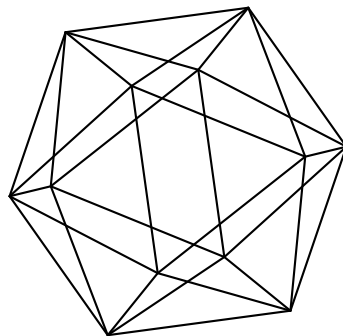


Max-Planck-Institut für Mathematik Bonn

Bounds on the number of connected components for
tropical prevarieties

by

Alex Davydow
Dima Grigoriev



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Dima Grigoriev

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Academic University
Ulitsa Khlopina 8
St. Petersburg 194021
Russia

CNRS, Mathématiques
Université de Lille
59655 Villeneuve d'Ascq
France

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ALEX DAVYDOW

ACADEMIC UNIVERSITY, ULITSA KHLOPINA 8,
SANKT-PETERSBOURG, 194021, RUSSIA

ADAVYDOW@GMAIL.COM

DIMA GRIGORIEV

CNRS, MATHÉMATIQUES, UNIVERSITÉ DE LILLE
VILLENEUVE D'ASCQ, 59655, FRANCE

DMITRY.GRIGORYEV@MATH.UNIV-LILLE1.FR

[HTTP://EN.WIKIPEDIA.ORG/WIKI/DIMA_GRIGORIEV](http://en.wikipedia.org/wiki/Dima_Grigoriev)

ABSTRACT. For a tropical prevariety in \mathbb{R}^n given by a system of k tropical polynomials in n variables with degrees at most d , we prove that its number of connected components is less than $\binom{k+7n-1}{3n} \cdot \frac{d^{3n}}{k+n+1}$. On a number of 0-dimensional connected components a better bound $\binom{k+4n}{3n} \cdot \frac{d^n}{k+n+1}$ is obtained, which extends the Bezout bound due to B. Sturmfels from the the case $k = n$ to an arbitrary $k \geq n$. Also we show that the latter bound is close to sharp, in particular, the number of connected components can depend on k .

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1. INTRODUCTION

Let a tropical prevariety $V \subset \mathbb{R}^n$ (see e.g. [1]) be given by k tropical polynomials f_1, \dots, f_k in n variables with (tropical) degrees at most d . The principal motivation of this paper is to bound the number c of connected components of V . Recall (see e.g. [1]) that V is a polyhedral complex. The main result (Corollary 9.3) states the bound

$$(1) \quad c \leq \binom{k + 7n - 1}{3n} \cdot \frac{d^{3n}}{k + n + 1}$$

For the number of isolated points of V (being its 0-dimensional connected components) we obtain (Corollary 9.4) a better bound

$$(2) \quad \binom{k + 4n}{3n} \cdot \frac{d^n}{k + n + 1}$$

It can be treated as a generalization of the Bezout inequality on the number of *stable* solutions (see [1], [2] and Section 5 below) proved in the case $k = n$ to the case of overdetermined (i. e. $k > n$) tropical systems. Recall that $k \geq n$ in order V to have an isolated point since the local codimension at any point of V is less or equal to k [3], see also Theorem 3.8. Moreover, [1] have proved a tropical Bezout theorem which states that the number of stable solutions (counted with multiplicities) of n tropical polynomials f_1, \dots, f_n with degrees d_1, \dots, d_n respectively, equals $d_1 \cdots d_n$.

In Section 8 we show that bound (2) is close to sharp by an explicit construction of tropical systems.

The observed phenomenon of dependency of the number of connected components on k in (1) and in (2) occurs similarly for real semialgebraic sets (moreover, for the sum of Betti numbers which strengthens the bounds established by Oleinik-Petrovskii, Milnor, Thom) [4], while due to a different reason.

Note that in the case of an algebraic variety given by a polynomial system $g_1 = \cdots = g_k = 0$ where the degrees of polynomials in n variables do not exceed d , the sum of the degrees of the irreducible components of the variety is bounded by d^n , i. e. does not depend on k . This holds because the variety remains the same if to replace g_1, \dots, g_k by their $n + 1$ generic linear combinations (see e.g. [5] and [6]).

Our conjecture is that the sum of Betti numbers of a tropical prevariety V is bounded by (1). In Theorems 6.5, 6.6 one can find somewhat weaker bounds on the sum of Betti numbers.

The important technical tool to study a system of tropical polynomials (see Section 4) is the *star table* (exploited in [7], [8]) consisting of

the set of monomials from the given tropical system in which the minimum is attained at a given point $v \in V$ (here a monomial is treated as a classical linear function). In these terms we define a *generalized vertex* v of V when the star table is maximal under inclusion. We produce a description of generalized vertices in terms of the exponents vectors of the starred monomials (Theorem 4.11). Then we prove that any connected component of a tropical prevariety given by a system of tropical polynomials of fixed degrees with all finite coefficients contains a generalized vertex (Theorem 6.1).

In Section 5 we study stable points of a tropical prevariety given by n tropical polynomials, and provide a criterion to be a stable point again in terms of the exponent vectors of the starred monomials (Theorem 5.10). This implies that a generalized vertex of V is a stable point of a suitable *multisubset* of $\{f_1, \dots, f_k\}$, consisting of n elements (Theorem 6.2). The established results provide a slightly better bound than (1) in case of finite coefficients (Corollary 6.4).

To get a bound slightly better than (2) in case of finite coefficients we prove in Section 7 that an isolated point of V is a stable point of an appropriate subset consisting of n elements among $\{f_1, \dots, f_k\}$ (Theorem 7.1). We emphasize that here we consider a subset, rather than a multisubset as in Theorem 6.2, this explains the difference between bounds (1) and (2).

In Section 9 we show that adding n extra variables and $2n$ extra tropical polynomials to $\{f_1, \dots, f_k\}$ we get a compact tropical prevariety being homotopy equivalent to V . Thus, the problem of bounding the number of connected and moreover, the sum of Betti numbers of V reduces to a compact tropical prevariety. Also in Section 9 we discuss systems of tropical polynomials with coefficients allowed to include infinity which allows one to complete the proofs of bounds (1) and (2).

2. TROPICAL SEMI-RING AND TROPICAL PREVARIETIES

Definition 2.1. Semi-fields \mathbb{R} and $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ endowed with operations $\oplus := \min$, $\otimes := +$, $\oslash := -$ are called tropical semi-rings with or without infinity correspondingly.

We will denote tropical semi-rings with or without infinity as \mathbb{K} and \mathbb{K}_∞ correspondingly.

As well as tropical addition and multiplication we will use tropical power: $x^{\otimes i} := x \otimes \dots \otimes x$.

In this paper we will study tropical polynomials and at first we have to define a tropical monomial:

Definition 2.2. *Tropical monomial* Q is defined as $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n} = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$, its *tropical degree* $\text{trdeg} = i_1 + \cdots + i_n$.

Note 2.1. *As in classic monomials we will often omit multiplication sign when it is clear if we speak about a tropical multiplication or a classic one. In addition we will omit multiplier of 0 as it is neutral element of tropical multiplication.*

Now we can define a tropical polynomial:

Definition 2.3. *Tropical polynomial* f is defined as $f = \bigoplus_j (a_j \otimes x_1^{\otimes i_{j1}} \otimes \cdots \otimes x_n^{\otimes i_{jn}}) = \min_j \{Q_j\}$;

$x = (x_1, \dots, x_n) \in \mathbb{R}_\infty^n$ is a **tropical zero** of f if at point x either minimum $\min_j \{Q_j(x)\}$ is attained for at least two different values of j if $\min_j \{Q_j(x)\}$ is finite or $Q_j(x) = \infty$ for all j . If $x \in \mathbb{R}^n$ we say that the tropical zero is finite. If all monomials with $\text{trdeg} \leq d$ present at f we say that polynomial f of tropical degree d has all its coefficients finite.

