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## 1. Introduction.

Let $G_n = \mathrm{Aff}(\mathbb{R}^n)$ be the group of affine transformations of $\mathbb{R}^n$. The group $G_n$ is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ is identified with its group of translations. A subsemigroup $S$ of $G_n$ is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset $K$ of $\mathbb{R}^n$ the set $\{g \in S : gK \cap K \neq \emptyset\}$ is finite. If a discrete group consists of isometries, then it acts properly discontinuously on $\mathbb{R}^n$. But this is not true for an arbitrary discrete subgroup of $G_n$, e.g. for an infinite discrete subgroup of $GL_n(\mathbb{R})$. A subsemigroup $S$ of $G_n$ is called *crystallographic* if $S$ acts properly discontinuously on $\mathbb{R}^n$ and there exists a compact subset $K_0$ of $\mathbb{R}^n$ such that $\bigcup_{s\in S} sK_0 = \mathbb{R}^n$.

If the signature of a nondegenerate quadratic form $B$ on $\mathbb{R}^n$ is $(n-1,1)$, then the form $B$ is called *hyperbolic*. Let $O(B)$ (resp. $SO(B)$) denote the orthogonal (resp. special orthogonal) group of $B$. Let $G_B$ be the subgroup of $G_n$ leaving the form $B$ invariant. It is clear that $G_B$ is the semidirect product $O(B) \ltimes \mathbb{R}^n$.

The motivating question here is the following:

***Question*** (H.Abels, G.Margulis, G.Soifer). *Is the Zariski closure of a crystallographic affine semigroup leaving a hyperbolic form invariant a virtually solvable group ?*

Our interest in this questions has two aspects. One is conceptual: Does the geometry of an action determine the algebraic properties of the acting semigroup? The second aspect came from our joint works with H.Abels and G.Margulis on the Auslander conjecture and our study of the dynamic of the action of affine transformations ([AMS1],[AMS2],[AMS3],[AMS4], [AMS 5]). The purpose of the present work is to relate Margulis' sign of an affine transformation to the study of the action of affine semigroups. We introduce the notion of subsets $X_\infty(S, K)$ and $X_\infty(S)$ of the unit sphere $S^n(0, 1) \subset \mathbb{R}^n$ for a semigroup $S$ of the affine group $G_n$ and a compact subset $K \subset \mathbb{R}^n$ (see 2.4 for definitions). These sets play an important role in the study of dynamics of affine transformations [AMS 5]. Roughly speaking $X_\infty(S)$ is the set of all possible directions *"at infinity"* for the translation part of an affine transformation $s$ of $S$. Remark, $X_\infty(S, K) \subseteq X_\infty(S)$ for every compact subset $K$ of $\mathbb{R}^n$.

The goal of the present work is to prove the following

***Main Theorem.*** *Let $S \subseteq Aff(\mathbb{R}^n), n \leq 3$ be a subsemigroup. Assume that there exists a compact subset $K$ of $\mathbb{R}^n$ such that $X_\infty(S, K) = S^3(0, 1)$. Then the Zariski closure of $S$ is a virtually solvable group.*

Using the same arguments and slightly changing the proof reader can show that if $X_\infty(S)$ is dense in $S^3(0, 1)$ then the Zariski closure of the semigroup $S$ acting properly discontinuously is a virtually solvable group.

By Lemma 2.5 (3), we have $X_\infty(S) = X_\infty(S, K) = S^n(0, 1)$. for a crystallographic semigroup $S$ and every compact subset $K \subset \mathbb{R}^n$. Hence

***Corollary 1*** *Let $S \subseteq Aff(\mathbb{R}^n), n \leq 3$ be a crystallographic semigroup. Then the Zariski closure of $S$ is a virtually solvable group.*

***Remark.*** There is no hypothesis about an invariant form in the main theorem and corollary 1. It is absolutely unclear if a semigroup which acts properly discontinuously on $\mathbb{R}^n$ such that $X_\infty(S) = S^n(0,1)$ is a crystallographic semigroup.

Obviously we have

***Corollary 2*** ( see [GF]) *Let $\Gamma \leq Aff(\mathbb{R}^n), n \leq 3$ be a crystallographic group. Then $\Gamma$ is a virtually solvable group.*

We remark that the proof in [GF] is based on completely different ideas. W.Goldman and Y. Kamishima proved in [GK] the following theorem

***Theorem*** *Let $\Gamma$ be a crystallographic group leaving a hyperbolic form invariant, then $\Gamma$ is virtually solvable.*

Let us state the following conjecture

***Conjecture*** *Let $S$ be a semigroup (or a group), $S \subseteq G_B$, where $B$ is a hyperbolic form. Assume that $X_\infty(S)$ is a dense subset of the unit sphere $S^n(0,1)$ of $\mathbb{R}^n$. Then the Zariski closure of $S$ is a virtually solvable group.*

Since the Zariski closure of $S$ is virtually solvable, using almost the same arguments as we have used in the final stages of the proof of Theorem 2 in ( [GS 2]), one can show that

*A crystallographic semigroup $S$ of $Aff(\mathbb{R}^n), n \leq 3$, leaving a hyperbolic form invariant is a group.*

The example below shows that there exists a semigroup $S$ which is not a group, such that $S$ acts properly discontinuously $\mathbb{R}^n$.

***Example.*** Let $T = \{v_1, \ldots, v_k\}$ be a set of vectors of $\mathbb{R}^n$. Assume that the convex hull $Conv(T)$ of $T$ does not contain the zero vector. Let $S$ be a subsemigroup of $Aff(\mathbb{R}^n)$ generated by the translations $\widetilde{T} = \{t_{v_1}, \dots, t_{v_k}\}$. Let us show that $S$ acts properly discontinuously on $\mathbb{R}^n$. Indeed, since $0 \notin Conv(T)$, there exists $v_0 \in \mathbb{R}^n$ such that the scalar product $(v_0, v_i) > 0, i = 1, \dots, k$. Put $n_0 = \min_{1 \leq i \leq k}(v_0, v_i)$. Clearly, $n_0 > 0$. Let $s$ be an element of the semigroup $S$. Then $s$ is a translation by a vector $v_s$, where $v_s = n_1 v_1 + \dots + n_k v_k, n_i \in \mathbb{Z}, n_i \geq 0, i = 1, \dots, k$. Assume that $\|v_s\| \leq c$. Then we have $c\|v_0\| \geq |(v_s, v_0)| \geq n_0 \sum_1^k n_i$. Thus, $0 \leq n_i \leq c\|v_0\|/n_0$ for all $i = 1, \dots, k$. Therefore the set of vectors $\{v \in S \,|\, \|v\| \leq c\}$ is finite for every constant $c$. Hence $S$ acts properly discontinuously on $\mathbb{R}^n$. In contrast to this, the group generated by the set $\widetilde{T}$ is not discrete in general and therefore does not act properly discontinuously on $\mathbb{R}^n$.

It is clear that if $\mathbb{R}^n = Conv(T)$ then for the semigroup $S$ generated by $T$ there exists a compact subset $K_0$ of $\mathbb{R}^n$ such that $\bigcup_{s \in S} sK_0 = \mathbb{R}^n$. By using the technique presented in our paper [AMS 5] and the ideas of the example above it is possible, but not obvious, to construct a free subsemigroup $S$ of $G_B$ acting properly discontinuously on $\mathbb{R}^n$ in case $B$ is a quadratic form of signature $(k, k-1)$ where $k$ is even and $2k - 1 = n$. On the other hand, by choosing a subset $T$ such that $0 \in Conv(T)$, it is possible to construct a free semigroup $S$ of $G_B$, such that there exists a compact subset $K_0$ of $\mathbb{R}^n$ such that $\bigcup_{s \in S} sK_0 = \mathbb{R}^n$. However it will not act properly discontinuously on $\mathbb{R}^n$.

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## *2. Preliminaries.*

In order to make the exposition as self-contained as possible, we first collect the information needed in the proofs.

In this section we introduce the terminology we will use throughout the paper and recall terminology and results from [A], [AMS 1], [AMS 2], [AMS 3], [AMS 4], [AMS 5] and [BG]. We will prove some basic lemmas about the geometry and dynamics of the action of an affine transformation under the assumption that its linear part is hyperbolic.

