LINEAR SYSTEMS ON QUASI - ABELIAN VARIETIES

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(F. Capocasa - F. Catanese)

Introduction.

Let Γ be a discrete subgroup of $\mathbb{C}^n$: then the study of the Γ-periodic meromorphic functions on $\mathbb{C}^n$ can be reduced to the study of the space of sections of holomorphic line bundles L on the quotient complex manifold $X = \mathbb{C}^n/\Gamma$, called a quasi-torus.

Furthermore, one can reduce (cf. e.g. [C-C], expecially pages 62-65) oneself to consider the case where

1) X is a Cousin quasi-torus (also called a toroidal group) : this means that $H^o(X,\mathcal{O}_X) = \mathbb{C}$, and implies in particular that the $\mathbb{Z}$-rank of Γ equals $n+m$, with $0 < m \leq n$,

and

2) L is a positive line bundle, i.e. , the alternating form $c_1(L) : \Gamma \times \Gamma \to \mathbb{Z}$ given by the first Chern class of L can be obtained as the imaginary part of a positive definite Hermitian form H on $\mathbb{C}^n$ (i.e., $c_1(L) = \mathrm{Im}(H)/_{\Gamma \times \Gamma}$) .

In fact, (cf. e.g. [A-G] and [C-C]) a quotient Cousin quasi-torus $X = \mathbb{C}^n/\Gamma$ is said to be a quasi-abelian variety if the following equivalent conditions are satisfied

i) there exists a positive line bundle L on X

ii) (Generalized Riemann bilinear Relations) there exists a Hermitian form H such that

 iia) H is positive definite on $\mathbb{C}^n$.

 iib) $\mathrm{Im}(H)/_{\Gamma \times \Gamma} \to \mathbb{Z}$

iii) X has the structure of a quasi-projective algebraic variety

iv) there exists an aperiodic meromorphic function on X.

The main difference with the classical case of Abelian varieties, which are the X's as above which are compact, is that here the Picard group of line bundles can be infinite dimensional (cf. [Ma], [Vo]) , and there are the so called non linearizable bundles. So, whereas in the classical case the sections of line bundles can be explicitly written down in terms of the so called theta functions, here it is not completely solved the problem of determining the line bundles which have non zero sections.

In [C-C] we showed that this problem would be solved if one could prove the following conjecture (3.22 in [C-C]) :

if L is a positive line bundle, then $H^o(X,L) \neq 0$.
Moreover, we showed that this conjecture would imply Lefschetz type embedding theorems for quasi-Abelian varieties .
This conjecture had been proven by Cousin ([Cou]) in the special case $m = 1$ by constructing explicit Weierstrass products.

With different methods (essentially Nakano's vanishing theorems ([Na]) for positive line bundles on weakly 1-complete manifolds) , Y.Abe ([A2]) reproved Cousin's result and showed moreover the existence of a (non explicitly given) positive integer r such that if L is positive and $c_1(L)$ is divisible by an integer $d \geq r$, then $H^o(X,L)$ is infinite dimensional in the case where X is not compact.

The main purpose of this paper is to obtain all the analogues, both of the classic embedding theorems of Lefschetz and of its recent improvements, in the non compact case of quasi-Abelian varieties.

First of all, using Abe' s result and a descent trick, we show that Abe's result can be improved to yield $r = 2$.

Secondly,we prove the best possible Lefschetz - type results :

Theorem A . If $d \mid c_1(L)$, with $d \geq 2$, then $H^o(L)$ is base point free.

Theorem B . If $d \mid c_1(L)$, with $d \geq 3$, then $H^o(L)$ gives a projective embedding of X.

The present theorem B is indeed even better than the "conditional" theorem 3.26 of ([C-C]) .
But we can do even more, showing the analogue of a result of Ohbuchi ([O], cf. also [L-B], page 88) :

Theorem C . If $2 \mid c_1(L)$ then $H^o(L)$ gives a projective embedding of X unless the linearized bundle B associated to a square root of L yields a polarization (X,B) which is reducible with one factor (X_1,L_1) yielding a principally polarized Abelian variety.

We would like to remark that in the non compact case, as we show in 2.4, we encounter the further difficulty that Poincare's complete reducibility theorem does not hold.

On the other hand, although we are able (see later) to solve the above conjecture only in special cases, we show new evidence for its validity.
Firstly, in fact, via the descent trick, for every subgroup $\Gamma' \subset \Gamma$ such that the pull back of L on $X' = \mathbb{C}^n/\Gamma'$ has Chern class $c_1(L)$ divisible by an integer $d \geq 2$, we show that there exists a representation $\chi: \Gamma/\Gamma' \to \mathbb{C}^*$ such that, if M_χ is the flat bundle on X associated to χ, then $H^0(L \otimes M_\chi) \neq 0$ (this follows from prop. 1.1, cor.1.2, and thm.1.3, as remarked in 1.4).
This shows that the positive line bundles with sections have a "positive probability" .
Secondly, we show that the conjecture holds true under some mild assumption (cf. theorem D).
We also hope that the conjecture can be attacked in this way from the point of view of representation theory of finite groups , although in some infinite dimensional spaces .
Finally , in section 2, we pose some new questions.

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1. Sections of line bundles on quasi-abelian manifolds.

Proposition 1.1 Let L be a positive line bundle on X and let $\Gamma' \subset \Gamma$ be a discrete subgroup s.t. $H^0(\pi^*(L)) \neq 0$ on $X' = \mathbb{C}^n/\Gamma'$ where $\pi : X' \to X$ is the canonical projection. Then there exists a representation $\chi: \Gamma/\Gamma' \to \mathbb{C}^*$ such that, if M_χ is the flat bundle on X associated to χ , $H^0(L \otimes M_\chi) \neq 0$ (and indeed infinite dimensional if $H^0(\pi^*(L))$ is such) .