Then we define a tropical hypersurface:

Definition 2.4. The set of tropical zeros from \mathbb{R}^n of a tropical polynomial is called a *tropical hypersurface*.

And finally a tropical prevariety:

Definition 2.5. *Tropical prevariety* is the intersection of a finite number of tropical hypersurfaces.

3. HAHN SERIES AND TROPICAL VARIETIES

To introduce tropical varieties it will be convenient to use a generalization of Puiseux series known as Hahn (or Hahn-Mal'cev-Neumann) series (see [9]).

Definition 3.1. The field of *Hahn series* $K[[T^\Gamma]]$ in the indeterminate T over a field K and with value group Γ (an ordered group) is the set of formal expressions of the form $f = \sum_{e \in \Gamma} c_e T^e$ with $c_e \in K$ such that

the support $\{e \in \Gamma : c_e \neq 0\}$ of f is well-ordered. The sum and product of $f = \sum_{e \in \Gamma} c_e T^e$ and $g = \sum_{e \in \Gamma} d_e T^e$ are given by $f + g = \sum_{e \in \Gamma} (c_e + d_e) T^e$ and $fg = \sum_{e \in \Gamma} \sum_{e' + e'' = e} c_{e'} d_{e''} T^e$ ($\sum_{e' + e'' = e}$ is finite as well-ordered set could not contain infinite decreasing sequence).

To define a tropical variety we have to introduce the operation of tropicalization.

Definition 3.2. *Tropicalization* of $x' \in K[[T^\Gamma]]$ is a point $x \in \Gamma \cup \{\infty\}$ equal to the least power of T in x' if x' is not equal to zero, or ∞ otherwise.

We will denote operation of tropicalization by trop .

Tropicalization V of a variety V' over the field of Hahn series $K[[T^\Gamma]]$ consists of the closure in the euclidean topology of the set of points $x \in \mathbb{R}^n$ for which there is a point $x' = (x'_1, x'_2, \dots, x'_n) \in V'$ with $x'_1 \cdots x'_n \neq 0$, such that $x = (\text{trop}(x'_1), \text{trop}(x'_2), \dots, \text{trop}(x'_n))$. Set V is referred to as a *tropical variety*.

While any tropical hypersurface is a tropicalization of a hypersurface over the field of Hahn series $\mathbb{C}[[T^{\mathbb{R}}]]$ (cf. [19]) some tropical prevarieties do not correspond to any varieties over $\mathbb{C}[[T^{\mathbb{R}}]]$. For example a tropical prevariety given by the linear system

$$A = \begin{cases} 0 \oplus x \oplus y \oplus z \\ 0 \oplus x \oplus 1y \oplus 1z \end{cases} .$$

is not a tropical variety. However, any tropical variety is a tropical prevariety and moreover the following theorem holds (see [1], [18]):

Theorem 3.3. *For any variety V' given by polynomial system A' in Hahn series $\mathbb{C}[[T^{\mathbb{R}}]]^n$ its tropicalization V is a tropical prevariety in \mathbb{R}^n , and V coincides with the intersection of tropical hypersurfaces being tropicalizations of all the polynomials from the ideal generated by A' . Moreover, V equals the tropical prevariety determined by the intersection of a finite number of tropicalizations of hypersurfaces provided by polynomials from the ideal generated by A' (such a finite subset is called a *tropical basis of the ideal*)*

To study tropical prevarieties we will use some properties of Hahn series.

Theorem 3.4 ([10]). *For any algebraically closed field K and ordered divisible group Γ field of Hahn series $K[[T^\Gamma]]$ is algebraically closed.*

Thus (see e.g. [11]) we can apply Bezout theorem to $\mathbb{C}[[T^{\mathbb{R}}]]$.

Definition 3.5. Let n projective hypersurfaces be given in $\mathbb{P}^n(\mathbb{C}[[T^{\mathbb{R}}]])$ by n homogeneous polynomials in $n + 1$ variables. Point x is a *stable intersection point of these hypersurfaces with multiplicity e* if under generic small perturbation of the coefficients of given polynomials corresponding hypersurfaces will have exactly e intersection points in a small neighborhood of x .

Theorem 3.6 (Bezout's theorem). *Let n projective hypersurfaces be given in $\mathbb{P}^n(\mathbb{C}[[T^{\mathbb{R}}]])$ by n homogeneous polynomials in $n + 1$ variables, of degrees d_1, d_1, \dots, d_n . Then the number of stable intersection points of these hypersurfaces is equal to $d_1 d_2 \cdots d_n$.*

Another important property of the field of Hahn series $\mathbb{C}[[T^{\mathbb{R}}]]$ implied by the fact that it is algebraically closed is

Theorem 3.7 (Dimension of intersection [3]). *Let variety V' be given by a polynomial system A in n variables over the field of Hahn series $\mathbb{C}[[T^{\mathbb{R}}]]$. Then if system A consists of k polynomials the codimension of each irreducible component of V' is less or equal to k .*

This properties of Hahn series are important for studying tropical varieties and prevarieties due to the following theorem:

Theorem 3.8 ([12], [18]). *For any irreducible variety V' of dimension m over the field of Hahn series $\mathbb{C}[[T^{\mathbb{R}}]]$ the local dimension at any point x of its tropicalization V is equal to m .*

Remark 3.9. While Theorem 3.8 was known for varieties over the field of Puiseux series, the proof can be literally extended to Hahn series.

4. GENERALIZED VERTICES

To study tropical prevarieties it will be convenient to use the following definition of vertex:

Definition 4.1. By a *vertex* of a tropical prevariety we will mean a point for which we could not choose a direction in such a way that there is a neighborhood of the point where prevariety can be represented as a generalized open ended cylinder with axis parallel to the chosen direction (a generalized open ended cylinder is a product of an arbitrary set of a smaller dimension and a line interval).

In addition we will need a generalization of this definition, and at first we have to give a definition of a star table of a tropical system similar to one introduced in [8]:

Definition 4.2. Let A be a tropical polynomial system of k polynomials in n variables with the greatest degree d . We associate with it a table A^{*x} of the size $k \times \binom{n+d-1}{d}$ with rows corresponding to polynomials and columns corresponding to all possible monomials of degree at most d in n variables. We put $*$ to the entry (i, j) iff the j -th monomial treated as a (classical) linear function attains a minimal value among all the monomials at the point x in i -th polynomial and we leave all others entries empty (see Example 4.3).

Example 4.3. Consider a tropical system

$$A = \begin{cases} 0 \oplus 1x \oplus y \\ 0 \oplus -2x \oplus -2y \oplus -2x^{\otimes 2} \oplus -3xy \oplus -1y^{\otimes 2} \end{cases} .$$

At point $(-1, 0)$ this system is equal to

$$A = \begin{cases} 0 \oplus 0 \oplus 0 \\ 0 \oplus -3 \oplus -2 \oplus -4 \oplus -4 \oplus -1 \end{cases} ,$$

so

$$A^{*(-1,0)} = \begin{bmatrix} 0 & x & y & x^{\otimes 2} & xy & y^{\otimes 2} \\ * & * & * & * & * & * \end{bmatrix} .$$

At point $(1, 0)$ this system is equal to

$$A = \begin{cases} 0 \oplus 2 \oplus 0 \\ 0 \oplus -1 \oplus -2 \oplus 0 \oplus -2 \oplus -1 \end{cases} ,$$

so

$$A^{*(1,0)} = \begin{bmatrix} 0 & x & y & x^{\otimes 2} & xy & y^{\otimes 2} \\ * & * & * & * & * & * \end{bmatrix} .$$

At point $(-2, -1)$ this system is equal to

$$A = \begin{cases} 0 \oplus -1 \oplus -1 \\ 0 \oplus -4 \oplus -3 \oplus -6 \oplus -6 \oplus -3 \end{cases} ,$$

so

$$A^{*(-2,-1)} = \begin{bmatrix} 0 & x & y & x^{\otimes 2} & xy & y^{\otimes 2} \\ * & * & * & * & * & * \end{bmatrix} .$$

All local properties of the tropical prevariety can be expressed in terms of this table (see the next theorem). In the next chapter we will show how to test stability of a solution of a tropical system using this table (Theorem 5.10), for another example see [8] where the star table is used to calculate the local dimension of a linear prevariety.

