Tie transformations of Dynkin graphs and singularities on quartic surfaces

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§ 0. Introduction

This article is the continuation of my previous one [9]. We continue to study possible combinations of rational double points on quartic surfaces in the projective space of dimension 3.

Now, in [9] I proposed a certain converting procedure on Dynkin graphs. It was called an elementary transformation. This notion was natural and simple. However, it had a limit in its application and we had to check certain arithmetic conditions for the application.

In this article we would like to propose another procedure. The new procedure is called a tie transformation. By one tie transformation we can make the number of vertices in the Dynkin graph larger by one. This is the different point from elementary transformations. By elementary transformations we can never make the number of vertices larger. Because of this property by tie transformations we can treat many interesting examples of K 3 surfaces whose Picard number is the maximal 20. (Crollary 0.3, Persson [6], Wall [16].)

A part of this article was announced in Urabe [8]. (In Urabe [8], [10] we called the above new procedure a "connection". However, this name is confusing. Therefore we would like to change the name and to call it a tie transformation.)

We assume that every variety is algebraic and is defined over the complex number field $\mathbb{C}$.

As in the previous article [9] we obey the following conventions on Dynkin graphs.

(1) A disjoint finite union of connected Dynkin graphs of type A , B , D or E is called a Dynkin graph.

(2) Any given Dynkin graph consisting of only a unique vertex is beforehand determined to be of type A_1 or of type B_1 .

(3) Assume that the Dynkin graph has a component of type B . If it is of type B_k with $k \geq 2$, the vertex at the top of the double edge with the arrow is called <u>the vertex corresponding to a short root</u>. If it is of type B_1 , the unique vertex is the one corresponding to a short root. The other vertices and any vertices of any graph ot type A , D or E are called <u>the vertices corresponding to long roots</u>.

(4) When we make a connected Dynkin graph G into the extended Dynkin graph of the corresponding type by adding a new vertex and

a few edges, the new vertex is regarded as the one corresponding to a long root, if G is not of type B_1 . If G is of type B_1 , the new vertex corresponds to a short root.

(5) Assume that a Dynkin graph G contains a_k of connected components of type A_k , b_ℓ of components of type D_ℓ , c_m of components of type E_m , and d_n of components of type B_n $(k \geq 1 , \ell \geq 4 , m = 6, 7, 8 , n \geq 1)$. Then we denote

$$G = \sum a_k A_k + \sum b_\ell D_\ell + \sum c_m E_m + \sum d_n B_n .$$

Definition 0.1. Assume that applying the following procedure to a Dynkin graph G , we have obtained the Dynkin graph $\bar{G}$. Then we call the following procedure a <u>tie transformation</u> of Dynkin graphs.

(1) Attach an integer to each vertex of G by the following rule: Now, let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the fundamental system of roots associated with a connected component G_0 of G . Let $\sum_{i=1}^{k} n_i \alpha_i$ be the associated maximal root. Then the attached integer to the vertex corresponding to α_i is n_i .

(2) Add one vertex and a few edges to each component of G and make it into the extended Dynkin graph of the corresponding type. Attach moreover the integer 1 to each new vertex.

(3) Choose in an arbitrary manner subsets A , B of the set

of vertices of the extended graph $\tilde{G}$ satisfying the following conditions:

<a> $A \cap B = \phi$.

<b> Choose arbitralily a component $\tilde{G}_1$ of the extended graph $\tilde{G}$ and let V be the set of vertices in $\tilde{G}_1$. Let N be the sum of the numbers attached to elements in $B \cap V$. (If $B \cap V = \phi$, $N = 0$.) Then, the greatest common divisor of N and the numbers attached to elements in $A \cap V$ is necessarily 1.

(4) Erase out all attached integers.

(5) Remove vertices belonging to A together with the edges issuing from them.

(6) Draw another new vertex called θ . Connect θ and each element in B following the rule below: If $v \in B$ corresponds to a long root, we connect θ and v by a single edge. $\overset{\theta}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\circ}$. If $v \in B$ corresponds, to a short root, then we connect θ and v by a double edge with an arrow in the direction from θ to v . $\overset{\theta}{\circ}\!\Longrightarrow\!\overset{v}{\circ}$.

Remarks. (1) The new vertex θ is regarded as the one corresponding to a long root.

(2) Often the resulting graph $\bar{G}$ after the above procedure (1) - (6) is <u>not</u> a Dynkin graph. We consider only the cases where the resulting graph $\bar{G}$ is a Dynkin graph and then we call the above procedure a <u>tie transformation</u>.

Theorem 0.2. Assume that G' is a Dynkin graph obtained by an elementary transformation or a tie transformation from one of the following 9 basic Dynkin graphs. Assume moreover that applying an elementary transformation or a tie transformation to G' once more, we have obtained a Dynkin graph G without any vertex corresponding to a short root. Then there exists a normal quartic surface in the projective space of dimension 3 whose combination of singularities just agrees with G .

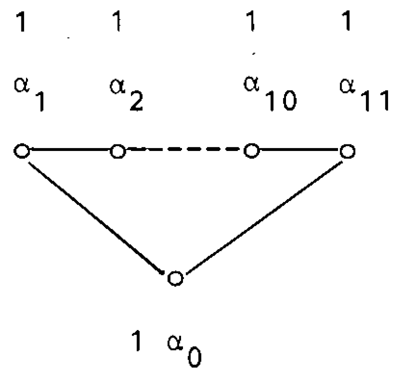
The basic Dynkin graphs:

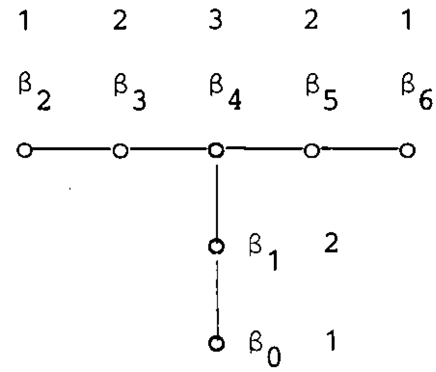
$A_{11} + E_6$, $2D_8 + B_1$, $D_{16} + B_1$, $2E_8 + B_1$, $A_{15} + B_2$, $2E_7 + B_3$, $D_{12} + B_5$, $E_8 + B_9$, B_{17} .

Corollary 0.3. There is a normal quartic surface with any one of the following combinations of singularities.

A_{19} , $D_{18} + A_1$, D_{18} , $A_{18} + A_1$, $A_{17} + A_2$, $2D_8 + A_2 + A_1$, $2E_8 + A_2 + A_1$, $A_{15} + 2A_2$, ... , etc.

Proof. Consider the Dynkin graph $A_{11} + E_6$. We apply a tie transformation to it. At the second step we get the following graph.





Set $A = \{\alpha_0, \beta_1\}$ and $B = \{\alpha_{11}, \beta_2\}$. For the component A_{11} the sum N of numbers attached to B is 1 and the number attached to A is 1. Their G.C.D. is 1. For E_6, $N = 1$, the number attached to A is 2 and their G.C.D. is 1. Under this choice we get the graph $A_{17} + A_1$ as the result of the tie transformation. Applying it once more to $A_{17} + A_1$, we can obtain A_{19}, $D_{18} + A_1$, $A_{18} + A_1$, $A_{17} + A_2$ etc. If we start from other basic graphs, we can obtain various other graphs.

Q.E.D.

We conjectured in Urabe [8] that the converse of Theorem 0.2. also holds. However, there are counter-examples for this conjecture. $G = A_{10} + A_7 + 2A_1$ etc. There is a normal quartic surface with the combination of singularities $A_{10} + A_7 + 2A_1$. But the Dynkin graph $A_{10} + A_7 + 2A_1$ cannot be obtained by tie transformations repeated twice from any one of the basic 9 graphs. Since $A_{10} + A_7 + 2A_1$ has 19 vertices, we can never obtain it from any one of the basic graphs, if we use an elementary transformation. In the last section of this article we discuss these counter-examples.

Any Dynkin graph obtained from one of the basic graphs by using a tie transformation has to satisfy a certain condition in the theory of lattices. (Section 1, Theorem 1.1.) We could not succeed in writing down this condition explicitly in terms of the arithmetic theory. At the present stage the best theorem giving a necessary and sufficient condition is the following.

Theorem 0.4. Let $G = \sum a_k A_k + \sum b_\ell D_\ell + \sum c_m E_m$ (a finite sum) be a Dynkin graph with components of type A , D or E only. Set $r = \sum a_k k + \sum b_\ell \ell + \sum c_m m$. Then the following conditions (A) , (B) and (C) are equivalent.

(A) There exists a normal quartic surface in the projective space of dimension 3 whose combination of singularities just agrees with G and moreover one of the following conditions <1> , <2> , <3> holds for the root lattice $Q = Q(G)$ of type G . By $d(Q)$ we denote the discriminant of A .

<1> $r = 18$, and for every prime number p , $\varepsilon_p(Q) = 1$.

<2> $r = 17$, and for every prime number p , $d(Q) \notin Q_p^{*2}$ or $\varepsilon_p(Q) = (-1\ ,\ d(Q))_p$.

<3> $r \leq 16$.

(B) One of the following (B - 1) , (B - 2) holds.

(B-1) G is a Dynkin graph obtained from one of the 9 basic Dynkin graphs in Theorem 0.2

<1> by elementary transformations repeated twice

<2> by an elementary transformation following after a tie transformation

or <3> by a tie transformation following after an elementary transformation

such that it has no vertex corresponding to a short root.

(B-2) G is a Dynkin graph obtained from one of the following 11 sub-basic Dynkin graphs by one elementary transformation such that it has no vertex corresponding to a short root.

The sub-basic Dynkin graphs:

$A_{11} + A_5 + B_2$, $A_9 + A_6 + A_1 + B_2$, $2A_8 + B_2$, $2A_7 + 2A_1 + B_2$

$A_{10}+A_6+A_2$, $2A_9$, $2A_7+A_3+A_1$, $A_7+A_6+A_3+A_2$,
$A_7+3A_3+A_2$, $3A_6$, $6A_3$.

(C) G has no vertex corresponding to a short root and G is a Dynkin graph obtained by one elementary transformation from one of Dynkin subgraphs of the modified Coxeter-Vinberg graph Σ'_{18} .

Remarks. (1) r = the number of vertices in G = rank Q .

(2) By $\varepsilon_p(Q)$ we denote the Hasse symbol of the inner product space $Q \otimes \mathbb{Q}$. For every prime number p , $\varepsilon_p(Q)$ has values ± 1 . The symbol $(a, b)_p$ $(a, b \in \mathbb{Q}, a \neq 0, b \neq 0)$ is the Hilbert symbol. For every prime number p , $(a, b)_p = \pm 1$. $\mathbb{Q}_p$ is the field of p-adic numbers, i.e., the quotient field of the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. $\mathbb{Q}_p^{*2} = \{a^2 | a \in \mathbb{Q}_p, a \neq 0\}$. (Serre [7], Urabe [9]).

(3) The sub-basic Dynkin graphs are the maximal graphs in the set of Dynkin graphs G satisfying the following conditions:

<1> G cannot be obtained from any one of the 9 basic graph by one tie transformation.

<2> The root lattice $Q = Q(G)$ of type G has a full embedding into the odd unimodular lattice of signature $(18, 1)$.

(4) The modified Coxeter-Vinberg graph Σ'_{18} is described as follows. (Vinberg-Kaplinskaja [15]). It has 24 vertices. The vertices $e_1, e_2, \dots, e_{22}$ correspond to long roots. Any two of them are either connected by a single edge or not connected. They form a tetrahedral subgraph Σ^*_{18} as is illustrated in the following Figure α. (The numbering is different from that in Vinberg-Kaplinskaja [15].)

