A new invariant for spin manifold and its application to the classification of (n-2)-connected 2n-dimensional almost parallelizable manifolds

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Abstract

For a $2n$-dimensional spin manifold M with an ϕ-orientation (w,h) (c.f: Definition 1.4 below), where $n \neq 3(mod4)$, we define a quadratic function $\phi_{M,h}$: $H^{n-1}(M, \mathbf{Z}_4) \to \mathbf{Q}/\mathbf{Z}$ satisfying

$$\phi_{M,h}(x+y) = \phi_{M,h}(x) + \phi_{M,h}(y) + j(x \cup Sq^2 y)[M],$$

where ϕ is a certain secondary cohomology operation and $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$ is the inclusion homomorphism. Using Gauss sum, we define an Arf invariant $\sigma(\phi_{M,h}) \in \mathbf{Z}_3$ which depends only on the equivalent class(Witt class) of the quadratic function and satisfies $\sigma(\phi_{-M,h}) = \sigma(\phi_{M,h})$ and $\sigma(\phi_{M\#M',h}) = \sigma(\phi_{M,h})\sigma(\phi_{M',h})$.

Assuming that the Wu classes $v_{n+2-2^i}(\nu_M) = 0$ for all i where ν_M is the stable normal bundle of M. When $n = 0,1(mod4)$, the equivalent class of $\phi_{M,h}$ and therefore $\sigma(\phi_{M,h})$ is a homotopy invariant of the spin manifold M. When $n = 2(mod4)$, the equivalent class of $\phi_{M,h}$ is invariant under homotopy equivalences fixing the Wu orientation(c.f. 1.4 for the definition of Wu orientation).

Using this new quadratic function we obtain a complete classification of $(n-2)$-connected $2n$-dimensional almost parallelizable manifolds up to homeomorphism and homotopy equivalence, where $n \geq 4$ and $n+2 \neq 2^i$ for some i. As a corollary of the classification, two such homotopy equivalent manifolds are homeomorphic.

§ 1. Introduction and Summary

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The purpose of this paper is two folds. The first, also the main part, is to define a $\mathbf{Q}/\mathbf{Z}$-valued function

$$\phi_{M,h} : H^{n-1}(M, \mathbf{Z}_4) \to \mathbf{Q}/\mathbf{Z}$$

for a spin manifold M of dimension $2n$, where $n \neq 3(mod4)$ and $n \geq 4$, with an additional structure, the ϕ-orientation (c.f: definition 1.1 below). This function is "quadratic" with respect to the symmetric bilinear form

$$\begin{array}{l} H^{n-1}(M,\mathbf{Z}_2) \otimes H^{n-1}(M,\mathbf{Z}_2) \to \mathbf{Z}_2 \\ x \otimes y \to x \cup Sq^2 y[M]. \end{array}$$

We will show that the equivalent class of $\phi_{M,h}$ depends only on the spin structure/Wu orientation(c.f: definition 1.1 below or [4]) when $n = 0, 1(mod4)/2(mod4)$, provided the Wu classes $v_{n+2-2^i} = 0$ for all i. Moreover, let $f : M \to N$ denote a homotopy equivalence preserving the spin structures/Wu orientations for $n = 0, 1(mod4)/2(mod4)$, we prove that $\phi_{M,h}(f^*x) = \phi_{N,h}(x)$ for all $x \in H^{n-1}(N, \mathbf{Z}_4)$, provided the Wu classes $v_{n+2-2^i} = 0$ for all i. Therefore the Arf invariant $\sigma(\phi_{M,h})$ is a homotopy invariant of the spin/Wu oriented manifold M when $n = 0, 1(mod4)/2(mod4)$, if $v_{n+2-2^i} = 0$ for all i.

In particular, if M is a framed manifold, this gives rise a nice homotopy invariant appliable to obtain the classification of the $(n-2)$-connected $2n$ dimensional almost parallelizable manifolds. This is the second part of the present paper, also the original motivation of this paper. Recall that the classification of this kinds of manifolds up to homeomorphism in the special case of the homology groups are all torsion free was accomplished by Ishimoto [9][10] . His method does not work in general. One corollary may be interesting is that, by our work(c.f: Theorem 1.11 below), the homotopy and the homeomorphism classification of the $(n-2)$-connected $2n$-dimensional almost parallelizable manifolds are in fact the same.

Throughout this paper, all homology/cohomology groups will be with integral coefficients unless otherwise stated. Usually spaces will have base points. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. S denotes suspension and $\{X, Y\} = lim[S^k X, S^k Y]$. K_n denotes $K(\mathbf{Z}_2, n)$. l will always denote the basic class for various Eilenberg-Maclane spaces by the context. $\{Y_k\}_{k \in \mathbf{Z}_+}$ will denote a connected spectrum with $U \in H^0(Y) \cong \mathbf{Z}$ a generator satisfying $i^*U \in H^0(S^0)$ a generator, where $i : S^0 \to Y$ is the inclusion map of the spectrum.

Definition 1.1. *(i) $\{Y_k\}_{k \in \mathbf{Z}_+}$ is called ϕ-orientable if $Sq^2 U = 0$, $\chi(Sq^{n+2})(U) = 0$ and $0 \in \phi(U)$, where ϕ is a secondary cohomology operator defined in §2 precisely which is associated with the Adem relation:*

$$\begin{array}{ll} \chi(Sq^n)Sq^3 + \chi(Sq^{n+2})Sq^1 + Sq^1\chi(Sq^{n+2}) = 0 & n = 2(mod4) \\ \chi(Sq^n)Sq^3 + Sq^1\chi(Sq^{n+2}) = 0 & n = 0(mod4) \\ \chi(Sq^{n+1})Sq^2 + Sq^1\chi(Sq^{n+2}) = 0 & n = 1(mod4) \end{array}$$

and $\chi : A_2 \to A_2$ is the anti-automorphism of the Steenrod algebra A_2(c.f: [1]).
(ii) A spherical fibration ξ is called ϕ-orientable if its Thom spectrum $T\xi$ is ϕ-orientable. A manifold is called ϕ-orientable if its stable normal bundle(fibration) is ϕ-orientable.
(iii) For the ϕ-orientable spin spherical fibration $\xi \searrow M$, a Wu orientation of ξ is a lifting of the classifying map $\xi : M \to Bspin_G$ to $BSpin_G\langle v_{n+2}\rangle$. A Wu orientation of ν_M, the stable normal bundle of M, is called a Wu orientation of M, where $BSpin_G\langle v_{n+2}\rangle \to BSpin_G$ is a principal fibration with $v_{n+2} \in H^{n+2}(BSpin_G, Z_2)$ as the k-invariant.

There is a ϕ-orientable spectrum $\widetilde{W}(n)$ as the follows such that for any ϕ-orientable spectrum Y, there exists a connected spectral map $f : Y \to \widetilde{W}(n)$. We say $\widetilde{W}(n)$ is a universal ϕ-orientable spectrum.

$\widetilde{W}(n)$ is a Ω-spectrum, where $\widetilde{W}_k(n)$ is the total space of the following Postnikov tower:

$$\begin{array}{ccc}
\widetilde{W}_k(n) & & \\
\downarrow \Pi_2 & & \\
W_k(n) & \xrightarrow{k_2} & K_{k+n+2} \\
\downarrow \Pi_1 & & \\
K(\mathbf{Z}, k) & \xrightarrow{Sq^2 \times \chi(Sq^{n+2})} & K_{k+2} \times K_{k+n+2}
\end{array}$$

where $k_2 \in \phi(\Pi_1{}^* \mathrm{l_k})$.

The universal property of $\widetilde{W}(n)$ implies that it is unique up to homotopy. It is easy to see that a spectrum Y is ϕ-orientable if and only if $U \in H^0(Y)$ can be lifted to a map $w : Y \to \widetilde{W}(n)$.

Example (i): By the definition, the sphere spectrum S^0 is ϕ-orientable. Thus every stable parallelizable manifold is ϕ-orientable.

Example (ii): For $n = 0, 1(mod 4)$, let $\gamma \searrow BSpin_G$ be the universal spin spherical fibration and let $U \in H^0(MSpin_G, \mathbf{Z}_2)$ be its Thom class. Notice that, $\chi(Sq^{n+2})U = \chi(Sq^{n+1})Sq^1U = 0$ if n is odd. $\chi(Sq^{n+2})U = \chi(Sq^n)Sq^2U = 0$ if $n = 0(mod 4)$. Thus U can be lifted to a map $f : MSpin_G \to W(n)$. By the Thom isomorphism, f^*k_2 gives an element of $\bar{k}_2 \in H^{n+2}(BSpin_G, \mathbf{Z}_2)$. Consider the principal fibration $\pi : BSpin_G\langle \bar{k}_2\rangle \to BSpin_G$ with k-invariant $\bar{k}_2$. It is easy to see that the fibration $\pi^*\gamma$ is ϕ-orientable. Note that $\pi^*\gamma$ is the universal ϕ-orientable fibration, i.e, the classifying map of any ϕ-orientable stable spherical fibration can be lifted to $BSpin_G\langle \bar{k}_2\rangle$.

Example (iii): For $n = 2(mod 4)$, the similar technique above gives a principal fibration $\pi : BSpin_G\langle \bar{k}_2\rangle \to BSpin_G\langle v_{n+2}\rangle$, where $BSpin_G\langle v_{n+2}\rangle \to BSpin_G$ is the fibration with fibre K_{n+1} and k-invariant v_{n+2}. $\pi^*\gamma$ is the universal ϕ-orientable spherical fibration.