Proof. Let $G = \Gamma/\Gamma'$ be the kernel of π . Then the bundle $L' = \pi^*(L)$ is G-linearized (cf. [Mu1]) and

$$H^o(L') \cong H^o(\pi^*(L)) \cong H^o(L \otimes \pi_*(\mathcal{O}_X)) \cong \oplus_{\chi \varepsilon G^\vee} H^o(L \otimes M_\chi)$$

where $G^\vee$ is the group of characters of G , $G^\vee$ = Hom $(G, \mathbb{C}^*)$ and M_χ is the flat line bundle associated to the representation of $\Pi_1(X) = \Gamma$ induced by χ: $G \to \mu_r \subset \mathbb{C}^*$ via the epimorphism $\Gamma : \to G = \Gamma/\Gamma'$.

We can conclude that there exists a $\chi \; \varepsilon \; G^\vee$ s.t. $H^0(L \otimes M_\chi) \neq 0$.

<u>Q.E.D.</u>

<u>Corollary 1.2.</u> Let L be a positive line bundle. Then there exists a flat torsion bundle F s.t. $H^0(L \otimes F) \neq 0$ (and it is indeed infinite dimensional if X is not compact).

<u>Proof.</u> Let $r : X \to X$ be the multiplication by an integer r. By the result of Abe ([A2], thm . 6.4) $L' = r^*(L)$ has non zero sections (and has infinite dimension if X is not compact) if r >> 0, since $c_1(L') = r^2$ $c_1(L)$. We can thus apply proposition 1.1.

<u>Q.E.D.</u>

With a similar technique we can prove the following result :

<u>Theorem 1.3.</u> Let L be a positive line bundle on X s.t. $d \mid c_1(L)$, with $d \geq 2$. Then $H^o(L) \neq 0$(and it is indeed infinite dimensional if X is not compact).

<u>Proof.</u> Recall first of all that any line bundle L is associated to some cocycle in $H^1(\Gamma, \mathcal{O}^*_{\mathbb{C}^n})$ of the Appell-Humbert normal form

$$k_\gamma(z) = \rho(\gamma) \exp(-i/_2 [H(z,\gamma) + 1/_2 H(\gamma,\gamma)] + \psi_\gamma(z))$$

where we have chosen coordinates in $\mathbb{C}^n$ such that $\Gamma = \mathbb{Z}^n \oplus \Lambda \mathbb{Z}^m$ and $\psi_\gamma(z)$ is $F + \mathbb{Z}^n$ periodic , F being the maximal complex subspace contained in $\mathbb{R} \otimes \Gamma \subset \mathbb{C}^n$ ($F = (\mathrm{Im}\, \Lambda)\, \mathbb{C}^m$).

The cocycle of L is said to be <u>linearized</u> if $\psi_\gamma \equiv 0$ for all $\gamma \; \varepsilon \; \Gamma$.

It is known (cf. e.g. [C-C] , thm 2.4.) that a linearized line bundle has an infinite dimensional space of sections if it is positive.

In particular, a flat line bundle is linearized with Hermitian form H = 0 (ρ is here a character and not merely a semicharacter).

Under our hypotheses, we can write $L = B^d \otimes \mathcal{L}$, where B is linearized with Chern class equal $(1/d)\, c_1(L)$.

By the above result $H^o(B^{d-1} \otimes \mathcal{L} \otimes M_\chi) \neq 0$ for a suitable flat torsion line bundle M_χ. But, being B linearized, $B \otimes (M_\chi)^{-1}$ is also linearized and thus, by the previously quoted results, it has non zero sections. Therefore also $B^{d-1} \otimes \mathcal{L} \otimes M_\chi \otimes B \otimes (M_\chi)^{-1} \cong L$ has non zero sections, as we wanted to show.

Q.E.D.

Remark 1.4. As a corollary, in proposition 1.1. , the hypothesis $H^o(\pi^*(L)) \neq 0$ can be replaced by the condition that $c_1(\pi^*(L))$ is divisible by $d \geq 2$.

This remark supplies us with many examples of subgroups of Γ which satisfy these hypotheses.
Let $\{ e_1 , \dots , e_l, g_1, \dots , g_{h+k} , f_1, \dots , f_{h+k} \}$ be a Frobenius basis of Γ, i.e., a basis in which $Im(H)/_{\Gamma \times \Gamma}$ is expressed by an alternating matrix A

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & -D & 0 \end{pmatrix}$$

where D is a diagonal matrix of order h+k , $D = diag(d_1, \dots , d_{h+k})$, s.t. $d_i \mid d_{i+1}$, and we assume $d_1, \dots , d_h = 1$, $d = d_{h+1} > 1$.
If Γ' is the subgroup of Γ generated by $\{ e_1, \dots , e_l, dg_1, \dots , dg_h, g_{h+1}, \dots, g_{h+k} , f_1, \dots , f_{h+k} \}$, then is easy to check that $c_1(\pi^*(L))$ is divisible by d.

2 Lefschetz type theorems.

Theorem A. Let L a positive line bundle on X. If $d \mid c_1(L)$, with $d \geq 2$ then $H^0(L)$ is base point free.