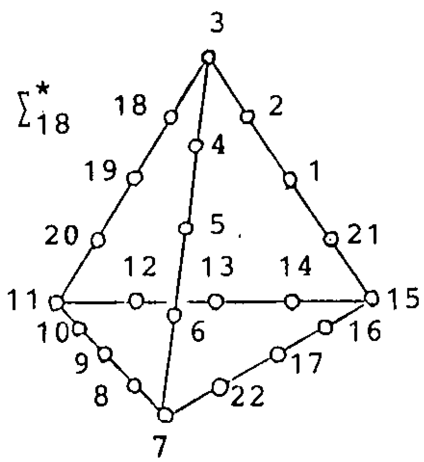


Figure α

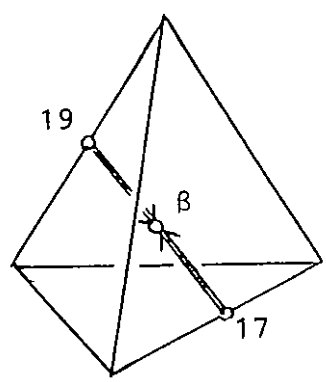


Figure β

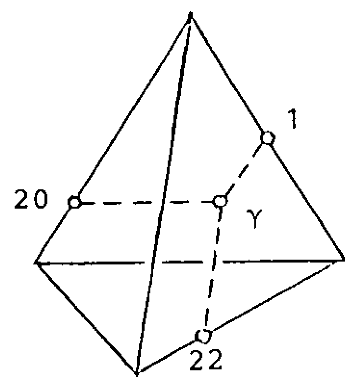


Figure γ

The remaining two vertices β, γ correspond to short roots. Figure β and Figure γ show their connections with Σ^*_{18}. The vertex β is connected with e_{17} and e_{19} only by a double edge with an arrow in the direction to β. The vertex γ is connected with three vertices e_1, e_{20}, e_{22} only by a broken edge. Moreover, the vertex β is connected with the vertex γ by a single edge accompanied with the mark ∞ (or by a heavy edge). $\overset{\beta}{\circ}\!\overset{\infty}{\text{——}}\!\overset{\gamma}{\circ}$.

We would like to explain what $\sum_{18}'$ is in the following.

First we explain the Coxeter-Vinberg graph $\sum_{18}$ associated with the integral orthogonal transformation group $O_{18,1}(\mathbb{Z})$ of the unimodular lattice with signature (18,1). (Vinberg-Kaplinskaja [15], Vinberg [14], Conway-Sloane [2]). The graph $\sum_{18}$ has 37 vertices and $\sum_{18}$ contains $\sum_{18}'$ as a subgraph. Among them 22 vertices correspond to long roots and they form a subgraph isomorphic to $\sum_{18}^*$ in Figure α . Therefore $\sum_{18}^* \subset \sum_{18}' \subset \sum_{18}$. Here note that $\sum_{18}^*$ has an action of the symmetric group S_4 of degree 4. The graph $\sum_{18}$ has the action of S_4 extending that on $\sum_{18}^*$. (However, $\sum_{18}'$ does not have an action.) The remaining 15 vertices of $\sum_{18}$ correspond to short roots. Three of 15 short vertices are conjugate to β with respect to S_4 and they are called the vertices of the first kind. The remaining 12 of 15 short vertices are conjugate to γ with respect to S_4 and they are called the vertices of the second kind.

(1) Vertices of the first kind are connected among themselves by single edges accompanied with the mark ∞ .

(2) Vertices of the second kind are connected among themselves by broken edges. o-----o .

(3) A vertex a of the first kind is connected to a vertex b of the second kind by a broken edge if in $\sum_{18}^*$ there is no vertex connected both to a and to b and there are no two

mutually connected vertices, one of which is connected to and the other to b. Otherwise a and b are connected by a single edge accompanied with the mark ∞.

Now, let us consider an unimodular lattice U of signature $(18,1)$. U is uniquely determined up to isomorphism. (Milnor-Husemoller [4].) The quadratic form on $U \cong \mathbb{Z}^{19}$ has the following form.

$$x^2 = (x, x) = -x_0^2 + x_1^2 + \ldots + x_{18}^2 , \quad x = (x_0, x_1, \ldots, x_{18}) \in \mathbb{Z}^{19} .$$

By $O_{18,1}(\mathbb{Z})$ we denote the group of integral linear transformations on U preserving the quadratic form. Let C denote the <u>negative cone</u> in $U \otimes \mathbb{R}$.

$$C = \{\xi \in U \otimes \mathbb{R} \mid \xi^2 < 0\} .$$

Let $\alpha \in U$ be an element with $\alpha^2 = 1$ or 2. Then α defines an integral orthogonal transformation $s_\alpha \in O_{18,1}(\mathbb{Z})$ by

$$s_\alpha(x) = x - 2(x, \alpha)\ \alpha/(\alpha, \alpha), \quad x \in U .$$

The transformation s_α is called the <u>reflection</u> with respect to α and has order 2. On $U \otimes \mathbb{R}$ it induces the reflection with respect to the hyperplane orthogonal to α. The subgroup Γ_{18} of $O_{18,1}(\mathbb{Z})$ generated by all reflections s_α with $\alpha \in U$ and $\alpha^2 = 1,2$ has a <u>fundamental polyhedron</u> P_{18} contained in C. P_{18} is a connected component of C minus the union of all hyperplanes orthogonal to some element $\alpha \in U$ with $\alpha^2 = 1$ or 2.

The graph Σ_{18} is the one describing P_{18}. The vertices of Σ_{18} have one-to-one correspondence with the facets of P_{18} (18-dimensional walls of P_{18}). Any two vertices of Σ_{18} are governed by the following rules:

(1) If the corresponding two facets are orthogonal, then they are not connected.

(2) If the facets have an angle $\pi/3$, then they are connected by a single edge.

(3) If the facets have an angle $\pi/4$, then they are connected by a double edge with an arrow in the direction to the vertex corresponding to a short root.

(4) If the facets are parallel in C, in other words, if the intersection of the two facets has a non-empty intersection with $\bar{C} - \{0\}$ ($\bar{C}$ denotes the closure of C), and if it has no intersection with C, then they are connected by a single edge accompanied with the mark ∞.

(5) If the facets have no intersection in $\bar{C}$, then they are connected by a broken edge.

The modified graph Σ'_{18} is a subgraph of Σ_{18} such that any Dynkin subgraph of Σ_{18} is conjugate to a subgraph of Σ'_{18} with respect to S_4.

We can also show the following theorem.

Theorem 0.5. Let G be a Dynkin graph without components of type B . If G can be obtained by one tie transformation from one of the 11 sub-basic Dynkin graphs in Theorem 0.4 , then there exists a normal quartic surface in the projective space of dimension 3 whose combination of singularities just agrees with G .

The plan of this article is like the following. In section 1 we develope the theory of tie transformations. We explain that the notion of tie transformations of Dynkin graphs is as natural and simple as that of elementary transformations. In section 2 we study the unimodular lattice with signature (18,1) . The reason why the sub-basic graphs are chosen is explained. In the last section section 3 we discuss the counter-examples to the converse of Theorem 0.2 and Theorem 0.5. We use Nikulin's lattice theory as the main tool in this section (Nikulin [5]).

I would like to express thanks to the Max-Planck-Institute, in particular to Professor F. Hirzebruch and Professor D. Zagier for warm hospitality. I would like to thank also my wife for making the exact list of maximal Dynkin subgraphs of Σ_{18} . This article is dedicated to my wife.

§ 1. The notion of tie transformations.

First of all, we recall the results in Urabe [9] .

By Urabe [9] Theorem 1.15 the problem on the combinations of rational double points is reduced to the problem on the embedding of the lattice $S = \mathbb{Z}\lambda \oplus Q(G)$ into the unimodular even lattice with signature $(19,3)$. Here λ is an element with $\lambda^2 = -4$ and $Q(G)$ is the root lattice associated with the Dynkin graph G .

Recall moreover several ideas in [9] in the lattice theory.

(1) We can pass from the unimodular even lattice Λ to the quotient quasi-lattice $\Lambda/\mathbb{Z}\lambda$ equipped with the canonical bilinear form with values in $\mathbb{Q}$, where $\lambda \in \Lambda$ is an element with $\lambda^2 = -4$.

(2) By the above passage the isotropic element u appearing in the condition (b) in Urabe [9] Theorem 1.15 corresponds to a short root μ in $\Lambda/\mathbb{Z}\lambda$. Because of this reason, we need to consider also Dynkin graphs of type B .

(3) Since $\Lambda/\mathbb{Z}\lambda$ contains an odd unimodular lattice with index 2 , we can apply the theory of odd unimodular lattices.

(4) After decomposing the indefinite even unimodular lattice $\bar{\Lambda}$ into a direct sum $\bar{\Lambda} = \Lambda \oplus H$ of another unimodular lattice Λ

and a hyperbolic plane H, we can pass from $\bar{\Lambda}$ to Λ.

The notion of tie transformations is based on the above fourth idea.

Recall that a submodule M of a root module L is said to be <u>full</u>, if any element η in the primitive hull of M in L such that $\eta^2 = 1$ or 2 belongs to M.

Theorem 1.1. Let $\bar{\Lambda}$ be an even unimodular lattice with signature $(16+N, N)$, $N \geqq 2$, and Λ be an even unimodular lattice with signature $(15+N, N-1)$. Let $\bar{\lambda} \in \bar{\Lambda}$ be an element with $\bar{\lambda}^2 = -4$. Let $\bar{G}$ be a Dynkin graph with at most one component of type B and $\bar{Q} = Q(\bar{G})$ be the root lattice of type $\bar{G}$. The following three conditions are equivalent.

(1) There is a full embedding $\bar{Q} \subset \bar{\Lambda}/\mathbf{Z}\lambda$ satisfying the following condition <*>.

> <*> There are a fundamental system $\bar{\Delta} \subset \bar{Q}$ of roots of $\bar{Q}$, a long root $\bar{\alpha} \in \bar{\Delta}$ in $\bar{\Delta}$, and an isotropic element $\bar{u} \in \bar{\Lambda}/\mathbf{Z}\bar{\lambda}$ such that $\bar{u} \cdot \bar{\beta} = 0$ for any $\bar{\beta} \in \bar{\Delta}$ with $\bar{\beta} \neq \bar{\alpha}$ and $\bar{u} \cdot \bar{\alpha} = 1$.

(2) There are an element $\lambda \in \Lambda$ with $\lambda^2 = -4$ and a Dynkin graph G with at most one component of type B satisfying the following conditions <a> and <b>.

<a> There is a full embedding $Q(G) \subset \Lambda/\mathbb{Z}\lambda$.

<b> $\bar{G}$ is obtained from G by one tie transformation.

(3) There are an element $\lambda \in \Lambda$ with $\lambda^2 = -4$ and a maximal positive-definite root sublattice $Q \subset \Lambda/\mathbb{Z}\lambda$ such that $\bar{G}$ is obtained by one tie transformation from the Dynkin graph G of Q .

Our Theorem 0.2 is a consequence of above Theorem 1.1. Indeed, let G'' be one of the 9 basic Dynkin graph and G' be a Dynkin graph obtained from G'' by one elementary transformation or a tie transformation.

By Urabe [9] Corollary 3.10 there is an element $\lambda \in Q(2E_8) \oplus H$ with $\lambda^2 = -4$ such that the root system $R(Q(2E_8) \oplus H/\mathbb{Z}\lambda)$ is of type G'' . In particular, there is a full embedding $Q(G'') \subset Q(2E_8) \oplus H/\mathbb{Z}\lambda$. If G' is obtained by an elementary transformation, then by Urabe [9] Proposition 4.2 there is a full embedding $Q(G') \subset (Q(2E_8) \oplus H/\mathbb{Z}\lambda) \oplus H$ $= Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$. (Here we identify $\lambda \in Q(2E_8) \oplus H$ with $\lambda \oplus 0 \in Q(2E_8) \oplus H \oplus H$.) If G' is obtained by a tie transformation, then by above Theorem 1.1 (2) $\Rightarrow$ (1) there is a full embedding $Q(G') \subset Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$.

Next assume that G is a Dynkin graph without components of type B and that G is obtained from G' by an elementary transformation or a tie transformation. Then by Urabe [9]

Proposition 4.2 or Theorem 1.1 we have a full embedding $Q(G) \subset Q(2E_8) \oplus H \oplus H \oplus H/\mathbb{Z}\lambda$. Set $S = \pi^{-1}(Q(G))$, where $\pi : Q(2E_8) \oplus H \oplus H \oplus H \longrightarrow Q(2E_8) \oplus H \oplus H \oplus H/\mathbb{Z}\lambda$ is the canonical surjective morphism. By Lemma 2.7, Corollary 2.8 and Lemma 4.1 in Urabe [9] we see that $S \cong \mathbb{Z}\lambda \oplus Q(G)$ and that S satisfies the condition (a) and (b) in Urabe [9] Theorem 1.15 (2). Thus there is a normal quartic surface with the combination of singularities G .