We denote by $MSpin_G\langle\bar{k_2}\rangle$ the Thom spectrum of this universal ϕ-orientable spherical fibration. Throughout the rest we fix a connected spectral map $\mathbf{u} : MSpin_G\langle\bar{k_2}\rangle \to \widetilde{W}(n)$.

Let $\kappa : K(\mathbf{Z}_4, n-1) \times K(\mathbf{Z}_4, n-1) \to K(\mathbf{Z}_4, n-1)$ denote the multiplication of $K(\mathbf{Z}_4, n-1)$. Write $H(\kappa)$ for the Hopf construction of κ.

Proposition 1.2. *The homomorphism*

$$H(\kappa)_* : \pi^s_{2n}(K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1)) \to \pi^s_{2n}(K(\mathbf{Z}_4, n-1))$$

is injective if $n \neq 3(mod4)$, *and zero if* $n = 3(mod4)$.

As one can read from Theorem 2.1 in §2, $\pi^s_{2n}(K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$ if $n \geq 4$, $\pi^s_{2n}(K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ if $n = 2(mod4)$. We set λ_0 for the generator of $Im(H(\kappa)_*)$ whenever $n \neq 2(mod4)$, and a specified generator of $\pi^s_{2n}(K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ otherwise. The following theorem is a key in this paper.

Theorem 1.3. *Suppose that* $\{Y_k\}_{k\in\mathbf{Z}_+}$ *is an* ϕ *orientable spectrum. Then there exists a homomorphism*

$$h : H_{2n}(K(\mathbf{Z}_4, n-1); Y) \to \mathbf{Q}/\mathbf{Z}$$

such that $h(\lambda) = \frac{1}{4}$ *and* $\frac{1}{2}$ *by* $n = 2(mod4)$ *and* $0, 1(mod4)$ *respectively, where* λ *is the image of* λ_0 *under the homomorphism* $i_* : H_{2n}(K(Z_4, n-1); S^0) \to H_{2n}(K(Z_4, n-1); Y)$.

Now we are ready to give the definition of ϕ-orientation for a Thom spectrum. We may make a more general definition but the following is enough for our purpose.

Definition 1.4. *Let* Y *be an* ϕ*-orientable Thom spectrum and let*

$$h : H_{2n}(K(Z_4, n-1); \widetilde{W}(n)) \to \mathbf{Q}/\mathbf{Z}$$

is a homomorphism as above. For each Thom map $w : Y \to MSpin\langle\bar{k_2}\rangle$, *we say that the pair* $(\mathbf{u} \circ w, h)$ *is an* ϕ*-orientation of* Y.

Recall that an $2n$-Poincaré triple (M, ξ, α) is
(i). A CW complex M with finitely generated homology.
(ii). A fibration ξ over M with fibre homotopy equivalent to S^{k-1}, k large.
(iii). $\alpha \in \pi_{2n+k}(T\xi)$ such that an $(2n+k)$ Spanier-Whitehead S-duality is given by

$$S^{2n+k} \xrightarrow{\alpha} T\xi \xrightarrow{\Delta} T\xi \wedge M^+$$

where Δ is the diagonal map.

For each $2n$-Poincaré triple (M, ξ, α), set $A_\alpha : \{M_+, K(\mathbf{Z}_4, n-1)\} \to \{S^{2n+k}, T\xi \wedge K(\mathbf{Z}_4, n-1)\}$ for the S-duality.

Definition and Property 1.5. *Suppose that (M,ξ,α) is a Poincaré triple of dimension $2n$, where ξ is ϕ-orientable and $n \neq 3(mod4)$. For each ϕ-orientation $(\mathbf{u}\circ w, h)$ of $T\xi$ and an $x \in H^{n-1}(M,\mathbf{Z}_4)$, we define*

$$f(x) := (\mathbf{u}\circ w \wedge id)\circ A_\alpha(x),$$
$$\phi_{M,h}(x) := h(f(x)).$$

The function $\phi_{M,h}$ satisfies

$$\phi_{M,h}(x+y) = \phi_{M,h}(x) + \phi_{M,h}(y) + j(x \cup Sq^2 y)[M],$$

where $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$ is the inclusion.

Remark 1.6: (i) From the defintion it is not hard to see that $\phi_{M,h}(x)$ depends only on the ϕ-oriented bordism class $[M,x]$.
(ii) When $n = 3(mod4)$, the analogue definition gives a linear function by Proposition 1.2 and the proof of 1.5.
(iii) When the Poincaré triple and the orientation is clear from the context, we write sometimes ϕ_h instead of $\phi_{M,h}$.

The following property follows immediately from the definition.

Proposition 1.7. *Suppose that ξ is a trivial fibration and $\mathbf{u}\circ w$ factors through $S^0 \to \widetilde{W}(n)$. Then the above function ϕ_h factors through $\mathbf{Z}_4 \subset \mathbf{Q}/\mathbf{Z}$ and $\mathbf{Z}_2 \subset \mathbf{Q}/\mathbf{Z}$ by $n = 2(mod4)$ and $n = 0,1(mod4)$.*

Recall that $\pi_n(SO(n)) \cong \mathbf{Z}_4$ when $n = 2(mod4)$. The following theorem gives a geometric property of the quadratic function $\phi_{M,h}$.

Theorem 1.8. *Let (M,ξ,α) be a Poincaré triple, where M is an ϕ-orientable $2n$-dimensional manifold, $n = 2(mod4)$ and the number of $1'$s in the binary expansion of $n+3$ is greater than 2. If w comes from the Thom construction of a map $(g',g) : (\xi, M) \to (\pi^*\gamma, BSpin_G\langle \bar{k_2}\rangle)$ and $\phi_{M,h}$ is the quadratic function associated with (M,ξ,α) and the orientation $(\mathbf{u}\circ T(g'),h)$. Suppose that $\beta : S^{n+1} \to M$ is an embedding representing a homology class $[\beta(S^{n+1})] \in H_{n+1}(M)$ such that $g\circ\beta \simeq *$. Let x denote the Poincaré dual of $[\beta(S^{n+1})]$. Then*

$$\phi_{M,h}(x) = j(\nu(\beta)\oplus\varepsilon),$$

where $j : \mathbf{Z}_4 \to \mathbf{Q}/\mathbf{Z}$ the inclusion and $\nu(\beta)\oplus\varepsilon$ is the normal bundle of β in $M \times \mathbf{R}$.

We say that two quadratic functions $\phi_{M_1,h_1} : H^{n-1}(M_1,\mathbf{Z}_4) \to \mathbf{Q}/\mathbf{Z}$ and $\phi_{M_2,h_2} : H^{n-1}(M_2,\mathbf{Z}_4) \to \mathbf{Q}/\mathbf{Z}$ as above are equivalent if there exists an isomorphism $\tau : H^{n-1}(M_1,\mathbf{Z}_4) \to H^{n-1}(M_2,\mathbf{Z}_4)$ such that $\phi_{M_2,h_2}(\tau x) = \phi_{M_1,h_1}(x)$ for all $x \in H^{n-1}(M_1,\mathbf{Z}_4)$.

The following result says that the quadratic function depends only on the spin structure(Wu orientation) of the stable normal bundle of the manifold and is independent of the normal invariant α and the ϕ-orientation. Moreover, it is a homotopy invariant of the spin (Wu oriented)manifold if $n = 0, 1(mod4)(n = 2(mod4))$.

Theorem 1.9. *Let (M_1, ξ_1, α_1) and (M_2, ξ_2, α_2) be $2n$-dimensional Poincaré triples where ξ_i, $i = 1, 2$, are ϕ-orientable. Suppose that $(\mathbf{u} \circ w_i, h)$ are ϕ-orientations of $T\xi_i$ and the Wu classes $v_{n+2-2^j}(\xi) = 0$ for all $2^j \leq n+2$. Assume that $f : M_1 \to M_2$ is a homotopy equivalence preserving the spin structures (Wu orientations) if $n = 0, 1(mod4)(2(mod4))$. Then*

$$\phi_{M_1,h}(f^*x) = \phi_{M_2,h}(x)$$

for all $x \in H^{n-1}(M_2, \mathbf{Z}_4)$.

In particular, if M is a $2n$-dimensional stable parallelizable manifold, the constant map $c : M \to BSpin_G/BSpin_G\langle v_{n+2}\rangle$ gives rise a standard spin structure/Wu orientation of the stable normal bundle of M. Thereby we have an associated function $\phi_{M,h}$. Fix a homomorphism

$$h : H_{2n}(K(\mathbf{Z}_4, n-1), \widetilde{W}(n)) \to \mathbf{Q}/\mathbf{Z}$$

as in 1.3, by Theorem 1.9 the equivalent class of $\phi_{M,h}$ is a well-defined homotopy invariant of M if $n + 2 \neq 2^i$ for some i, which provides an exact invariant for the classification of $(n-2)$-connected $2n$-dimensional almost parallelizable manifolds up to homeomorphism and homotopy equivalent.

To phrase our classification theorem we now fix some notations.

Let H be a finitely generated abelian group, and

$$\mu : Hom(H, \mathbf{Z}_2) \otimes Hom(H, \mathbf{Z}_2) \to \mathbf{Z}_2$$

be a symmetric bilinear form. We say that μ is of <u>*diagonal zero*</u> if $\mu(x, x) = 0$ for each $x \in Hom(H, \mathbf{Z}_2)$. A function $\phi : Hom(H, \mathbf{Z}_4) \to \mathbf{Q}/\mathbf{Z}$ is called <u>*quadratic*</u> with respect to μ if

$$\phi(x + y) = \phi(x) + \phi(y) + j(\mu(x, y))$$

where $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$ is the inclusion. This gives a triple (H, μ, ϕ). We say triples (H_1, μ_1, ϕ_1), and (H_2, μ_2, ϕ_2) are *isometric* if there exists an isomorphism $\tau : H_1 \to H_2$ such that $\mu_1(x, y) = \mu_2(\tau x, \tau y)$ and $\phi_1(x) = \phi_2(\tau x)$ for all x, y. We denote by $[H, \mu, \phi]$ for the isometry class of a triple.