Theorem 4.4. *Let a tropical prevariety V be given by a tropical system A . If $A^{*y} = A^{*z}$ then there is an ϵ , such that ϵ -neighborhood of point y of V is homeomorphic to ϵ -neighborhood of point z of V , moreover this homeomorphism is given by a shift of coordinates which sends y to x .*

Proof. Let d be the maximal degree of polynomials in A and $x = (x_1, x_2, \dots, x_n)$ be a set of variables of these polynomials. Let's denote by Δ_y and Δ_z the minimal differences between the values of the starred and non-starred monomials from the same polynomials at point y and point z correspondingly. Let $\Delta = \min(\Delta_y, \Delta_z)$. Denote $\epsilon = \frac{\Delta}{3d}$.

Now we will prove that ϵ fits the requirements of the theorem. Let's make a change of variables $x'_i = x_i - y_i$ which corresponds to a shift of tropical prevariety in such a way that y is shifted to 0. The resulting

tropical system we denote by B . Let's denote the shifted prevariety by W . Due to our choice of ϵ in ϵ -neighborhood of 0 only monomials which are starred at 0 can be starred (they are not greater than $0 + d\epsilon = \frac{\Delta}{3}$, while others are not lesser than $\Delta - d\epsilon = \frac{2\Delta}{3}$), so while studying B in ϵ -neighborhood of 0 we can w.l.o.g. assume that all non-starred monomials are infinite. Moreover w.l.o.g. we can assume that all coefficients in starred monomials in B are equal to zero, otherwise we can tropically multiply corresponding polynomials to change them to zero (see Example 4.5).

Now if we repeat the same operation replacing all occurrences of y by z we will obtain the system which will be the same as B up to assumptions we made in the end of the previous paragraph. So ϵ -neighborhood of z can be obtained from ϵ -neighborhood of y by a shift (as both of them can be obtained by a shift from ϵ -neighborhood of 0 of W). \square

Example 4.5. Consider system

$$\left\{ 2x^{\otimes 2} \oplus x \oplus 0 \quad . \right.$$

Assume that we want to study the prevariety given by this system in the neighborhood of the point $x = -2$. First we make a change of variable $x' = x + 2$:

$$\left\{ -2x'^{\otimes 2} \oplus -2x' \oplus 0 \quad , \right.$$

then tropically multiply the polynomial by 2:

$$\left\{ x'^{\otimes 2} \oplus x' \oplus 2 \quad . \right.$$

And in $\frac{1}{3}$ -neighborhood of 0 this system can be replaced by:

$$\left\{ x'^{\otimes 2} \oplus x' \quad . \right.$$

Now we can give a definition of a generalized vertex:

Definition 4.6. A point x is a *generalized vertex* of a tropical polynomial system A iff the star table A^{*x} is strictly maximal with respect to inclusion, i.e. for any other point $y \neq x$ the star table A^{*y} does not contain A^{*x} .

Example 4.7. Point $(-2, -1)$ is not a generalized vertex for a system A from Example 4.3 as $A^{*(-1,0)}$ is greater than $A^{*(-2,-1)}$ with respect to inclusion.

Theorem 4.8. *A point x is a generalized vertex of a tropical polynomial system A iff there is no vector along which the directional derivative of every starred monomial in A^{*x} in every polynomial is the same (starred monomials from different polynomials can have different directional derivatives, see Example 4.9), i. e. there is no line that passes through the point x along which we can move while preserving star table the same in some neighborhood of x .*

Proof. (1) First we prove, that if there is a vector along which the directional derivative of every starred monomial in A^{*x} in every polynomial is the same, then x is not a generalized vertex. It's so because if we will move from point x in the direction of this vector there will be a neighborhood where we will preserve the star table (so initial star table was not strictly maximal).
 (2) Now we prove the converse: if x is not a generalized vertex then there is a vector along which directional derivative of every starred monomial in A^{*x} in every polynomial is the same. If x is not a generalized vertex, then there is a point y whose star table A^{*y} contains A^{*x} . Directional derivative along the vector (x, y) will be the same for all points stared in A^{*x} in each polynomial, because difference between these monomials' values in the same polynomial is the same (they are equal both in point x and point y).

□

Example 4.9. Consider system A from Example 4.3. Point $(-2, -1)$ is not a generalized vertex, because we can choose a vector $(1, 1)$, and directional derivatives of all starred monomials in the first polynomial along this vector will be the same and equal to $\frac{1}{\sqrt{2}}$, while directional derivatives of starred monomials in the second polynomial along this vector will be the same and equal to $\sqrt{2}$.

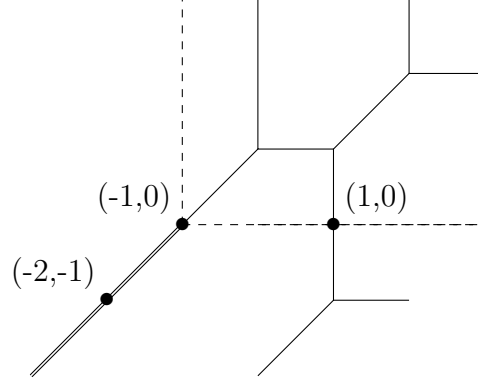
Points $(1, 0)$ and $(-1, 0)$ are generalized vertices because we could not find a vector with the required property.

The corresponding prevariety is drawn on Figure 1. Prevariety is depicted with double lines, the first hypersurface with dashed and the second one with solid.

Let's define function $p_n(m_1)$ which takes a tropical monomial in n variables x_1, x_2, \dots, x_n as arguments and returns a vector in \mathbb{R}^n in the following way: $p_n(cx_1^{\otimes a_1} x_2^{\otimes a_2} \dots x_n^{\otimes a_n}) = (a_1, a_2, \dots, a_n)$.

Let's define function $v_n(m_1, m_2)$ which takes two tropical monomials in n variables as arguments and returns a vector in \mathbb{R}^n in the following way: $v_n(m_1, m_2) = p_n(m_2) - p_n(m_1)$.

FIGURE 1. Illustration for Example 4.9



Example 4.10.

- $v_2(0, x_1) = (1, 0)$,
- $v_2(x_1^{\otimes 2}x_2, x_1x_2^{\otimes 2}) = (-1, 1)$,
- $v_3(0, 2x_1x_2x_3) = (1, 1, 1)$.

Now we can give a criterion of a point being a generalized vertex in terms of the star table just of this point invoking also functions v_n .

Theorem 4.11. *Assume that for a tropical polynomial system A of k polynomials in n variables with a solution at x we can choose $2n$ monomials $m_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq 2$ with the following properties:*

- *one monomial can be chosen several times.*
- *the monomials $m_{i,1}$ and $m_{i,2}$ are marked with a star in A^{*x} in the same line,*
- *the linear span of vectors $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{n,1}, m_{n,2})$ has dimension equal to n .*

Then x is a generalized vertex of this system, conversely if x is a generalized vertex we can always choose $2n$ monomials in the described way.