In the rest of this section we give the proof of Theorem 1.1. Note that the implication (3) $\Rightarrow$ (2) is obvious, since maximality implies fullness.

First in the following we show the implication (2) $\Rightarrow$ (1) . We assume the condition (2). We decompose G into a sum of connected Dynkin graphs $G = G_1 + G_2 + \dots + G_m$. Let $\Delta_i = \Delta(G_i)$ be a fundamental system of roots of the root lattice $Q(G_i)$ associated with the connected Dynkin graph G_i . Let η_i be the maximal root associated with Δ_i . It can be written in the form $\eta_i = \sum_{\alpha\in\Delta_i} n_{i\alpha}\alpha$. $n_{i\alpha}$ is a positive integer depending on the type of G . Set $n_{i,-\eta_i} = 1$. Set $\Delta_i^+ = \Delta_i \cup \{-\eta_i\}$, $\Delta = \bigcup_{i=1}^{m} \Delta_i$ and $\Delta^+ = \bigcup_{i=1}^{m} \Delta_i^+$.

The vertices of the extended graph $\tilde{G}$ have one-to-one correspondence with elements in Δ^+ . Thus corresponding to the procedure of the tie transformation, we have subsets $A \subset \Delta^+$, $B \subset \Delta^+$ satisfying the following (1), (2) and (3).

(1) $A \cap B = \phi$

(2) For every i with $1 \leq i \leq m$, $A \cap \Delta_i^+ \neq \phi$.

(3). For every i with $1 \leq i \leq m$, the greatest common divisor of the set of numbers

$$\sum_{\alpha \in B \cap \Delta_i^+} n_{i\alpha}, \quad n_{i\beta} \ (\beta \in A \cap \Delta_i^+)$$

is necessarily 1.

Let $H = \mathbb{Z}u + \mathbb{Z}v \ (u^2 = v^2 = 0, \ u \cdot v = v \cdot u = 1)$ be the hyperbolic plane. The lattices $\Lambda \oplus H$ and $\bar{\Lambda}$ are isomorphic and moreover we have an isomorphism $\varphi : \Lambda \oplus H \longrightarrow \bar{\Lambda}$ of lattices such that $\varphi(\lambda \oplus 0) = \bar{\lambda}$, since any two elements $\xi \in \bar{\Lambda}$ with $\xi^2 = -4$ are conjugate with respect to the orthogonal transformation group. (Milnor-Husemoller [4], Nikulin [5] Theorem 1.14.4.) Via this φ we identify $\Lambda \oplus H$ and $\bar{\Lambda}$. Then $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ is identified with $(\Lambda/\mathbb{Z}\lambda) \oplus H$. Here note that $\Delta^+ \subset Q(G) \subset \Lambda/\mathbb{Z}\lambda$. We define a sublattice $\bar{Q}$ of $\Lambda/\mathbb{Z}\lambda \oplus H$ as follows.

$$\bar{Q} = \sum_{\alpha \in \Delta^+ - (A \cup B)} \mathbb{Z}\alpha + \sum_{\alpha \in B} \mathbb{Z}(\alpha - u) + \mathbb{Z}(u+v) \subset \Lambda/\mathbb{Z}\lambda \oplus H .$$

Set $\bar{\Delta} = [\Delta^+ - (A \cup B)] \cup \{\alpha - u \mid \alpha \in B\} \cup \{u+v\}$. We can check that $\bar{\Delta}$ is a fundamental system of roots, whose Dynkin graph is $\bar{G}$. $\bar{Q}$ is a root lattice of type $\bar{G}$. It is easy to see that the condition <*> is satisfied. Therefore we only have to show that $\bar{Q}$ is full in $\Lambda/\mathbb{Z}\lambda \oplus H$.

Set $R' = \{\alpha \in P(\bar{Q}, \Lambda/\mathbb{Z}\lambda \oplus H) | \alpha^2 = 1 \text{ or } 2\}$ and $R = \{\alpha \in \bar{Q} | \alpha^2 = 1 \text{ or } 2\}$ = the root system generated by $\bar{\Delta}$. Here for a submodule M in a free $\mathbb{Z}$ - module L of finite rank, by $P(M, L)$ we denote the primitive hull of M in L . $P(M, L) = \{x \in L | mx \in M \text{ for some non-zero integer } m .\}$.

Under the above definition obviously $R \subset R'$. We would like to show $R = R'$.

Lemma 1.2. If $R \neq R'$, then there is an element $\bar{\beta} \in R' - R$ with the following form.

$$\bar{\beta} = \sum_{\alpha \in \Delta^+ - (A \cup B)} a_\alpha \alpha + \sum_{\beta \in B} a_\beta (\beta - u) , \quad (a_\alpha , a_\beta \in \mathbb{Q}) .$$

Proof. The last condition is equivalent to $\bar{\beta} \cdot u = 0$.

Let Q' be the submodule of $\Lambda/\mathbb{Z}\lambda \oplus H$ generated by R' . Q' is the root lattice of R' . Set $P = \{x \in Q' | x \cdot u = 0\}$. P is a primitive sublattice of Q' . Let $\{\beta_1 , \beta_2 , \ldots , \beta_\ell\}$ be a fundamental system of roots of the root system of P . $\ell \leq \operatorname{rank} P = \operatorname{rank} Q' - 1 = \operatorname{rank} \bar{Q} - 1$. On the other hand since $\bar{\Delta} - \{u + v\} \subset P$, the number of elements in $\bar{\Delta} = \operatorname{rank} \bar{Q} \leq \ell + 1$. Thus $\ell + 1 = \operatorname{rank} Q'$. By this equality and by the primitiveness of P one knows that there is a root $\beta_0 \in Q'$ such that

$\{\beta_0, \beta_1, \ldots, \beta_\ell\}$ is a fundamental system of roots of Q'.

If $\beta_i \notin R$ for some i with $1 \leq i \leq \ell$, then setting $\bar{\beta} = \beta_i$ for this i, this $\bar{\beta}$ satisfies the condition in the lemma. Therefore in what follows we deduce a contradiction assuming $\beta_i \in R$ for every i with $1 \leq i \leq \ell$. Since Q' has a basis $\beta_0, \beta_1, \ldots, \beta_\ell$ and since $Q' \neq \bar{Q}$, $\beta_0 \notin R$ under this assumption.

On the other hand setting

$$\beta_0 = \sum_{\alpha \in \Delta^+ - (A \cup B)} a_\alpha \alpha + \sum_{\alpha \in B} a_\alpha (\alpha - u) + b(u+v) \ (a_\alpha, b \in \mathbb{Q}),$$

we have $b = \beta_0 \cdot u \in \mathbb{Z}$ since $\beta_0 \in \Lambda / \mathbb{Z}\lambda \oplus H$. Moreover we have

$$b\mathbb{Z} = \{x \cdot u \mid x \in Q'\} \supset \{x \cdot u \mid x \in \bar{Q}\} \supset \{c(u+v) \cdot u \mid c \in \mathbb{Z}\} = \mathbb{Z}.$$

Thus $b = \pm 1$ and $a_{\alpha_0} \notin \mathbb{Z}$ for some $\alpha_0 \in \Delta^+ - A$ since $\beta_0 \notin R$.

Now, since $u + v \in \bar{Q} \subset Q'$, we have integers $c_0, c_1, \ldots, c_\ell$ such that $u + v = \sum_{i=0}^{\ell} c_i \beta_i$. Since $1 = (u+v) \cdot u = c_0 \beta_0 \cdot u = c_0 b$, we have $c_0 = b = \pm 1$. Moreover, since $\beta_i \in R$ for $1 \leq i \leq \ell$, we can write it in the form

$$\beta_i = \sum_{\alpha\in\Delta^+-(A\cup B)} e_{i\alpha}\alpha + \sum_{\alpha\in B} e_{i\alpha}(\alpha-u)\ (e_{i\alpha}\in\mathbb{Z}) .$$

By these equalities we have

$$\pm\{\sum_{\alpha\in\Delta^+-(A\cup B)} a_\alpha\alpha + \sum_{\alpha\in B} a_\alpha(\alpha-u)\} = \sum_{\alpha\in\Delta^+-(A\cup B)} e_\alpha\alpha + \sum_{a\in B} e_\alpha(\alpha-u)$$

where $e_\alpha = \sum_{i=1}^{\ell} c_i e_{i\alpha}$. Since $\bar{\Delta}$ is linearly independent, we have $\pm a_\alpha = e_\alpha$ for every $\alpha \in \Delta^+ - A$. In particular $\pm a_{\alpha_0} = e_{\alpha_0}$. However, the left-hand side of this equality is not an integer, while the right-hand side is an integer. This is a contradiction.

Q.E.D.

Now, assume $R \neq R'$. Choose an element $\bar{\beta}$ as in Lemma 1.2. Let β be the $\Lambda/\mathbb{Z}\lambda$ - component of $\bar{\beta}$.

$$\beta = \sum_{\alpha\in\Delta^+-(A\cup B)} a_\alpha\alpha + \sum_{\alpha\in B} a_\alpha\alpha = \sum_{\alpha\in\Delta^+-A} a_\alpha\alpha .$$

This β belongs to the primitive hull of $Q(G)$ in $\Lambda/\mathbb{Z}\lambda$. Since $Q(G)$ is full in $\Lambda/\mathbb{Z}\lambda$, β is an element in $Q(G)$. Corresponding to the direct sum decomposition $Q(G) = \bigoplus_{i=1} Q(G_i)$, we can write it in the form $\beta = \sum_{i=1}^{m} \beta_i$, $\beta_i \in Q(G_i)$. Then we have $\beta_i = \sum_{\alpha\in\Delta_i^+-A} a_\alpha\alpha$.

Since $\beta^2 = \bar{\beta}^2 = 1$ or 2 and since G has at most one component of type B , $\beta = \beta_j$ for some unique j with

$1 \leq j \leq m$ and $\beta_i = 0$ if $i \neq j$, $1 \leq i \leq m$. Let us fix a number i with $i \neq j$ and $1 \leq i \leq m$ for a while. We have $\sum_{\alpha \in \Delta_i^+ - A} a_\alpha \alpha = 0$.

Case 1. $-\eta_i \in A$.

Then we have $\Delta_i^+ - (A \cup B) = \Delta_i - (A \cup B)$ and $B \cap \Delta_i^+ = B \cap \Delta_i$. Thus the set $[\Delta_i^+ - (A \cup B)] \cup [B \cap \Delta_i^+] = \Delta_i^+ - A = \Delta_i - A$ is a part of a basis of $Q(G_i)$. It implies that $a = 0$ for every $\alpha \in \Delta_i^+ - A$.

Case 2. $-\eta_i \notin A$.

Set $a_i = a_{-\eta_i}$ for simplicity.

$$
\begin{aligned}
0 &= \sum_{\alpha \in \Delta_i - A} a_\alpha \alpha + a_i(-\eta_i) \\
&= \sum_{\alpha \in \Delta_i - A} a_\alpha \alpha - a_i \sum_{\alpha \in \Delta_i} n_{i\alpha}\alpha \\
&= \sum_{\alpha \in \Delta_i - A} (a_\alpha - a_i n_{i\alpha})\alpha - \sum_{\alpha \in \Delta_i \cap A} a_i n_{i\alpha}\alpha .
\end{aligned}
$$

Since Δ_i is a basis of $Q(G_i)$, $a_i n_{i\alpha} = 0$ for $\alpha \in A \cap \Delta_i$ and $a_\alpha - a_i n_{i\alpha} = 0$ for $\alpha \in \Delta_i - A$. Besides since $A \cap \Delta_i \neq \phi$ and $n_{i\alpha} \neq 0$, we have $a_i = 0$. Thus $a_\alpha = 0$ for every $\alpha \in \Delta_i^+ - A = (\Delta_i - A) \cup \{-\eta_i\}$

Here note that by the above arguments until here,

$$
\bar{\beta} = \sum_{\alpha \in \Delta_j^+ - (A \cup B)} a_\alpha \alpha + \sum_{\alpha \in B \cap \Delta_j^+} a_\alpha(\alpha - u) \quad \text{and}
$$

$$\beta = \beta_j = \sum_{\alpha \in \Delta_j^+ - A} a_\alpha \alpha \in Q(G_j) .$$

In particular $\quad - \bar{\beta} \cdot v = \sum_{\alpha \in B \cap \Delta_j^+} a_\alpha \quad$ is an integer,

since $\quad \bar{\beta} \in \Lambda/\mathbb{Z}\, \lambda \oplus H$.