For a spin manifold M of dimension 2n, let μ_M denote the symmetric bilinear form(c.f [16])

$$\begin{aligned} \mu_M : H^{n-1}(M, \mathbf{Z}_2) \otimes H^{n-1}(M, \mathbf{Z}_2) &\to \mathbf{Z}_2 \\ x \otimes y &\to \langle x \cup Sq^2 y, [M]\rangle \end{aligned}$$

It is obvious that the equivalence class of μ_M is a homotopy invariant of M.

Consider μ_M as a matrix over $\mathbf{Z}_2$. If $H_{n-1}(M) \cong \mathbf{Z}_{2^i} \oplus \cdots \oplus \mathbf{Z}_{2^i}$, then the rank $r(M)$ of μ_M and the homology group $H_{n-1}(M)$ determine the isometry class $[H_{n-1}(M), \mu_M]$. In general, these two datas can not determine the isometry class. A simple example is, taking $H = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_4$, let e_i, $i = 1,2,3$, is a basis of H. Set $\mu_1(e_i, e_j) = 1$ if $(i,j) = (1,2),(2,1)$, and 0 for other entries. Set $\mu_2(e_i, e_j) = 1$ if $(i,j) = (2,3),(3,2)$, and 0 otherwise. These have the same rank 2 and the homology group. But they are not isometric.

Proposition 1.10. (i): *Let M be an almost parallelizable manifold of dimension $2n$. If $n = 2(mod4)$, $\mu_M(x,x) = 0$, $\forall x \in H^{n-1}(M, \mathbf{Z}_2)$.*
(ii): If n is odd, then $\mu_M(x,x) = 0$, $\forall x \in Im(\rho_2 : H^{n-1}(M, \mathbf{Z}_4) \to H^{n-1}(M, \mathbf{Z}_2))$.
(iii): If $n = 0(mod4)$, then there is a S^{n-1}-bundle over S^{n+1} so that $\mu_{S^{n+1} \times_\theta S^{n-1}}(x,x) \neq 0$, where x is a generator of the $(n-1)$-th dimension cohomology group.

In the case (ii) above, I do not know if $\mu_M(x,x) = 0$ in general.

Let M be a $2n$ dimensional framed manifold where $n = 1(mod2)$. Set q_M : $H^n(M, \mathbf{Z}_2) \to \mathbf{Z}_2$ for the Kervaire quadratic function associated with M. By [4], q_M is independent of the framing if $\alpha(n+1) \geq 2$(In fact its equivalent class is a homotopy invariant). Let 2^i denote the maximal exponent of the 2-torsion of $H_{n-1}(M)$, denote by $Sq_i^1 \in H^n(K(Z_{2^i}; n-1), \mathbf{Z}_2) \cong \mathbf{Z}_2$ for the generator. Consider Sq_i^1 as a cohomology operation we get a function

$$q_M(Sq_i^1) : H^{n-1}(M, \mathbf{Z}_{2^i}) \to \mathbf{Z}_2.$$

This gives a homomorphism since $Sq_i^1 x \cup Sq_i^1 y = Sq_i^1(x \cup Sq_i^1 y) = 0$ for $x, y \in H^{n-1}(M, \mathbf{Z}_2)$. We denote by $[H_{n-1}(M), \mu_M, q_M(Sq_i^1)]$ for the isometry class of the triple. By [6], the Kervaire invariant of a smooth framed manifold of dimension $2n$, where $n \neq 2^i - 1$, is zero. For $i \leq 5$, there are smooth manifold of dimension $2^{i+1} - 2$ of Kervaire invariant 1. It is still open whether there is such a manifold for $i \geq 6$.

Note that the Kervaire invariant does not depend on the framings of the underlied $2n$-manifold if $n \neq 1,3,7$ and the manifold is highly connected, e.g, $(n-2)$-connected. Moreover, by [4] it is not hard to show that the Kervaire form is a homotopy invariant if $n \neq 1,3,7$ and $(n-2)$-connected. one should compare this with 1.9 for a proof.

Theorem 1.11. *Let $n \geq 4$, $\alpha(n+2) \geq 2$. The homeomorphism types(homotopy types) of $(n-2)$-connected $2n$-dimensional smooth almost parallelizable manifolds are in (1-1) correspondence with the following algebraic data*

(a) $\wp_n = \{[H, \mu, \phi], b \in 2\mathbf{Z}_+, 2^{n+1}(2^{n-1}-1)a_{n/2}Num(\frac{B_n}{2n})|sign : \mu$ is of diagonal zero, $sign \leq b\}$ if $n = 2(mod4)$;
(b) $\wp_n = \{[H, \mu, \phi], b \in 2\mathbf{Z}_+, 2^{n+1}(2^{n-1}-1)a_{n/2}Num(\frac{B_n}{2n})|sign : \phi$ factors through $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}, sign \leq b\}$ if $n = 0(mod4)$;

(c) $\wp_n = \{[H,\mu,\phi,\omega], b \in 2\mathbf{Z}_+, k \in \mathbf{Z}_2 : \omega \in Hom(tor(H) \otimes \mathbf{Z}_{2^i}, \mathbf{Z}_2), \phi$ *factors through* $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$ *and* $\mu(x,x) = 0$ *if* x *can be lifted to an* $\mathbf{Z}_4$ *class,* $\mu(x,x) = \delta\omega(x)$ *if* x *is of order* 2 *where* $\delta \in \{0,1\}$ *is ambiguous.* $k \equiv 0$ *if there is no a framed manifold of dimension* $2n$ *of Kervaire invariant* 1. $2^i =$ *the maximal exponent of the* 2*-group in* H $\}$ *if* $n = 1(mod4)$;

(d) $\wp_n = \{[H,\mu,\omega], b \in 2\mathbf{Z}_+, k \in \mathbf{Z}_2 : \omega \in Hom(torH \otimes \mathbf{Z}_{2^i}, \mathbf{Z}_2), \mu(x,x) = \delta\omega(x)$ *if* x *is of order* 2 *where* $\delta \in \{0,1\}$ *is ambiguous.* $k \equiv 0$ *if* $n = 7$ *or there is no a framed manifold of dimension* $2n (n \neq 7)$ *of Kervaire invariant* 1. $2^i =$ *the maximal exponent of the* 2*-group in* $H\}$ *if* $n = 3(mod4)$;

via assigning a manifold M *to* $[H_{n-1}(M), \mu_M, \phi_M], b_n(M)$ *and the signature* $SignM$;
$[H_{n-1}(M), \mu_M, \phi_M], b_n(M)$ *and the signature* $SignM$;
$[H_{n-1}(M), \mu_M, \phi_M, q(Sq^1_{2^i})], b_n(M)$ *and the Kervaire invariant of* M;
$[H_{n-1}(M), \mu_M, q(Sq^1_{2^i})], b_n(M)$ *and the Kervaire invariant of* M
by $n = 2, 0, 1(mod4)$ *and* $n = 3(mod4)$ *respectively, where* $a_l = 2$ *for* l *odd,* 1 *for* l *even.* B_n *the* n^{th} *Bernoulli number.* $\alpha(n+2)$ *is the number of* 1*'s in the binary expansion of* $n+2$.

Now let us consider the algebra of the invariants arised in the above Theorem 1.11. Let $\phi : V \to \mathbf{Q}/\mathbf{Z}$ be a quadratic function where V is a $\mathbf{Z}_4$ module with a $\mathbf{Z}_2$-inner product. We say that the inner product is *nonsingular* if the determinant of a matrix representation of the inner product is nonzero. The inner product is of *diagonal zero* if $x \cdot x = 0$ for all $x \in V$. Note that the bilinear form μ_M above for a spin manifold gives rise an inner product on the $\mathbf{Z}_4$-module $H^{n-1}(M, \mathbf{Z}_4)$. Proposition 1.9 says that, when either $n = 2(mod4)$ or $H^{n-1}(M, \mathbf{Z}_4)$ is a free module, this inner product is of diagonal zero. Take the Gauss sum $\lambda(\phi) = \sum_{x \in V} e^{2\pi\phi(x)i} \in \mathbf{C}$ and let $\sigma(\phi) = \sum_{x \in V} \frac{e^{2\pi\phi(x)i}}{8^{\frac{dimV}{2}}}$. The following theorem shows that σ gives rise a perfect invariant.

Theorem 1.12. *Let* V *be a* $\mathbf{Z}_4$*-module with an inner product "$\cdot$" as above. For every quadratic function* ϕ*, define the Arf invariant* $\sigma(\phi) = \sum_{x \in V} \frac{e^{2\pi\phi(x)i}}{8^{\frac{dimV}{2}}}$ *which satisfies*

- (i) $\sigma(\phi) \in \mathbf{Z}_3 \subset \mathbf{C}$
- (ii) $\sigma(\phi_1 \oplus \phi_2) = \sigma(\phi_1)\sigma(\phi_2)$
- (iii) $\sigma(-\phi) = \sigma(\phi)$
- (iv) $\sigma(\phi) = 0$ *if and only if* $2\phi \neq 0$
- (v) $\sigma(\phi) \in \mathbf{Z}_2 \subset \mathbf{Z}_3^*$ *if* $2\phi = 0$
- (vi) *If* V *is nonsingular and of diagonal zero,* $\phi_1 \cong \phi_2$ *if and only if* $\sigma(\phi_1) = \sigma(\phi_2)$
- (vii) $\sigma(\phi) = 0$ *if* V *is singular and there is an* $x \in V$ *such that* $\phi(x) \neq 0$ *and* $x \cdot V = 0$.