Proof. Write $L = B^d \otimes \mathcal{L}$, where B is linearized and $\mathcal{L}$ is a topologically trivial line bundle.
By virtue of Prop. 1.2., there exists a character χ s.t.
$H^0(B\otimes\mathcal{L} \otimes M_\chi) \neq 0$; moreover also $H^0(B\otimes M_\chi{}^{-1}) \neq 0$ and $H^0(B) \neq 0$, since they are positive linearized bundles.
Let F as above be the maximal complex subspace contained in $\mathbb{R}\otimes\Gamma$.
Here and in the following we shall often, by slight abuse of notation, use the same symbol for vectors and subsets of $\mathbb{C}^n$ and and their image in X under π.
If $a \in F$, let $T_a : X \to X$ be the traslation by a and set $(T_a)^*(B) = B_a$.
Notice that $(T_a)^*(\mathcal{L}) \cong \mathcal{L}$ because the cocycle of $\mathcal{L}$ is given by F-periodic functions, and also $(T_a)^*(M_\chi) \cong M_\chi$.
Let σ be a non zero section in $H^0(B\otimes\mathcal{L} \otimes M_\chi)$, s_2 a non zero section in $H^0(B\otimes M_\chi{}^{-1})$, and s_3 , ... s_d non zero sections in $H^0(B)$.
It is straightforward to see that $\prod_{i=2,..d} s_i\,(z-a_i) \cdot \sigma(z+a_2 + ..a_d)$ is a section of L for each choice of a_2 , .. a_d in F .
Assume that exists a point z s.t.
(*) $\prod_{i=2,..d} s_i\,(z-a_i) \cdot \sigma(z+a_2 + ..a_d) = 0$ for all $(a_2 , .. a_d)$ in F^{d-1}
Then one of the d above holomorphic functions must be identically zero on z+F, call it s . But, since F has dense image in $K = \mathbb{R}\Gamma / \Gamma$ (cf. [Mo]), and the smallest (complex) analytic set containing z + K is X , s should be identically zero on X, which is a contradiction.

Q.E.D.

Theorem B. If L is positive and $d \mid c_1(L)$, with $d \geq 3$, then $H^o(L)$ gives an embedding of X in a projective space.

Proof. We can first decompose L as $L = B^d \otimes \mathcal{L}$ with B linearized and positive and then , setting $L' = B^i \otimes \mathcal{L} \otimes M_\chi$ where i=1or 2 , $i \equiv d$ (mod2), and χ is a suitable character , decompose $L \cong L' \otimes N^2$ where N is linearized and positive and $H^o(L') \neq 0$.
Note again that , for a in F, if $i + 2j = d$, then
$L \cong (T_{2ja})^* L' \otimes ((T_{-ia})^* N)^2$.

Claim I. Let $\mathcal{B}_a$ be the base locus of $H^o((T_{-a})^*(L'))$, $a \varepsilon F$: then the intersection $\Delta = \cap_{a\varepsilon F} \mathcal{B}_a = \emptyset$.

Proof. $\mathcal{B}_a = \mathcal{B}_0 + a$, whence, if $z \varepsilon \Delta$, then $z+F \subset \Delta$, $[z] + K \subset [\Delta]$ and finally $[\Delta]= X$, a contradiction.

□

Claim II. The sections of L give a local embedding at any point x of X.

Proof. We can assume that a is generic, thus if s is a non zero section of $H^o(L')$,by claim I , s (x + 2ja) is non zero.
It suffices thus to show that for generic a , the sections of $H^o((T_{-ia})^* N)^2)$ give a local embedding at x .
By the same argument as in claim I, it suffices therefore to know that the locus Σ of points where the sections of $H^o(N^2)$ do not give a local embedding is a proper analytic subset of X.

Otherwise the sections of $H^o(N^2)$ would give a map with positive dimensional fibres . This is contradicted by the following

Claim II'. The sections of $H^o(N^2)$ give a map with no positive dimensional fibre .

Proof. Let x and y be points in one such fibre.
Since $N^2 \cong (T_b)^*N \otimes (T_{-b})^* N$, for each b in X (N being linearized), we find that for each divisor D of a section of N , for each b in X , if x lies in D + b , then either D - b contains y , or D + b contains y.
I.e., for each d in D (set b=x-d) then D contains either y + x -d , or y - x + d . Whence either $(T_{y+x}) (-D) = D$, or $(T_{y-x}) (D) = D$.
The conclusion is that the group of translations of D , $\{ t \mid D + t = D \}$ has positive dimension for all such D.
Let now X' be a quotient Abelian variety of X such that the linearized bundle N pulls back from X' , and let D be the pull back of a divisor D' of a section of N' on X' (cf. e.g., prop. 2.8 of [C-C]). Then also D' should have a positive dimensional group of translations, contradicting the fact that this group is well known to be finite, N' being a positive line bundle . □

Claim III. The sections of L separate pairs of points $x \neq y$ of X.

Proof. Similarly, it suffices to show that for generic a the sections of $H^o((T_{-ia})^* N)^2)$ separate x and y, i.e., the sections of $H^o(N^2)$ separate x+a and y+a for generic a in F.

Otherwise, for every section f of $H^o(N)$, let D be its divisor.
Then one would have that for each a in F and each b, if D +b contains x+a, then either
i) D + b contains y + a, or
ii) D - b contains y + a.
In case ii), for each d in D, and each a in F, if we set b = x +a -d, then y+a+b = y+x+2a -d lies in D, whence D contains a translate of F, what is a contradiction as usual.
Thus i) holds, thus for each d in D, d + y -x lies in D, that is, D is (y-x) periodic.
If X is compact, this is not possible, otherwise (cf. [L-B]) all sections of $H^o(N)$ would pull back from a quotient of X, contradicting the Riemann-Roch formula. In the non compact case, we use prop. 2.8 of [C-C], by which there exist quotients X_1, X_2, such that
1) N is a pull back of a line bundle N_i on X_i
2) ker ($X \to X_1 \times X_2$) = 0.

Since X_i is compact, there exists an aperiodic section of $H^o(N_i)$ for i=1,2 ; whence (x-y) maps to 0 in X_i, and it follows from 2) that x=y.

Q.E.D.