Next we consider the number j .

Case 1. $\; -\eta_j \in A$.

The set $\Delta_j^+ - A = \Delta_j - A$ is a part of a basis of $Q(G_j)$. Thus $a_\alpha \in \mathbb{Z}$ for every $\alpha \in \Delta_j^+ - A$.

Case 2. $\; -n_j \notin A$.

Set $\; b = a_{-\eta_j} \;$ for simplicity .

$$\beta = \sum_{\alpha \in \Delta_j - A} a_\alpha \alpha + b(-\eta_j)$$

$$= \sum_{\alpha \in \Delta_j - A} (a_\alpha - b\, n_{j\alpha})\alpha - \sum_{\alpha \in \Delta_j \cap A} b\, n_{j\alpha}\alpha .$$

Since Δ_j is a basis of $Q(G_j)$, one knows the following $*$.

$$* \quad \begin{cases} a_\alpha - b n_{j\alpha} \in \mathbb{Z} & \text{for} \quad \alpha \in \Delta_j - A \\ b n_{j\alpha} \in \mathbb{Z} & \text{for} \quad \alpha \in \Delta_j \cap A = \Delta_j^+ \cap A . \end{cases}$$

Here consider the following equality.

$$\sum_{\alpha\in\Delta_j\cap B}(a_\alpha - bn_{j\alpha}) = \sum_{\alpha\in\Delta_j^+\cap B} a_\alpha - b\sum_{\alpha\in\Delta_j^+\cap B} n_{j\alpha}$$

If $-\eta_j \notin B$, since $\Delta_j \cap B = \Delta_j^+ \cap B$, this equality holds. In case $-\eta_j \in B$, since $\Delta_j^+ \cap B = (\Delta_j \cap B) \cup \{-\eta_j\}$ and since $n_{j,-\eta_j} = 1$, it holds. Namely this equality always holds. Now, since the left-hand side and the first term of the right-hand side of this equality are integers, $b\sum_{\alpha\in\Delta_j^+\cap B} n_{j\alpha}$

is an integer. On the other hand by condition (3) we have integers p , q_α such that

$$p\sum_{\alpha\in\Delta_j^+\cap B} n_{j\alpha} + \sum_{\alpha\in\Delta_j^+\cap A} q_\alpha n_{j\alpha} = 1 .$$

Thus $b = p(b\sum_{\alpha\in\Delta_j^+\cap B} n_j) + \sum_{\alpha\in\Delta_j^+\cap A} q_\alpha(b\, n_{j\alpha})$ is an integer. By this fact and by $*$, one knows that $a_\alpha \in \mathbb{Z}$ for every $\alpha \in \Delta_j^+ - A$.

Consequently

$$\bar\beta = \sum_{\alpha\in\Delta_j^+-(A\cup B)} a_\alpha \alpha + \sum_{\alpha\in\Delta_j^+\cap B} a_\alpha(\alpha - u), \quad (a_\alpha \in \mathbf{Z})$$

and we have $\bar\beta \in R$, which contradicts the choice of $\bar\beta$. Therefore $R = R'$ and $\bar Q$ is full in $\Lambda/\mathbf{Z}\lambda \oplus H$. It concludes the proof of the implication (2) $\Rightarrow$ (1) .

Next we will show the implication (1) $\Rightarrow$ (3) in Theorem 1.1. We assume the condition (1) . We take $\bar{\Lambda}$, $\bar{\alpha}$ and $\bar{u}$ satisfying <*> and fix them. Set $\bar{\Delta} := \{\bar{\alpha}, \bar{\beta}_1, \dots, \bar{\beta}_\ell\}$.

Lemma 1.3. Let L be the orthogonal complement of $\mathbb{Z}\bar{\lambda}$ in $\bar{\Lambda}$. We identify L itself with the image of the composition of the natural morphisms $L \longrightarrow \bar{\Lambda} \longrightarrow \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$.

(1) $L = \{x \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda} \mid x^2 \text{ is an even integer.}\}$
$= \{x \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda} \mid x \cdot y \in \mathbb{Z} \text{ for every } y \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}.\}$

(2) There is a canonical isomorphism of quasi-lattices between $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ and the dual module $L^* = \mathrm{Hom}(L,\mathbb{Z})$ such that under this isomorphism the canonical bilinear form $L^* \times L \longrightarrow \mathbb{Z}$ is identified with the restriction $(\bar{\Lambda}/\mathbb{Z}\bar{\lambda}) \times L \longrightarrow \mathbb{Q}$ of the bilinear form on $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$.

Proof. Easy.

Lemma 1.4. (1) There is an element $\bar{v}_1 \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ with $\bar{v}_1{}^2 = 0$ and $\bar{v}_1 \cdot \bar{u} = 1$.
(2) For every element $\bar{x} \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$, $\bar{x} \cdot \bar{v}_1$ and $\bar{x} \cdot \bar{u}$ are integers.

Proof. (1) Assume $\bar{u} = a\,\bar{u}_1$ with $a \in \mathbb{Z}$ and $\bar{u}_1 \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$.

Obviously $a \neq 0$ and $\bar{u}_1{}^2 = 0$ since $a^2 \bar{u}_1{}^2 = \bar{u}^2 = 0$. Thus $\bar{u}_1 \in L$ by Lemma 1.3 (1). We have $1 = \bar{\alpha} \cdot \bar{u} = a\bar{\alpha} \cdot \bar{u}_1$ and $a = \pm 1$ since $\bar{\alpha} \cdot \bar{u}_1 \in \mathbb{Z}$ by Lemma 1.3 (1) again. It implies that $\mathbb{Z}\bar{u}$ is primitive in $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$. By Lemma 1.3 (2) we have an element $\bar{v}_2 \in L$ with $\bar{v}_2 \cdot \bar{u} = 1$. Since L is even, we can write $\bar{v}_2{}^2 = 2b$ with $b \in \mathbb{Z}$. The element $\bar{v}_1 = \bar{v}_2 - b\bar{u}$ satisfies the condition.

(2) It is obvious by Lemma 1.3 (1).

Q.E.D.

The sublattice $\bar{H}_1 = \mathbb{Z}\bar{u} + \mathbb{Z}\bar{v}_1$ of rank 2 with the basis $\bar{u}, \bar{v}_1$ is a hyperbolic plane. Let M_1 be the orthogonal complement of $\bar{H}_1$ in $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$. Since $\bar{H}_1$ is a unimodular lattice and since $\bar{H}_1 \subset L$, one knows $\bar{\Lambda}/\mathbb{Z}\bar{\lambda} = M_1 \oplus \bar{H}_1$ (orthogonal direct sum). We have decompositions in the following form.

$$\bar{\alpha} = \alpha + a\bar{u} + \bar{v}_1$$

$$\bar{\beta}_i = \beta_i + b_i\bar{u} \qquad (1 \leq i \leq \ell)$$

$$(\alpha, \beta_i \in M_1 , \quad a, b_i \in \mathbb{Z}) .$$

Set $\varepsilon_i = \bar{\alpha} \cdot \bar{\beta}_i$. Note that $\varepsilon_i = 0$ or -1 .

We have the following equalities.

$$\alpha^2 + 2a = \bar{\alpha}^2 = 2 , \alpha^2 = \alpha \cdot \bar{\alpha} = 2 - 2a$$

$$\alpha \cdot \beta_i + b_i = \varepsilon_i , \alpha \cdot \beta_i = \alpha \cdot \bar{\beta}_i \quad (1 \leq i \leq \ell) .$$

Next we define the Eichler-Siegel transformation $\psi : \bar{\Lambda}/\mathbb{Z}\bar{\lambda} \longrightarrow \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ associated with the isotropic element $\bar{u}$ and the element α with $\alpha \cdot \bar{u} = 0$ by

$$\psi(x) = x + (x\cdot\alpha)\bar{u} - (x \cdot \bar{u})\alpha + (a-1)(x \cdot \bar{u})\bar{u} .$$

(Ebeling [3] pp. 331). We can check that ψ is an isomorphism preserving the bilinear form. The inverse of ψ is given by the following.

$$\psi^{-1}(y) = y - (y \cdot \alpha)\bar{u} + (\bar{y} \cdot \bar{u})\alpha + (a-1)(y \cdot \bar{u})\bar{u} .$$

It is easy to show:

$$\psi(\bar{u}) = \bar{u}$$

$$\psi(\bar{\alpha}) = \bar{u} + \bar{v}_1$$

$$\psi(\bar{\beta}_i) = \beta_i + \varepsilon_i\bar{u} \quad (1 \leq i \leq \ell) .$$

Setting $\bar{v} = \psi^{-1}(\bar{v}_1)$, we have the following lemma.

Lemma 1.5. There is an element $\bar{v} \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ satisfying the following conditions:

$$\bar{u} \cdot \bar{v} = 1 , \bar{v}^2 = 0 , \bar{\alpha} \cdot \bar{v} = 1 ,$$

$$\bar{\beta}_i \cdot \bar{v} = \bar{\beta}_i \cdot \bar{\alpha} \quad (1 \leq i \leq \ell) .$$

Set $\bar{H} = \mathbb{Z}\bar{u} + \mathbb{Z}\bar{v}$. $\bar{H}$ is a hyperbolic plane. Let M be the orthogonal complement of $\bar{H}$ in $\bar{\Lambda}/\mathbb{Z}\bar{\lambda}$. $\bar{\Lambda}/\mathbb{Z}\bar{\lambda} = M \oplus \bar{H}$ (orthogonal direct sum). $\bar{\alpha} = \bar{u} + \bar{v}$. If $\bar{\beta}_i \cdot \bar{\alpha} = 0$, then $\bar{\beta}_i \in M$. If $\bar{\beta}_i \cdot \bar{\alpha} = -1$, then $\bar{\beta}_i = \beta_i' - u$ with $\beta_i' \in M$.

Let $\pi : \bar{\Lambda} \longrightarrow \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ denote the canonical surjective morphism.

Lemma 1.6. For every element $\bar{x} \in \bar{\Lambda}/\mathbb{Z}\bar{\lambda}$ such that $\bar{x}^2$ is an even integer, there is a unique element $x \in \bar{\Lambda}$ with $\pi(x) = \bar{x}$ and $x \cdot \bar{\lambda} = 0$. Moreover, for this element the equality $x \cdot y = \bar{x} \cdot \pi(y)$ holds for every $y \in \bar{\Lambda}$.

Proof. It is obvious by Lemma 1.4.

By Lemma 1.6 we have elements $u, v \in \bar{\Lambda}$ with $\pi(u) = \bar{u}$, $\pi(v) = \bar{v}$, $u^2 = v^2 = 0$, $u \cdot v = 1$ and $u \cdot \bar{\lambda} = v \cdot \bar{\lambda} = 0$. $H = \mathbb{Z}u + \mathbb{Z}v$ is a hyperbolic plane. Let Λ_1 be the orthogonal complement of H in $\bar{\Lambda}$. One knows $\bar{\Lambda} = \Lambda_1 \oplus H$ (orthogonal direct sum), $\bar{\lambda} \in \Lambda_1$ and $\Lambda_1 = \pi^{-1}(M)$. Λ_1 is an even unimodular lattice with signature $(15+N, N-1)$. We have an isomorphism $\rho : \Lambda_1 \longrightarrow \Lambda$ of lattices. Setting $\lambda = \varphi(\bar{\lambda})$, we identify Λ_1 and Λ, $\bar{\lambda}$ and λ via φ . Then M is identified with $\Lambda/\mathbb{Z}\lambda$. $M = \Lambda/\mathbb{Z}\lambda$.

