples of 91. The number 91 itself seems to be an artifact of the precise choice of terms used to construct the ladders; a different choice would lead to a different number. However, the fact that the coefficients of the leading terms contain a substantial common factor seems to be a more general property, and is exhibited strongly in the results for the Salem numbers described later.

3. Ladder Combinations and the Case $x^{11} + x^7 = 1$

3.1 If x satisfies an irreducible equation of degree n with $2n_-$ complex roots and $(n_+ - n_-)$ real roots, then Zagier has indicated that, at the level of the dilogarithm, n_- component-ladders are independent, so that $(n_- + 1)$ are needed to form a valid ladder result. This helps to explain why some of the cylotomic equations in [1] were considered "barren": they needed to be combined with further results to form a valid ladder, and if these are not forthcoming, no results will ensue, supernumary cyclotomic equations notwithstanding. A case in point is the base determined by [1]:

$$x^{11} + x^7 = 1 \tag{19}$$

for which two supernumary and two available generic results exist. Equation (19) factorises as

$$x^{11} + x^7 - 1 = \left(x^2 - x + 1\right)\left(x^9 + x^8 - x^6 + x^4 + x^3 - x - 1\right)$$

and the irreductible $9th$ degree polynomial has one real root and four complex pairs. Thus $(n_- + 1) = 5$, so an additional relation is needed to produce a valid result.

3.2 A new cyclotomic equation of index 20 has now been generated, making possible a test of the above assertion. It is found that

$$1 - x^{20} = (1 - x^{10})(1 - x^4)^2(1 - x^2)(1 - x^3)^{-1}(1 - x)^{-1}x^{-10} \tag{21}$$

with a corresponding dilogarithmic component-ladder:

$$L\,(20) = li\,(20) - li\,(10) \; - 2\,li\,(4) + li\,(3) - li\,(2) + li\,(1) - 5\,\log^2(x) \tag{22}$$

where $li\,(N) = Li_2(x^N)/N$. The remaining equations were found in [1] and lead to defining

$$L\,(10) = li\,(10) - li\,(5) + li\,(3) + li\,(2) - li\,(1) + (5/2)\log^2 x \tag{23}$$

$$L\,(9) = li\,(9) + li\,(3) - li\,(2) + 2\,\log^2 x \tag{24}$$

$$L\,(7) = li\,(7) + (11/2)\log^2 x \tag{25a}$$

$$L\,(30) = li\,(30) - li\,(15) - li\,(6) + li\,(3) + 2\,\log^2 x \tag{25b}$$

Of these the last two are generic and the others are supernumary. With the required five relations the resulting valid ladder can now be readily found:

$$15\,L\,(30) - 10\,L\,(20) - 5\,L\,(10) + 9\,L\,(9) - 14\,L\,(7) + 2\,\zeta\,(2) = 0 \tag{26}$$

Since, according to Zagier, $(n_+ + n_-)$ further results are needed to reach a valid result at the third order, these formulas cannot be expected to extend beyond the dilogarithm.

3.3. The new cyclotomic equation (21) of index 20 was generated by Zagier's program, which is, in principle, exhaustive. The same program was applied to the equation $x^5 + x^3 = 1$, and it confirmed that, for this base, no supernumary results exist, though several were found in [1] for $p = 5,\; q = 1,\; 2$ and 4.

4. Smallest Degree Salem Number

4.1 The Salem numbers have their complex roots on the unit circle, and have been extensively studied by Boyd [3].

The smallest degree is given by the quartic

$$\xi^4 - \xi^3 - \xi^2 - \xi + 1 = 0 \tag{27}$$

with real root in $(0, 1)$

$$x = \frac{\sqrt{13} + 1 - \sqrt{2\sqrt{13} - 2}}{4} = .5807..... \tag{28}$$

The remaining real root is $1/x$, and there are two complex roots on the unit circle.

Eleven cyclotomic equations have been located, of indices from 3 to 36 :

$$1 - x^3 = x^2(1-x)^{-1} \tag{29a}$$

$$1 - x^6 = \left(1-x^2\right)^3(1-x)^{-2}x \tag{29b}$$

$$1 - x^8 = \left(1-x^4\right)(1-x)^{-2}x^3 \tag{29c}$$

$$1 - x^{10} = \left(1-x^5\right)^2\left(1-x^2\right)x^{-1} \tag{29d}$$

$$1 - x^{11} = (1-x)^{-5}x^8 \tag{29e}$$

$$1 - x^{12} = \left(1-x^6\right)\left(1-x^4\right)^2\left(1-x^2\right)^{-2}x \tag{29f}$$