One may ask whether the above results (Theorems A,B) are the best possible ones. The answer is yes, and it suffices to look at the case where X is compact and L gives a principal polarization, i.e., when all the elementary divisors d_i are equal to 1.

In fact, in this case, $H^o(L)$ has only one non zero section, whereas $H^o(L^2)$ yields a 2 to 1 map (more precisely,c.f. [L-B], pages 99-101, for suitable choice of the origin, an embedding of the quotient of X by multiplication by -1).
This exception reproduces itself as follows :

Definition 2.1 Let (X,L) be a pair, consisting of a quasi-Abelian variety and a positive line bundle. Then (X,L) is called a polarized quasi-Abelian variety, and is said to be reducible if there exists two

similar pairs (X_i, L_i) i=1,2 such that $(X,L) \cong (X_1,L_1)\otimes(X_2,L_2)$, what means that there exist homomorphisms $\pi_i : X \to X_i$ such that $\pi_1 \times \pi_2$ is an isomorphism , and $L \cong \pi_1{}^*(L_1)\otimes\pi_2{}^*(L_2)$.
Otherwise, (X,L) is said to be irreducible.

Remark 2.2 Let (X,L) be $\cong(X_1,L_1)\otimes(X_2,L_2)$, where X_1 is compact and L_2 is trivial . Since X_2 is quasi-Abelian , there are no noncostant functions on it, and by the Kunneth formula (or by the Leray theorem) $H^o(X,L) \cong H^o(X_1,L_1)$ which is a finite dimensional vector space. This example shows that theorem 8.2 of [A2] is incorrect, the error lying in the unproven assertion that , for each n, L is the pull back from an Abelian variety of a line bundle with Pfaffian bigger than n .

It follows easily that any (X,L) can be written as a product of irreducibles $(X_1,L_1)\otimes \ldots\ldots \otimes(X_r,L_r)$. Moreover, by the Kunneth formula, such a decomposition gives to the space of global sections the structure of completed tensor product

$H^o(X,L)\cong H^o(X_1,L_1)\hat{\otimes} \ldots\ldots \hat{\otimes} H^o(X_r,L_r)$.

Therefore, if one such factor is a principally polarized Abelian variety, then again the generalized Lefschetz theorems A, B cannot be improved. To obtain improvements , it suffices in the classical case to assume that the pair (X,L) is irreducible (or that it does not have a reduction with a principally polarized factor).
But, in order to show that in some cases a polarization is reducible, an important tool to study Abelian varieties is Poincare's reducibility theorem . We show here a partial extension of it , and in the following example we show that it does not hold in general for quasi-Abelian varieties.

Proposition 2.3 Let (X,L) be a polarized quasi-Abelian variety, and assume that X" is a sub-Abelian variety. Then there is a closed sub quasi-Abelian variety X' of X such that the natural map of X"×X' to X is onto and with finite kernel.
Proof. Let $X = V/\Gamma$, $X'' = V''/\Gamma''$, and let V' be the orthogonal to V" with respect to a positive definite Hermitian form representing $c_1(L)$. Since Γ'' spans V" as a real vector space, if we set $\Gamma' = \Gamma \cap V'$, Γ' equals to the orthogonal to Γ'' with respect to $c_1(L)$. H being positive

definite on V'' , the rank of Γ'' plus the rank of Γ' add up to the rank of Γ , thus $\Gamma / \Gamma' + \Gamma''$ is finite and our assertion is proven.

Q.E.D.

Example 2.4 Let X be a generic $\mathbb{C}^*$ -extension of the product of two elliptic curves, and consider the projection $f : X \to E$, where E is the second elliptic curve. Then

i) X is a quasi-Abelian variety

ii) $X'' = \ker(f)$ is a Cousin quasi-torus , whence quasi-Abelian

iii) f does not split

iv) if L' is a line bundle of degree 1 on E , and L is the pull-back of L', then $h^o(L) = 1$.

v) X is not isogenous to $X'' \times E$.

Proof. We can assume that $X = \mathbb{C}^3 / \Gamma$, where $\Gamma = \mathbb{Z}^3 \oplus \mathbb{Z}(b,\tau', 0)^t \oplus \mathbb{Z}(d,0,\tau)^t$. Then f is induced by the third coordinate function in $\mathbb{C}^3$. By pulling back a polarization on $E' \times E$, we get a linearized bundle whose Hermitian form is positive definite on the maximal complex subspace F of $\mathbb{R}\Gamma$ (spanned by the imaginary parts of the last two given basis vectors of Γ) . Whence, i) and ii) are verified if both X, and X'' are Cousin, which follows (cf. e.g. [C-C] , 1.5) if b and τ' are linearly independent over $\mathbb{Q}$.

iii) : if f would split, then there would be two vectors in Γ whose third coordinates would respectively equal $1,\tau$, and which should be $\mathbb{C}$-linearly dependent. In particular ,if we assume that multiplication by τ does not give an endomorphism of E', there should exist integers n,n' such that $n' \tau = n + d$, which can be excluded by assuming $\tau,1,d$ to be $\mathbb{Q}$-linearly independent.

iv) follows now from ii) since L is trivial on the fibres of f (isomorphic to X'') , and every holomorphic section is constant on the fibres of f , whence every section of L pulls back from a section of L' on E.

v) : otherwise X is isomorphic to $X'' \times E''$ where E'' is a finite covering of E, and then there would exist an integer h such that there would be two vectors in Γ whose third coordinates would respectively equal $h, h\tau$, and which should be $\mathbb{C}$-linearly dependent. We proceed as in step iii).

□

The preceding example shows moreover (by iv)) that in the next proposition neither the result of 2) can be improved,nor can be relaxed the hypothesis of positivity in 3).