Set $\bar{P} = \{\bar{x} \in \bar{Q} | \bar{x} \cdot \bar{u} = 0\}$. $\bar{P}$ is a root lattice and $\bar{\Delta}_- = \bar{\Delta} - \{\bar{\alpha}\} = \bar{\Delta} - \{\bar{u}+\bar{v}\}$ is a fundamental system of roots of $\bar{P}$. $\bar{P} \subset M + \mathbb{Z}\bar{u}$. Let $\rho : M + \mathbb{Z}\bar{u} \longrightarrow M$ denote the projection to the M-factor. The restriction $\rho|\bar{P}$ is an isomorphism of lattices onto the image. Let $\hat{P}$ be the primitive hull of the image $\rho(\bar{P})$ in M. Then the quotient $\hat{P}/\rho(\bar{P})$ is a finite cyclic group. (Urabe [9] Proposition 2.9 (3)). Let Q be a maximal positive-definite root sublattice of M containing $\hat{P}$. The torsion part of $Q/\rho(\bar{P})$ is cyclic. Thus $\rho(\bar{P})$ is obtained from Q by an elementary transformation. (Urabe [9] Proposition 2.9 (4)). Namely there is a fundamental system Δ of roots of Q such that $\rho(\bar{\Delta}_{-})$ is a subset of Δ^{+}. Here Δ^{+} is the extended fundamental system, which is the union of Δ and (-1) times maximal roots associated with the irreducible components of Δ. Set

$$A = \Delta^{+} - \rho(\bar{\Delta}_{-})$$

$$B = \{\rho(\bar{\beta}) \mid \bar{\beta} \in \bar{\Delta}_{-},\ \bar{\alpha}\cdot\bar{\beta} \neq 0\}\ .$$

If we recall the rule which we use when we make the Dynkin graph or the extended Dynkin graph from the fundamental system of roots, and if we ignore the condition (3) <b> in Definition 0.1, then we know that the Dynkin graph $\bar{G}$ of the fundamental system $\bar{\Delta}$ of roots of $\bar{Q}$ is obtained from the Dynkin graph G of Q by one tie transformation. The subsets A, B which we have to choose on the way of the tie transformation are the ones corresponding to the above A, B. The new vertex θ corresponds to $\bar{\alpha} = \bar{u} + \bar{v}$.

Thus we only have to show that the condition on G.C.D. in Definition 0.1 (3) <b> is satisfied. This follows from the following proposition, since $\bar{Q}$ is full.

Proposition 1.7. Let Q be an <u>irreducible</u> root lattice of type A,B,D or E, $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be a fundamental system of roots of Q, and $\eta = \sum n_{\alpha_i} \alpha_i$ be the maximal root associated with Δ. (Every n_{α_i} is a positive integer.) Set $n_{-\eta} = 1$ and set $\Delta^+ = \Delta \cup \{-\eta\}$. Let $A \subset \Delta^+$, $B \subset \Delta^+$ be subsets such that $A \neq \phi$, $A \cap B = \phi$ and B contains at most three elements. Let $H = \mathbb{Z}u + \mathbb{Z}v$ $(u^2 = v^2 = 0,\ u \cdot v = v \cdot u = 1)$ be the hyperbolic plane. We define a sublattice $\bar{Q}$ of $Q \oplus H$ as follows:

$$\bar{Q} = \sum_{\alpha \in \Delta^+ - (A \cup B)} \mathbb{Z}\alpha + \sum_{\alpha \in B} \mathbb{Z}(\alpha - u) + \mathbb{Z}(u+v) .$$

Finally by n we denote the greatest common divisor of the numbers

$$\sum_{\alpha \in B} n_\alpha,\ n_\beta (\beta \in A) .$$

Under these assumptions, if $n \neq 1$, then there is an element γ in the primitive hull of $\bar{Q}$ in $Q \oplus H$ such that $\gamma^2 = 1$ or 2, $\gamma \cdot u = 0$ and $\gamma \notin \bar{Q}$.

This proposition is a consequence of the following property of root systems.

Fact 1.8. Let $Q, \Delta, \eta, n_\alpha$ and Δ^+ be the same as in Proposition 1.7. Let $p > 1$ be a prime number and $S = \{\alpha \in \Delta^+ | n_\alpha$ is a multiple of $p.\}$. If $S \neq \phi$, then there are integers m_α $(\alpha \in \Delta^+)$ such that $(\sum_{\alpha \in \Delta^+} m_\alpha \alpha)^2 = 1$ or 2 and such that

$pm_\alpha = n_\alpha$ for $\alpha \in S$.

It is easy to see that if $S \neq \phi$, then Q is neither of type A nor of type B_1 and $p = 2,3$ or 5. With the help of the tables at the end of Bourbaki [1] we can check Fact 1.8.

Proof of Proposition 1.7. Assume $n \neq 1$. Let p be a prime number dividing n , and set $S = \{\alpha \in \Delta^+ | n_\alpha$ is a multiple of $p.\}$. We have $\phi \neq A \subset S$. Thus we have integers $m_\alpha (\alpha \in \Delta^+)$ as in Fact 1.8. Set $E = B \cap (\Delta^+ - S)$, $F = (\Delta^+ - S) - E$. Set $s_\alpha = n_\alpha - pm_\alpha$ $(\alpha \in \Delta^+)$ and set

$$\gamma = -\frac{1}{p}\left(\sum_{\alpha \in F} s_\alpha \alpha + \sum_{\alpha \in E} s_\alpha (\alpha - u)\right) .$$

Note that $E \subset B$ and $F \subset \Delta^+ - (A \cup B)$ since $A \subset S$. Thus $\{\alpha | \alpha \in F\} \cup \{\alpha - u | \alpha \in E\}$ is a part of a basis of $\bar{Q}$. The above γ is an element in $\bar{Q} \otimes \mathbb{Q}$. Besides for $\alpha \in E \cup F$, $s_\alpha \equiv n_\alpha \not\equiv 0$ (mod p) . Thus $\gamma \notin \bar{Q}$.

On the other hand $\sum_{\alpha \in E} s_\alpha = \sum_{\alpha \in E} (n_\alpha - pm_\alpha) \equiv \sum_{\alpha \in E} n_\alpha \equiv \sum_{\alpha \in B} n_\alpha \equiv 0$ (mod p) . Thus $e = \sum_{\alpha \in E} s_\alpha / p$ is an integer.

$$\gamma = -\frac{1}{p} \sum_{\alpha \in \Delta^+ - S} (n_\alpha - pm_\alpha)\alpha + eu$$

$$= -\frac{1}{p} \sum_{\alpha \in \Delta^+} (n_\alpha - pm_\alpha)\alpha + eu$$

$$= \sum_{\alpha \in \Delta^+} m_\alpha \alpha + eu \text{ (since } \sum n_\alpha \alpha = 0 \text{) .}$$

One knows that $\gamma \in Q \oplus H$, $\gamma \cdot u = 0$ and $\gamma^2 = (\sum m_\alpha \alpha)^2 = 1$ or 2.

Q.E.D.

We complete all the proof of Theorem 1.1.

§ 2. Coxeter groups on a hyperbolic space

In this section we would like to verify Theorem 0.4.

First of all, the following proposition treats the arithmetic conditions in Theorem 0.4 (A). By $d(Q)$ we denote the discriminant of a lattice Q. The symbol $\varepsilon_p(Q)$ is the Hasse symbol and $(\ ,\ )_p$ is the Hilbert symbol. (Serre [7])

Proposition 2.1. Let Q be a positive-definite lattice of rank r and λ be an element with $\lambda^2 = -4$. Set $S = \mathbb{Z}\lambda \oplus Q$ (orthogonal direct sum). Assume that there is an embedding $S \subset Q(2E_8) \oplus H \oplus H \oplus H = \Lambda$. The following conditions are equivalent.

(1) The orthogonal complement of S in Λ contains an isotropic element.

(2) One of the following <1>, <2>, <3> holds.

<1> $r = 18$, and for every prime number p, $\varepsilon_p(Q) = 1$.

<2> $r = 17$, and for every prime number p, $d(Q) \notin \mathbb{Q}_p^{*2}$ or $\varepsilon_p(Q) = (-1, d(Q))_p$.

<3> $r \leq 16$.

We omit the proof because it is the same as that of Urabe [9] Corollary 3.3.

By Proposition 2.1 and the theory developed in Urabe [9] one knows that the condition (A) in Theorem 0.4 is equivalent to the following (C'). Let $\lambda \in Q(2E_8) \oplus H \oplus H$ be an element with $\lambda^2 = -4$. Such an element λ is unique up to orthogonal transformations.

(C') There is a positive-definite primitive submodule T' of $Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$ such that G is obtained from the Dynkin graph of the root system $R(T')$ of T' by one elementary transformation.

Set $U = \{x \in Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda \mid x^2 \in \mathbb{Z}\}$. U is a sublattice of the quasi-lattice. $Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$, and is odd, unimodular and with signature (18,1). Under a suitable isomorphism $U \cong \mathbb{Z}^{19}$ the quadratic form on U is given by

$$x^2 = (x,x) = -x_0{}^2 + x_1{}^2 + \dots + x_{18}{}^2$$
$$\text{for } x = (x_0, x_1, \dots, x_{18}) \in \mathbb{Z}^{19} .$$

Since $R(T') \subset U$, the above (C') is equivalent to the following (C").

(C") There is a positive-definite primitive submodule T of U such that G is obtained from the Dynkin graph of the root system $R(T)$ of T by one elementary transformation.

By the following proposition one knows that the condition (C) in Theorem 0.4 is equivalent to (C"). Therefore we have the equivalence (A) $\Longleftrightarrow$ (C) .

Proposition 2.2. The following three sets of Dynkin graphs coincide.

(1) The set of the Dynkin graph of the root system $R(T)$ where T runs over all positive-definite primitive submodules of the unimodular lattice U of signature $(18,1)$.

(2) The set of any Dynkin graph isomorphic to a subgraph of the Coxeter-Vinberg graph $\sum_{18}$.

(3) The set of any Dynkin graph isomorphic to a subgraph of the modified Coxeter-Vinberg graph $\sum'_{18}$.

Proof. First we show that the set (1) contains the set (2). Assume that the Dynkin graph G is a subgraph of $\sum_{18}$. Associated with vertices of G , we have a set S of facets of the fixed fundamental polyhedron $P = P_{18}$. Let Δ be the set of roots in U perpendicular to some member of S and directing outwards from P . Let R be the root system generated by Δ . Δ is a fundamental system of roots of R . Let Q be the submodule generated by R in U . Q is a root lattice and Q,R and Δ are of type G . Let T be the primitive hull of Q in U . Obviously $R \subset R(T)$. Let $V = \{x \in U \otimes \mathbb{R} \mid x \cdot y = 0$ for every $y \in Q.\}$ be the orthogonal complement of Q in $U \otimes \mathbb{R}$. V coincides with the orthogonal complement of T in $U \otimes \mathbb{R}$. V intersects with the negative cone C . Let

$\alpha \in R(T)$ be an arbitrary element. The associated reflection s_α fixes every point on $C \cap V$. By the theory of Coxeter groups (Bourbaki [1] Chap.5 § 3) the subgroup of Γ_{18} of all elements fixing every point on $C \cap V$ coincides with the group generated by s_β's with $\beta \in \Delta$. It implies $\alpha \in R$. Thus $R = R(T)$ and $R(T)$ is of type G.

Conversely, the set (1) is contained in the set (2). Let T be a positive-definite primitive submodule of U. Let $V = \{x \in U \otimes \mathbb{R} | x \cdot y = 0 \text{ for every } y \in T.\}$ be the orthogonal complement of T in $U \otimes \mathbb{R}$. Let $p_0 \in V \cap C$ be a general point. Choosing p_0 generally enough, one can assume $T = \{y \in U | p_0 \cdot y = 0\}$. Thus the subgroup $\Gamma(p_0)$ of Γ_{18} generated by all reflections whose mirrors pass through p_0 coincides with the Weyl group (of the root system) of T. Now, we apply Vinberg's algortihm with respect to the point p_0 and try to construct a fundamental polyhedron of Γ_{18}. (Vinberg [12] section 3 Proposition 4.) By his algorithm we obtain Σ_{18} by extending the Dynkin graph of $\Gamma(p_0)$. Thus the Dynkin graph of T is a subgraph of Σ_{18}.

Here note that the root system $R(T)$ in (1) has at most one component of type B. Thus any Dynkin subgraph of Σ_{18} has at most one component of type B.

The equivalence between (2) and (3) is obvious by the definition of Σ'_{18} and by the above property of Dynkin subgraphs of Σ_{18}. Any Dynkin subgraph of Σ_{18} is conjugate to a subgraph of Σ'_{18} with respect to the action of S_4.

Q.E.D.