Proof. (1) First we prove in one direction: if we can not choose $2n$ monomials with the required properties, then x is not a generalized vertex. If we could not choose $2n$ monomials with the required properties, then it would mean that the linear span of vectors $v_n(y, z)$ where y and z are arbitrary starred monomials from the same polynomial has dimension (over all the polynomials) lesser than n . But if we choose a vector orthogonal to this linear span, directional derivatives of any pair of starred monomials from the same polynomial along this vector will be the same (as directional derivative of their difference will be

equal to zero). And by Theorem 4.8 this means that x is not a generalized vertex.

- (2) Now we prove the converse: if x is not a generalized vertex then we can not choose $2n$ monomials with the required properties. By Theorem 4.8 we can choose a vector v along which all directional derivatives of starred monomials from the same polynomial will be the same (for all the polynomials from A). And this means that v is orthogonal to $v_n(y, z)$ for any monomials y and z starred in the same polynomial. But the latter contradicts to that the dimension of the linear span of vectors $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{n,1}, m_{n,2})$ equals n .

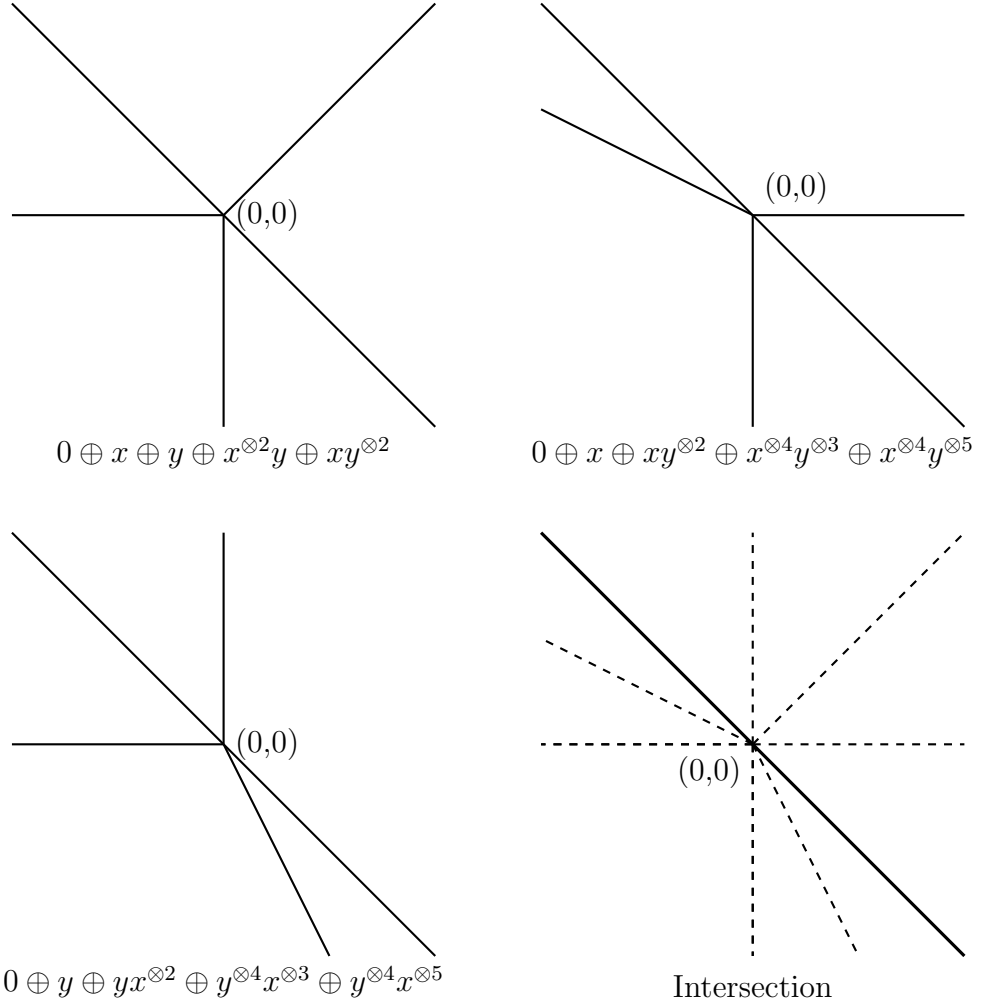
□

Generalized vertex is indeed a generalization of a vertex:

Theorem 4.12. *If x is a vertex point of the prevariety V given by a tropical polynomial system A then x is a generalized vertex point of A .*

Proof. Assume the contrary. Then by Theorem 4.8 we can choose a vector along which all directional derivatives of monomials starred in A^{*x} will be the same. That means that if we choose a line passing through the point x and directed by this vector, then we can move along it in both directions while keeping star table the same in some neighborhood of point x . And by Theorem 4.4 in this neighborhood of point x prevariety V is a generalized open-ended cylinder. □

However, converse is not true:



Example 4.13. For a system of tropical polynomials

$$\begin{cases} 0 \oplus x \oplus y \oplus x^{\otimes 2}y \oplus xy^{\otimes 2} \\ 0 \oplus x \oplus xy^{\otimes 2} \oplus x^{\otimes 4}y^{\otimes 3} \oplus x^{\otimes 4}y^{\otimes 5} \\ 0 \oplus y \oplus yx^{\otimes 2} \oplus y^{\otimes 4}x^{\otimes 3} \oplus y^{\otimes 4}x^{\otimes 5} \end{cases}$$

0 is a generalized vertex, but it is not a vertex (this prevariety is equal to a line directed by vector $(-1, 1)$).

5. STABILITY OF SOLUTIONS CRITERIA

In this paper it will be convenient to use the following definition:

Definition 5.1. By the *amplitude of a perturbation* we will denote the maximal difference between corresponding finite coefficients of the

initial system and the perturbed one (the infinite coefficients are not perturbed).

In this section we always consider tropical system of n equations in n variables.

Following Sturmfels, and others [1] we will use the following definition of stability:

Definition 5.2. A point x is a stable point of multiplicity of tropical polynomial system A of n equations in n variables if any sufficiently small generic perturbation leads to a prevariety being a finite set of points, and among them will be exactly k points in a neighborhood of x . (More formally: there is Δ , such that for any $\delta < \Delta$ there is $\epsilon > 0$ such that for any generic perturbation with amplitude not greater than ϵ there will be k solutions in δ -neighborhood of x).

This definition can be extended to faces of higher dimension in the following way:

Definition 5.3. A g -face L is a stable g -face of multiplicity k of tropical polynomial system A of $n - g$ equations in n variables if any sufficiently small generic perturbation leads to a prevariety with exactly k g -faces in a neighborhood of x . (More formally: there is Δ , such that for any $\delta < \Delta$ there is $\epsilon > 0$ such that for any generic perturbation with amplitude not greater than ϵ the δ -neighborhood of x will be intersected by k g -faces).

Our results will be heavily based on the tropical Bezout's equality, which states the following:

Theorem 5.4 (Tropical Bezout's Equality[1]). *Every n tropical polynomials with finite coefficients in n variables have D stable finite solutions counted with multiplicities where D is the product of degrees of the given polynomials.*

As it was mentioned by Tabera [2] from this theorem the following property of stable points of tropical prevariety can be obtained:

Theorem 5.5. *Given n tropical hypersurfaces in n -dimensional space the stable points of the prevariety being their intersection form a well-defined set that varies continuously under perturbations of the given hypersurfaces.*

In this chapter we will always consider systems with finite coefficients (this means that no monomial of degree at most d in a polynomial can be omitted), unless we set some of the coefficients to infinity explicitly.

For effective usage of Theorems 5.4, 5.5 we have to introduce several simple criteria of stability. While proving theorems we will often w. l. o. g. study stability at point 0, and we will consider all minimal coefficients (i. e. coefficients of the starred monomials in A^{*0}) to be equal to 0 too (this specific case can be obtained from the general case by the change of variables to shift point under consideration to 0 and by tropical multiplication of equations by constants, see Example 4.5).