***2.0.*** Let $V$ be a finite dimensional vector space over a local field $k$ with absolute value $|\cdot|$, and let $P = \mathbb{P}(V)$ be the projective space corresponding to $V$. Let $g \in GL(V)$ and let $\chi_g(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) \in k[\lambda]$ be the characteristic polynomial of the linear transformation $g$. Set $\Omega(g) = \{\lambda_i : |\lambda_i| = \max_{1\leq j\leq n} |\lambda_j|\}$. Put $\chi_1(\lambda) = \prod_{\lambda_i\in\Omega(g)}(\lambda - \lambda_i)$ and $\chi_2(\lambda) = \prod_{\lambda_i\notin\Omega(g)}(\lambda - \lambda_i)$. Then $\chi_1$ and $\chi_2$ belong to $k[\lambda]$ since the absolute value of an element is invariant under Galois automorphisms, . Therefore $\chi_1(g) \in GL(V)$ and $\chi_2(g) \in GL(V)$. Let us denote by $V(g)$ (resp. $W(g)$) the subspace of $V$ corresponding to $\ker(\chi_1(g))$ (resp. $\ker(\chi_2(g))$). We will often use for an element $g \in GL(V)$ the following notation, $V(g) = V^+(g)$, $W(g) = W^-(g)$, $V(g^{-1}) = V^-(g)$ and $W(g^{-1}) = W^+(g)$. Let $\lambda_-(g) = \max\{|\lambda\ | :\ \lambda$ is an eigenvalue of $g$ of absolute value less than 1 $\}$. Let $\lambda_+(g) = \min\{|\lambda\ | :\ \lambda$ is an eigenvalue of $g$ of absolute value more than $1\}$. Put $\lambda(g) = \max\{\lambda_+^{-1}(g), \lambda_-(g)\}$. It is clear that $\lambda(g) = \lambda(g^{-1})$.

***2.1.*** Recall that $g \in GL(V)$ is called *proximal* if $\dim(V^+(g)) = 1$. A proximal element $g$ has a unique eigenvalue of maximal absolute value. Hence this eigenvalue has algebraic and geometric multiplicity one. For $S \subseteq GL(V)$ set $\Omega_0(S) = \{g \in S :$ $g$ and $\mathrm{g}^{-1}$ are proximal$\}$. A semisimple element $g \in \Omega_0(GL(V))$ is called *dipole.*

Let $g$ be a semisimple element in $GL(\mathbb{R}^n)$. Then the space $\mathbb{R}^n$ can be decomposed into the direct sum of three subspaces $A^+(g)$, $A^-(g)$ and $A^0(g)$ determined by the condition that all eigenvalues of the restriction $g \mid A^+(g)$ (resp. $g \mid A^-(g)$, $g \mid A^0(g)$ ) have an absolute value more than 1 (resp. less than 1, equal to 1). Put $D^+(g) = A^+(g) \oplus A^0(g)$ and $D^-(g) = A^-(g) \oplus A^0(g)$. Obviously $D^+(g) \cap D^-(g) = A^0(g)$. Let $G$ be a subgroup of $GL(V)$. If $\dim(A^0(g)) = \min\{\dim A^0(t) | \, t \in G\}$, then $g$ is called *regular* of $G$.

***2.2.*** Let $\| \centerdot \|$ and $d$ denote the norm and metric on $\mathbb{R}^n$ corresponding to the standard inner product on $\mathbb{R}^n$. Let $P = \mathbb{P}(\mathbb{R}^n)$ be the projective space corresponding to $\mathbb{R}^n$. Let $\|g\|_+$ be the norm of the restriction $g|_{A^-(g)}$. Denote by $\|g\|_- = \|g^{-1}\|_+$ and put $s(g) = \max\{\|g\|_+, \|g\|_-\}$. A regular element $g$ is called *hyperbolic* if $s(g) < 1$. Let $\pi : \mathbb{R}^n \diagdown \{0\} \to P$ be the natural projection. For a subset $X$ of $\mathbb{R}^n$ not containing 0, we put $\pi(X) = \pi(X \diagdown \{0\})$ .

The metric $\| \centerdot \|$ on $\mathbb{R}^n$ induces the metric $\widehat{d}$ on the projective space $P = \mathbb{P}(\mathbb{R}^n)$. Thus for any point $p \in P$ and a subset $A \subseteq P$, we can define

$$\widehat{d}(p, A) = \inf_{a \in A} \widehat{d}(p, a).$$

Let $A_1$ and $A_2$ be two subsets of $P$. We define

$$\widehat{d}(A_1, A_2) = \inf_{a_1 \in A_1, a_2 \in A_2} \widehat{d}(a_1, a_2).$$

For two subspaces $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ we put $\widehat{d}(V, W) = \widehat{d}(\pi(V/\{0\}), \pi(W/\{0\}))$. Let $B$ be a quadratic non-degenerated form. et where $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$. A hyperbolic element $g \in SO(B)$ is called *$\varepsilon$-hyperbolic*,

$$\widehat{d}(A^+(g), D^-(g)) \geq \varepsilon$$

and

$$\widehat{d}(A^-(g), D^+(g)) \geq \varepsilon.$$

Two different hyperbolic elements $g_1$ and $g_2$ are called *transversal* if $A^{\pm}(g_1) \bigcap D^{\mp}(g_2) = \{0\}$ and $A^{\pm}(g_2) \bigcap D^{\mp}(g_1) = \{0\}$. Let us make a simple but very useful remark. Let $g \in SO(B)$. For a regular element $g \in SO(B)$, the space $A^+(g)$ ( resp. $A^-(g)$) is the unique maximal isotropic subspace of $D^+(g)$ ( resp. $D^-(g)$) Therefore two hyperbolic elements $g_1$ and $g_2$ are transversal, if and only if, $A^+(g_1) \cap A^-(g_2) = \varnothing$ and $A^-(g_1) \cap A^+(g_2) = \varnothing$. We prove now the following useful proposition

***Proposition 2.3*** *Let $S$ be a Zariski dense semigroup of $SO(B)$, where $B$ is a quadratic from of the signature $(2,1)$. Let $A = \{A_1, \dots, A_r\}$ be a finite set maximal $B$–isotropic subspaces of $\mathbb{R}^3$. Then there exists a hyperbolic element $s \in S$ such that $A^+(s) \notin A$ and $A^-(s) \notin A$.*

*Proof.* . Let $\Omega$ be the set of regular elements of $SO(B)$. This set is Zariski open. It is clear, that the set $\Omega_1 = \{w \in SO(B) |\, wA_i \cap A_i = \{0\}$ for all $i = 1, \dots, r\}$ is non-empty and Zariski open. Therefore $\Omega_1 \cap \Omega \cap S \neq \varnothing$. Let $s \in \Omega_1 \cap \Omega \cap S$. Assume that $(A^+(s) \cup A^-(s)) \cap A \neq \varnothing$. Then for some $i, 1 \leq i \leq r$ we have $sA_i = A_i$. Contradiction. Clearly, there exists $N$ such that $s^n$ is a hyperbolic element for $n > N$, $A^+(s^n) = A^+(s)$, and $A^-(s^n) = A^-(s)$. This proves the statement. □

Two transversal hyperbolic elements $g_1$ and $g_2$ are called $\varepsilon$- *transversal*,

$$\min_{1 \leq i \neq j \leq 2} \{\widehat{d}(A^+(g_i), D^-(g_j)), \widehat{d}(A^-(g_i), D^+(g_j))\} \geq \varepsilon.$$

Let $l : G_n \to GL_n(\mathbb{R})$ be the natural homomorphism (see [A]). Recall that $l(g)$ is called the *linear part* of an affine transformation $g$. Let $X \subseteq G_n$, then the set $l(X) = \{l(x), x \in X\}$ is called the *linear part* of $X$. It is clear that $G_B = \{x \in G_n \mid l(x) \in O(B)\}$ and that $l(G_B) = O(B)$. An affine transformation is called *dipole* (respectively *hyperbolic*, $\varepsilon -$ *hyperbolic*) if $l(g)$ is dipole (respectively $l(g)$ is hyperbolic, $l(g)$ is $\varepsilon$- hyperbolic). Two affine transformations $g_1$ and $g_2$ are called *transversal* (respectively $\varepsilon -$ *transversal*) if the linear parts $l(g_1)$ and $l(g_2)$ are transversal (respectively $\varepsilon$ -transversal).