Lemma 2.3. (1) Any Dynkin subgraph of $\sum_{18}$ has at most one component of type B.
(2) Any maximal Dynkin subgraph G of $\sum_{18}$ has either 17 or 18 vertices. If G has 17 vertices, then there is a subgraph F of $\sum_{18}$ containing G such that every connected component of F is an extended Dynkin graph of some type. (The type of G and the type of F do not necessarily coincide.)

Proof. (1) is a corollary of Proposition 2.2.
(2) Let C_+ be one of the two connected components of the negative cone C containing the fixed fundamental polyhedron $P = P_{18}$. The quotient $C_+/\mathbb{R}_+$ can be regarded as the Lobačevskiǐ space containing $P/\mathbb{R}_+$. By Vinberg [14] section 3 number 2 and Vinberg [15] Theorem 1 $P/\mathbb{R}_+$ is a finite polyhedron with finite volume. Namely there are a finite number of non-zero vectors $v_1, v_2, \ldots, v_\ell$ belonging to the closure of C_+ and P is the interior of the minimal convex body containing the set $\bigcup_{j=1}^{\ell} \mathbb{R}_+ v_j$, where $\mathbb{R}_+ v_j = \{tv_j \mid t \in \mathbb{R},\ t > 0\}$.

Note that by definition of P, for each v_j, $1 \leq j \leq \ell$, there are linearly independent 18 roots $\alpha_1, \alpha_2, \ldots, \alpha_{18} \in U$ depending on v_j such that $v_j \cdot \alpha_i = 0$ for $1 \leq i \leq 18$.

Let G be a maximal Dynkin subgraph of $\sum_{18}$. The set of vertices of G corresponds to a set S of facets of P. Let Q be the sublattice in U generated by all roots perpendicular to some facet in S. Q is a positive-definite root lattice of type G and the orthogonal complement V of Q in $U \otimes \mathbb{R}$ intersects with C. Thus the intersection of V with the closure of P is a non-empty convex body in V. In particular V contains v_j for some j with $1 \leq j \leq \ell$. Take and fix such a j with $v_j \in V$. Let S' be the set of facets of P passing through v_j. We have $S \subset S'$. Let F be the subgraph of $\sum_{18}$ whose set of vertices corresponds to S'. By definition F contains G. We have 2 cases.

Case 1. $v_j \in C$.

The orthogonal complement A of $\mathbb{Z}v_j$ in U is a positive-definite lattice of rank 18. F is the Dynkin graph of the root system of A. Since A contains 18 linearly independent roots $\alpha_1,\ldots,\alpha_{18}$, the number of vertices in F is 18. On the other hand by maximality $F = G$. Thus G has 18 vertices.

Case 2. $v_j \in \partial C = \bar{C} - C$.

Note that $v_j^2 = 0$. Let A be the orthogonal complement of $\mathbb{Z}v_j$ in U. A is a positive semi-definite lattice of rank 18 with $\mathbb{Z}v_j \subset A$. Set $I = \mathbb{Q}v_j \cap A$. The quotient A/I is a positive definite lattice of rank 17. Let F' be the Dynkin graph of A/I. Now, the image of $\alpha_i \in A$ $(1 \leq i \leq 18)$ is also a root in A/I and they span a root sublattice of rank 17. Thus F' has 17 vertices. The graph F is the corresponding extended Dynkin graph to F'. Since G is a maximal Dynkin subgraph contained in F, G has 17 vertices.

Q.E.D.

By Lemma 2.3 and by the symmetry of $\sum_{18}$, it is not difficult to make the list of all conjugacy classes with respect to S_4 of maximal Dynkin subgraphs of $\sum_{18}$.

The list contains 178 classes with 18 vertices and 27 classes with 17 vertices.

Among them we have 4 conjugacy classes of type $2D_8 + B_1$, and for any one of the following 8 types we have 2 conjugacy classes.

$$
\begin{array}{ll}
B_1 + E_6 + A_7 + A_4 , & B_1 + E_6 + A_{11} \\
B_1 + E_7 + A_9 + A_1 , & B_1 + E_7 + A_7 + A_2 + A_1 \\
B_1 + D_{12} + D_5 , & B_1 + A_7 + 2D_5 \\
A_{11} + A_3 + 2A_2 , & A_{17} + A_1 .
\end{array}
$$

For the other $162 (= 178 - 2 \times 8)$ classes with 18 vertices and $23 (= 27 - 4 \times 1)$ classes with 17 vertices, the conjugacy class is uniquely determined by the type of the Dynkin graph.

The number of items in the list is too large for the list to be contained in this article.

Next we show that the condition (C) in Theorem 0.4 implies (B).

Assume that a Dynkin graph G is obtained from a Dynkin graph G' by one elementary transformation and G' has an embedding $\varphi : G' \longrightarrow \sum_{18}$ as a subgraph of $\sum_{18}$. Let G^* be a maximal Dynkin graph of $\sum_{18}$ containing the image $\varphi(G')$. By Urabe [9] Proposition 2.9 one knows that G can be obtained also from the Dynkin graph G^* by one elementary transformation. Thus replacing $\varphi(G')$ by G^* , one can assume that $\varphi(G')$ is a maximal Dynkin subgraph in $\sum_{18}$, in other words, φ is a <u>maximal embedding</u>. In the following we assume this assumption.

We have a positive-definite primitive submodule T of U whose root system $R(T)$ is of type G' by the proof of Proposition 2.2 (2) $\Rightarrow$ (1). The module T is associated with $\varphi(G')$ and depends on the choice of the fundamental polyhedron $P = P_{18} \subset C \subset U \otimes \mathbb{R}$. For a fixed P, T is uniquely determined depending on $\varphi(G')$. If one chooses another P , then the obtained T is conjugate to the original T with respect to the group Γ_{18} .

We have three cases.

(1) For some maximal embedding φ , the orthogonal complement of T in U contains an isotropic element.

(2) For some maximal embedding φ , the sublattice $\bar{Q}$ of T generated by $R(T)$ satisfies the condition <*> in Theorem 1.1 (1). (Note that $\bar{Q} \subset T \subset U \subset Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$.)

(3) For any maximal embedding φ , for any fundamental system of roots $\bar{\Delta} \subset R(T)$ and for any isotropic element $u \in Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$, the inequality $\sum_{\alpha \in \Delta} |\alpha \cdot u| \geq 2$ holds.

In case (1) by the results in Urabe [9] G' is obtained from one of the 9 basic Dynkin graphs by one elementary transformation. Thus we have the case (B-1) <1> .

In case (2) by Theorem 1.1 we have an element $\lambda' \in Q(2E_8) \oplus H$ with $\lambda'^2 = -4$ and a maximal positive-definite root sublattice $Q \subset Q(2E_8) \oplus H/\mathbb{Z}\lambda'$ such that G' is obtained by one tie transformation from the Dynkin graph G'' of Q . Besides by Urabe [9] Corollary 3.10 G'' is one of the basic 9 Dynkin graphs. Thus we have the case (B-1) <2> .

Now, we have to consider case (3). We would like to reduce (3) to a simpler condition. Let us consider about the choice of φ . Recall the action of S_4 on $\sum_{18}$. By Vinberg-Kaplinskaja [15] Theorem 2 any automorphism of the graph $\sum_{18}$ is induced by an orthogonal transformation of U . Any orthogonal transformation on U can be extended to $Q(2E_8) \oplus H \oplus/\mathbb{Z}\lambda$. Thus in order to check (3) it suffices to check it only for the representatives of S_4-conjugacy classes of the embedding $\varphi : G' \longrightarrow \sum_{18}$.

Secondly we consider the choice of $\bar{\Delta}$. Let S be the set of facets of P corresponding to the vertices of $\varphi(G')$. Let $\bar{\Delta}_0$ be the set of roots perpendicular to some facet in S and directing outwards from P . The set $\bar{\Delta}_0 = \bar{\Delta}(\varphi)$ is a fundamental system of roots of T and is uniquely determined by $\varphi(G')$. For any fundamental system of roots $\bar{\Delta} \subset R(T)$, we have an element σ of the Weyl group of T with $\sigma(\bar{\Delta}) = \bar{\Delta}_0$, and σ induces an orthogonal transformation of $Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$. Thus we only have to check the condition on $\bar{\Delta}_0$.

(3) is equivalent to the following (3*).

(3*) Let $\varphi_1,\dots,\varphi_\ell$ be the representatives of S_4-conjugacy classes of the maximal embeddings $\varphi : G' \longrightarrow \sum_{18}$. For any φ_i, $1 \leq i \leq \ell$ and for any isotropic element u in $Q(2E_8) \oplus H \oplus H/\mathbb{Z}\lambda$, $\sum|\alpha\cdot u| \geq 2$, where the sum is taken over all elements α in the fundamental system of roots $\bar{\Delta}_0 = \bar{\Delta}(\varphi_i)$ associated with φ_i .

Now, we have special isotropic elements, namely, the vectors spanning 1-dimensional egdes of the fundamental polyhedrom P lying on the boundary of C . We call an element $u \in U$ an isotropic element at infinity of P , if $u \neq 0$, $u^2 = 0$, and $\pm u$ belongs to the closure $\bar{P}$.

We consider the following condition depending on an embedding $\varphi : G' \longrightarrow \Sigma_{18}$.

(3'-φ) For any isotropic element u at infinity of P , $\sum|\alpha \cdot u| \geq 2$, where the sum is taken over all elements in $\bar{\Delta}_0 = \bar{\Delta}(\varphi)$.

One can calculate any isotropic element u at infinity of P by the following method. (Vinberg [11] § 3)

(1) Choose a subgraph F of Σ_{18} isomorphic to an extended Dynkin graph of rank (= the number of vertices minus the number of connected components) 17.

(2) Let F_0 be a connected component of F . Let S_0 be the set of facets of Σ_{18} corresponding to F_0 and Δ^+ be the set of roots perpendicular to some facet in S_0 and directing outwards from P .

(3) Then $u = m \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ for some integer m , where n_α's are the coefficients of the maximal root, which are positive integers uniquely determined depending on the type of F_0 .

The resulting element $\sum n_\alpha \alpha$ depends on the subgraph F , but it does not depend on the choice of F_0 .

It is easy to see that any S_4-conjugacy class of subgraphs of Σ_{18} isomorphic to extended Dynkin graphs of rank 17 is uniquely determined by its type and its type is one of the following 9. They coincide with our basic Dynkin graphs (Vinberg [13])

$A_{11} + E_6$, $2E_8 + B_1$, $D_{16} + B_1$
$2D_8 + B_1$, $A_{15} + B_2$, $2E_7 + B_3$
$D_{12} + B_5$, $E_8 + B_9$, B_{17} .

Consequently, it is possible to check the condition $(3'\text{-}\varphi)$ for all maximal Dynkin subgraph of $\sum_{18}$. Note that if it satisfies $(3'\text{-}\varphi)$, then it has 18 vertices by Proposition 2.3 (2). Thus it suffices to check it only for representatives of S_4-conjugacy classes of maximal Dynkin subgraphs of $\sum_{18}$ with 18 vertices.

Proposition 2.4. (1) A maximal Dynkin subgraph of $\sum_{18}$ satisfies the condition $(3'\text{-}\varphi)$ if and only if it is S_4-conjugate to one of the 13 graphs explained below. (We write below the vertices to be omitted to obtain the desired graph. For vertices in the tetrahedral subgraph $\sum_{18}^{*}$ we write only the number of the vertex.)

$B_2 + A_{11} + A_5$: 4,10,16,19,22 , all of the 1st kind except β , all of the 2nd kind.

$B_2 + A_9 + A_6 + A_1$: 3,10,16,19,22 , all of the 1st kind except β , all of the 2nd kind.

$B_2 + 2A_8$: 2,10,16,19,22 , all of the 1st kind except β , all of the 2nd kind.

$B_2 + 2A_7 + 2A_1$: 3,11,16,19,22 , all of the 1st kind except β , all of the 2nd kind.

$A_{10} + A_6 + A_2$: 2,8,15,20 , all of the 1st and the 2nd kind.

$2A_9$: 4,14,20,22 , all of the 1st and the 2nd kind.

$2A_7 + A_3 + A_1$: 3,8,10,15 , all of the 1st and the 2nd kind.

$A_7 + A_6 + A_3 + A_2$: 3,8,12,15 , all of the 1st and the 2nd kind.

$A_7 + 3A_3 + A_2$: 3,8,11,15 , all of the 1st and the 2nd kind.