Consequently, if h is a homomorphism as in Theorem 1.3, for every $2n$-dimensional ϕ-orientable manifold M, fix a spin structure/Wu orientation by $n = 0, 1(mod4)/2(mod4)$. Endow M with an ϕ-orientation $(\mathbf{u} \circ w, h)$, $n \neq 3(mod4)$. We set $\sigma_h(M) = \sigma(\phi_{M,h}) \in \mathbf{Z}_3$ for the Arf invariant of the quadratic function $\phi_{M,h}$ defined above. If $v_{n+2-2^i} = 0$ for all

i, $\sigma_h(M)$ does not depend on the ϕ-orientation and so gives rise a homotopy invaraint of the spin/Wu oriented manifold M when $n = 0, 1(mod4)/n = 2(mod4)$. If M is stable parallelizable, we always put the standard spin/Wu orientation on M as we mentioned above.

By 1.12 it is readily to obtain

Corollary 1.13. *Let ℓ denote the monoid of the $2n$-dimensional ϕ-orientable manifolds with connected sum as the addition. Then*

$$\sigma_h : \ell \to \mathbf{Z}_3$$

is a nontrivial homomorphism satisfying $\sigma_h(-M) = \sigma(M)$, where the addition on $\mathbf{Z}_3$ is the multiplication.

Let $\Delta_{2n}(G)$ denote the set of homeomorphism(homotopy) types of $(n-2)$-connected $2n$-dimensional stable parallelizable manifolds with nonsingular diagonal zero bilinear forms μ, the $(n-1)$-th homology groups $G = G_0 \oplus \mathbf{Z}_{2^i} \oplus \cdots \oplus \mathbf{Z}_{2^i}$ $(i \geq 2)$and n-th rational Betti number zero, where G_0 is a group of odd order.

Corollary 1.14. *(i) If $n = 2(mod4)$, $\sigma_h : \Delta_{2n}(G) \to \mathbf{Z}_3$ is a bijection.*
(ii) If $n = 0(mod4)$, $\sigma_h : \Delta_{2n}(G) \to \mathbf{Z}_2 \cong \mathbf{Z}_3^* \subset \mathbf{Z}_3$ is a bijection.
(iii) If $n = 3(mod4)$, $\#\Delta_{2n}(G) = 2$.
(iv) If $n = 1(mod4)$, $\#\Delta_{2n}(G) = 5$.

As an example of the application of Theorem 1.11 and the above algebraic facts, we want to construct several manifolds so that they are the annihilator under the connected sum when $n = 2, 3(mod4)$.

(i) If $n = 2(mod4)$, let $S(\zeta)$ be the S^{n-1}-bundle over S^{n+1} with charateristic class $\zeta \in \pi_n(SO(n))$ a generator. We denote by K the resulting manifold of a framed surgery on $S(\zeta)$ to kill $4[S^{n-1}] \in H_{n-1}(S(\zeta))$. Thus for the generator $x \in H^{n-1}(K, \mathbf{Z}_4) \cong \mathbf{Z}_4$, $\phi_M(x) = \frac{1}{4} \in \mathbf{Q}/\mathbf{Z}$ by Theorem 1.8.

(ii) If $n = 3(mod4)$, let K be the resulting manifold of a framed surgery on $2\Delta \subset S^n \times S^n$ where Δ is the diagonal embedding. Note that $q_K(Sq^1 x) = 1$ if $x \in H^{n-1}(K, \mathbf{Z}_2) \cong \mathbf{Z}_2$ is a generator.

Corollary 1.15. *Let $n = 2, 3(mod4)$. Suppose that M and M' are $(n-2)$-connected $2n$-dimensional stable parallelizable manifolds such that $[H_{n-1}(M), \mu_M] \cong [H_{n-1}(M'), \mu_{M'}]$ and $b_n(M) = b_n(M')$. Then $M\#K$ is homeomorphic to $M'\#K$.*

The organization of this paper is as the follows. In §2, we give some preparations on stable homotopy theory of the Eilenberg-Maclane spaces. In §3, we are addressed to show the results 1.2 through 1.9 in §1. §4 is devoted to the classification of $(n-2)$-connected $2n$-manifolds and prove the Theorem 1.10 through 1.14. The last section, §5, is going to discuss some algebraic invariants arised from these quadratic functions defined here.

§2. Some preliminary on the stable homotopy of $K(\pi, n-1)$

The purpose of this section is to calculate the stable homotopy groups $\pi_{2n}^s(K(\pi, n-1))$(see theorem 2.1), build the 2-stage Postnikov tower for $\Sigma^q K(\mathbf{Z}_4, n-1)$ where q is large(see Prop 2.8). This leads to several secondary cohomology operations mentioned in §1 which is crucial in this paper.

Theorem 2.1. *The $2n$-th stable homotopy group of $K(\pi, n-1)$ for $n \geq 4$ is as the following table:*

$n \geq 4$	$0(mod4)$	$1(mod4)$	$2(mod4)$	$3(mod4)$
$\pi_{2n}^s(K(\pi, n-1))$	$(\mathbf{Z}_2)^{2(m+k)+s+p}$	$(\mathbf{Z}_2)^{m+2k+s+p}$	$(\mathbf{Z}_4)^{m+k} \oplus (\mathbf{Z}_2)^{s+p}$	$(\mathbf{Z}_2)^{k+s+p}$

where $p = \binom{m+k+s}{2}$ and $\pi = G_0 \times \mathbf{Z}^m \times \mathbf{Z}_{2^{i_1}} \times \cdots \times \mathbf{Z}_{2^{i_k}} \times \mathbf{Z}_2^s$, here $i_j \geq 2$ for $1 \leq j \leq k$ and $G_0 \otimes \mathbf{Z}_2 = 0$.

Remark 2.2. In the case of $\pi = \mathbf{Z}$, Theorem 2.1 can be read out from [16].

Recall that for each locally finite connected CW complex X, one can form a space, namely

$$\Gamma_m X = S^{m-1} \propto_T X \wedge X = S^{m-1} \times (X \wedge X)/\{(x,y,z) \sim (-x,z,y); (x,*) \sim *\}$$

for every $m \in \mathbf{Z}_+$. By Milgram [18] Theorem 1.11, for a $(n-1)$-connected X, $\Gamma_m X$ is $(2n-1)$-connected. Moreover, if $X = K(\pi, n)$, we have a fibration

$$G_m \to \Sigma^m K(\pi, n) \to K(\pi, m+n)$$

where $G_m \simeq \Sigma^m \Gamma_m(K(\pi, n))$ through dimension $(3n+m-1)$. Thus $\pi_i^s(K(\pi, n)) \cong \pi_i^s(\Gamma_m(K(\pi, n))$ for $n < i < 3n-1$. There are maps

$$J : X \wedge X \to S^m \propto_T X \wedge X,$$
$$K : S^m \propto_T X \wedge X \to \Sigma^m X \wedge X.$$

where J is the inclusion map and K is defined by identifying $S^{m-1} \propto_T X \wedge X$ to a point in $S^m \propto_T X \wedge X$. For reader's convienence we recall that

Proposition 2.3(Milgram[18])*(i) J^* is surjective onto the invariant subalgebra under T^* of $H^*(X \wedge X, Z_p)$ for p an odd prime. Moreover, $ker J^* = im K^*$ and the following sequence is exact:*

$$H^*(\Sigma^m X . \wedge X, Z_p) \xrightarrow{1+(-1)^m(\Sigma^m T)^*} H^*(\Sigma^m X \wedge X, Z_p) \to im K^* \to 0.$$

(ii) (mod2) J^* is surjective as in (i), the sequence above is again exact, but there are additional elements $e^i \cup (\theta \otimes \theta)$ for $1 \le i \le m$ where $\theta \in \widetilde{H}^*(X, \mathbf{Z}_2)$, and these completely describe $H^*(\Gamma_m X, \mathbf{Z}_2)$.

Proof of Theorem 2.1: Throughout this proof we assume that m is large with respect to n. We divide the proof into the following five steps.
Step (I): $\pi^s_{2n}(K(G_0, n-1)) = 0$.
Consider the Atiyah-Hirzebruch Spectral sequence converging to $\pi^s_{2n}(\Gamma_m(K(G_0, n-1)) \cong \pi^s_{2n}(K(G_0, n-1))$. By the Proposition 2.3 one knows easily that all E_2-terms of the AHSS are zero and so $\pi^s_{2n}(K(G_0, n-1)) = 0$.

Step(II): For $i \ge 2$ and n even, $\pi^s_{2n}(K(\mathbf{Z}_{2^i}, n-1)) \cong (\mathbf{Z}_2)^2$, $\mathbf{Z}_4$ by $n = 0(mod4)$ and $2(mod4)$ respectively.

The E_2-terms of the AHSS converging to $\pi^s_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))$ are:

$$\begin{aligned} E^2_{2n-2,2} &= H_{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \cong \mathbf{Z}_2 \\ E^2_{2n-1,1} &= H_{2n-1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ E^2_{2n,0} &= H_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}) \cong (\mathbf{Z}_2)^2 \end{aligned}$$

if n is even. The former two isomorphisms follow from the Proposition 2.3 directly. To see the last one, note that

(1) $H_{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))) \cong \mathbf{Z}_2 \cong \pi_{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))) \cong \pi^s_{2n-2}(K(\mathbf{Z}_{2^i}, n-1))$.
(2) $Sq^2 : H^{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \to H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$ is nonzero and $\pi^s_{2n-1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))) \cong Z_{2^{i+1}}$(c.f [18]). By the Whitehead exact sequence(c.f: [23] p555) we have

$$H_{2n-1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))) \cong Z_{2^{i+1}}.$$

By the Bockstein exact sequence associated to $0 \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to 0$ we obtain that the order

$$|H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_4)| = 2|H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)|.$$

Combining these with the universal coefficients Theorem we conclude that

$$H_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))$$

consists only some $\mathbf{Z}_2$-direct summands and so we obtain that $E^2_{2n,0} \cong (\mathbf{Z}_2)^2$.