Let $g \in G_B$ be a hyperbolic element without fixed points. Then there exists a $g$–invariant line $L_g$, and the restriction of $g$ to $L_g$ is a translation by a non- zero vector $t_g$. Let us note that all such lines are parallel; $t_g$ does not depend on the choice of $L_g$ and $l(g)t_g = t_g$. We will assume that we have fixed some point in the affine space $\mathbb{R}^n$ as a point of origin and we define $L_g$ to be the $g$ -invariant line closest to it . Define affine subspaces $C_g^0 = L_g + A^0(g)$, $C_g^+ = D^+(g) + L_g$, $C_g^- = D^-(g) + L_g$. Clearly, $C_g^+ \cap C_g^- = C_g^0$. Let as recall the following useful observation. If a subsemigroup $S \subseteq G_B$ acts properly discontinuously and $g \in S$ is a hyperbolic element, then $g$ acts without fixed points. Then the linear part $l(g)$ of every hyperbolic element $g \in S$ has 1 as an eigenvalue ([A], Lemma 6.1) and thus $t_g \neq 0$ and $l(g)t_g = t_g$. Actually Lemma 6.1 [A] says that every element of $S$ of infinite order has 1 as an eigenvalue.

***2.4.*** For a non-zero vector $v$, $v \in \mathbb{R}^n$, we denote by $L_v^+ = \{tv, t \in \mathbb{R}, t > 0\}$. Let $S$ be a semigroup of $G_n$ and $M \subset S$. Let $K$ be a compact subset of $\mathbb{R}^n$. We consider the set of norm one vectors $X_\infty(M,K)$ defined as follows: $v \in X_\infty(M,K)$ if $\|v\| = 1$ and there exist a constant $C = C(v,K)$ and a sequence of points $\{p_i\}_{i\in\mathbb{N}} \subseteq K$ and a sequence of elements $\{s_i\}_{i\in\mathbb{N}} \subseteq M$ such that $d(s_ip_i, p_i) \to \infty$ and $d(s_ip_i, L_v^+) \leq C$. Obviously, $s_ip_i - p_i/\|s_ip_i - p_i\| \to v$ when $i \to \infty$

Clearly $X_\infty(M,K_1) \subseteq X_\infty(M,K_2)$ if $K_1 \subseteq K_2$. It is easy to see, that for every compact $K$ and element $s$ of $S$ we have $X_\infty(M,K) = X_\infty(Ms, s^{-1}K)$. Let $U(0,R)$ be the closed ball of $\mathbb{R}^n$ with center at 0 of radius $R$. Let $X_\infty(S)$ be the closure of the set $\bigcup_{N\in\mathbb{N}} X_\infty(S, U(0,N))$

***Lemma 2.5.*** *Let $S$ be a semigroup $S \subseteq G_n$. Then*

1. *For every two compact subsets $K_1$ and $K_2$ in $\mathbb{R}^n$ $K_1 \subset K_2$ and $M \subset S$ we have $X_\infty(M,K_1) \subset X_\infty(M,K_2)$*
2. *For every $v \in X_\infty(S)$ and $s \in S$, we have $sv/\|sv\| \in X_\infty(S)$.*

*3. If $S$ is a crystallographic semigroup, then for every compact subset $K$ we have $X_\infty(S,K) = S^n(0,1)$. Therefore $X_\infty(S) = S^n(0,1)$.*

*Proof.* The proof is straightforward. □

Assume that $S \subset G_B$ is a semigroup such that the linear part $l(S)$ is Zariski dense in $SO(B)$ where $B$ is a non-degenerated quadratic form of signature $(p,q), p \geq q$ Denote by $\Omega_\varepsilon(S) = \{s \in S | \, s \text{ is an } \varepsilon\text{–hyperbolic element } \}$. Let us recall the following result ([AMS 1], Theorem1). Let $\Gamma$ be a Zariski dense semigroup of $SO(B)$. Then there exist an $\varepsilon = \varepsilon(\Gamma)$ and a finite set of elements $\Gamma_0 = \{\gamma_1, \ldots, \gamma_r\} \subset S$, $r \leq (p+q)^2$ such that for every $\gamma \in \Gamma$ there exists a suitable element $\gamma_i$ of the set $\Gamma_0$ such that $\gamma\gamma_i \in \Omega_\varepsilon(\Gamma)$

***Lemma 2.6*** Let $K$ be a compact subset. Then there exists a compact set $K_1$ such that

$X_\infty(S,K) \subset X_\infty(\Omega_\varepsilon(S), K_1)$.

*Proof.* Let $v$ be a vector of $X_\infty(S,K)$. Then there exist two sequences $\{g_n\}_{n\in\mathbb{N}} \subset S$ , $\{k_n\}_{n\in\mathbb{N}} \subset K$ and a constant $C = C(K,v)$ such that $g_nk_n - k_n/\|g_nk_n - k_n\| \to v$ when $n \to \infty$. By the theorem above, there exists a finite subset $S_0 \subset S$ such that for every $g_n$ we have $g_ns_i \in \Omega_\varepsilon(S)$ for a suitable $s_i \in S_0, i = i(n)$. The set $K_1 = \bigcup_{s\in S_0} s^{-1}K$ is compact, since the set $S_0$ is finite. Clearly, $v \in X_\infty(\Omega_\varepsilon(S), K_1)$. □

Now we will recall an important definition first introduced by G. Margulis [GM 1] for $n = 3$, generalized in [AMS 2] for the case when the signature of the quadratic form is $(k+1,k)$ and finally for an arbitrary quadratic form in [AMS 4]. We will follow along the lines of [AMS 4]. Let $B$ be a quadratic form of signature $(p,q)$, $p \geq q, p+q = n$. Let $v_1, v_2, \ldots, v_p$, $w_1, w_2, \ldots, w_q$ is a basis of $\mathbb{R}^n$ such that for a $v$ of $\mathbb{R}^n$ $v = x_1v_1 + \cdots + x_pv_p + y_1w_1 + \cdots + y_qw_q$, we have

$$B(v,v) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2.$$

Consider the set $\Psi$ of all maximal $B$–isotropic subspaces. Let $X$ be the subspace spanned by $\{v_1, v_2, \dots, v_p\}$ and $Y$ be the subspace spanned by $\{w_1, w_2, \dots, w_q\}$. It is clear that $\mathbb{R}^n = X \oplus Y$. Define the cone

$$\mathbb{C}_B = \{v \in \mathbb{R}^n | B(v, v) < 0\}.$$

Clearly $Y \subset \mathbb{C}_B$. We have the two projections

$$\pi_X : \mathbb{R}^n \longrightarrow X \text{ and } \pi_{\mathrm{Y}} : \mathrm{\mathbb{R}^n} \longrightarrow \mathrm{Y}$$

along $Y$ and $X$, respectively. The restriction of $\pi_Y$ to $V \in \Psi$ is a linear isomorphism $V \longrightarrow Y$. Hence if we fix an orientation on $Y$, then we have also fixed an orientation on each $V \in \Psi$. For $V \in \Psi$ let us denote the $B$–orthogonal s of $V$ by $V^\perp = \{z \in \mathbb{R}^n \ ; \ B(z, V) = 0\}$. We have $V \subset V^\perp$ since $V$ is $B$–isotropic. We also have

$$\dim V^\perp = \dim V + (p - q) = p.$$

The restriction of $\pi_X$ to $V^\perp$ is a linear isomorphism $V^\perp \longrightarrow X$. Hence if we fix an orientation on $X$, then we have also fixed an orientation on $V^\perp$ for each $V \in \Psi$. Thus we have orientations on both $V$ and $V^\perp$ and we have naturally induced an orientation on any subspace $W$, such that $V^\perp = W \oplus V$. If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal, then $W = V_1^\perp \cap V_2^\perp$ is a subspace that is transversal to both $V_1$ and $V_2$; therefore $W \oplus V_1 = V_1^\perp$ and $W \oplus V_2 = V_2^\perp$. So there are two orientations $\omega_1$ and $\omega_2$ on $W$, where $\omega_i$ is defined if we consider $W$ as a subspace in $V_i^\perp$. We have [AMS 3,Lemma 2.1]

***Lemma 2.7.*** *The orientations defined above on $W$ are the same if $q$ is even and they are opposite if $q$ is odd.*

***2.8.*** Assume now that $B$ is of signature $(k+1, k)$. Let $V$ be a maximal $B$–isotropic space and $W = V^\perp$. Following along the procedure above we choice and fix a positive orientation on $W$, namely, we have an oriented basis $v_1, v_2 \dots, v_k$ in $V$ and a vector $v_{k+1} \in W$ such that the bases $v_1, v_2 \ldots, v_k, v_{k+1}$ is positively oriented. Hence an anisotropic vector $w \in W$ is called *positive* (resp. *negative*)if $B(v, v_{k+1}) > 0$ (resp $B(v, v_{k+1}) < 0$.)