$3A_6$: 6,10,15,18 , all of the 1st and the 2nd kind.

$6A_3$: 3,7,11,15 , all of the 1st and the 2nd kind.

$B_1 + E_6 + A_7 + A_4$: 4,10,14,17,19 , all of the 1st kind except β , all of the 2nd kind.

$B_1 + E_6 + A_{11}$: 4,10,16,17,19 , all of the 1st kind except β , all of the 2nd kind.

(2) For the last 2 Dynkin graphs $B_1 + E_6 + A_7 + A_4$, $B_1 + E_6 + A_{11}$ in (1), there are maximal embeddings into $\sum_{18}$ which are not conjugate to the ones in (1). They satisfy the condition <*> in Theorem 1.1 for the isotropic element at infinity of P associated with the extended Dynkin subgraph of type $2E_8 + B_1$.

$B_1 + E_6 + A_7 + A_4$; 1,4,12,20,22 , all of the 1st kind, all of the 2nd kind except γ .

$B_1 + E_6 + A_{11}$: 1,2,12,20,22 , all of the 1st kind, all of the 2nd kind except γ .

By the above proposition one knows that in case (3) we have case (B-2) in Theorem 0.2. Thus one can conclude that (C) implies (B) in Theorem 0.2.

Remark. It is easy to check that any one of the 11 sub-basic Dynkin graph satisfies not only the condition (3'-φ) but also the stronger condition (3*). Obviously the corresponding module T to such a graph has no isotropic element in the orthogonal complement. Assume that we have a fundamental system of roots $\bar{\Delta}_0$ of T, a long root $\alpha \in \bar{\Delta}_0$ and an isotropic element u such that $u \cdot \beta = 0$ for $\beta \in \bar{\Delta}_0$ with $\beta \neq \alpha$ and $u \cdot \alpha = 1$. Then the discriminant of the root lattice generated by $\bar{\Delta}_0 - \{\alpha\}$ is a square number. Thus the choice of α is much restricted. For any possible choice of α one can check that there is no isotropic element satisfying the condition by direct calculation using some concrete information in Vinberg-Kaplinskaja [15].

Lastly we show that condition (B) in Theorem 0.2 implies (A).

In case (B-1)<1>, applying Urabe [9] Corollary 3.10 at the beginning and applying succeedingly Urabe [9] Proposition 4.2 twice, one has a full embedding $Q(G) \subset Q(2E_8) \oplus H \oplus H \oplus H/\mathbf{Z}\lambda$, such that the orthogonal complement of $Q(G)$ has an isotropic element, where λ is an element $\lambda^2 = -4$. The inverse image S of $Q(G)$ in $Q(2E_8) \oplus H \oplus H \oplus H$ satisfies the conditions (a), (b) in Urabe [9] Theorem 1.15. Thus a normal quartic surface with singularities G exists. Moreover, since the orthogonal complement of S contains an isotropic element by Lemma 1.5, one can conclude also the arithmetic condition in (A) by Proposition 2.1.

In case (B-1)<2>, instead of applying Urabe [9] Proposition 4.2 twice, we can apply Urabe [9] Proposition 4.2

once and Theorem 1.1 once. Then, by the same argument as above we have (A).

In case (B-1)<3> by the same argument as in case (B-1)<2> , one can conclude the existence of the desired quartic surface. As for the arithmetic condition, one can discuss as follows. In case (B-1)<3> we have a decomposition $Q(2E_8) \oplus H \oplus H \oplus H/\mathbb{Z}\lambda \cong (Q(2E_8) \oplus H)/\mathbb{Z}\lambda) \oplus H_1 \oplus H_2$ where $\lambda \in Q(2E_8) \oplus H$ and $H_i = \mathbb{Z}u_i + \mathbb{Z}v_i$ $(i = 1,2)$ is a hyperbolic plane with $u_i^2 = v_i^2 = 0$, $u_i \cdot v_i = 1$. $Q = Q(G)$ is a full sublattice of this quotient quasi-lattice. We have a fundamental system of roots $\Delta = \{\alpha, \beta_1, \ldots, \beta_{17}\}$ of Q such that $\alpha = u_2 + v_2$, $\beta_j = \bar{\beta}_j + \varepsilon_j u_1 + \delta_j u_2$ $(1 \leq j \leq 17)$ with $\bar{\beta}_j \in Q(2E_8) \oplus H/\mathbb{Z}\lambda$, $\varepsilon_j, \delta_j = 0$ or -1 . Thus we have an isotropic element u_1 in the orthogonal complement of Q . By Proposition 2.1 one has the arithmetic condition.

In case (B-2) applying Proposition 2.2 and Proposition 2.4 instead of Urabe [9] Corollary 3.10 one can conclude (A) by the same argument as in the case (B-1)<1> .

We complete all the proof of Theorem 0.4.

As for Theorem 0.5, by Proposition 2.4, Proposition 2.2 and Theorem 1.1, it is obvious.

§ 3. Nikulin's lattice theory

In this section we would like to give examples of Dynkin graphs G such that there is a normal quartic surface with singularities G but G cannot be obtained from any Dynkin subgraph of Σ_{18} by an elementary transformation or a tie transformation.

To tell the truth, I was very disappointed to find these examples. However, it is very challenging to find out the missing part of the theory and we should not be worried over them.

There might be a series of transformations of graphs, and the elementary one and the tie one might be the 0-th part and the first part of it.

By Urabe [9] Theorem 1.15 and by Theorem 1.1 it suffices to show the following.

Proposition 3.1. Let G be one of the following Dynkin graphs with vertex 19.

$$\begin{array}{ll} A_{10} + A_7 + 2A_1, & 2A_6 + A_4 + A_2 + A_1 \\ 2A_4 + A_2 + A_1 + E_8, & A_{13} + A_4 + 2A_1 \\ A_{12} + A_4 + A_2 + A_1, & A_{11} + A_5 + 3A_1 \\ A_{10} + A_4 + A_3 + A_2 . & \end{array}$$

Let $\Lambda = Q(2E_8) \oplus H \oplus H \oplus H$ and set $S = \mathbb{Z}\lambda \oplus Q(G)$ (orthogonal direct sum) where λ is an element with $\lambda^2 = -4$ and $Q(G)$ is the root lattice of type G. Then the following (I) and (II) hold.

(I) S has an embedding $S \subset \Lambda$ of lattices satisfying the following conditions (a) and (b). Let $\hat{S}$ denote the primitive hull of S in Λ.

(a) If $\eta \in \hat{S}$, $\eta^2 = 2$ and $\eta \cdot \lambda = 0$, then $\eta \in S$.

(b) $\hat{S}$ has no element u with $u^2 = 0$ and $u \cdot \lambda = -2$.

(II) For any embedding $S \subset \Lambda$ of lattices satisfying (a) and (b), for any fundamental system of roots Δ of $Q(G)$, for any long root $\alpha \in \Delta$ and for any isotropic element $u \in \Lambda$, the following condition (c) never holds.

(c) $u \cdot \alpha = 1$, $u \cdot \lambda = 0$ and $u \cdot \beta = 0$ for any element $\beta \in \Delta$ with $\beta \neq \alpha$.

Remark. There seems to be several Dynkin graphs with 17 vertices satisfying (I) and (II) other than the 7 ones in Proposition 3.1.

To show the above proposition we use Nikulin's lattice theory (Nikulin [5]). It is the advantage of his theory that for any given Dynkin graph G we can always determine whether the above condition (I) is satisfied or not after a finite calculations. However, we cannot deduce a law dominating all possible combinations of singularities on quartic surfaces from his theory. Moreover, sometimes it is tiresome to search an appropriate overlattice, if we use his theory.

Let M be a non-degenerate even lattice. We can identify the dual module $M^* = \mathrm{Hom}(M,\mathbb{Z})$ of M with a submodule in $M \otimes \mathbb{Q}$ defined by $\{x \in M \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z}$ for every $y \in M\}$. Then, $M \subset M^*$ and the quotient M^*/M is a finite abelian group whose order is equal to the absolute value of the discriminant of M . We can define a finite bilinear form

$b_M : M^*/M \times M^*/M \longrightarrow \mathbb{Q}/\mathbb{Z}$ and a finite quadratic form $q_M : M^*/M \longrightarrow \mathbb{Q}/2\mathbb{Z}$ by b_M (x mod M, y mod M) $\equiv x \cdot y$ mod $\mathbb{Z}$ q_M(x mod M) $\equiv x^2$ mod $2\mathbb{Z}$ for $x, y \in M^*$. We call b_M <u>the discriminant bilinear form</u> of M and q_M the <u>discriminant quadratic form</u> of M . They play important roles in Nikulin's theory.

For example, for the root lattice $Q = Q(A_k)$ of type A_k, Q^*/Q is a cyclic group of order $k+1$ and the image $\bar{\omega}_1$ of the first fundamental weight $\omega_1 \in Q^*$ is the generator. We have $q_Q(a\bar{\omega}_1) \equiv a^2 \cdot k/(k+1) \bmod 2\mathbb{Z}$.

If a lattice N contains a lattice M with finite index, N is said to be an <u>overlattice</u> of M .

We use the following two lemmas to check the condition (II). Let G be an arbitrary Dynkin graph and $\Lambda = Q(2E_8) \oplus H \oplus H \oplus H$.

Lemma 3.2. (Nikulin [5] Corollary 1.6.2, Corollary 1.9.3.) Let $\hat{S}$ be an overlattice of $S = \mathbb{Z}\lambda \oplus Q(G)$ with a primitive embedding $\hat{S} \subset \Lambda$. Let T' be the orthogonal complement of $\hat{S}$ in Λ and T be the lattice obtained from T' by reserving the sign of the bilinear form.

(1) T has the same discriminant quadratic form as $\hat{S}$.

(2) Let $\mathbb{Z}_p$ denote the ring of p-adic integers for a prime number p . For any prime number p the isomorphism class of the $\mathbb{Z}_p$-lattices $T \otimes \mathbb{Z}_p$ depends on the overlattice $\hat{S}$ but does not depend on the primitive embedding $\hat{S} \subset \Lambda$.

Lemma 3.3. Assume that we have an isotropic element u satisfying (c) for some overlattice $\hat{S}$ of S , for some primitive embedding $\hat{S} \subset \Lambda$, for some fundamental system of roots $\Delta \subset Q(G)$ and for some long root $\alpha \in \Delta$. Let ω_α be the fundamental weight corresponding to the pair (Δ,α) . Namely $\omega_\alpha \in Q(G) \otimes \mathbb{Q}$, $\omega_\alpha \cdot \alpha = 1$ and $\omega_\alpha \cdot \beta = 0$ for $\beta \in \Delta$ with $\beta \neq \alpha$.

(1) For every element $x \in \hat{S}$, $\omega_\alpha \cdot x$ is an integer.

(2) The $\mathbb{Z}_p$-quasi-lattice $T^* \otimes \mathbb{Z}_p = \mathrm{Hom}(T,\mathbb{Z}) \otimes \mathbb{Z}_p$ represents the rational number $\omega_\alpha{}^2$. (In other words, there is a non-zero element $x \in T^* \otimes \mathbb{Z}_p$ with $x^2 = \omega_\alpha{}^2$.)

(3) Moreover, if $\omega_\alpha = \sum_{\beta\in\Delta} n_\beta \beta/m$ ($n_\beta, m \in \mathbb{Z}$) and if m is not a multiple of p , then the $\mathbb{Z}_p$-lattice $T \otimes \mathbb{Z}_p$ represents $\omega_\alpha{}^2$.

Proof. Let T' and T be the same as in Lemma 3.2. By * we denote the dual module. Since $\hat{S} \oplus T' \subset \Lambda \subset \hat{S}^* \oplus T'^*$ we have $\sigma \in \hat{S}^*$ and $\tau \in T'^*$ with $u = \sigma+\tau$. Since $\sigma\cdot\lambda = u\cdot\lambda = 0$, $\sigma\cdot\alpha = u\cdot\alpha = 1$ and $\sigma\cdot\beta = u\cdot\beta = 0$ for any $\beta \in \Delta$ with $\beta \neq \alpha$, we conclude $\sigma = \omega_\alpha$. We have $\omega_\alpha \cdot x = u\cdot x \in \mathbb{Z}$ for $x \in \hat{S}$. On the other hand since $0 = u^2 = \sigma^2 + \tau^2$, $\tau^2 = -\omega_\alpha{}^2$. Thus T'^* represents $-\omega_\alpha{}^2$, and T^* and $T^* \otimes \mathbb{Z}_p$ represents $\omega_\alpha{}^2$.