$$1 - x^{14} = \left(1-x^7\right)\left(1-x^2\right)(1-x)^{-3}x^4 \tag{29g}$$

$$1 - x^{18} = \left(1-x^9\right)\left(1-x^6\right)^2\left(1-x^3\right)^{-1}\left(1-x^2\right)^{-1}x \tag{29h}$$

$$1 - x^{20} = \left(1-x^{10}\right)\left(1-x^5\right)\left(1-x^4\right)\left(1-x^2\right)^{-1}(1-x)^{-1}x^2 \tag{29i}$$

$$1 - x^{30} = \left(1-x^{15}\right)\left(1-x^5\right)\left(1-x^3\right)\left(1-x^2\right)^3(1-x)^{-3}x^2 \tag{29j}$$

$$1 - x^{36} = \left(1-x^{18}\right)\left(1-x^4\right)^3\left(1-x^3\right)\left(1-x^2\right)^{-3}(1-x)^{-3}x^6 \tag{29k}$$

These are arbitrary, though convenient, combinations of factors. Clearly (29f), for example, could be combined into (29k) to give alternative forms. Doing so would affect the corresponding numbers appearing in the ladders, but would not change their basic character. However, there may be combinations that yield slightly smaller coefficients. In the ensuing printout the leading terms at the fifth order, for example, contain a common factor 7349; conceivably a superior choice at the lower levels could eliminate it, though at the sixth order only a modest factor 13 is common to the leading terms.

4.2 According to Zagier's construction, the ladders should combine in pairs, with $\zeta(2)$, to give $11 - 1 = 10$ independent results at the second order. They should also combine in pairs, with $\zeta(3)$, to give $10 - 1 = 9$ results at the third order, or 8 without $\zeta(3)$– this is a requirement for extension to the fourth order; and so on. The reason that only 2 rather than 3 ($= n_+$) are used up is that the real roots occur in inverse pairs for this symmetrical base equation, so the number of independent real roots is halved, leading to a net loss of only

$(n_+ - n_-)/2 + n_- = (n_+ + n_-)/2 = 2$. This schema is in fact found, and is shown in the following printout, wherein $z\,m \equiv \zeta\,(m)$ and

item 3 defines the modified polylogarithm, along the lines of (3).

item 4 defines a logarithmic term similar to (5)

items 5 to 15 define 11 polylogarithmic ladders $l\,N\,(n)$ of index N, (3 to 36), and order n.

items 16 to 25 define the 10 ladder combinations $m\,N\,(n)$ which, along with $\zeta\,(2)$, are zero at $n = 2$.

items 26 to 33 define the 8 ladder combinations $n\,N\,(n)$ which are absent any $\zeta\,(3)$ term, and are all zero at $n = 3$.

items 34 to 40 define the 7 ladder combinations $p\,N\,(n)$ which, along with $\zeta\,(4)$, are zero at $n = 4$.

items 41 to 45 define the 5 ladder combinations $q\,N\,(n)$ which are absent any $\zeta\,(5)$ term, and are all zero at $n = 5$.

items 46 to 49 define the 4 ladder combinations $rN\,(n)$ which, along with $\zeta\,(6)$, are zero at $n = 6$.

items 50 and 51 define the 2 ladder combinations $s\,N\,(n)$ which are absent any $\zeta\,(7)$ term, and are each zero at $n = 7$.

item 52 defines the sole ladder combination $t\,N\,(n)$ which, along with $\zeta\,(8)$, is zero at $n = 8$.

The arbitrary choice of factors entering the component-ladders notwithstanding, this final result is unique, since the leading terms containing $Li_8(x^{36})$ and $Li_8(x^{30})$ must occur here, with the coefficients as specified. Any common factors have already been eliminated in this result. Thus the 26 digit multiplier of $\zeta\,(8)$, enormous though it is, seems unavoidable. However, it can be written as a product involving the highest common factors of the leading terms of the previous three ladders, namely 7349, 13 and 11033267 to give a net coefficient $13 \times 7349 \times 11033267 \times M$ where