Let $g$ be a hyperbolic element without fixed points, $g \in G_B$. Then, $D^+(g) = (A^+(g))^\perp$ and $D^-(g) = (A^-(g))^\perp$, $\dim A^+(g) = \dim A^-(g) = k$ and $\dim A^0(g) = 1$. We define an orientation on the space $A^0(g)$ induced by an orientation on $D^+(g)$ (see [AMS 2], [AMS 4]). Let $v_0(g)$ be the corresponding vector, with $B(v_0(g), v_0(g)) = 1$. Then $v_0$ is a positive vector of $D^+(g)$. On the other hand,$A^0(g) \subset D^-(g)$. Let $w_0(g)$ be a positive vector of $D^-(g),\ w_0(g) \in A^0(g), B(w_0, w_0) = 1$. Then by Lemma 2,7, we have $B(v_0, w_0) = (-1)^k$. Clearly $B(v_0, w_0) = -1$ when the signature of the form $B$ is $(2, 1)$.

Thus $C^0(g)$ is a $g$-invariant line and the restriction $g \mid C^0(g)$ is a translation by a non-zero vector $t_g, t_g \in A^0(g)$ . Since $t_g \in A^0(g)$, we have $B(t_g, t_g) > 0$. It is easy to check that if $p$ is an arbitrary point in $\mathbb{R}$ and $t_p = gp - p$, then $B(t_p, v_0(g)) = B(t_g, v_0(g))$. Note that there exist two non-zero constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that for every $\varepsilon$ and an $\varepsilon$-hyperbolic element $g$, and any vector $v \in A^0(g)$, we have $c_1(\varepsilon)B(v, v) \leq \|v\| \leq c_2(\varepsilon)B(v, v)$. As in [AMS 3], define the sign $\alpha(g)$ of a hyperbolic affine transformation $g$ by

$$\alpha(g) = B(t_g, v_0(g)).$$

Clearly,

$$\alpha(g) = B(t_p, v_0(g)),$$

since $B(t_p, v_0(g)) = B(t_g, v_0(g))$ for an arbitrary point $p \in \mathbb{R}^n$. Let us recall the following important observation called the opposite sign lemma [A, Lemma 8.4]: if a semigroup $S \subseteq$ *Aff*$(\mathbb{R}^n)$ contains two hyperbolic transversal elements $g_1$ and $g_2$ such that $\alpha(g_1)\alpha(g_2) < 0$, then $S$ does not act properly discontinuously on $\mathbb{R}^n$.

***2.9.*** Let us explain main ideas of the proof of Main Theorem. The crucial point in the proof is to show that $l(S)$ is not Zariski dense in $SO(B)$. On the contrary we suppose that $l(S)$ is Zariski dense in $SO(B)$. Let $v_1, v_2, w_1$ be a basis of $\mathbb{R}^3$ such that for every vector $v = x_1v_1 + x_2v_2 + y_1w_1$ we have $B(v, v) = x_1^2 + x_2^2 - y_1^2$. Let $g$ and $h$ be two hyperbolic elements, $g \in G_B, h \in G_B$. Assume that $A^+(g) = A^+(h)$ and $A^-(g)$ (resp. $A^-(h)$) spanned by the vector $w_1 + v_1$ ( resp. $w_1 - v_1$). Suppose that $gq_0 - q_0 = hq_0 = q_0 = -w_1$. (see Figure 1)

Let us show that $\alpha(g) > 0$ and $\alpha(h) < 0$. Indeed, let $v(g) \in D^-(g) \cap X$ (resp. $v(h) \in D^-(h) \cap X$) be a vector such that $w_1 + v_1, v(g)$ (resp.$w_1 - v_1, v(h)$) forms a positively oriented basis in $D^-(g)$ (resp. $D^-(h)$ ). It follows from our definition ( see 2.7) that $v(g) = -v(h)$, since $\pi_X(w_1 + v_1) = v_1 = -\pi_X(w_1 - v_1)$. Denote by $v$ the vector of $A^+(g) = A^+(h)$ such that $\pi_Y(v) = w_1$. Let $v_0(g)$ (resp. $v_0(h)$) be the vector of $A^0(g)$ (resp. $A^0(h)$) such that $v, v_0(g)$ ( resp. $v, v_0(h)$) is the positively oriented basis of $D^+(g)$ and $B(v_0(g), v_0(g)) = 1$ (resp. $D^+(h)$ and $B(v_0(h), v_0(h))$). By Lemma 2.7, $w_1 + v_1, -v_0(g)$ (resp. $w_1 - v_1, -v_0(h)$) is positively oriented basis of $D^-(g)$ (resp. $D^-(h)$. By the explanations given in the beginning of (2.8) the vectors $v_0(g)$ is a positive vector of $D^+(g)$. Therefore $v_0(g)$ is a negative vector of $D^-(g)$. Hence we have $B(v_0(g), v(g)) < 0$ since $v(g)$ is a positive vector of $D^-(g)$. By the same arguments, $B(v_0(h), v(h)) < 0$. Thus $B(v_0(h), v(g)) > 0$, since $v(g) = -v(h)$. Hence $\pi_Y(v_0(g)) = -\pi_Y(v_0(h))$. Consequently we have $\alpha(g) = B(gq_0 - q_0, v_0(g)) = B(w_1, v_0(g)) > 0$ and $\alpha(h) = B(hq_0 - q_0, v_0(h)) = B(w_1, v_0(h)) < 0$.

Assume now that there exist $\varepsilon$ and two sequences $\{g_n\}_{n\in\mathbb{N}} \subseteq S$ and $\{h_n\}_{n\in\mathbb{N}} \subseteq S$ of $\varepsilon$-hyperbolic, $\varepsilon$ -transversal elements with the properties that for $n \to \infty$ we have $A^+(g_n) \to A^+(g), A^+(h_n) \to A^+(h) = A^+(g),\ A^-(g_n) \to A^-(g),\ A^-(h_n) \to A^-(h)$. Suppose that there exist a compact set $K$ and two sequences $\{q_n\}_{n\in\mathbb{N}} \subset K$ and $\{p_n\}_{n\in\mathbb{N}}$ such that $g_nq_n - q_n/d(g_nq_n, q_n) \to w_1$ , and $h_np_n - p_n/d(h_np_n, p_n) \to w_1$. It is easy to see that there exists $N$ such that for $n > N$ we have $\alpha(g_n) > 0$ and $\alpha(h_n) < 0$. Thus by Lemma 2.7 $S$ does not act properly discontinuously and consequently, $l(S)$ is not Zariski dense in $SO(B)$.