Theorem 5.6. *Given a tropical polynomial system A with a solution in x , let us replace all the coefficients of monomials which are starred in A^{*x} by arbitrary set of real numbers and the rest of coefficients by infinity (the resulting system denote by C). Point x is a stable solution of A iff C will have a finite tropical solution for any set of chosen real numbers.*

Example 5.7. Consider tropical system:

$$\begin{cases} 0 \oplus 3x \oplus 0xy \oplus 0x^{\otimes 2} \\ 3 \oplus 0x \oplus 0y^{\otimes 3} \end{cases} .$$

0 is a stable solution of this system as system:

$$\begin{cases} a_1 \oplus a_2xy \oplus a_3x^{\otimes 2} \\ a_4x \oplus a_5y^{\otimes 3} \end{cases} ,$$

has a finite solution for any real a_1, a_2, a_3, a_4, a_5 .

Proof. W. l. o. g. we can assume that x is a zero and the coefficients of all starred monomials in A^{*0} are equal to 0.

Let d be the maximal tropical degree of polynomials in the system and let the smallest nonzero coefficient in the initial equation be equal to Δ .

- (1) First we prove in one direction: if 0 is a stable point of A , then we can find a finite solution of C for any set of coefficients taken as in the theorem. We will prove that for a fixed set of coefficients there is a solution. W. l. o. g. we can assume that all the coefficients are positive (otherwise we can tropically multiply equations by a constant). Let the greatest coefficient be equal to M . As 0 is a stable solution of the initial system we can choose a δ with the following properties:

- $0 < \delta < \frac{\Delta}{4d}$,
- for any perturbation of parameters of the initial system with amplitude less or equal to δ there will be a stable solution in a $\frac{\Delta}{4d}$ neighborhood of 0.

Let's consider a perturbation B of the initial system with nonzero coefficients unchanged and zero coefficients replaced by the corresponding coefficients from C multiplied by $\frac{\delta}{M}$.

By our choice of δ we can find a solution y of B in a $\frac{\Delta}{4d}$ neighborhood of 0. Monomials which were nonzero in A could not be minimal in this solution as they are too large. Indeed, as coefficients change is not greater than δ and solution coordinates are less than $\frac{\Delta}{4d}$ the value of monomial which was zero in A after the perturbation in y is not greater than $\delta + d\frac{\Delta}{4d} < \frac{\Delta}{2}$, while the value of monomial which was nonzero in A , after perturbation in y are at least $\Delta - \delta - d\frac{\Delta}{4d} > \frac{\Delta}{2}$.

If we classically multiply the solution and coefficients of the equation from B by $\frac{M}{\delta}$, y will be a solution for a multiplied system, and if we change all coefficients which were nonzero by infinity, solution still remains a solution as all the monomials we have changed were not minimal. So we have found a solution for C .

- (2) Now we prove the converse: if we can find a solution of a system C for any replacement of the coefficients then 0 is a stable point. We will prove that we can choose such a monotone function p that for any perturbation with amplitude $\delta < \min(p^{-1}(\frac{\Delta}{4d}), \frac{\Delta}{4d})$ there is a solution in $p(\delta)$ neighborhood of 0. Let's denote the perturbed system by E .

Replace by infinity all monomials in E which are nonzero in the initial system A . By our assumption this system will have a solution, and as it has a solution, it has a solution which can be bounded by $2Mn!d^n$, where M is a maximal coefficient: by Theorem 6.1 a tropical prevariety has at least one generalized vertex and this vertex can be obtained as an intersection of n linearly independent hyperplanes (classical), given by equality of n pairs of tropical monomials taken as in Theorem 4.11. If we know which hyperplanes intersect in this vertex then we can calculate its position using Cramer's rule. As all powers in tropical equations are integral and do not exceed d , as the system of equations for the vertex we will obtain a linear system with integral coefficients for variables and with a constant part which does not exceed $2M$. We can estimate determinant of this system as at least 1 (it's integral and system is not degenerate by Theorem 4.11) and the determinants in the numerators of the Cramer's formula can be estimated by $2Mn!d^n$. So we can choose $p(\delta) = 2\delta n!d^n$.

This solution will be a solution of E , as monomials corresponding to non-starred monomials of A are too large to be minimal (as coefficients change is not greater than δ and solution coordinates are less than $p(\delta)$, the value of a monomial which was starred in A^{*0} after perturbation in the new solution point is at most $\delta + dp(\delta) < \frac{\Delta}{2}$ and the value of a non-starred monomial after perturbation in the new solution point is at least $\Delta - \delta - dp(\delta) > \frac{\Delta}{2}$). For any small perturbation we found a solution in a neighborhood of 0, so 0 is a stable point. \square

Using this theorem we can prove the following lemma:

Lemma 5.8. *If x is a stable solution of a tropical system A , then for any tropical system F and point y , if $F^{*y} = A^{*x}$, then y is a stable point of F .*

Proof. W. l. o. g. we can assume that x and y are equal to 0, and that the coefficients of the monomials starred in A^{*0} and F^{*0} are equal to zero.

As 0 is a stable point of A , by Theorem 5.6 if we set all coefficients in the monomials which are non-starred in A^{*0} to infinity and replace all the coefficients of the starred monomials by arbitrary values the obtained system C will have a solution.

But the result of replacement (system C) is the same for systems A and F , so if in system F we set all coefficients in the monomials which are non-starred in F^{*0} to infinity and replace all the coefficients of the starred monomials by arbitrary values the obtained system will have a solution. And by Theorem 5.6 this means that 0 is a stable solution of F . \square

This proposition can be strengthened to the following theorem:

Theorem 5.9. *If x is a stable solution of a tropical system A , then for any tropical system F and point y , if F^{*y} contains A^{*x} , then y is a stable point of F .*

Proof. W. l. o. g. we can assume that x and y are equal to 0, and that the coefficients of the monomials starred in A^{*0} and F^{*0} are equal to zero.

Due to the Theorem 5.5 and Theorem 5.4 we can refer to a stable points movement under a parameter perturbation. We will prove that if we change one of nonzero coefficients to zero (the resulting system we denote by G), still 0 remains a stable solution of G . The rest will immediately follow from Lemma 5.8. We will prove by contradiction.

Let 0 be an unstable solution of G . Let Δ be a minimal distance from 0 to stable solutions of G . We can choose $\epsilon > 0$ such that if we perturb G with amplitude less than ϵ then every stable solution will move by distance less than Δ .

Now consider a perturbation of G with new zero coefficient replaced by ϵ and other coefficients unchanged. By Lemma 5.8 perturbed system has 0 as a stable solution, but by choice of ϵ we get a contradiction as no stable solution could move to 0 . \square

Now we can formulate the last criterion of stability we needed (we will use functions v_n defined in Section 2):

Theorem 5.10. *Assume that for a tropical polynomial system A of n equations in n variables with a solution at x we can choose $2n$ monomials $m_{i,j}, 1 \leq i \leq n, 1 \leq j \leq 2$ with the following properties:*

- *monomials $m_{i,1}$ and $m_{i,2}$ are from i -th polynomial and they are starred in A^{*x} ,*
- *linear span of vectors $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{n,1}, m_{n,2})$ has dimension equal to n .*

Then x is a stable solution of system A , conversely if x is a stable solution we can always choose $2n$ monomials in the way described.

Proof. W. l. o. g. we can assume that x is equal to 0 , and that the coefficients of the monomials starred in A^{*0} are equal to zero.