Proposition 2.5 Let D be the divisor of a linearized line bundle L on a quasi-Abelian variety. Then

1) if D is reducible as $D'_1 + \dots + D'_r$, then there is another divisor in $|D|$ reducible as $D_1 + \dots + D_r$, with D_i the divisor of the linearized bundle B_i associated to D'_i .

2) If $|D| = \{D\}$ ($h^o(L) = 1$) , then there is a principally polarized Abelian variety (X',D'), and a holomorphic map $f : X \to X'$ with connected fibres such that D is the pull back of D' .

3) If D is positive and $|D|$ has a fixed part, then the polarization (X,D) is reducible with one factor being a principally polarized Abelian variety.

<u>Proof.</u> 1) : if D'_i is the divisor of a bundle L_i , then (cf. [C-C], pp. 64-65), there is a semi-positive definite Hermitian form H_i of maximal rank , representing the first Chern class of L_i , such that the image Γ_i of Γ into $V/\ker H_i$ is discrete. Then , if $X'_i = (V/\ker H_i)/\Gamma_i$, L_i pulls back from a positive bundle L_i' on X'_i (the pair (X'_i, L'_i) is called the reduction of L_i) : the same holds for its linearization B_i ,which therefore has a non zero section, whose divisor we choose as D_i .

2) : As in 1), consider the reduction (X',L') of L. L' is linearized and positive, and with $h^o(L') = 1$,therefore (X',L') is a principally polarized Abelian variety. Assume that the projection f of X to X' has disconnected fibres. Then (the fundamental group of X' being Abelian) f factors through an Abelian variety Y which is a finite covering of X' of positive degree . This contradicts $h^o(L) = 1$.

3) : let H be a positive definite Hermitian form corresponding to the linearized bundle of L .

By prop. 2.8 of [C-C], there exist lattices Γ', Γ'' , whose intersection is Γ, such that the imaginary part of H is integral on those lattices.

It follows that there are projections of X to polarized Abelian varieties X' (resp. : X") such that L is a pull-back of the respective polarization.

Let Θ be the fixed part of $|D|$. By 1) , Θ is a linearized divisor and if $p : X \to Y$ is the reduction of Θ , then, by 2), Y is a principally polarized Abelian variety.

Since p factors through both projections to X' , resp. X", it follows by the decomposition theorem (cf. [L-B] , pp. 77 and foll.) , that there are splittings of both the projections $\pi' : X' \to Y$, $\pi'' : X'' \to Y$.

Write $X = V/\Gamma$, $Y = W/\Lambda$: then these splittings give an isomorphism of W with the H-orthogonal of V° (V° being the tangent

space to $Z = \ker p$) and this isomorphism carries Λ into the intersection of Γ' with Γ'' .
Whence, we have obtained a splitting $Y \to X$ of p , giving an isomorphism of X with $Z \times Y$. So L is a tensor product of two pull backs of linearized bundles from the two factors . The second one must be the given principal polarization, whence 3) is proven.

Q.E.D.

Theorem C . If L is a positive line bundle, then the sections of $H^0(L^2)$ give a projective embedding of X if and only if the linearized bundle B associated to L does not yield a polarization (X,B) which is reducible with one factor (X_1,L_1) being a principally polarized Abelian variety.

Remark 2.6 The corresponding result for Abelian varieties was proved by Obuchi ([O], cf. also [L-B], page 88).

Proof of theorem C As usual, by virtue of Prop. 1.2., there exists a character χ s.t. $H^0(B\otimes\mathcal{L}\otimes M_\chi) \neq 0$, and $H^0(B\otimes M_\chi{}^{-1}) \neq 0$.
Moreover, by our assumption, both spaces have dimension at least 2 (infinite in the non compact case). We let D' be the divisor of a generic section of $H^0(B\otimes\mathcal{L}\otimes M_\chi) \neq 0$, and D the divisor of a generic section of $H^0(B\otimes M_\chi{}^{-1})$.
By our usual argument, for fixed x and generic a in F , x does not belong to $D + a$ (neither to $D' + a$).
Step I : let x and y be distinct points of X . If x and y are not separated by the sections of $H^0(L^2)$, then, for every a in F, and D,D' as above, it follows that if $D+a$ contains x , then either
i) $D'-a$ contains y , or
ii) $D+a$ contains y .

Sublema 2.7 : Let Λ be a k-dimensional linear system of divisors $\subset |D|$,where $k \geq 1$, and let F be as usual the maximal complex subspace of $\mathbb{R}\Gamma$. Then if $\Delta = \{ (D,a) \mid D + a \text{ contains } x \} \subset \Lambda \times F$, Δ is irreducible if Λ restricted to $x + F$ has a fixed part $\mathcal{B}'$ which contains no divisor.

In any case, Δ contains a unique irrreducible component Δ^{hor} mapping onto F.

<u>Proof</u> : Δ is a $\mathbb{P}^{k-1}$-bundle over F - $\mathcal{B}'$. Let $\mathcal{B}$ be the base locus of Λ and set $\mathcal{B}_x$ = F ∩ (x - $\mathcal{B}$) . Since Δ is a divisor in $\Lambda \times F$, the only possibility that Δ is reducible is that $\mathcal{B}_x$ has a component which has codimension 1 in F .
In this case, though,where $\mathcal{B} \cap (x + F) = \mathcal{B}'$ contains a divisor, we let Δ^{hor} be the closure of the above $\mathbb{P}^{k-1}$-bundle over F - $\mathcal{B}'$.

□

There are only two cases :

<u>Case A)</u> : Δ is irreducible .

<u>Case B)</u> : Δ is reducible and the analytic Zariski closure of $\mathcal{B}'$ in X does not contain a divisor (else , |D| would have a fixed part, contradicting 3) of proposition 2.5) .