The proof falls naturally into the following steps. First we will show that there ex-

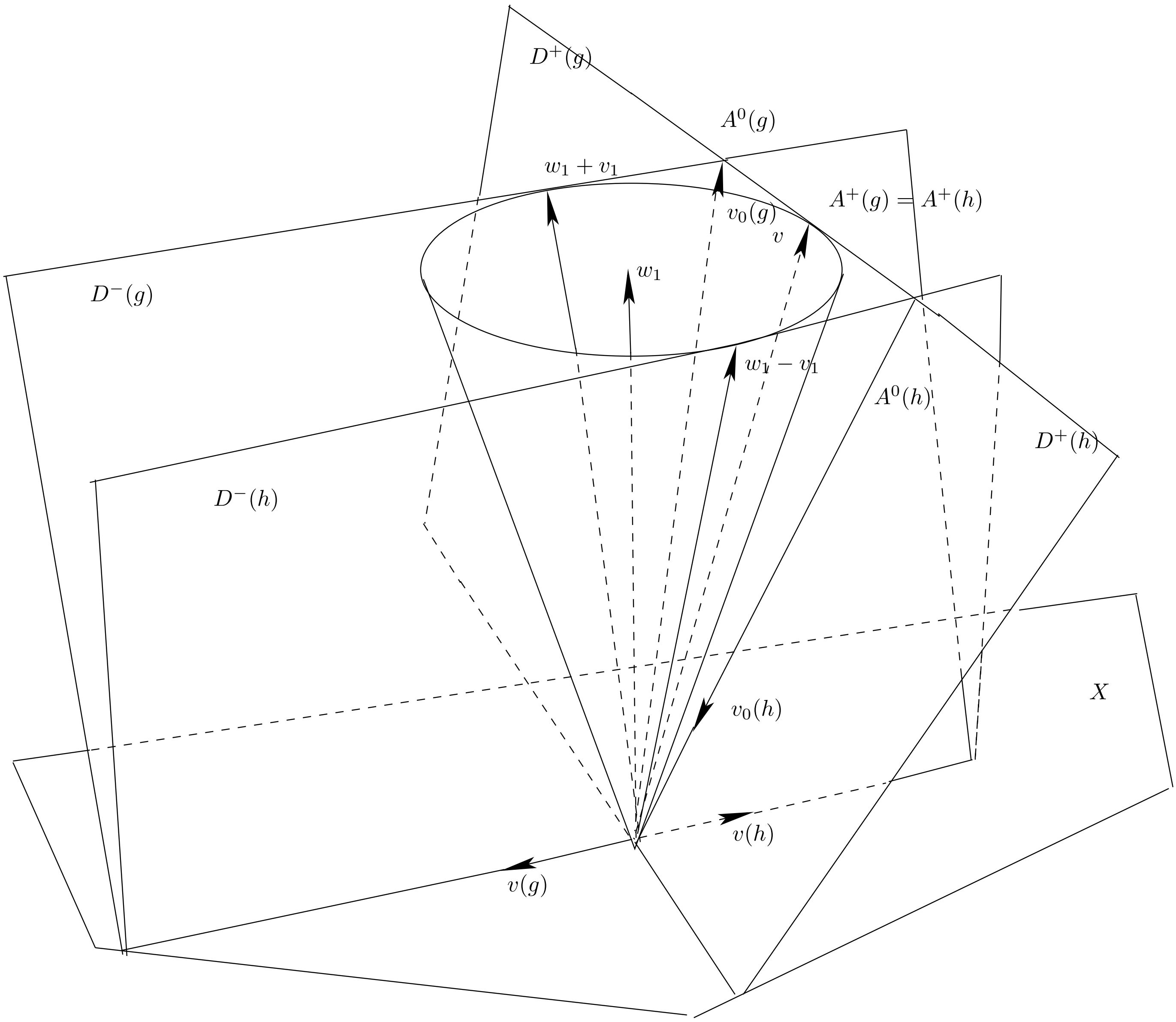


Figure 1: Opposite sign

ist four hyperbolic, transversal elements $g_1, g_2, g_3, g_4$ such that $w_1 \in A^-(g_1) + A^-(g_2) \cap A^-(g_1) + A^-(g_2)$. Then we will show that there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ is a sequence of $\varepsilon$-hyperbolic elements , $h_n q_0 - q_0/|B(h_n q_0 - q_0, h_n q_0 - q_0)|^{1/2} \to w_1$ and $\|h_n q_0 - q_0\| \to \infty$ such that it is be possible to choose two elements $g_i, g_j$ of $g_1, g_2, g_3, g_4$ in a way that the elements $h_n g_i$ and $h_n g_j$ have an opposite sign for sufficiently big $n$.

## *3. Main results.*

We begin by recalling known facts on hyperbolic elements in $G_B$ [AMS 3], [AMS4]. Assume that $g$ and $h$ are two hyperbolic elements. Let $\varepsilon$ be a positive number such that $g$ and $h$ are $\varepsilon$- hyperbolic and $\widehat{d}(A^+(g), A^+(h)) \geq \varepsilon$. Since $D^{\pm}(g) = (A^{\pm}(g))^{\perp}$ it is easy to see, that there exists a constant $c(\varepsilon)$ such that $\widehat{d}(A^+(g), D^+(h)) \geq c(\varepsilon)$. Thus we can conclude:

***Lemma 3.1*** *Let $g$ and $h$ be two $\varepsilon$-hyperbolic elements such that*

$$\min\{\widehat{d}(A^+(g), A^-(h)), \widehat{d}(A^-(g), A^+(h))\} \geq \varepsilon.$$

*Then there exists a constant $c(\varepsilon)$ such that the two $\varepsilon$-hyperbolic elements $g$ and $h$ are $c(\varepsilon)$-transversal*

***Remark 3.2*** *Let $B$ be a hyperbolic form of signature $(2,1)$, and let $g$ be an element of $G_B$. Then $g$ is hyperbolic if and only if $g$ is dipole. Hence in this case we have $A^{\pm}(g) = V^{\pm}(g)$ and $D^{\pm}(g) = W^{\pm}(g)$. Taking into account Lemma 3.1, we will say that two hyperbolic, transversal elements $g$ and $h$ of $G_B$ are $\varepsilon$–hyperbolic, $\varepsilon$–transversal if $\widehat{d}(A^+(g), A^-(g)) > \varepsilon$, $\widehat{d}(A^+(h), A^-(h)) > \varepsilon$ and $\widehat{d}(A^+(g), A^-(h)) > \varepsilon$ , $\widehat{d}(A^+(h), A^-(g)) > \varepsilon$ respectively.*

***Lemma 3.3.[AMS 3]*** *There exists $s(\varepsilon) < 1$ and $c(\varepsilon)$ such that for any two $\varepsilon$–hyperbolic*

*$\varepsilon$–transversal elements $g, h \in SO(B)$ with $s(g) < s(\varepsilon)$ and $s(h) < s(\varepsilon)$, we have*

*(1) the element $gh$ is $\varepsilon/2$–hyperbolic and is $\varepsilon/2$–transversal to both $g$ and $h$;*

*(2) $\widehat{d}(A^+(gh), A^+(g)) \leq c(\varepsilon)s(g))$;*

*(3) $\widehat{d}(A^-(gh), A^-(h)) \leq c(\varepsilon)s(h))$;*

*(4) $s(gh) \leq c(\varepsilon)s(g)s(h)$.*

From now on we will assume that n=3 and $B$ is a quadratic form of signature $(2,1)$. Assume that the linear part $l(S)$ is Zariski dense in $SO(B)$. It follows from Lemma 2.3 that there exist a pair of hyperbolic transversal elements $g$ and $h$ in $S$ such that $A^+(g) \neq A^+(h)$ and $A^-(g) \neq A^-(h)$ . Fix these elements and consider the two subspaces $C(g) = A^+(g) \oplus A^-(g)$ and $C(h) = A^+(h) \oplus A^-(h)$ of $\mathbb{R}^3$. It follows from Lemma 3.3 that for any $\delta > 0$ there exists $N, N \in \mathbb{Z}$, $N > 0$ such that $\widehat{d}(A^+(g^m h^n), A^+(g)) \leq \delta$, $\widehat{d}(A^+(h^n g^m), A^+(h)) \leq \delta$, $\widehat{d}(A^-(g^m h^n), A^-(h)) \leq \delta$ and $\widehat{d}(A^-(h^n g^m), A^-(g)) \leq \delta$ for $n > N$, $m > N$. Any two chosen elements in $\mathbb{C}_B$ with the same $B$ -norm are conjugate by an element from $SO(B)$. Hence we can conjugate $S$ by an element from $G_B$ such that $C(g) \bigcap C(h) = Y$. Thus we have

***Lemma 3.4.*** *If the Zariski closure of the linear part $l(S)$ is $SO(B)$, then there are two hyperbolic transversal elements $g$ and $h$ in $S$ such that $C(g) \bigcap C(h) = Y$. Choosing a suitable positive number $\varepsilon$, we can and will assume that the elements $g$ and $h$ are $\varepsilon$-hyperbolic and $\varepsilon$- transversal.*