Note the differential $d_2 : E^2_{2n,1} \to E^2_{2n-2,2}$ is dual to

$$Sq^2 : H^{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \to H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$$

which is nonzero by [18] Proposition 3.7 and so $E^2_{2n-2,2}$ does not survive in the E^∞-term. Thus $E^\infty_{2n-2,2} = 0$.

Similarly, the differential $d_2 : E^2_{2n,0} \to E^2_{2n-2,1}$ is the composition of the dual $(Sq^2)_*$ of Sq^2 and $\rho_2 : H_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), Z) \to H_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$, the mod 2 reduction. Let $\theta \in H^{n-1}(K(\mathbf{Z}_{2^i}, n-1), \mathbf{Z}_2)$ be the generator, $Sq^2(\theta \otimes \theta) \neq 0$, thus the element in $E^2_{2n,0}$ reduced to the dual of $Sq^2\theta \otimes \theta + \theta \otimes Sq^2\theta$ does not survive in E^∞ and so $E^\infty_{2n,0} \cong \mathbf{Z}_2$.

The differential $d_2 : E^2_{2n+1,0} \to E^2_{2n-1,1}$ is the composition of the dual of

$$Sq^2 : H^{2n-1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \to H^{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$$

and the mod 2 reduction

$$\rho : H_{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), Z) \to H_{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2).$$

Note that $Sq^2(Sq^1_i\theta \otimes \theta + \theta \otimes Sq^1_i\theta)$ evaluates on $H_{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z})$ is nonzero, and therefore the dual of $Sq^1_i\theta \otimes \theta + \theta \otimes Sq^1_i\theta$ does not survive in E^∞-term. On the other hand, the dual of $e^1 \cup (\theta \otimes \theta)$ survives in E^∞. This follows from the fact

$$Sq^2(e^1 \cup (\theta \otimes \theta)) = \binom{n}{2} e^3 \cup (\theta \otimes \theta)$$

and when $n = 0(mod4)$, $\binom{n}{2} = 0$.
When $n = 2(mod4)$, the projection of $e^3 \cup (\theta \otimes \theta) \in H^{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$ to the summand $Hom(H_{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1))), \mathbf{Z}_2)$ in the universal coefficients theorem evaluates at $H_{2n+1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)))$ is zero.

In sum, $\pi^s_{2n}(K(\mathbf{Z}_{2^i}, n-1))$ is of order 4 whenever $i \geq 2$. By comparing the Atiyah-Hirzebruch spectral sequences of $\pi^s_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)))$ and $\pi^s_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)))$ one knows further that the forgetful homomorphism

$$\rho_i : \pi^s_{2n}(\Gamma_m(K(\mathbf{Z}, n-1))) \to \pi^s_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)))$$

is an isomorphism. Recall that[17] $\pi^s_{2n}(\Gamma_m(K(\mathbf{Z}, n-1))) \cong \mathbf{Z}_4$ or $(\mathbf{Z}_2)^2$ by $n = 2(mod4)$ or $0(mod4)$. This concludes the Step (II).

Step (III): For $i \geq 2$ and n odd, $\pi^s_{2n}(K(\mathbf{Z}_{2^i}, n-1)) \cong (\mathbf{Z}_2)^2$, $\mathbf{Z}_2$ by $n = 1(mod4)$ and $3(mod4)$ respectively.

We give only the proof in the case of $n = 1(mod4)$. The rest case is easier and similar.
First of all, by [18] p77-80, $H_{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)) \cong \mathbf{Z}_{2^i}$ and $H_{2n-1}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)) \cong Z_2$ if n odd. By Proposition 2.3

$$H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \cong (Z_2)^3$$

with generators $e^2 \cup (\theta \otimes \theta)$, $Sq_i^1\theta \otimes Sq_i^1\theta$ and $Sq^2\theta \otimes \theta + \theta \otimes Sq^2\theta$. The relations $Sq^1(e^1 \cup (\theta \otimes \theta)) = e^2 \cup (\theta \otimes \theta)$, where $\dim\theta = n-1$ even, $Sq^1(Sq_i^1\theta \otimes Sq_i^1\theta) = e^1 \cup Sq_i^1\theta \otimes Sq_i^1\theta \neq 0$ and the Bockstein exact sequence shows that $H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_4)$ is of order 8, same as the order of $H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$ and we conclude that

$$H_{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}) \cong (\mathbf{Z}_2)^2.$$

Note that the term $E^2_{2n-2,2} = H_{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \cong \mathbf{Z}_2$ does not survive in the E^3-term since the differential $d_2 : E^2_{2n,1} \to E^2_{2n-2,2}$ is dual to

$$Sq^2 : H^{2n-2}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2) \to H^{2n}(\Gamma_m(K(\mathbf{Z}_{2^i}, n-1)), \mathbf{Z}_2)$$

which is nonzero. Similarly, $E^\infty_{2n,0} \cong \mathbf{Z}_2$ since the differential $d_2 : E^2_{2n,0} \to E^2_{2n-2,1} \cong \mathbf{Z}_2$ is nonzero. $E^\infty_{2n-1,1} \cong \mathbf{Z}_2$ or 0 by $i \geq 2$ or $i = 1$ respectively. By comparing the Atiyah-Hirzebruch spectral sequences it follows that, the forgetful homomorphism

$$\rho : \mathbf{Z}_2 \cong \pi^s_{2n}(K(\mathbf{Z}, n-1)) \to \pi^s_{2n}(K(\mathbf{Z}_{2^i}, n-1))$$

is an injection with image consisting of the $E^\infty_{2n-1,1}$-term in the AHSS if $i \geq 2$, and zero if $i = 1$.

$$i_2 : \pi^s_{2n}(K(\mathbf{Z}_2, n-1)) \to \pi^s_{2n}(K(Z_{2^i}, n-1))$$

is an injection with image consists of the $E^\infty_{2n,0}$-term. Thus $\pi^s_{2n}(K(Z_{2^i}, n-1)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Step (IV): $\pi^s_{2n}(K(\mathbf{Z}_2, n-1)) \cong \mathbf{Z}_2$ for $n \geq 4$.
This can be read out from the table in Milgram [18]page 77.

Step (V): $\pi^s_{2n}(K(\mathbf{Z}_{2^i}, n-1) \wedge K(Z_{2^j}, n-1)) \cong \mathbf{Z}_2$ if $i, j \geq 1$ or ∞, where $\mathbf{Z}_\infty := \mathbf{Z}$.

$K(\mathbf{Z}_{2^i}, n-1) \wedge K(_{2^j}, n-1)$ is $(2n-3)$-connected. For $n \geq 4$, the $2n$-th homotopy group of $K(\mathbf{Z}_{2^i}, n-1) \wedge K(\mathbf{Z}_{2^j}, n-1)$ is already in the stable range. By AHSS it is easy to be verified.

Combining the steps (I) to (V) the proof of Theorem 2.1 follows.

Remark 2.4. By the AHSS as in the proof of Theorem 2.1 it is readily to see that: if $\alpha \in \pi_{2n}(K(\mathbf{Z}_{2^i}, n-1) \wedge K(\mathbf{Z}_{2^j}, n-1))$ is a generator, then $\alpha^*(l \otimes Sq^2 l) \neq 0$ where l is the basic class.

Now we want to build the Postnikov towers for $\Sigma^q K(\mathbf{Z}_4, n-1) \to K(\mathbf{Z}_4, q+n-1)$ which will be used in §3 to show the Theorem 1.3 and 1.9, where $n \neq 3(mod 4)$. The case of $n = 3(mod 4)$ is not interested for our purpose. Similar cases were considered in [17].

Proposition 2.5. For q large, the 2-stage Postnikov tower of $\Sigma^q K(\mathbf{Z}_4, n-1)$ is as the follows:

(1): $n = 2(mod4)$

$$\begin{array}{ccccc}
K_{q+2n} & \xrightarrow{i_2} & E_2 & & \\
 & & \downarrow \Pi_2 & & \\
K_{q+2n-2} \times K_{q+2n} & \xrightarrow{i_1} & E_1 & \xrightarrow{\omega_2} & K_{q+2n+1} \\
 & & \downarrow \Pi_1 & & \\
\Sigma^q K(\mathbf{Z}_4, n-1) & \xrightarrow{\Sigma^q l_{n-1}} & K(\mathbf{Z}_4, q+n-1) & \xrightarrow{Sq^n \times Sq^{n+2}} & K_{q+2n-1} \times K_{q+2n+1}
\end{array}$$

where $i_1^*(\omega_2) = Sq^2 Sq^1 l_{q+2n-2} + Sq^1 l_{q+2n}$.

(2): $n = 0(mod4)$

$$\begin{array}{ccccc}
K_{q+2n} & \xrightarrow{i_2} & E_2 \times K_{q+2n} & & \\
 & & \downarrow \Pi_2 & & \\
K_{q+2n-2} & \xrightarrow{i_1} & E_1 \times K_{q+2n} & \xrightarrow{\omega_2} & K_{q+2n+1} \\
 & & \downarrow \Pi_1 & & \\
\Sigma^q K(\mathbf{Z}_4, n-1) & \xrightarrow{\Sigma^q l_{n-1} \times \Sigma^q (Sq^2 l_{n-1} \cup l_{n-1})} & K(\mathbf{Z}_4, q+n-1) \times K_{q+2n} & \xrightarrow{(Sq^n, 0)} & K_{q+2n-1}
\end{array}$$

where $i_1^*(\omega_2) = Sq^2 Sq^1 l_{q+2n-2}$.