- (1) First we will prove in one direction: if we could find monomials with described properties, then 0 is a stable point. By Theorem 5.9 if we prove that 0 is a stable point of a system with all the coefficients of monomials except $m_{i,j}, 1 \leq i \leq n, 1 \leq j \leq 2$, replaced by say 1 (this system will have only 2 monomials with zero coefficients in each equation), then 0 is a stable point of system A . And by Theorem 5.6 0 is stable iff the system with coefficients in nonzero monomials replaced by infinity will have a solution for any set of coefficients replacing zeros. Now we can notice that tropical polynomial system with two monomials in each polynomial is just a classical linear system (see Example 5.11) and restriction on monomials we gave is just a criterion of this system to have rank n . So as required this system will have a solution for any set of coefficients.
- (2) Now we will prove the converse: if we could not find monomials with described properties, then 0 is not a stable point. We will prove that we can replace all nonzero coefficients by infinity and zero coefficients by arbitrary real numbers in such a way

that the obtained system will have no solution and thus, by Theorem 5.6 0 is not a stable point.

Let's replace all zero coefficients by arbitrary real numbers which are linear independent over \mathbb{Q} and nonzero coefficients by infinity. Consider a solution of this system. Let's choose $f_{i,j}$, $1 \leq j \leq 2$ as pairs of starred monomials from i -th equation (if there are more than two starred monomials, we will choose just two arbitrary among them). Linear span of $v_n(f_{i,1}, f_{i,2})$, $1 \leq i \leq n$ has dimension lesser than n by assumption, so the system of classical linear equations, expressing that $f_{i,1} = f_{i,2}$, $1 \leq i \leq n$ will have the rank lesser than n . This system has rational coefficients of the variables, while free terms from its' equations are linear independent over \mathbb{Q} , so it has no solutions, as otherwise a rational linear dependency between these constants could be found. So we come to a contradiction and this means that 0 is not a stable point. □

Example 5.11. Tropical polynomial system:

$$\begin{cases} x_1 x_2 \oplus x_1^{\otimes 3} \\ 6x_1^{\otimes 5} \oplus 4x_2^{\otimes 2} \end{cases}$$

is equivalent to classical linear system:

$$\begin{cases} x_1 + x_2 = 3x_1 \\ 6 + 5x_1 = 4 + 2x_2 \end{cases}$$

The criterion from Theorem 5.10 of a point being a stable solution of a tropical system will be used further in our paper. In fact, this criterion can be tested in polynomial time, by means of an algorithm which produces a maximal rank subset of an intersection of two matroids, see e.g. [13], [14].

6. ESTIMATING THE NUMBER OF CONNECTED COMPONENTS FOR TROPICAL SYSTEMS WITH FINITE COEFFICIENTS

Using the theorems from the Section 5 we can bound the number of connected components of a tropical prevariety.

As in the previous section we assume that all the coefficients in a tropical system are finite.

At first we will show that every connected component contains at least one generalized vertex.

Theorem 6.1. *If a tropical prevariety V is given by a tropical polynomial system A with finite coefficients, then in any connected component of V there is at least one generalized vertex.*

Proof. Consider a point x of V . If it is not a generalized vertex then by Theorem 4.8 there is a vector along which all the directional derivatives of starred monomials in each polynomial are the same. Let's look at the star table while moving from point x forward and backward along this vector. In some neighborhood of x the star table will not change, but at some point a new star has to appear (as the coefficients of A are finite all monomials are present, so there will be at least one non-starred monomial whose derivative along the chosen vector differs from the derivatives of the starred monomials in the same polynomial, as there is no vector along which the derivatives of all the monomials are the same). Let's choose this point as a new x . By this procedure we have increased the number of stars in A^{*x} . Now we can repeat the described process. But as there is a finite number of cells in the star table, we can't repeat this process up to infinity, so at some step the chosen point x must be a generalized vertex. \square

To estimate the number of generalized vertices we will prove the following theorem:

Theorem 6.2. *For any generalized vertex x of a tropical polynomial system in n variables we can choose a multiset of n polynomials from this system in such a way that x is a stable solution for a tropical system given by the chosen polynomials (one polynomial can be chosen several times). Moreover, if there were $k \geq n$ polynomials in the initial system, then we can choose at least $k - n + 1$ different multisets of n polynomials with the described properties.*

Proof. Existence of one multiset with the described properties immediately follows from Theorem 4.11 and Theorem 5.10.

The second part of the theorem can be proved in the following way: consider a multiset $\{p_1, p_2, \dots, p_n\}$, of n polynomials with the described properties. As x is a stable point of these polynomials we can choose a set of monomials $m_{1..n,1..2}$ as described in Theorem 5.10. Now we will with each polynomial associate one vector: $a_i = v_n(m_{i,1}, m_{i,2})$. By Theorem 5.10 this vectors form a basis in \mathbb{R}^n . Now we will prove that for any polynomial p_{n+1} which is not represented in the multiset $\{p_1, p_2, \dots, p_n\}$ we can choose a polynomial p_i in such a way that x will be a stable point of the system given by polynomials $\{p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_n, p_{n+1}\}$.

As x is a solution of p_{n+1} there are at least two monomials which are starred in p_{n+1} at the point x . Let's denote them by $m_{n+1,1}$ and $m_{n+1,2}$ (if there are more than two starred monomials in p_{n+1} at the point x we will choose an arbitrary pair of starred monomials). As $\{a_1, a_2, \dots, a_n\}$ is a basis, there should be a linear combination of this vectors which will be equal to $v_n(m_{n+1,1}, m_{n+1,2})$, this means that $v_n(m_{n+1,1}, m_{n+1,2}) = ca_i + L(a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ for $c \neq 0$ and a certain $1 \leq i \leq n$, where L is a linear function. So $\{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n, v_n(m_{n+1,1}, m_{n+1,2})\}$ is a basis, and by Theorem 5.10, this means that x is a stable point of $\{p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_n, p_{n+1}\}$. As we can choose at least $k - n$ polynomials which are not included in the initial multiset, there are at least $k - n + 1$ multisets with the properties required in the theorem. \square

As a consequence from this theorem we can obtain:

Theorem 6.3. *The number of generalized vertices of a tropical prevariety in \mathbb{R}^n given by k polynomials with tropical degrees bounded by d and with finite coefficients is not greater than $\frac{d^n}{k-n+1} \binom{k+n-1}{n}$.*

Proof. We can choose up to $\binom{k+n-1}{n}$ different multisets of equations and by Theorem 5.4 system formed by each of these multisets will have at most d^n stable points. Moreover each point will be calculated at least $k - n + 1$ times due to Theorem 6.2. This implies the required bound. \square

By Theorem 6.1 we can obtain:

Corollary 6.4. *The number of connected components of tropical prevariety in \mathbb{R}^n given by k polynomials with tropical degrees bounded by d and with finite coefficients is not greater than $\frac{d^n}{k-n+1} \binom{k+n-1}{n}$.*

However this bound is not sharp, and while it's rather precise for considerably overdetermined system (in Theorem 8.1 we will show that for overdetermined systems a bound $\frac{d^n}{k-n+1} \binom{k}{n}$ can be achieved), for underdetermined systems a better bound can be proved:

Theorem 6.5. *The number of connected components of a tropical prevariety in \mathbb{R}^n given by k polynomials with tropical degree bounded by d is not greater than $\binom{d+n}{d}^{2k}$.*

Proof. A tropical prevariety is a union of at most $\binom{d+n}{d}^{2k}$ convex polyhedra, each of them given by a star table with exactly two stars in every row. \square

While this bound is not interesting for overdetermined system, for small k and d comparatively to n it can be much better than the bound from Corollary 6.4.

Now we can obtain a bound on the sum of Betti numbers (discrete Morse's theory states that in compact tropical prevariety it is bounded by the number of faces of all dimension, see e. g. [15]).