***Lemma 3.5.*** *Let $S$ be a semigroup, $S \subseteq G_B$ such that $l(S)$ is Zariski dense in $SO(B)$. Then there exists a set $\{g_1, g_2, g_4, g_4\}$ of hyperbolic transversal elements of $S$ such that for every non-zero vector $v \in A^-(g_1) \oplus A^-(g_2) \cap A^-(g_3) \oplus A^-(g_4)$ we have $B(v,v) < 0$.*

*Proof.* It follows from Proposition 2.3 that there exists a set $A = \{s_i, 1 \leq i \leq 4\}$ of a hyperbolic transversal elements of $S$. We can order the set $A$ such that we have $B(v,v) <$

0 for every non-zero vector $v$ of $A^-(s_1) \oplus A^-(s_2) \cap A^-(s_3) \oplus A^-(s_4)$. $\square$

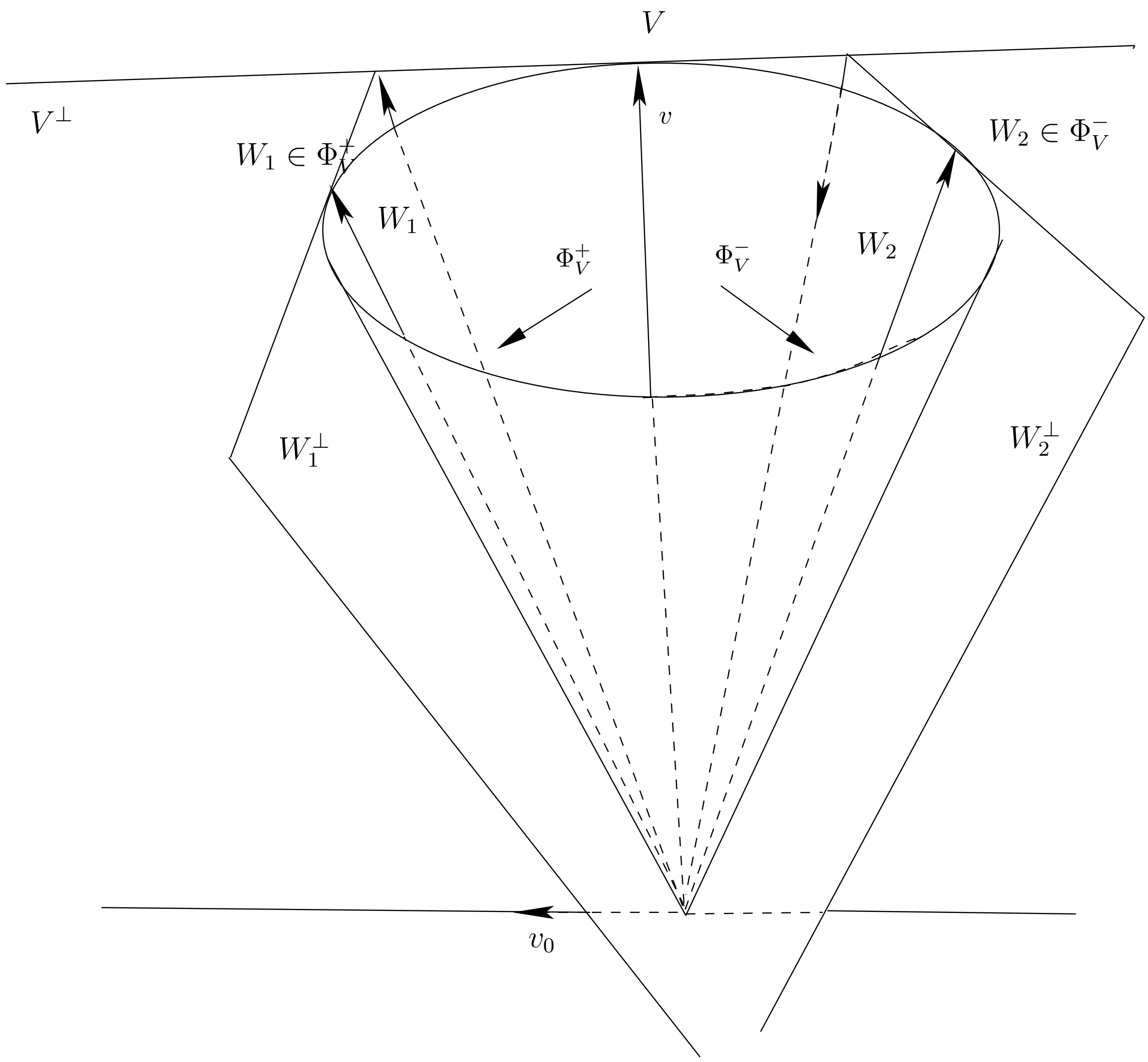


Figure 2: Positive and negative parts

***3.6.*** We will use notations and definitions from ***2.6***. Let $V$, $V \subseteq \mathbb{R}^n$ be a maximal $B$-isotropic subspace and let $v$ be a vector from $V$ such that $V$ is spanned by $v$ and $\pi_Y(v) = w_1$ (see Figure 2 below ). Let $v_0$ be a vector from $V^\perp \cap X$ such that $B(v_0, v_0) =$

1 and the basis $\pi_X(v), v_0$ has the same orientation as $v_1, v_2$. Let $W$ be a maximal $B$-isotropic subspace, $W \neq V$ . Then $\dim(V^{\perp} \bigcap W^{\perp}) = 1$. There exists a unique vector $w_0(W)$, $w_0(W) \in V^{\perp} \bigcap W^{\perp}$ such that $w_0(W) = v_0 + \alpha(W)v$. Set $\Phi_V^+ = \{W \in \Phi | \, \alpha(W) > 0\}$ and $\Phi_V^- = \{W \in \Phi | \, \alpha(W) < 0\}$. Since $v_0 \in X$, we have $B(v_0, w_1) = 0$. Therefore $B(w_0(W)), w_1) = \alpha(W)$, $B(v, w_1) = -\alpha(W)$. We conclude that for every vector $w \in \Phi_V^+$ (resp. $w \in \Phi_V^-$) we have $B(w, w_1) < 0$ (resp.$B(w, w_1) > 0$). From Lemma 2.7 and the choice of hyperbolic elements $g$ and $h$, it immediately follows that $\Phi_{a_1}^{\pm} = \Phi_{a_3}^{\mp}$ and $\Phi_{a_2}^{\pm} = \Phi_{a_4}^{\mp}$. Assume that we choose two different $B$–isotropic spaces $W_1$ and $W_2$ such that $w_1 \in W_1 \oplus W_2$ and $V \cap W_1 \oplus W_2 = \{0\}$. Clearly, if $W_1$ belongs to $\subset \Phi_V^+$ then $W_2 \subset Phi_V^-$ and viceversa.

***Proposition 3.7.*** *Let $S$ be a semigroup, $S \subseteq G_B$. Assume that $S$ acts properly discontinuously on $\mathbb{R}^n$ and that there exists a compact subset $K_0$ of $\mathbb{R}^n$ such that $X_\infty(S, K_0) = S^n(0, 1)$. Then the linear part $l(S)$ of $S$ is not Zariski dense in $SO(B)$.*

*Proof.* Assume that $l(S)$ is Zariski dense in $SO(B)$. Let $A = \{s_1, s_2, s_3, s_4\}$ be the set of hyperbolic transversal elements which fulfill the requirements of Lemma 3.5.Let $d_1 = \min_{1 \le i,j \le 4, i \neq j}\{\widehat{d}(A^+(s_i), A^+(s_j))\}$, $d_2 = \min_{1 \le i,j \le 4, i \neq j}\{\widehat{d}(A^-(s_i), (A^-(s_j))\}$, $d_3 = \min_{1 \le i,j \le 4}\{\widehat{d}(A^+(s_i), A^-(s_j))\}$. Set $d_0 = \min\{d_1, d_2.d_3\}$. Let $K_0$ be a compact subset of $\mathbb{R}^n$ such that $X_\infty(S, K_0) = S^n(0, 1)$. Then $w_1 \in X_\infty(S, K_0)$. Therefore there are two infinite subsets $\{l_n, n \in \mathbb{N}\} \subseteq K_0$ and $\{g_n, n \in \mathbb{N}\} \subseteq S$ such that

*(1)* $\|g_n l_n - l_n\| \to \infty$ when $n \to \infty$;

*(2)* $g_n l_n - l_n / d(g_n l_n, l_n) \to w_1$ when $n \to \infty$.