$$M = 16450174632074 \tag{30}$$

This strongly suggests that there may be alternative formulations with more modest coefficients.

```
\precision=100
x=real((roots(y^4-y^3-y^2-y+1))[1]);
li(n,k)=polylog(n,x^k)/k^(n-1)
l(n)=log(x)^n/(n!)
l3(n)=li(n,3)+li(n,1)+2*l(n)
l6(n)=li(n,6)-3*li(n,2)+2*li(n,1)+l(n)
l8(n)=li(n,8)-li(n,4)+2*li(n,1)+3*l(n)
l10(n)=li(n,10)-2*li(n,5)-li(n,2)-l(n)
l11(n)=li(n,11)+5*li(n,1)+8*l(n)
l12(n)=li(n,12)-li(n,6)-2*li(n,4)+2*li(n,2)+l(n)
l14(n)=li(n,14)-li(n,7)-li(n,2)+3*li(n,1)+4*l(n)
l18(n)=li(n,18)-li(n,9)-2*li(n,6)+li(n,3)+li(n,2)+l(n)
l20(n)=li(n,20)-li(n,10)-li(n,5)-li(n,4)+li(n,2)+li(n,1)+2*l(n)
l30(n)=li(n,30)-li(n,15)-li(n,5)-li(n,3)-3*li(n,2)+3*li(n,1)+2*l(n)
l36(n)=li(n,36)-li(n,18)-3*li(n,4)-li(n,3)+3*li(n,2)+3*li(n,1)+6*l(n)
m6(n)=3*l6(n)-9*l3(n)+4*z2*l(n-2)
m8(n)=8*l8(n)-9*l3(n)-3*z2*l(n-2)
m10(n)=10*l10(n)-9*l3(n)+8*z2*l(n-2)
m11(n)=11*l11(n)-36*l3(n)-8*z2*l(n-2)
m12(n)=12*l12(n)+15*l3(n)-13*z2*l(n-2)
m14(n)=7*l14(n)-18*l3(n)+z2*l(n-2)
m18(n)=3*l18(n)+2*l3(n)-2*z2*l(n-2)
m20(n)=20*l20(n)-9*l3(n)-8*z2*l(n-2)
m30(n)=30*l30(n)-99*l3(n)+32*z2*l(n-2)
m36(n)=18*l36(n)-37*z2*l(n-2)
n10(n)=40*m10(n)-63*m6(n)+22*m8(n)
n11(n)=22*m11(n)-21*m6(n)-78*m8(n)
n12(n)=12*m12(n)+33*m6(n)-10*m8(n)
n14(n)=14*m14(n)-15*m6(n)-14*m8(n)
n18(n)=36*m18(n)+17*m6(n)-2*m8(n)
n20(n)=16*m20(n)+15*m6(n)-22*m8(n)
n30(n)=40*m30(n)-457*m6(n)-166*m8(n)
n36(n)=36*m36(n)+165*m6(n)-218*m8(n)
p11(n)=33*n11(n)-75*n10(n)+826*z4*l(n-4)
p12(n)=114*n12(n)+30*n10(n)-637*z4*l(n-4)
p14(n)=798*n14(n)-330*n10(n)+3131*z4*l(n-4)
p18(n)=171*n18(n)+65*n10(n)-275*z4*l(n-4)
p20(n)=1425*n20(n)-1935*n10(n)+3647*z4*l(n-4)
p30(n)=855*n30(n)-4545*n10(n)+45044*z4*l(n-4)
p36(n)=684*n36(n)-2700*n10(n)+11825*z4*l(n-4)
q14(n)=51443*p14(n)-549670*p11(n)-448632*p12(n)
q18(n)=66141*p18(n)+468787*p11(n)+568482*p12(n)
q20(n)=29396*p20(n)-3672757*p11(n)-4510188*p12(n)
q30(n)=7349*p30(n)+634524*p11(n)+1318488*p12(n)
q36(n)=22047*p36(n)-3300946*p11(n)-3812480*p12(n)
r18(n)=1381536*q18(n)+1726844*q14(n)+2009631208533*z6*l(n-6)
r20(n)=650*q20(n)-5880*q14(n)-5244121467*z6*l(n-6)
r30(n)=959400*q30(n)+2736580*q14(n)+3500076239803*z6*l(n-6)
r36(n)=921024*q36(n)-6189316*q14(n)-7172856338635*z6*l(n-6)
s30(n)=275831675*r30(n)-697481757*r18(n)-83660467200*r20(n)
s36(n)=99299403*r36(n)+601178907*r18(n)+94027038400*r20(n)
t36(n)=155097*s36(n)-9083232*s30(n)+17339886100374329220502046*z8*l(n-8)
```

5. Salem Number with Smallest Mahler Measure

5.1 It was conjectured [4], and this has been supported more recently by calculations of Boyd, that the Salem number with the smallest Mahler measure is a root of the equation

$$\xi^{10} + \xi^9 - \xi^7 - \xi^6 - \xi^5 - \xi^4 - \xi^3 + \xi + 1 = 0 \qquad (31)$$

This polynomial possesses two real roots, $x = \cdot 8501...$ and its inverse, and 4 complex pairs on the unit circle. Thus $n_- = 4$, $n_+ - n_- = 2$, and at the even order $4 + 1 = 5$ ladders combine to give a valid result. At the odd order there is a net loss of 5 in going from odd to even orders (including the elimination of $\zeta\,(n)$). Since there exists a grand total of 71 cyclotomic equations, there should exist $71 - 4 = 67$ valid dilogarithmic ladders, $67 - 5 = 62$ results (free of $\zeta\,(3)$) at $n = 3$, and so on, leading eventually to 4 results at $n = 16$. It is conjectured that this is the highest order for which valid ladder results for any base may exist. Results for $n = 2$ are reported in [5], and the generation of the higher order formulas is currently under investigation, and has led to the expected 4 ladders at the sixteenth order.