(3): $n = 1(mod4)$

$$\begin{array}{ccccc}
K_{q+2n} & \xrightarrow{i_2} & E_2 & & \\
 & & \downarrow \Pi_2 & & \\
\Omega C & \xrightarrow{i_1} & E_1 & \xrightarrow{\omega_2} & K_{q+2n+1} \\
 & & \downarrow \Pi_1 & & \\
\Sigma^q K(\mathbf{Z}_4, n-1) & \xrightarrow{\Sigma^q l_{n-1}} & K(\mathbf{Z}_4, q+n-1) & \xrightarrow{\beta \times Sq^{n+1} \times Sq^{n+1} Sq_2^1} & C
\end{array}$$

where $C = K(\mathbf{Z}_4, q+2n-1) \times K_{q+2n} \times K_{q+2n+1}$, $i_1^*(\omega_2) = Sq^2 l_{q+2n-1}$ and $\beta(mod2) = Sq^n l_{q+n-1}$.

Proof: We give only a complete proof in the case of $n = 2(mod4)$. Other cases are similar. There is a fibration $\Sigma^q \Gamma_q \xrightarrow{h} \Sigma^q K(\mathbf{Z}_4, n-1) \xrightarrow{\Sigma^q l_{n-1}} K(\mathbf{Z}_4, n+q-1)$(c.f: page 10 for the notation and reference), Γ_q is $(2n-3)$-connected. Let $\theta \otimes \theta \in H^{2n-2}(\Gamma_q, \mathbf{Z}_2)$ be a generator(c.f: Prop 2.3) which is spherical. The transgression of $\theta \otimes \theta$ is $Sq^n l_{n+q-1}$ by the Serre exact sequence. Recall that $H_{2n}(\Gamma_q) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. The Whitehead exact sequence(c.f: [23] p555) applies to show that, there is exactly a spherical element in $H_{2n}(\Gamma_q)$. It is not hard to check that, there is a dual of this elemnt in $H^{2n}(\Gamma_q, \mathbf{Z}_2)$, namely z, so that its transgression is $Sq^{n+2} l_{n+q-1}$. Therefore we may build the first stage tower as

$$\begin{array}{ccccc}
K_{q+2n-2} \times K_{q+2n} & \xrightarrow{i_1} & E_1 & & \\
 & & \downarrow \Pi_1 & & \\
\Sigma^q K(\mathbf{Z}_4, n-1) & \xrightarrow{\Sigma^q l_{n-1}} & K(\mathbf{Z}_4, q+n-1) & \xrightarrow{Sq^n \times Sq^{n+2}} & K_{q+2n-1} \times K_{q+2n+1}
\end{array}$$

Note that, $H^{q+2n}(E_1)/im\Gamma_1^* \cong \mathbf{Z}_2$ with a representative ω_1, $i_1^*\omega_1 = Sq^2 l_{q+2n-2}$. ω_1 gives rise a secondary cohomology operation ϕ_0 associated with the Adem relation $Sq^2Sq^n = 0$ with $\mathbf{Z}_4$-coefficients. We want to prove that, there is a lifting f_1 of $\Sigma^q l_{n-1}$ so that $f_1^*(\omega_1) = \Sigma^q(Sq^2 l_{n-1} \cup l_{n-1})$. To see this, consider the fibre inclusion map $h : \Sigma^q\Gamma_q \to \Sigma^q K(\mathbf{Z}_4, n-1)$, by Peterson-Stein [20] we obtain that

$$Sq^2 Sq^n_h(\Sigma^q l_{n-1}) = \phi_0(h^*\Sigma^q l_{n-1}) \in H^{q+2n}(\Sigma^q\Gamma_q, \mathbf{Z}_2)/Sq^2(imh^*).$$

Obviously $\Sigma^q(\theta \otimes \theta) \in Sq^n_h(\Sigma^q l_{n-1})$ and so $Sq^2(\Sigma^q(\theta \otimes \theta)) \in \phi_0(h^*\Sigma^q l_{n-1})$. Notice now the indeterminacy is zero. By the naturality of the cohomology operation ϕ_0 we have that $h^*\phi_0(\Sigma^q l_{n-1})$ is nonzero and therefore $\Sigma^q(Sq^2 l_{n-1} \cup l_{n-1}) \in \phi_0(\Sigma^q l_{n-1})$ since it is the only nontrivial element in the cokernel of $(\Sigma^q l_{n-1})^*$. Thus there is a lifting f_1 so that $f_1^*(\omega_1) = \Sigma^q(Sq^2 l_{n-1} \cup l_{n-1})$ as we claimed.

Notice that $H^{q+2n+1}(E_1, \mathbf{Z}_2)/im\Gamma_1^* \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with representatives $Sq^1\omega_1$ and ω_2, where $i_1^*\omega_2 = Sq^2Sq^1 l_{2n+q-2} + Sq^1 l_{2n+q}$. ω_2 gives rise a secondary cohomolgy operation, namely ψ. By using the Perterson-Stein formula as above we have $0 \in \psi(\Sigma^q l_{n-1})$ and so we may choose the lifting f_1 so that ω_2 lies in the kernel of f_1^*. The second stage of the Postnikov tower may be built by killing ω_2. This completes the proof for the case of $n = 2(mod4)$.
Notice in the case of $n = 0(mod4)$, there is no a similar secondary cohomology operation to capture the element $\Sigma^q(Sq^2 l_{n-1} \cup l_{n-1})$ and so it must be added at the first stage. The rest and the case of $n = 1(mod4)$ are similar. ♣

The 2^{nd} k-invariant ω_2 in the Postnikov tower above gives a unique secondary cohomology operator ψ (with $\mathbf{Z}_4$-coefficients) associated with the Adem relation ·

$$\begin{array}{ll} Sq^2Sq^1Sq^n + Sq^1Sq^{n+2} = 0 & n = 2(mod4) \\ Sq^2Sq^1Sq^n = 0 & n = 0(mod4) \\ Sq^2Sq^{n+1} = 0 & n = 1(mod4) \end{array}$$

If n is even, E_1 is the universal example of the operator ψ. If $n = 1(mod4)$, we write E_1' for the universal example of ψ which is the 1-stage Postnikov tower over $K(\mathbf{Z}_4, q+n-1)$ with k-invariant $Sq^{n+1}l_{q+n-1}$. Also we let E_2' for the fibre space over E_1' with an k-invariant given by the operator ψ.

By Peterson-Stein[19], there are operators ϕ which are S-dual to ψ(which is unique determined by ψ) so it is a secondary operator associated with the Adem relations:

$$\begin{array}{ll} \chi(Sq^n)Sq^3 + \chi(Sq^{n+2})Sq^1 + Sq^1\chi(Sq^{n+2}) = 0 & n = 2 \bmod 4 \\ \chi(Sq^n)Sq^3 + Sq^1\chi(Sq^{n+2}) = 0 & n = 0 \bmod 4 \\ \chi(Sq^{n+1})Sq^2 + Sq^1\chi(Sq^{n+2}) = 0 & n = 1 \bmod 4 \end{array}$$

as we stated in §1.

Let $q_2 : \Sigma^q K(\mathbf{Z}_4, n-1) \to E_2$ denote a factor in the Moore-Postnikov factorization of $\Sigma^q K(\mathbf{Z}_4, n-1) \to K(\mathbf{Z}_4, q+n-1)$ above. Notice that q_2 induces an isomorphism at the $(q+2n)$-th homotopy groups if $n \neq 0, 3(mod4)$, and an epimorphism if $n = 0(mod4)$.

Consider the following immediate commutative diagram:

$$\begin{array}{ccccc} \Sigma^q K(\mathbf{Z}_4, n-1) & \longrightarrow & K(\mathbf{Z}_4, q+n-1) & \xrightarrow{Sq^{n+1}Sq_i^1} & K_{q+2n+1} \\ \downarrow \Sigma^q(Sq_i^1) & & \downarrow Sq_2^1 & & \| \\ \Sigma^q K_n & \longrightarrow & K_{q+n} & \xrightarrow{Sq^{n+1}} & K_{q+2n+1} \end{array}$$

Let $E_0 \to K_{q+n}$ be the principal fibration with k-invariant Sq^{n+1}. Recall that [4] E_0 is the first stage Postnikov-Moore factorization of $\Sigma^q K_n \to K_{q+n}$. When $n = 1(mod4)$, by the above Proposition $Sq_2^1 : K(\mathbf{Z}_4, n+q-1) \to K_{n+q}$ can be covered by a map $f : E_1 \to E_0$ which induces an epimorphism at the $(2n+q)$-th homotopy groups. By the commutative diagram above, $\Sigma^q(Sq_2^1)_* : \pi_{2n+q}(\Sigma^q K(\mathbf{Z}_4, n-1)) \to \pi_{2n+q}(\Sigma^q K_n)$ is an epimorphism. The follows gives geometric proof of this fact more generally.

Proposition 2.6. *Let n be odd and $Sq_i^1 \in H^n(K(Z_{2^i}, n-1), \mathbf{Z}_2)$ denote the generator. Then*

$$(Sq_i^1)_* : \pi_{2n}^s(K(\mathbf{Z}_{2^i}, n-1)) \to \pi_{2n}^s(K(\mathbf{Z}_2, n)) \cong \mathbf{Z}_2$$

is an epimorphism.