By Lemma 2.6, we can and will additionally assume that $g_n$ is $\varepsilon = \varepsilon(S)$-hyperbolic for all $n \in \mathbb{N}$. The projective space $P$ is compact. Hence we can and will assume that the sequences $\{\pi(A^+(g_n))\}_{n \in \mathbb{N}}$ and $\{\pi(A^-(g_n))\}_{n \in \mathbb{N}}$ converge. Denote $\lim_{n\to\infty} \pi(A^+(g_n)) = p_1 \in P$ and $\lim_{n\to\infty} \pi(A^-(g_n)) = p_2 \in P$. Note that $\widehat{d}(p_1, p_2) \ge \varepsilon$.

It follows from Proposition 2.3 that there exists a hyperbolic element $s_0$ transversal to every element from $A$ such that $\{\pi(A^+(s_0)), \pi(A^-(s_0))\} \subseteq P \setminus \{p_1, p_2\}$. Set $\delta_1 = \min\{\widehat{d}((p_1, ), A^-(s_0)), \widehat{d}((p_2, ), A^+(s_0))\}$ and $\delta_2 = \min_{1\le i\le 4} \widehat{d}(A^+(s_i), , A^-(s_0))$. Put $\delta = \min\{\delta_1, \delta_2\, , \varepsilon(S)\}$. There exists $N_0 \in \mathbb{N}$ such that $\widehat{d}((p_1, A^+(g_n)) < \delta/4$ and $\widehat{d}((p_2, A^-(g_n)) < \delta/4$ for $n > N_0$. Thus we conclude, that for $n > N_0$ the hyperbolic elements $g_n$ and $s_0$ are $\delta_1$–transversal and $\delta_1$–hyperbolic. Put $q = s(s_0)$. Clearly, $s(s_0^k) = q^k$. Thus by Lemma 3.3 we conclude, that there exists $N_1 = N_1(\delta)$ such that the element $g_n s_0^k$ is $\delta/2$–hyperbolic, $\widehat{d}(A^-(g_n s_0^k), A^-(s_0)) \le q^{k/2}$ and $s(g_n s_0^k) \le q^{k/2}$ for all $n \ge N_1$. For every $k > N_1$ we consider the sequence $\{g_n^{(k)} s_0^k\}_{n\in\mathbb{N}}$ such that

(1) the sequence $\pi(A^+(g_n^{(k)} s_0^k))$ converges in $P$ to a point $a_k$ for every $k \ge N_1$

(2) $\{g_n^{(k+1)}\}_{n\in\mathbb{N}} \subset \{g_n^{(k)}\}_{n\in\mathbb{N}}$

Clear, that there exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ such that $k_i < k_{i+1}$ and the sequence of points $\{a_{k_i}\}_{i\in\mathbb{N}}$ converges. Set $a_0 = \lim_{i\to\infty} a_{k_i}$.

Let $V$ be the subspace of $\mathbb{R}^3$ such that $\pi(V) = a_0$ (See Fig. 2). It it easy to see that $\min\{\widehat{d}(a_0, A^-(s_1)), \widehat{d}(a_0, A^-(s_2))\} > d_0/2$ or $\min\{\widehat{d}(a_0, A^-(s_2)), \widehat{d}(a_0, A^-(s_4))\} > d_0/2$. Clearly that without lost of generality we can assume that $\min\{\widehat{d}(a_0, A^-(s_1)), \widehat{d}(a_0, A^-(s_2))\} > d_0/2$. Therefore (see (3.6)), since the vector $w_1 \in A^-(s_1) \oplus A^-(s_2)$ we have $A^-(s_1) \cup A^-(s_2) \subset \Phi_V^+ \cup \Phi_V^-$, We will suppose that $A^-(s_1) \subset \Phi_V^+$ and $A^-(s_2) \subset \Phi_V^-$. On account of (3.6) for a positive vector $w_0(A^-(s_1))$ we have $B(w_0(A^-(s_1)), w_1) > 0$ and for a positive vector $w_0(A^-(s_2))$ we have $B(w_0(A^-(s_2)), w_1) < 0$. It is obvious that there exists an $\varepsilon_0$ such that for every maximal $B$–isotropic spaces $V_0, W_1, W_2$ such that $\widehat{d}(a_0, V_0) < \varepsilon_0$, $\widehat{d}(A^-(s_1), W_1) < \varepsilon_0$, and $\widehat{d}(A^-(s_2), W_2) < \varepsilon_0$, we have $W_1 \subset \Phi_{V_0}^+$ and $W_1 \subset \Phi_{V_0}^+$.

Put $\delta_0 = \min\{d_0/4, \delta/2\ \varepsilon_0/4\}$. Let $c(\delta_0)$ and $c(\delta_0/2)$ be the constants which fulfil the requirements of (2), (3) Lemma 3.3. Let $N_3 \in \mathbb{N}$ be a positive number such that $q^r(c(\delta_0) + c(\delta_0/2)) \le \delta_0/4$ and $\widehat{d}(a_{k_i}, a_0) \le \delta_0/8$ for $r > N_3$ and $k_i > N_3$. Choose $k_i >$

$N_3$ and denote $r_0 = k_i$. From $\widehat{d}(a_{k_i}, a_0) \leq \delta_0/8$ and $a_{k_i} = \lim_{n\to\infty} \pi(A^+(g_n^{(r_0)} s_0^{r_0})$ follows that there exists $N_4, N_4 \in \mathbb{N}$ such that $\widehat{d}(A^+(g_n^{(r_0)}), a_{r_0}) < \delta_0/8$. Denote $\widetilde{g}_n = g_{n+N_4}^{(r_0)}$. Clearly, $\widehat{d}(A^+(\widetilde{g}_n), a_0) \leq \widehat{d}(A^+(g_n^{(r_0)}), a_{r_0}) + \widehat{d}(a_{r_0}, a_0) \leq \delta_0/4 < \varepsilon$. Recall that the elements $g_n$ and $s_0$ are $\delta_1$–transversal and $\delta_1$–hyperbolic. Hence they are $\delta_0$–transversal and $\delta_0$–hyperbolic. Thus from (3) Lemma 3.3 and the choice of $r_0$ we obtain $\widehat{d}(A^-(\widetilde{g}_n), A^-(s_0)) = \widehat{d}(A^-(g_{n+N_4}^{(r_0)}), A^-(s_0)) = \widehat{d}(A^-(g_{n+N_4}^{r_0} s_0^{r_0}), A^-(s_0)) < c(\delta_0)q^{n_0} < \delta_0/4$. Hence we have $\widehat{d}(A^-(\widetilde{g}_n), A^+(s_i)) > \widehat{d}(A^-(s_0), A^+(s_i)) - \widehat{d}(A^-(\widetilde{g}_n), A^-(s_0)) \geq \delta_0/2$. Recall, that $\min\{\widehat{d}(a_0, A^-(s_1)), \widehat{d}(a_0, A^-(s_2)) > d_0/2 > \delta_0/2$. Therefore the elements $g_0$ and $s_i$ are $\delta_0/2$–hyperbolic and $\delta_0/2$–transversal. From Lemma 3.3 we obtain

(1) $\widehat{d}(A^+(\widetilde{g}_n s_1^k), A^+(\widetilde{g}_n)) \leq c(\delta_0/2)s(g_0) \leq c(\delta_0)s(s_0)^{r_0} \leq \delta_0/4 < \varepsilon_0$.

(2) $\widehat{d}(A^-(\widetilde{g}_n s_i^r), A^-(s_i)) \leq c(\delta_0/2)s(s_i)^{r_i}$ for $i = 1, 2$.

Thus from (2) we obtain

(3) There exists $R_0$ such that $\widehat{d}(A^-(\widetilde{g}_n s_i^{R_0+1}), A^-(s_i)) \leq \delta_0/4 < \varepsilon_0$ where $i = 1, 2$

Set $\widehat{g}_n^{(i)} = \widetilde{g}_n s_i^{R_0+1}$ for $i = 1, 2$ and $l_n^{(i)} = s_0^{-r_0} s_i^{-(R_0+1)} l_n$. Then

(4) $\widehat{g}_n^{(i)} l_n^{(i)} - l_n^{(i)} / d(\widehat{g}_n^{(i)} l_n^{(i)}, l_n^{(i)}) \to w_1$ for $n \to \infty$.