6. Combination of Different Bases

6.1 As a rule, attempts to combine different bases do not lead to interesting ladder results; no cyclotomic equations are produced, the matter ending there. An exception exists for the bases ψ and ϕ introduced in section 2, since the two are closely related. Their product $x = \psi\,\phi$ can be shown to satisfy

$$\left(1-x^6\right)^3 = \left(1-x^4\right)^4\left(1-x^3\right)^3\left(1-x^2\right)(1-x)^{-1}x^3 \tag{32}$$

but this is insufficient to generate a valid ladder. More fruitful is the ratio $y = \phi/\psi$. It satisfies the sextic

$$y^6 + y^4 + y^3 - y^2 - 1 = 0 \tag{33}$$

for which $n_- = 2$. Hence three cyclotomic equations lead to three component-ladders which should combine to give a valid result.

Algebraic manipulation of (33) yields

$$\left(1-y^4\right)^2 = \left(1-y^2\right)y^3 \tag{34}$$

$$1 - y^6 = \left(1-y^3\right)^2\left(1-y^4\right)(1-y)^{-1}y^{-4} \tag{35}$$

$$1 - y^{10} = \left(1-y^5\right)^2\left(1-y^4\right)^{-1}\left(1-y^2\right)(1-y)^{-1}y^3 \tag{36}$$

$$\left(1-y^{24}\right)^2 = \left(1-y^{12}\right)^2\left(1-y^8\right)^3\left(1-y^4\right)^{-2}\left(1-y^3\right)^2(1-y)^{-1}y^{-1} \tag{37}$$

From these, four component-ladders, of indices $4, 6, 10$ and 24, can be constructed. The index 24 ladder is somewhat simpler if the index 6 and 4 ladders are subtracted from it, leading to defining

$$L\,(4) = 2\,li\,(4) - li\,(2) + 3\,l \tag{38a}$$

$$L\,(6) = li\,(6) - 2\;li\,(3) - li\,(4) + li\,(1) - 4\,l \tag{38b}$$

$$L\;(10) = li\,(10) - 2\;li\,(5) + li\,(4) - li\,(2) + li\,(1) + 3\,l \tag{38c}$$

$$L\;(24) = 2\,li\,(24) - 2\,li\,(12) - 3\,li\,(8) - li\,(6) + li\,(4) + li\,(2) \tag{38d}$$

with $li\,(N) = Li_2(y^N)/N$; $l = \frac{1}{2}\log^2(y)$. Then it is found numerically that

$$10\,L\,(10) - 6\,L\,(6) - 8\,L\,(4) = 3\,\zeta\,(2) \tag{39}$$

$$12\,L\,(24) + L\,(4) = 3\,\zeta\,(2) \tag{40}$$

The former of these can be obtained from (2.40) of [5] for which the base equation is (2.42):

$$u^p + u^n - u^q + u^{n+m} - u^{q+p+m} = 1 \tag{41}$$

With $m = 0, p = 3, n = 4, q = 5$ this becomes $1-2u^4+u^8-u^3+u^5 = 0$ which, on abstracting a factor $\left(1 - u^2\right)$, reduces to (33). But no analytic derivation of (40) is currently known.

Acknowledgments Many of the results reported here were obtained with the help of the PARI computer program, and I am very grateful to Henri Cohen for his assistance in the computations.

References

1. L. Lewin and M. Abouzahra, Supernumary polylogarithmic ladders and related functional equations, Aequationes Mathematicae 39 (1990), 210–253.
2. D. Zagier, Appendix A in "The Structural Properties of Polylogarithms" (ed. L. Lewin), American Mathematical Society, 1991.
3. D. W. Boyd, Small Salem Numbers, Duke Mathematical Journal, 44 (1977), 315–328.
4. D. H. Lehmer, Factorization of Certain Cyclotomic Functions, Ann. Math. 34 (1933), 461–479.
5. L. Lewin, "The Structural Properties of Polylogarithms" (ed. L. Lewin), American Mathematical Society, 1991.