In both cases, either i) holds for each pair (D,a) in Δ^{hor} , or ii) does.
i) is absurd, since then y-a belongs to the base locus of |D'| for each a , and this base locus should be the whole of X, a contradiction.

We can thus assume that ii) holds for each pair (D,a) in Δ^{hor} .
In case A), this means that for each D in |D| , if x-a is in D , then also y-a is in D . Equivalently, translation by (y-x) carries D ∩ (x+F) to D ∩ (y+F) . By lemma 2.8, both sets are analytically Zariski dense in D. Therefore, any such divisor D is (y-x) periodic. As we saw in claim III of theorem B, this is impossible.
In case B) , the above holds if we replace D ∩ (x+F) by D ∩ (x+F) - $\mathcal{B}'$. But, since the analytic Zariski closure of $\mathcal{B}'$ does not contain a divisor, this smaller set is again Zariski dense in D.

□

<u>Step II</u> : Assume that the sections of $H^0(L^2)$ do not give a projective embedding of X at x,and let v be a tangent vector at x which is in the kernel of the differential . Since for generic D' and a , D' + a does not contain x, it follows that for all a in F in case A) , for all points in F - $\mathcal{B}_x$ in case B), it holds that if D contains x-a, then v is tangent to D at x-a. But since we have thus ,as we saw, a Zariski dense set in D, it follows that D is invariant by translation by the subgroup exp(v). This should hold for any D in the linear system |D|. But since D is linearized, it suffices to take a divisor D which is a pull back of a positive divisor on an Abelian variety to derive a contradiction.

<u>Q.E.D.</u>

Lemma 2.8. If D is an irreducible divisor in a quasi-Abelian variety, then $D \cap (\pi(F))$ is analytically Zariski dense in D (whence the same conclusion holds for every divisor).

Proof The proof will be carried out in three steps .
Step 1. $D \cap (\pi(F))$ is not empty.
Step 2. If K is the maximal compact subgroup of X , $K = \mathbb{R}\Gamma/\Gamma$, $D\cap (\pi(F))$ is dense in $D\cap K$.
Step 3. If $D \cap K$ is not empty, its analytic Zariski closure is a divisor, whence it equals D (by irreducibility) .

1) : it suffices to show that the volume of $F \cap \pi^{-1}(D)$ is infinite. To this purpose, we consider the line bundle L associated to D , and we consider a cocycle in $H^1 (\Gamma, \mathcal{O}^*_{\mathbb{C}^n})$ in Appell-Humbert normal form

$$k_\gamma(z) = \rho(\gamma) \exp(-i/_2 [H(z,\gamma) + 1/_2 H(\gamma,\gamma)] + \psi_\gamma(z)) .$$

D is the divisor of a section of L , i.e. , of a function f(z) solving the functional equation $f(z + \gamma) = k_\gamma(z) f(z)$.

The existence of a non zero section f implies that the Hermitian form is semi-positive definite and non zero on the maximal complex subspace F contained in $\mathbb{R}\otimes\Gamma \subset \mathbb{C}^n$ (cf. e.g. [C-C] , thm .3.20) . In particular, the trace of a Hermitian matrix representing the restriction of H to F is strictly positive.

We choose now (cf. [C-C], page 49) apt linear coordinates (u,v) in $\mathbb{C}^n$, i.e., such that F is the subspace {u=0} , $\mathbb{R}\Gamma$ = { Imu = 0 } , and $\Gamma = \mathbb{Z}^{n-m} \oplus \Omega\mathbb{Z}^{2m}$, with the matrix $\Omega = (\Omega_u{}^t, \Omega_v{}^t)^t$ such that Ω_v defines a lattice in F .

We let, for q in $(\mathbb{Z}^{2m})^{2m}$, $P(q) \subset F$ to be the fundamental parallelotope $\Omega_v \cdot q \cdot (Q)$, Q being the unit cube .

$F \cap \pi^{-1}(D)$ is defined by the equation $f(0,v) = 0$, and if P' is a parallelotope, the function $w(P') = \mathrm{vol}(P' \cap F \cap \pi^{-1}(D))$ is calculated, by the Poincare' Lelong equation , by

(2.9) $w(P') = (1/2\pi i) \int_{\partial P'} (d\log (f) \wedge \eta^{m-1})$, η being the standard Kahler form on F .

Now , call F_k the k-th initial face of $P = P(q)$, and F_{ik} the codimension 2 initial face of P (that is, the image of the points of Q where the i-th and k-th coordinates are zero) .

We can split the integral w(P) as a sum over the corresponding initial and final k-th faces of P, F_k and F'_k (they differ by translation by $\Omega_v \cdot q_k = (\gamma_k)_v$). We can actually find a γ_k in Γ such that $\gamma_k = (\gamma_u, \Omega_v \cdot q_k)$ with γ_u in the unit cube.
We can write our volume as follows

$$w(P) = \Sigma_{k=1,\ldots 2m} (1/2\pi i) [-\int_{Fk} + \int_{F'k}] (\mathrm{dlog}(f) \wedge \eta^{m-1})$$

and the individual integrals can be written as

$$(1/2\pi i) \int_{Fk} ((\mathrm{dlog}(f(0,v + \Omega_v q_k) - \mathrm{dlog} f(0,v)) \wedge \eta^{m-1}).$$