Proof: We give a proof by using differentail topological method here.

Identifying the reduced framed bordism group $\tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_2, n-1))$ with $\pi_{2n}^s(K(\mathbf{Z}_2, n-1))$, and $\tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_2, n))$ with $\pi_{2n}^s(K(\mathbf{Z}_2, n))$. Under the Thom Pontryagin construction it is directly to see that the homomorphism

$$(Sq^1)_* : \pi_{2n}^s(K(\mathbf{Z}_2, n-1)) \to \pi_{2n}^s(K(\mathbf{Z}_2, n))$$

can be identified with the homomorphism

$$\tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_2, n-1)) \to \tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_2, n)) \cong \mathbf{Z}_2$$

$$[M, x] \to [M, Sq^1 x]$$

By Brown [4], if M is the boundary of a framed manifold V and $i : M \to V$ is the inclusion, the Kervaire quadratic form $q : H^n(M, \mathbf{Z}_2) \to \mathbf{Z}_2$ is zero on $Im i^*$. Thus

$$q(Sq^1) : \tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_2, n-1)) \to \mathbf{Z}_2,$$

$$[M, x] \to q(Sq^1 x)$$

is well defined and a homomorphism since $Sq^1 x \cup Sq^1 y = Sq^1(x \cup Sq^1 y) = v_1 \cup (x \cup Sq^1 y) = 0$ by the Wu class $v_1 = 0$. Therefore the composition $q(Sq^1)$ is a homomorphism. To show the Proposition, it suffices to prove that $q(Sq^1)$ defined above is an isomorphism.

Assuming that $[M, x]$ is a framed bordism class, where M denote a framed manifold of dimension $2n$ and $x \in H^{n-1}(M, \mathbf{Z}_2)$. By using framed surgery we can assume that M is $(n-2)$-connected, and $H_{n-1}(M)$ is a cyclic group of order 2 and $x \in H^{n-1}(M, \mathbf{Z}_2)$ is a generator. Note that $\beta(x) \in H^n(M, \mathbf{Z})$ is the generator of the torsion subgroup, here β is the Bockstein homomorphism. By [4], the Poincaré dual of $\beta(x)$ is represented by an embedded n-sphere with trivial normal bundle if and only if $q(Sq^1 x) = 0$ as $Sq^1 x = \beta(x)(mod 2)$. In case it can be represented, we may do framed surgery on M by using this embedded sphere to obtain a $(n-2)$-connected manifold, namely N, so that $H_{n-1}(N) \cong \mathbf{Z}$. By Theorem 2.1, $\pi_{2n}^s(K(\mathbf{Z}, n-1)) = 0$ and $\mathbf{Z}_2$ by $n = 3(mod4)$ and $1(mod4)$. Thus the homomorphism $q(Sq^1)$ is injective if $n = 3(mod4)$. For $n = 1(mod4)$, as we have seen in the proof of Theorem 2.1, the forgetful homomorphism $\pi_{2n}^s(K(\mathbf{Z}, n-1)) \to \pi_{2n}^s(K(\mathbf{Z}_2, n-1))$ is zero. Therefore $[M, x]$ is bordant zero if $q(Sq^1 x) = 0$ and the above homomorphism $q(Sq^1)$ is injective too. Note that the group $\pi_{2n}^s(K(\mathbf{Z}_2, n-1)) \cong \mathbf{Z}_2$ and so $q(Sq^1)$ is an isomorphism. Note that $Sq^1 = Sq_i^1 \circ i$, where $i : K(\mathbf{Z}_2, n-1) \to K(\mathbf{Z}_{2^i}, n-1)$ represents the nonzero homology class. Thus $q(Sq_i^1) : \Omega_{2n}^{fr}(K(Z_{2^i}, n-1)) \to \mathbf{Z}_2$ is an epimorphism. This completes the proof.

From the proof of the above Proposition we have an immediate corollary
Corollary 2.7. *Let M be a framed manifold of dimension $2n$. If $q_M : H^n(M, \mathbf{Z}_2) \to \mathbf{Z}_2$ is the Kervaire quadratic form. For $x \in H^{n-1}(M, Z_{2^i})$,*
(i): $n = 3(mod4)$, $[M, x]$ is reduced bordant to zero if and only if $q(Sq_i^1)x = 0$.
(ii): $n = 1(mod4)$, $[M, x]$ is reduced bordant to $[M', x']$ where $x' \in H^{n-1}(M', \mathbf{Z})$ if and only if $q_M(Sq_i^1)(x) = 0$.

Now we are going to explain that, for a reduced framed bordism class $[M, x] \in \tilde{\Omega}_{2n}^{fr}(K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where M is a framed manifold of dimension $2n(n = 0, 1(mod4))$ and $x \in H^{n-1}(M, \mathbf{Z}_4)$, there is a $\mathbf{Z}_2$-component may be detected by a well-known invariant and the rest is detected by the quadratic function defined in §1. Let $PT : \tilde{\Omega}_{2n}(K(\mathbf{Z}_4, n-1)) \to \pi_{2n}^s(K(\mathbf{Z}_4, n-1))$ denote the Thom-Pontryagin isomorphism. If $n = 0(mod4)$, by the Postnikov tower in 2.5, $PT([M, x])$ has a component detected by $x \cup Sq^2 x[M]$ and the other corresponds to the nontrivial k-invariant. If $n = 1(mod4)$, by 2.7, $PT([M, x])$ has a component detected by $q(Sq_2^1 x)$ where q is the Kervaire quadratic form. The rest is detected by the nontrivial k invariant corresponding to the secondary cohomology operation ψ.

Recall that $\lambda_0 \in Im(H(\kappa)_*) \subset \pi_{2n}^s(K(\mathbf{Z}_4, n-1))$ is a generator of order 2 if $n = 0, 1(mod4)$. We shall prove that $\lambda_0^*(\sigma(Sq^2 l \cup l)) = 0$ if $n = 0(mod4)$, and $(Sq_2^1)_*(\lambda_0) = 0$ if $n = 1(mod4)$. Therefore λ_0 and the component detected by the cup product $x \cup Sq^2 x$ or $q(Sq_2^1)$ form a basis for the group $\pi_{2n}^s(K(\mathbf{Z}_4, n-1))$ if $n = 0(mod4)$ or $1(mod4)$.

Proposition 2.8. *Let $n = 0, 1(mod4)$ and λ_0 be the generator of $Im(H(\kappa)_*)$, where $H(\kappa)$ is the Hopf construction of the multiplication $\kappa : K(\mathbf{Z}_4, n-1) \times K(\mathbf{Z}_4, n-1) \to K(\mathbf{Z}_4, n-1)$. Then $\lambda_0^*(\sigma(Sq^2 l \cup l)) = 0$ if $n = 0(mod4)$ and $(Sq_2^1)_*(\lambda_0) = 0$ if $n = 1(mod4)$, where l*

is the basic class of $K(\mathbf{Z}_4, n-1)$, σ is the suspension and $Sq_2^1 \in H^n(K(\mathbf{Z}_4, n-1), \mathbf{Z}_2)$ is the generator.

Proof: Note that $H(\kappa)^*(\sigma(Sq^2 l \cup l)) = \sigma(Sq^2 l \otimes l + l \otimes Sq^2 l)$. By the Remark 2.4, $\lambda_0^*(\sigma(Sq^2 l \cup l)) = \alpha^*(\sigma(Sq^2 l \cup l)) + \alpha^*(\sigma(l \cup Sq^2 l)) = 0$. To show $(Sq_2^1)_*(\lambda_0) = 0$ for $n = 1(mod 4)$, note that λ_0 lies in the image of the forgetful homomorphism $\rho_* : \pi_{2n}^s(K(Z, n-1)) \to \pi_{2n}^s(K(\mathbf{Z}_4, n-1))$ by the commutative diagram

$$\begin{array}{ccc} \sum K(\mathbf{Z}, n-1) \wedge K(\mathbf{Z}, n-1) & \xrightarrow{\rho \wedge \rho} & \sum K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1) \\ \downarrow H(\kappa) & & \downarrow H(\kappa) \\ \sum K(\mathbf{Z}, n-1) & \xrightarrow{\Sigma\rho} & \sum K(\mathbf{Z}_4, n-1). \end{array}$$

Combining corollary 2.7 we conclude the proof.

§3. Proofs

Proof of Proposition 1.2: Recall that, for any space X, the sequence of maps

$$S^1 \wedge \Omega X \wedge \Omega X \xrightarrow{h} S^1 \wedge \Omega X \xrightarrow{\alpha} X$$

is a fibration, where h is the Hopf construction of the multiplication $\Omega X \wedge \Omega X \to \Omega X$, and α is the adjoint of the identity on ΩX. Thus

$$S^1 \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1) \xrightarrow{h} S^1 \wedge K(\mathbf{Z}_4, n-1) \xrightarrow{\alpha} K(\mathbf{Z}_4, n)$$

is a fibration and so

$$h_* : \pi_{2n+1}(S^1 \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1)) \to \pi_{2n+1}(S^1 \wedge K(\mathbf{Z}_4, n-1))$$

is an isomorphism. By theorem 2.1, $\pi_{2n+1}(S^1 \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$.