Next assume that m is not a multiple of p . Then, $\omega_\alpha \in \hat{S} \otimes \mathbb{Z}_p$ and $\tau \in (T'^* \otimes \mathbb{Z}_p) \cap (\Lambda \otimes \mathbb{Z}_p) = T' \otimes \mathbb{Z}_p$. Thus $T' \otimes \mathbb{Z}_p$ represents $\tau^2 = -\omega_\alpha{}^2$ and $T \otimes \mathbb{Z}_p$ represents $\omega_\alpha{}^2$.

Q.E.D.

In what follows we give the proof for $G = A_{10} + A_7 + 2A_1$. We assume that $G = A_{10} + A_7 + 2A_1$ and $S = \mathbb{Z}\lambda \oplus Q(A_{10} + A_7 + 2A_1)$. Let $\omega_1, \omega_2, \ldots, \omega_7$ denote the fundamental <u>weights</u> of the A_7-component and $\chi_1, \chi_2, \dots, \chi_{10}$ be the fundamental <u>weights</u> of the A_{10}-component. Let α_1, α_2 be the fundamental <u>roots</u> of the $2A_1$-component.

Lemma 3.4. (1) Let $\hat{S}$ be an overlattice of S satisfying the condition (a) and (b). Then, either $\hat{S} = S$ or $\hat{S} = S_1$, where $S_1 = S \cup (S+(\omega_4 + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\lambda))$.
(2) S has no primitive embedding into Λ. S_1 has a primitive embedding into Λ.

Proof. It is easy to see that

$$S^*/S \cong \mathbf{Z}/11 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/4$$

and $\bar{\chi}_1 = \chi_1 \bmod S$, $\bar{\omega}_1 = \omega_1 \bmod S$, $\alpha_1/2 \bmod S$, $\alpha_2/2 \bmod S$ and $\lambda/4 \bmod S$ are the generators of the respective components. We have

$$q_S(a,b,c,d,e) \equiv (10a^2/11)+(7b^2/8)+(c^2+d^2/2)-(e^2/4) \bmod 2\mathbf{Z}$$

for $(a,b,c,d,e) \in \mathbf{Z}/11 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/4$.
(1) By Nikulin [5] Proposition 1.4.1, solutions of $q_S \equiv 0$ correspond to overlattices. Note that by the condition (b) $\hat{S}$ cannot contain $(\alpha_1+\alpha_2+\lambda)/2$, the element corresponding to $(a,b,c,d,e) = (0,0,1,1,2)$. Besides if $\hat{S}$ contains an element corresponding to $(a,b,c,d,e) = (0,\pm 2,1,0,0)$ or $(0,\pm 2,0,1,0)$, then $\hat{S}$ contains a root system of type E_8. If $\hat{S}$ contains a corresponding element to $(a,b,c,d,e) = (0,4,0,0,0)$, then

$\hat{S}$ contains a root system of type E_7. By the condition (a) such cases are excluded. Thus we have only two cases corresponding to the solutions $(a,b,c,d,e) = (0,0,0,0,0)$ and $(0,4,1,1,2)$ of $q_S \equiv 0$.

(2) $S_1{}^*/S_1 \cong \mathbf{Z}/11 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/4$ and for $(x,y,z) \in \mathbf{Z}/11 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/4$

$$q_{S_1}(x,y,z) \equiv (10x^2/11)-(5y^2/8) + (z^2/4) \mod 2\mathbf{Z} .$$

Thus by Nikulin [5] Theorem 1.12.2 we have the conclusion.

Q.E.D.

By Lemma 3.4 it suffices to consider $\hat{S} = S_1$ and primitive embeddings of S_1 into Λ. Assume that we have an isotropic element $u \in \Lambda$ and a long root $\alpha \in \Lambda$ satisfying (c). We will deduce a contradiction.

By easy calculation one knows that if a fundamental weight ω_α satisfies $\omega_\alpha \cdot x \in \mathbf{Z}$ for any $x \in S_1$, then it is one of $\omega_2, \omega_4, \omega_6, \chi_1, \chi_2, \ldots, \chi_{10}$. Thus by Lemma 3.3(1) ω_α has to coincide with one of them. Their squares are $\omega_2{}^2 = \omega_6{}^2 = 3/2$, $\omega_4{}^2 = 2$, $\chi_i{}^2 = \chi_{11-i}{}^2 = i(11-i)/11$ $(1 \leq i \leq 10)$, respectively.

Now, let R be an integral domain, K be the quotient field of R and $\theta \in K$ be an element. We define a bilinear form $(\ , \ ) : R \times R \longrightarrow K$ by $(x,y) = \theta xy$. When we regard R itself as a quasi-lattice equipped with this bilinear form with values in K, we denote it by $R(\theta)$.

Consider the prime number $p = 2$. By Nikulin [5] Proposition 1.8.2 one knows $T \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2(3\times 8) \oplus \mathbf{Z}_2(4)$. Thus $T^* \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2(3/8) \oplus \mathbf{Z}_2(1/4)$. By easy calculations one knows

$T^* \otimes \mathbb{Z}_2$ does not represent values $\omega_4{}^2 = 2$, $\chi_1{}^2 = \chi_{10}{}^2 = 2\times 5/11$, $\chi_3{}^2 = \chi_8{}^2 = 2^3 \times 3/11$, $\chi_4{}^2 = \chi_7^2 = 2^2 \times 7/11$. On the other hand the assumption of Lemma 3.3(3) is satisfied for $p = 2$ and for $\chi_1, \chi_2, \ldots, \chi_{10}$. Since any 2-adic integer represented by $T \otimes \mathbb{Z}_2$ is a multiple of 2^2, $T \otimes \mathbb{Z}_2$ does not represent values $\chi_1{}^2 = \chi_{10}{}^2 = 2\times 5/11$, $\chi_2{}^2 = \chi_9{}^2 = 2\times 3^3/11$, $\chi_5{}^2 = \chi_6{}^2 = 2\times 3\times 5/11$. By Lemma 3.3 either the equality $\omega_\alpha = \omega_2$ or $\omega_\alpha = \omega_6$ has to hold.

Next consider $p = 11$. One has $T \otimes \mathbb{Z}_{11} \cong \mathbb{Z}_{11}(-11) \oplus \mathbb{Z}_{11}(1)$ and $T^* \otimes \mathbb{Z}_{11} \cong \mathbb{Z}_{11}(-1/11) \oplus \mathbb{Z}_{11}(1)$. By calculation one knows that $T^* \otimes \mathbb{Z}$ does not represent $\omega_2{}^2 = \omega_6{}^2 = 3/2$, $\omega_4{}^2 = 2$. Thus by Lemma 3.3 neither $\omega_\alpha = \omega_2$ nor $\omega_\alpha = \omega_6$ holds. It is a contradiction. We have the condition (II).

We can check that $\hat{S} = S_1$ satisfies (a) and (b), and we have the condition (I), too.

For the remaining 6 graphs in Proposition 3.1 the reasoning is similar. In what follows we sketch it.

$G = 2A_6 + A_4 + A_2 + A_1$: S has no proper overlattice, and has a primitive embedding into Λ. Considering prime numbers $p = 2,7$, we can get the conclusion.

$G = 2A_4 + A_2 + A_1 + E_8$: S has 2 proper overlattices except S itself. Both of them has the root system of type $2E_8 + A_2 + A_1$. Therefore they do not satisfy (a). On the other hand S has a primitive embedding into Λ. By considering $p = 5,3$, we get the conclusion.

$G = A_{13} + A_4 + 2A_1$: S has 5 overlattices including S itself. Only 3 of them including S itself satisfy (a) and (b). However, S itself has no primitive embedding into Λ. The other 2 overlattices are isomorphic to each other and have a primitive embedding into Λ . Applying Lemma 3.3(1) we have only 11 possibilities as the choice of $\alpha \in \Delta$. Considering $p = 2,5$, we have the conclusion.

$G = A_{12} + A_4 + A_2 + A_1$: S has no proper overlattice but it has a primitive embedding into Λ . Considering $p = 2,5$, we get the conclusion.

$G = A_{11} + A_5 + 3A_1$: There are only 3 overlattices of S satisfying (a), (b) and having a primitive embedding into Λ . If we exchange the order of 3 A_1-components, these overlattices are exchanged with each other. Thus such an overlattice $\hat{S}$ is unique up to isomorphisms. The 6th fundamental weight ω_6 of the A_{11}-component is the unique fundamental weight ω with $\omega \cdot x \in \mathbf{Z}$ for every $x \in \hat{S}$. $T \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2(4) \oplus \mathbf{Z}_2(4)$ and $T^* \otimes \mathbf{Z}_2$ does not represent $\omega_6{}^2 = 3$.

$G = A_{10} + A_4 + A_3 + A_2$: S has a unique proper overlattice but it does not satisfy (b). S itself satisfies (a) and (b) and it has a primitive embedding into Λ . Considering the prime numbers $p = 2,3$, we get the conclusion.

Before concluding this article, we would like to give a proposition worth mentioning, one more.

Proposition 3.5. There is not a normal quartic surface in $\mathbb{P}^3$ with D_{19} singularity.

Proof. For the lattice $S = \mathbf{Z}\lambda \oplus Q(D_{19})$ $(\lambda^2 = -4)$, $S^*/S \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4$ and $q_S(a,b) \equiv -a^2/4 + 19b^2/4 \pmod{2\mathbf{Z}}$, for $(a,b) \in \mathbf{Z}/4 \oplus \mathbf{Z}/4$. Thus $q_S \equiv 0 \bmod 2\mathbf{Z}$ if and only if

$a \equiv b \equiv 0 \pmod 4$ or $a \equiv b \equiv 2 \pmod 4$. Let ω be the fundamental weight associated with one of the two vertices at the end of the two-forked part of the Dynkin graph D_{19} . An overlattice $\hat{S}$ of S coincides either with S itself or $S_1 = S \cup (S+(\frac{1}{2}\lambda + 2\omega))$. By Nikulin [5] Theorem 1.12.2, S has no primitive embedding into Λ and S_1 has a primitive embedding into Λ . However, S_1 contains an isotropic element u with $u\cdot\lambda = -2$. Thus the condition (b) is not satisfied. By Urabe [9] Theorem 1.15 a normal quartic surface with singularity D_{19} cannot exist.

Q.E.D.

Remark. Consider the situation in the above proof. By the surjectivity of the period mappings, there is a K3 surface Z with an isomorphism $\alpha : S_1 \longrightarrow \mathrm{Pic}(Z)$ preserving bilinear forms such that $L = \alpha(\lambda)$ is a numerically effective line bundle of degree 4. The orthogonal complement of L in $\mathrm{Pic}(Z)$ is generated by 19 smooth rational curves on Z with self-intersection number -2 , and they form the configuration D_{19} . The complete linear system $|L|$ associated with L has no base point, since S_1 has no isotropic element u with $u\cdot\lambda = -1$ (Urabe [9] Proposition 1.6). Let $\varphi_L : Z \longrightarrow \mathbb{P}^3$ be the morphism associated with $|L|$. Since any isotropic element U in $\mathrm{Pic}(Z)$ with $U\cdot L = -2$ intersects with one of the rational curves in the configuration D_{19} , the image of φ_L is an irreducible quadratic surface $\sum_0$ with a unique singular point. Let $\rho : Z \longrightarrow X$ be the contraction morphism sending the D_{19}-configuration to a rational double point of type D_{19} . The morphism $\varphi_L : Z \longrightarrow \Sigma_0$ factors through ρ and the induced morphism $\bar{\varphi} : X \longrightarrow \Sigma_0$ defines a branched double covering such that the image of the unique singular point of type D_{19} on X is the singular point of Σ_0 .

Last of all I would like to ask the following question: Does our theory of Dynkin graphs and quartic surfaces have a connection with the representation theory of Lie groups? At the present stage in the both theories we can only find the same notions - Dynkin graphs, Weyl groups, ..., etc. - at the key points. If we try to find a path from our theory to Lie groups, we lose sight of the path at the point where the Hodge theory comes in. I would like to know the connection, if it exists.

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