Whence, rewriting the last integral as

$(1/2\pi i) \int_{Fk} ([\mathrm{dlog}\, f(0,v+\Omega_v q_k) - \mathrm{dlog} f((0,v+\Omega_v q_k)-\gamma_k)] + \{\mathrm{dlog} f(-(\gamma_k)_v , v) - \mathrm{dlog} f(0,v)\}) \wedge \eta^{m-1}$, we see (as in [Cou]) that the second term is bounded by a constant times the volume of F_k times the diameter of F_k times the norm $|(\gamma_k)_v|$ (this follows by the functional equation and uniform continuity on the unit cube) .
Whereas, by the functional equation, the first term equals

$$(1/2\pi i) \int_{Fk} ([\mathrm{dlog}\, k_{-\gamma}((0,v+\Omega_v q_k)] \wedge \eta^{m-1}) =$$

$$(1/4\pi) \int_{Fk} dH((0,v+(\gamma_k)_v),\gamma_k) \wedge \eta^{m-1}) =$$

$$(1/4\pi) \int_{Fk} dH((0,v),\gamma_k) \wedge \eta^{m-1}).$$

In turn , we can use again Stokes' theorem and write the last integral as a sum on the codimension 2- faces of F_k , thus we get

$$\Sigma_{i \neq k,\ i=1,\ldots 2m} (1/4\pi) \int_{Fik} H((\gamma_i)_v , \gamma_k) \wedge \eta^{m-1}) .$$

We multiply now the matrix q by an integer h , and we look at the asymptotic behaviour of the volume w (P(hq)) .
Then , since by the density of $\pi(F)$ the lim inf of $|(\gamma_k)_v|$ is zero, the first term is such that its lim inf is o (h^{2m}).
Whereas the first term is asymptotic to

$$\Sigma_{i \neq k,\ i,k=1,\ldots 2m} (1/4\pi)\ H((\gamma_i)_v , (\gamma_k)_v) \mathrm{vol}(F_{ik}) h^{2m}.$$

The leading term is homogeneous in the vectors $(\gamma_i)_v$,and semipositive definite being a volume calculation (minus a lower degree term , when we take the lim inf).

We only need to show that it is not identically zero.

But if it were so, then it would be identically zero for all choice of 2m vectors (v_i) in F . In particular , we choose an orthonormal basis (v_i) for the Euclidean metric on F for which the symmetric bilinear form S given by the real part of H is diagonalized.

In this particular case, our expression reduces to

$\Sigma_{i \neq k\ ,\ i,k=1,\ ..\ 2m}\ (1/4\pi)\ H((v_i)\ ,\ (v_k))$, whose real part is just the trace of the symmetric semidefinite form S .

But since H is non zero, also S is non zero, whence this trace is strictly positive.

□

2) : assume that we have a point of $D \cap K$. By changing the u-coordinates up to translation, we assume this point to be (0,0).

Since f(0,v) is not identically zero, by the Weierstrass preparation theorem we can assume that f is a pseudopolynomial around the origin ($v' = (v_1, \ldots v_{m-1})$) , $f(u,v) = v_m{}^{d} + \Sigma\, a_i(u,v')\, v_m{}^{d-i}$, with the $a_i(u,v')$'s vanishing at the origin .

By the density of $\pi(F)$ in K, there are points $x_\nu = (u_\nu , v_\nu)$ in $\pi(F)$ tending to (0,0), i.e., locally at the origin the subspaces $\{ u = u_\nu \}$ belong to $\pi(F)$.

Fix now any u_ν ,and v' : since f is monic in v_m ,there is a root which tends to 0 as soon as u_ν ,and v' tend to zero.

But this proves that our point in $D \cap K$ belongs to the closure of $\pi(F) \cap D$.

3) : Take a point in $D \cap K$, notation being as above.

Let $\delta(u,v')$ be the discriminant of f(u,v) with respect to v_m .

Since D is reduced, δ is not identically zero, whence it is not identically zero for u real and v' arbitrary.

Assume that g (u,v) is holomorphic and vanishes on $D \cap K$.

We know that, modulo (f), g is equivalent to a pseudopolynomial r of degree $\leq$ d-1. Now r is identically zero on $D \cap K$, but for generic u real, v' arbitrary, f has d distinct roots : whence r is identically zero.

This shows that the analytic Zariski closure of $D \cap K$ contains an open piece of D, and thus the whole of D being D irreducible.

Q.E.D.

Problem 2.9 Given a positive line bundle L on a quasi-Abelian variety, does there exist a section of L whose divisor of zeroes is aperiodic ?

Problem 2.10 Let (X,L) be an irreducible polarized non compact quasi-Abelian variety . Do the sections of $H^0(L)$ give a projective embedding of X ? Or, at least, a generically injective map ?

Remark. The last problem is motivated by recent interesting work of Debarre, Hulek and Spandaw, ([DHS]) concerning polarizations of type (1,..1,d) on Abelian varieties.

3 Again on the existence of sections.

For every subgroup Γ' of Γ as in 1.3, there exists, as we saw, a flat torsion bundle M_χ s.t. $H^o(L\otimes M_\chi) \neq 0$. If one believes in some sort of continuos dependence upon L of this χ , this implies evidence for the above quoted conjecture 3.22 of [C-C].

At least this fact shows that in the set of line bundles with a given positive Chern class (which, in the non linearizable case, is parametrized by a complex quasi-torus times an infinite dimensional vector space (cf. [Vo])) , the set $\mathcal{H}$ of bundles admitting a non zero section is such that its translates by certain points of 2-torsion fill the whole space , thus its "probability" is bigger than 2^{-n} .

To prove the conjecture, it would suffice to show that

$H^o(\pi^*(L)) = \oplus_{\chi \varepsilon G^\vee} H^o(L\otimes M_\chi)$, as a representation of the abelian group G, contains all the characters of $G^\vee = (\Gamma/\Gamma')^\vee$.

We can prove the conjecture under a mildly restrictive hypothesis.