Theorem 6.6. *The sum of Betti numbers of a compact tropical prevariety given by a system of k polynomials of maximal degree at most d in n variables does not exceed $2^{\frac{d^n}{k-n+1}} \binom{k+n-1}{n}$.*

Proof. This result immediately follows from the fact that any k -dimensional face of a compact tropical prevariety contains at least $k+1$ vertices. \square

However this result is far from sharp, for example for small d it can be improved by the bound on the number of faces for *arrangements* (arrangement is a union of several hypersurfaces, see e.g. [16])

Theorem 6.7. *The sum of Betti numbers of a compact tropical prevariety given by a system of k polynomials of maximal degree d in n variables does not exceed $3^n + 2^n \binom{n+d}{n} + o((k \binom{n+d}{n})^2)$.*

Proof. The number of all faces in the arrangement could be estimated as $3^n + 2^n \binom{m}{n} + o(m^n)$ where n is a dimension and m is the number of hypersurfaces (see e.g. Buck's formula in [16]). Faces of tropical prevariety is a subset of faces of arrangement of hypersurfaces, where for every pair of monomials from the same polynomial we add a hypersurface where they are equal. Thus we obtain the required bound (the number of monomials in each polynomial does not exceed $\binom{n+d}{n}$). \square

7. TROPICAL BEZOUT INEQUALITY FOR OVERDETERMINED SYSTEMS

While the bound on the number of connected components obtained in Corollary 6.4 can be used as a bound on the number of isolated points, in this particular case it can be slightly improved. Throughout of this section we consider an algebraic variety over Hahn series $V' \in (\mathbb{C}[[T^{\mathbb{R}}]] \setminus 0)^n$

Now we can prove, that:

Theorem 7.1. *Given an overdetermined tropical polynomial system A of $k \geq n$ equations in n variables with an isolated solution at x we can always choose $2k$ monomials $m_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq 2$ with the following properties:*

- *monomials $m_{i,1}$ and $m_{i,2}$ are taken from i -th polynomial and starred in A^{*x} .*
- *the linear span of vectors $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{k,1}, m_{k,2})$ has dimension equal to n .*

Proof. W. l. o. g. we can assume that $x = 0$ (we can always shift a prevariety in such a way that it is). In the proof we will refer to $v_n(x, y)$ as a vector given by the pair (x, y) .

Let's prove by contradiction. Consider $2k$ monomials $m_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq 2$ and number l with the following properties:

- monomials $m_{i,1}$ and $m_{i,2}$ are taken from i -th polynomial and starred in A^{*0} ,
- the linear span of vectors $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{k,1}, m_{k,2})$ has dimension equal to l ,
- for any other $2k$ monomials $m'_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq 2$, where $m'_{i,1}, m'_{i,2}$ are taken from i -th polynomial and starred in A^{*0} , the linear span of vectors $v_n(m'_{1,1}, m'_{1,2}), v_n(m'_{2,1}, m'_{2,2}), \dots, v_n(m'_{k,1}, m'_{k,2})$ has dimension equal or less than l .

Let's denote the linear span of $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{k,1}, m_{k,2})$ by \mathcal{L} . W. l. o. g. let's assume, that $\{v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{l,1}, m_{l,2})\}$ is a basis of \mathcal{L} . Let's notice that if $i > l$ and $m'_{i,1}$ and $m'_{i,2}$ are taken from i -th polynomial and starred in A^{*0} then $v_n(m'_{i,1}, m'_{i,2})$ is contained in \mathcal{L} (otherwise the linear span of $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{l,1}, m_{l,2}), v_n(m'_{i,1}, m'_{i,2})$ will be equal to $l + 1$ which contradicts our assumption).

As $v_n(m'_{i,1}, m'_{i,2})$ is contained in \mathcal{L} we can express it as a linear combination of the basis vectors. Let's notice that if $v_n(m_{j,1}, m_{j,2})$ occurs in the expression of $v_n(m'_{i,1}, m'_{i,2})$ with a non-zero coefficient we can swap j -th and i -th polynomials and at the same time swap $m_{j,1}, m_{j,2}$ and $m'_{i,1}, m'_{i,2}$ and the linear span of $v_n(m_{1,1}, m_{1,2}), v_n(m_{2,1}, m_{2,2}), \dots, v_n(m_{k,1}, m_{k,2})$ will remain the same. So we can choose a set S of $g \leq l$ polynomials from the first l polynomials in such a way that a vector given by any pair of monomials taken from polynomials from S and starred in A^{*0} or polynomials standing after l -th in A will lay in the linear span of the vectors $v_n(m_{i,1}, m_{i,2}), i \in S$.

Let's replace all the monomials, except $m_{i,j}, i \in S, 1 \leq j \leq 2$ by infinity and consider system B of the first l polynomials (some of them changed at the previous step). Obviously it has a solution at 0. And locally any solution of this system is a solution of the initial system (every vector given by a pair of monomials starred in A^{*0} from the polynomials from S in any polynomial can be represented as a linear combination of vectors $v_n(m_{i,1}, m_{i,2}), i \in S$). So the local dimension of the prevariety given by system B at 0 is zero. But it contradicts Theorem 3.8 as $l < n$. \square

We note that the proof of the latter theorem is somewhat similar to the proof of Theorem 41.1 on the intersection of matroids in [13].

Which leads to:

Theorem 7.2. *The number of isolated solutions of an overdetermined tropical polynomial system of $k \geq n$ polynomials in n variables is not greater than $\frac{\binom{k}{n}}{(k-n+1)}D$, where D is the product of n greatest degrees of the given polynomials.*

Proof. By theorems 7.1, 5.6 we can state that any isolated solution is a stable solution of some subsystem of the size n (by a suitable shift of variables we can always shift the solution to 0). There are less or equal than $\binom{k}{n}$ subsystems and by Bezout's equality each of them has at most D stable solutions. Moreover each solution is counted at least $k - n + 1$ times (the reasoning is the same as in Theorem 6.2). \square

Observe that (similar to the end of Section 5) the algorithm which produces a maximal rank subset of intersection of two matroids allows one to test with polynomial complexity, whether a given solution of a tropical polynomial system is isolated.

As we will show in the next section, this bound is close to sharp.

8. LOWER BOUNDS ON THE NUMBER OF ISOLATED TROPICAL SOLUTIONS

In this section we will build an example, which shows that Bezout inequality in case of tropical polynomial systems is close to sharp. While we will omit some monomials (i.e. we will use infinite coefficients), example like this can be build with finite coefficients only (if infinite coefficients are replaced by sufficiently large finite numbers).

Theorem 8.1. *Given n we can build a series of tropical systems of $k(n - 1), k \geq 3$ equations in $n \geq 2$ variables of degree $4d, d \geq 1$ in such a way that the number of solutions of systems from this set is $2(k - 1)^{n-1}d^n$.*

FIGURE 2. Hypersurface (curve) H_1 given by tropical polynomial $3 \oplus 1x_1 \oplus x_1x_2 \oplus 1x_2 \oplus x_1x_2^{\otimes 2} \oplus 2x_2^{\otimes 2}$ and its Newton's polygon. α is equal to 2 and β is equal to 3 in the picture.