Let $v_0(\widehat{g}_n^{(i)})$ be the positive norm one vector of $D^+(\widehat{g}_n^{(i)})$. Thus from (2),(3) for all $n$ we have $B(v_0(\widehat{g}_n^{(1)}), w_1) > 0$ and $B(v_0(\widehat{g}_n^{(2)}), w_1) < 0$. Moreover $\lim_{n\to\infty} B(v_0(\widehat{g}_n^{(1)}), w_1) > 0$ and $\lim_{n\to\infty} B(v_0(\widehat{g}_n^{(2)}), w_1) < 0$. Hence there exists $\widetilde{\varepsilon}$ such that for any vector $\widetilde{w}$, $\|w\| = 1$ and $\|w_1 - \widetilde{w}/\| \leq \widetilde{\varepsilon}$ we have $\lim_{n\to\infty} B(v_0(\widehat{g}_n^{(1)}), \widetilde{w}) > 0$ and $\lim n \to \infty B(v_0(\widehat{g}_n^{(2)}), \widetilde{w}) < 0$. Thus there exists a positive integer $M$ such that $n \geq M$ we have

$$B(v_0(\widehat{g}_n^{(1)}), \widehat{g}_n^{(1)} l_n^{(1)} - l_n^{(1)}) > 0$$

and

$$B(v_0(\widehat{g}_n^{(2)}), \widehat{g}_n^{(2)} l_n^{(2)} - l_n^{(2)}) < 0$$

Hence there are two transversal hyperbolic elements in $S$ of opposite sign. Hence $S$ does not act properly discontinuously on $\mathbb{R}^n$ by opposite sign lemma (see 2.8) . This contradiction completes the proof. □

*Proof.* (***Main Theorem***) Suppose that the Zariski closure of $S$ is not virtually solvable. Let $S_0$ be a semisimple part of the Zariski closure of $G = l(S)$. If $S_0$ is a compact group, then by [GS 2], this group is trivial. Contradiction. Therefore we can assume that $S_0$ is a non-compact simple group. Recall (see ***2.3***) that every element from the Zariski closure of $l(S)$ has 1 as an eigenvalue. Hence, $S_0$ is isomorphic to $SL_2(\mathbb{R}$ since $S_0$ is a subgroup of $SL_3(\mathbb{R})$. Thus we have a representation $\rho : SL_2(\mathbb{R}) \to SL_3(\mathbb{R})$. There are two possible cases: either $\rho(SL_2(\mathbb{R})) = SO(2,1)$ and $G = SO(2,1)$, or $\rho$ is the direct sum of the standard and trivial representations of $SL_2(\mathbb{R})$. It follows from Proposition 3.5 that the first case is impossible. Then we have two possibilities:

***(a)*** there is an one-dimensional subspace $V$ such that $l(s)v = v$ for every $s \in S$;

***(b)*** there is a $l(S)$-invariant subspace $V$, $\dim V = 2$.

***(a)***. Fix a hyperbolic element $g$. Let $L_g$ be the unique $g$-invariant line and let $v_0(g)$ be the unique Euclidean norm one vector such that for a point $p, p \in L_g$, we have $gp - p = \alpha^2 v_0(g)$. Let $K_0$ be a compact subset such that $X_\infty(S, K_0) = S^3(0,1)$. Hence $v_0(g), -v_0(g) \in X_\infty(S, K_0)$. It follows from Lemma 2.6 that there exist a positive number $\varepsilon$, a set of $\varepsilon$- hyperbolic elements $g_n, g_n \in S$ and a subset $\{p_n, p_n \in K_0, n \in \mathbb{N}\}$ such that

*( 1)* $g_n p_n - p_n / \|g_n p_n - p_n\| \to -v_0(g)$ when $n \to \infty$;

*(2)* $\|g_n p_n - p_n\| \to \infty$ when $n \to \infty$.

For every element $s \in S$, we have $l(s)v_0 = v_0$. Therefore $sg_n p_n - p_n / \|sg_n p_n - p_n\| \to -v_0(g)$ when $n \to \infty$ and for $q_n = s^{-1}p_n$, we have $g_n s q_n - q_n / \|g_n s q_n - q_n\| \to -v_0(g)$ when $n \to \infty$. Note that $\{q_n, n \in \mathbb{N}\}$ is a subset of a compact set $s^{-1}K_0$. Thus, we can assume that the sequence $\{A^+(g_n)\}_{n\in\mathbb{N}}$ (resp. $\{A^-(g_n)\}_{n\in\mathbb{N}}$) converges to $A^+$ (resp. $A^-$) and $\widehat{d}(A^+, A^\pm(g)) > 0$ (resp. $\widehat{d}(A^-, A^\pm(g)) > 0$. Hence there exists a $\varepsilon$- hyperbolic element

$h$ transversal to $g$ such that for a point $p$ from the unique $h$ invariant line $L_h$, we have $hp - p = -\beta^2 v_0(g)$. Then, using the same arguments as in opposite sign lemma [A, Lemma 8.4], we conclude that there exist infinite sets $N$ and $M$ such that $h^m g^n K_0 \bigcap K_0 \neq \emptyset$ for all $m \in M$, $n \in N$. This is impossible because $S$ acts properly discontinuously.

***(b)***. Consider an affine space $A = \mathbb{R}^3/V$, a projection $\pi_V : \mathbb{R}^3 \to A$ and an induced homomorphism $\rho_V : \mathrm{Aff}(\mathbb{R}^3) \to \mathrm{Aff}(A)$.Obviously, $\dim A = 1$. Let $K$ be a compact subset of $K$. It is clear that there exist a positive number $\delta$ and sequences $\{g_t\}_{t\in\mathbb{N}}$ and $\{h_t\}_{t\in\mathbb{N}}$ of $\delta$-hyperbolic elements such that we have for a point $k \in K$

*(1)* $|g_t k - k| \to \infty$, $|h_t k - k| \to \infty$ when $t \to \infty$;

*(2)* $(g_t k - k)(h_t k - k) < 0$ for all $t \in \mathbb{N}$;

*(3)* $g_t$ and $h_t$ are $\delta$-transversal for all $t \in \mathbb{N}$.

On the other hand the representation $\rho$ of the linear part $l(S)$ determines the following representation of $S$.

$$s \mapsto \begin{pmatrix} a_{11} & a_{12} & a_{13} & * \\ a_{21} & a_{22} & a_{23} & * \\ 0 & 0 & 1 & \alpha_s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\rho(l(s)) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that $\rho_V(s) = \alpha_s$. Hence by *(1)*,*(2)* and *(3)* above there exist two $\delta$-hyperbolic, $\delta$-transversal elements $\widehat{g}$ and $\widehat{h}$ such that $\alpha_{\widehat{g}}\alpha_{\widehat{h}} < 0$. Let $L_{\widehat{g}}$ (resp. $L_{\widehat{h}}$) be the unique $\widehat{g}$-invariant (resp $\widehat{h}$-invariant) line. Fix a point $p$ and a Euclidian distance $d$ in

$\mathbb{R}^3$. Then by [AMS 2, Lemma 2.4] there exists a positive $c = c(\widehat{g}, \widehat{h})$ such that

$$d(p, L_{\widehat{h}^m \widehat{g}^n}) \leq c[d(p, L_{\widehat{g}}) + d(p, L_{\widehat{h}})]$$

for all positive numbers $m$ and $n$. Since $\alpha_s^n = n\alpha_s$ and $\alpha_{\widehat{g}}\alpha_{\widehat{h}} < 0$, there are infinite sets of positive numbers $N$ and $M$ such that $|\alpha_{\widehat{g}^n} + \alpha_{\widehat{h}^m}| \leq c/2$ for $n \in N, m \in M$. Consider the ball $U(p, 2c)$. Then for $n \in N, m \in M$ the set $\{\widehat{h}^m \widehat{g}^n U(p, 2c) \bigcap U(p, 2c)\}$ is non-empty. This is a contradiction, which proves the theorem. □

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