Given a positive line bundle L , with elementary divisors $d_1, \ldots , d_{h+k}$, with $d_1, \ldots , d_h = 1$, $d = d_{h+1} > 1$, we choose a Frobenius basis $\{ e_1 , \ldots , e_l, g_1, \ldots , g_{h+k} , f_1, \ldots , f_{h+k} \}$ as we did after remark 1.3, and we let Γ' be the subgroup of Γ generated by $\{ e_1, \ldots , e_l, dg_1, \ldots ,$

$dg_h, g_{h+1}, \dots, g_{h+k}, f_1, \dots, f_{h+k}\}$, so that, if $X' = \mathbb{C}^n/\Gamma'$ and $\pi : X' \to X$ is the natural projection with kernel $G = \Gamma/\Gamma' \cong (\mathbb{Z}/d)^h$ then $c_1(\pi^*(L))$ is divisible by d.

Definition 3.1 Given a positive line bundle L on X call a covering $\pi : X' \to X$ as above a big covering .

If we have a big covering with kernel G as above, associated to a Frobenius basis, we have an embedding of the dual group $G^\vee$ of G inside the maximal compact subgroup K' of X' and ,via π , in $K \subset X$. In fact, it suffices to let

$\Gamma^* = \{ e_1, \dots, e_l, g_1, \dots\dots, g_{h+k}, d^{-1} f_1, \dots, d^{-1} f_h, f_{h+1}, \dots, f_{h+k}\}$, and then

$\Gamma^*/\Gamma \cong G^\vee$, $\Gamma^*/\Gamma' \cong G \times G^\vee$.

If the line bundle L is linearized, then $L' = \pi^*(L)$ is isomorphic to $T_y^*(L')$ for y in $G \times G^\vee \subset X'$.

In the general case ,this holds in particular if, given a cocycle in Appell-Humbert normal form

$$k_\gamma(z) = \rho(\gamma) \exp(-i/_2 [H(z,\gamma) + 1/_2 H(\gamma,\gamma)] + \psi_\gamma(z)),$$

the additive cocycle $\psi_\gamma(z)$ is $d^{-1} f_i$ -periodic for $i=1, \dots, h$.

Theorem D. Let L be a positive line bundle on X such that there exists a Frobenius basis of Γ for whose associated big covering $\pi : X' \to X$, with kernel G, the line bundle $L' = \pi^*(L)$ is isomorphic to $T_y^*(L')$ for y in $G \times G^\vee \subset X'$. Then $H^0(L) \neq 0$ (indeed, infinite dimensional if X is not compact).

Proof. As usual, consider the standard Heisenberg group $H'(G) = \{ (y,\varphi) \mid y \text{ in } G \times G^\vee \subset X', \varphi \text{ an isomorphism of } L' \text{ with } T_y^*(L')\}$.

We have a central extension $1 \to \mathbb{C}^* \to H'(G) \to G \times G^\vee \to 0$. Correspondingly, we have the finite Heisenberg group H(G) ,a subgroup of H'(G) for which, $\mu_d \subset \mathbb{C}^*$ being the group of d-th roots of unity , we have another central extension

$$1 \to \mu_d \to H(G) \to G \times G^\vee \to 0.$$

Clearly $H^o(L') \neq 0$ is a representation of $H'(G)$ and of $H(G)$. Every non zero section of $H^o(L')$ lies in a (finite dimensional) irreducible representation W of $H(G)$. Since the centre of $H'(G)$ acts on $H^o(L')$ by the standard character , it follows , by the Stone-von Neumann theorem (cf. [Mu3]) , that W is the representation $L^2(G)$.
In particular, there is a non zero G-invariant section in W . This section descends to a non zero section of $H^o(L)$, as we wanted.

Q.E.D.

References

[A1] Abe Y. : Holomorphic sections of line bundles over (H,C) groups . Manuscripta Math. **60** 1988 , pp 379-385

[A2] Abe Y. : Sur les fonctions periodiques de plusiers variables. Nagoya Math. J. **122** 1991, pp 82-114 .

[A-G] Andreotti A. , Gherardelli F. : Estensioni commutative di varieta' abeliane. Aldo Andreotti : Selecta **2** 1992 , Scuola Normale Superiore Pisa, pp. 433-461.

[C-C] Capocasa F. , Catanese F. : Periodic meromorphic functions. Acta Math. **166** 1991, pp 27-78 .

[Cou] Cousin P. : Sur les fonctions triplement periodiques de deux variables. Acta Math. **33** 1910, pp 135-232 .

[DHS] Debarre O. , Hulek K. , Spandaw J. : Very ample linear systems on Abelian varieties. Schriftenreihe Heft Nr 176 1993 , Hannover.

[L-B] Lange H. , Birkenhake C. : Complex Abelian Varieties. Grundlehren der mathematischen Wissenschaften **302** 1992 , Springer Verlag .

[Ma] Malgrange B. : La cohomologie d' une variete' analytique a' bord pseudoconvexe n'est pas necessairement separee' . C.R. Acad.Sci. Paris **280** 1975 , pp 93-95 .

[Mo] Morimoto A. : Non compact complex Lie groups without constant holomorphic functions. Conference in complex analysis 1966 , Minneapolis .

[Mu1] Mumford D. : Geometric invariant theory. 1965 Springer Verlag .

[Mu2] Mumford D. : Abelian varieties . 1970 Oxford Univ. Press .

[Mu3] Mumford D. : Tata lectures on Theta I,II. Prog. in Math. **28** , **43** 1983 ,1984 , Birkhauser.

[Na] Nakano S. : Vanishing theorems for weakly 1-complete manifolds II. Publ. Res. Inst. Math. Sci. Kyoto Univ. **10** 1974 , pp 101-110.

[O] Ohbuchi A. : Some remarks on simple line bundles on abelian varieties ,Man. Math. 57, 1987, pp. 225-238.

[Vo] Vogt C. : Line bundles on toroidal groups. J. Reine Angew. Math. **21** 1982 , pp 197-215 .

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