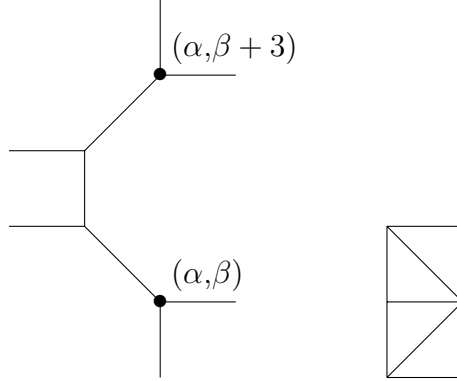
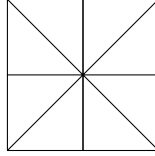


FIGURE 3. Newton's polygon used in Theorem 8.1



Proof. Consider a tropical polynomial system in 2 variables:

$$A = \begin{cases} 3 \oplus 1x_1 \oplus x_1x_2 \oplus 1x_1 \oplus x_1x_2^{\otimes 2} \oplus 2x_2^{\otimes 2} \\ 3 \oplus 1x_1 \oplus 3x_1x_2 \oplus 4x_2 \oplus 6x_1x_2^{\otimes 2} \oplus 8x_2^{\otimes 2} \\ 3 \oplus 1x_1 \oplus 6x_1x_2 \oplus 7x_2 \oplus 12x_1x_2^{\otimes 2} \oplus 14x_2^{\otimes 2} \\ \dots \\ 3 \oplus 1x_1 \oplus (3k-3)x_1x_2 \oplus (3k-2)x_2 \oplus (6k-6)x_1x_2^{\otimes 2} \oplus (6k-4)2x_2^{\otimes 2} \end{cases}$$

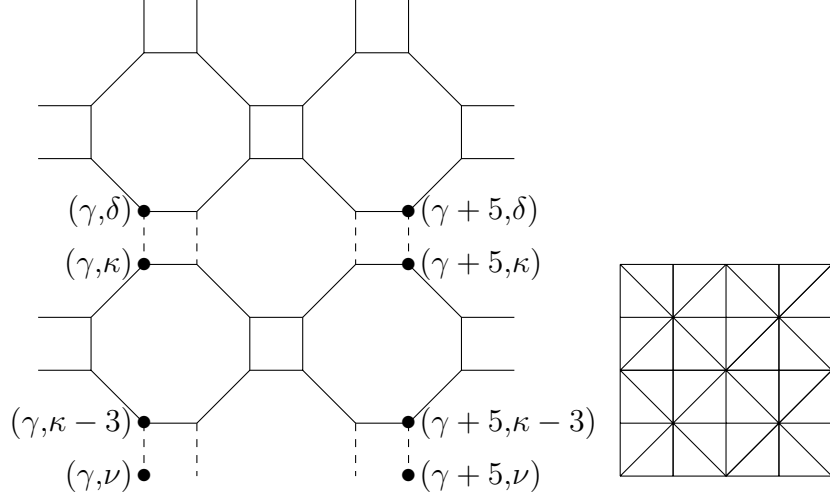
The graph of the hypersurface H_1 given by the first polynomial is depicted on Figure 2. The prevariety (curve) H_i of the i -th polynomial of A is obtained from H_1 by a vertical shift down by $3i - 3$.

Therefore, the points $(\alpha, \beta - 3j)$, $0 \leq j \leq k - 2$ are solutions of A .

Moreover, these points are isolated solutions since the prevariety of A consists of these points and of two vertical half-lines.

Now we construct a tropical system B in 2 variables consisting of k polynomials of degrees $4d$ for any $d \geq 1$. The Newton's polygon of each of these polynomials is a square with the mesh $2d$ which is obtained from the 2×2 square depicted in Figure 3 by replicating it. The coefficients of these polynomial are chosen with suitable conditions imposed

FIGURE 4. Example for Theorem 8.1



on the distances which follow. The curve of the first polynomial of B is depicted on Figure 4. The curve consists of d horizontal layers of d hexagons each of a height and a width equal to 3 each obtained from the previous one by a vertical shift. We impose the condition that the first shift (which is equal $\delta - \kappa$) is greater than $3k$. In a similar way the second shift $(\kappa - 3) - \nu$ is also greater than $3k$ and so on. The other polynomials are chosen in the way similar to system A : they give curves which are vertical shifts of the curve given by the first polynomial. The second curve is shifted down by 3, the third curve is shifted down by 6, \dots , the k -th curve is shifted down by $3(k - 1)$.

Solutions of B form d series of isolated points, each series consist of $2(k - 1)d$ points in each series and $2d$ half-lines (for each $0 \leq i \leq d - 1$) a series has isolated points: $(\gamma + 4i, \delta)$, $(\gamma + 4i + 1, \delta)$, $(\gamma + 4i, \delta - 3)$, $(\gamma + 4i + 1, \delta - 3)$, \dots , $(\gamma + 4i, \delta - 3k + 3)$, $(\gamma + 4i + 1, \delta - 3k + 3)$; $(\gamma + 4i, (\kappa - 3) - 3)$, $(\gamma + 4i + 1, (\kappa - 3) - 3)$, $(\gamma + 4i, (\kappa - 3) - 6)$, $(\gamma + 4i + 1, (\kappa - 3) - 6)$, \dots , $(\gamma + 4i, (\kappa - 3) - 3k + 3)$, $(\gamma + 4i + 1, (\kappa - 3) - 3k + 3)$; and so on).

Now consider system C_n in n variables which consists of polynomials from B repeated $n - 1$ times with x_2 replaced by unchanged in the first copy, replaced by x_3 in the second, by x_4 in the third, \dots , by x_n in the last, respectively. The isolated solutions of system C_n form a n -dimensional lattice consisting of $2(k - 1)^{n-1}d^n$ points (there are $2d$ series each with a fixed value of coordinate x_1 containing $((k - 1)d)^{n-1}$ isolated points).

Remark 8.2. System A in 2 variables consisting of k cubic tropical polynomials, has a linear in k number of solutions. On the contrary, one can prove that a system in 2 variables consisting of an arbitrary number of quadratic tropical polynomials, has at most 72 solutions.

□

9. COMPATIFICATION OF TROPICAL PREVARIETIES

In this section we will show that for any tropical prevariety V we can build a compact tropical prevariety being homotopy equivalent to V . This technique can be used to reduce the problem of estimating the number of connected components of tropical prevarieties to the case of compact prevarieties.

We will use the following theorem:

Theorem 9.1. *Given a tropical prevariety V we can find a constant s such that the intersection of V and a cube with the side equal to s and centered at the origin would be homotopy equivalent to V . In this theorem we allow the prevariety to be given by a system with infinite coefficients.*

This theorem can be viewed as a simplification of Lemma 9 proved in [17], or it can be proved directly with the help of Cramer's rule in the same way as it was used in Theorem 5.6.

Theorem 9.2. *Consider a tropical prevariety V given by a tropical system A in n variables. We can add $2n$ extra variables and $4n$ extra polynomials which being added to system A will form a system B which determines a compact tropical prevariety W being a homotopy equivalent to V .*

Proof. Let $2s$ be a side of the cube from Theorem 9.1. For each variable x_i we will add two variables: u_i and v_i ; and four (linear) polynomials: $x_i \oplus u_i$, $x_i \oplus u_i \oplus s$, $v_i \oplus -s$ and $x_i \oplus v_i \oplus -s$. The first two polynomials will guarantee that $u_i = x_i \leq s$, and the last two will guarantee that $x_i \geq v_i = -s$. Therefore, W is homeomorphic to $V \cap [-s, s]^n$. Prevariety W is compact and by Theorem 9.1 it is homotopy equivalent to V . While there are infinite coefficients in the added polynomials this is not a problem as we can assume that all infinite coefficients are equal to $M + 2s^d$, where M is a maximal coefficient occurring in the polynomials from A and d is a maximal degree of polynomials from A . In that case monomial with this coefficient could not be minimal as all variables are guaranteed to be less or equal than s . □

All results from the previous sections required all the coefficients in a tropical system to be finite (as they were based on Bezout's theorem, which was proved only in the case of finite coefficients). However Theorem 9.2 gives us the following generalizations to the case of infinite coefficients:

Corollary 9.3. *The number of connected components of a tropical prevariety given by k polynomials with tropical degrees bounded by d and with allowed infinite coefficients is not greater than $\frac{d^{3n}}{k+n+1} \binom{k+7n-1}{3n}$.*

Corollary 9.4. *The number of isolated solutions of an overdetermined tropical polynomial system of k equations in n variables with allowed infinite coefficients is not greater than $\frac{\binom{k+4n}{3n}}{(k+n+1)} D$, where D is the product of n greatest degrees of the given polynomials.*

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