Consider the suspension map $S^1 \wedge K(\mathbf{Z}_4, n-1) \to S^l \wedge K(\mathbf{Z}_4, n-1)$ for l large. Applying the generalized EHP sequence(c.f: [18]) in our range

$$\cdots \to \pi_{2n+1}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \xrightarrow{E} \pi_{2n+1}^s(S^1 \wedge K(\mathbf{Z}_4, n-1)) \xrightarrow{H} \pi_{2n+1}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1))$$
$$\xrightarrow{\partial} \pi_{2n}(S^1 \wedge K(\mathbf{Z}_4, n-1) \xrightarrow{E} \pi_{2n}^s(S^1 \wedge K(\mathbf{Z}_4, n-1)),$$

Notice that $H_{2n}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ and $H_{2n+1}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$ when n is even. Thereby $\pi_{2n+1}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$ and consequently E is injective by the exact sequence above as $\pi_{2n+1}^s(S^1 \wedge K(\mathbf{Z}_4, n-1))$ is of order 4 when n is even.

When $n = 1(mod 4)$, by Theorem 2.1 $\pi_{2n}^s(K(\mathbf{Z}_4, n-1)) \cong (\mathbf{Z}_2)^2$. On the other hand, it is readily to show that $\pi_{2n+1}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1))$ is cyclic. Thus the conclusion follows by applying the exact sequence again.

When $n = 3(mod4)$, E is zero by comparing that $\pi_{2n+1}(\Gamma_{l-1}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$, $\pi_{2n}(S^1 \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ and $\pi^s_{2n-1}(K(\mathbf{Z}_4, n-1) \cong \mathbf{Z}_2$ in the exact sequence. This completes the proof. ♣

Remark. By a similar argument one can show that, the Hopf construction $h(\kappa) : S^1 \wedge K(\mathbf{Z}_2, n-1) \wedge K(\mathbf{Z}_2, n-1) \to S^1 \wedge K(\mathbf{Z}_2, n-1)$ induces always a zero homomorphism on the $2n$-th stable homotopy groups. Thus the analogous definition of the quadratic function ϕ_h in §1 gives a linear function if $K(\mathbf{Z}_4, n-1)$ is replaced by $K(\mathbf{Z}_2, n-1)$.

Proof of Theorem 1.3. We give a proof in the case of $n = 2(mod4)$. Others are similar by using except one need to use Proposition 2.8. First note that the theorem is equivalent to say that

$$i_* : \mathbf{Z}_4 \cong H_{2n}(K(\mathbf{Z}_4, n-1), S^0) \to H_{2n}(K(\mathbf{Z}_4, n-1), Y)$$

is a monomorphism. Also it suffices to show this for the universal spectrum $\tilde{W}(n)$ since the map $i : S^0 \to \tilde{W}(n)$ factors through $i : S^0 \to Y$. Notice that $\tilde{W}_k(n)/S^k$ is $(k+2)$-connected. Thus in the following proof, we may assume that Y_k/S^k is $(k+2)$-connected for k large. Assuming k large, without lossing generality we can assume that Y_k is a finite complex. Write Y_k^* for the m S-dual of Y_k and $g : Y_k^* \to S^{m-k}$ for the S-dual of the inclusion $i : S^k \to Y_k$. Note that $g^*(\varsigma_{S^{m-k}}) \neq 0$, where $\varsigma_{S^{m-k}}$ is the cohomology fundamental class of the sphere. By the S-duality we get a commutative diagram

$$\begin{array}{ccc}
\{S^{2n+k}, S^k \wedge K(\mathbf{Z}_4, n-1)\} & \xrightarrow{i_*} & \{S^{2n+k}, Y_k \wedge K(\mathbf{Z}_4, n-1)\} \\
\downarrow \cong & & \downarrow \cong \\
\{S^{2n+m}, S^m \wedge K(\mathbf{Z}_4, n-1)\} & \xrightarrow{g^*} & \{S^{2n+k} \wedge Y_k^*, S^m \wedge K(\mathbf{Z}_4, n-1)\} \\
\downarrow \cong & & \downarrow q_{2*} \\
[S^{2n+m}, \tilde{E}_{n+m-1}] & \xrightarrow{g^*} & [S^{2n+k} \wedge Y_k^*, \tilde{E}_{n+m-1}],
\end{array}$$

where $\tilde{E}_{m+n-1}$ is the 2-stage Postnikov tower in Proposition 2.5 and $q_2 : S^m \wedge K(\mathbf{Z}_4, n-1) \to \tilde{E}_{n+m-1}$ is a factor of the Moore-Postnikov decomposition of $\Sigma^m l_{n-1}$. From the diagram above it suffices to show that the homomorphism g^* at the bottom line is injective. Let $i_0 : F \to \tilde{E}_{n+m-1}$ be the fibre of the 2-stage Postnikov tower. Note that F can be viewed as a fibration over K_{2n+m-2} with fibre $K(\mathbf{Z}_4, 2n+m)$ and k-invariant $j_*(Sq^2Sq^1)(l)$; where

$$j_* : H^{m+2n+1}(-, \mathbf{Z}_2) \to H^{m+2n+1}(-, \mathbf{Z}_4)$$

is the homomorphism induced by the inclusion $\mathbf{Z}_2 \subset \mathbf{Z}_4$ and l is the basic class of K_{m+2n-2}.

Consider the following commutative diagrams

$$\begin{array}{ccccc}
 & & [S^{2n+m}, F] & \xrightarrow{\cong_*} & [S^{2n+m}, \tilde{E}_{n+m-1}] \\
 & & \downarrow J := g^* & & \downarrow g^* \\
[S^{2n+k} \wedge Y_k^*, K(\mathbf{Z}_4, n+m-2)] & \xrightarrow{i_{1*}} & [S^{2n+k} \wedge Y_k^*, F] & \xrightarrow{i_{0*}} & [S^{2n+k} \wedge Y_k^*, \tilde{E}_{n+m-1}]
\end{array}$$

and

$$\begin{array}{ccccc} & & [S^{2n+m}, K(\mathbf{Z}_4, 2n+m)] & \xrightarrow{\cong} & [S^{2n+m}, F] \\ & & \downarrow g^* & & \downarrow J \\ [S^{2n+k} \wedge Y_k^*, K_{2n+m-3}] & \xrightarrow{j_*(Sq^2Sq^1)} & [S^{2n+k} \wedge Y_k^*, K(\mathbf{Z}_4, 2n+m)] & \xrightarrow{\cong} & [S^{2n+k} \wedge Y_k^*, F] \to 0 \end{array}$$

where $i_1 : K(\mathbf{Z}_4, n+m-2) \to F$ is the homotopy fibre of i_0. The bottom line in the above two diagrams are exact. To see the exactness in the second diagram, we need to note that, $H^{2n+m-2}(S^{2n+k} \wedge Y_k^*, \mathbf{Z}_2) \cong H^{k+2}(Y_k) = 0$ since Y_k/S^k is $(k+2)$-connected. To show $j_*(Sq^2Sq^1)$ is zer in the second diagram above, note that $Sq^3U_k = 0$, the duality implies that $\chi(Sq^3)H^{m-k-3}(Y_k^*) = Sq^2Sq^1H^{m-k-3}(Y_k^*) = 0$. Thus the second diagram implies that J is a monomorphism. The proof can be deduced if we can show $Ker(i_0)_* \cap Im(J) = Im(i_1)_* \cap Im(J) = 0$ in the first diagram above.

Let $q = m-n-k-1$, the tower

$$\begin{array}{cccccc} K_{q+n+1} & \to \Omega^{2n+k}F & \to \tilde{E}_q & & & \\ & \downarrow p & \downarrow & & & \\ & K_{q+n-1} \times K_{q+n+1} & \to E_q & \xrightarrow{\omega_3} & K_{q+n+2} & \\ & & \downarrow & & & \\ & & K(\mathbf{Z}_4, q) & \xrightarrow{Sq^n \times Sq^{n+2}} & & K_{q+n} \times K_{q+n+2} \end{array}$$

gives a diagram(not exact)

$$\begin{array}{ccccccc} [Y_k^*, \widetilde{E}_{q-1}] & \longrightarrow & [Y_k^*, E_{q-1}] & \longrightarrow & [Y_k^*, K(\mathbf{Z}_4, q-1)] & \xrightarrow{=} & [Y_k^*, K(\mathbf{Z}_4, q-1)] \\ & & & & \downarrow i_{1*} & & \downarrow Sq^n \times Sq^{n+2} \\ & & & & [Y_k^*, \Omega^{2n+k}F] & \xrightarrow{p_*} & [Y_k^*, K_{q+n-1} \times K_{q+n+1}] \end{array}$$

If $x \in H^{q-1}(Y_k^*, \mathbf{Z}_4)$ such that $i_{1*}(x) \in Im(J)$, $Sq^n(x) \in H^{n+q-1}(Y_k^*, \mathbf{Z}_2) \cong (H^{k+2}(Y_k, \mathbf{Z}_2))^* = 0$. On the other hand, by duality $\chi(Sq^{n+2})U_k = 0$ implies that $Sq^{n+2}H^{q-1}(Y_k^*, \mathbf{Z}_2) = 0$. Thus $Sq^{n+2}(x) = 0$ and

$$Im(i_1)_* \cap Im(J) \subset (i_1)_*(\{x \in H^{q-1}(Y^*, \mathbf{Z}_4) | Sq^n(x) = Sq^{n+2}(x) = 0\}).$$

Since Y_k is ϕ-orientable, i.e, $0 \in \phi(U_k)$. By [19] that $0 \in \psi(x)$. Thus x can be lifted to $\widetilde{E}_{q-1}$ and so $(i_1)_*(x) = 0$. This completes the proof. ♣

Proof of Proposition 1.4: By the definition, for k large, $f(x+y)$ is the following composition of maps

$$S^1 \wedge S^{2n+k} \xrightarrow{id \wedge \Delta\alpha} S^1 \wedge T\xi \wedge M_+ \xrightarrow{id \wedge u \circ w \wedge (x \times y)} S^1 \wedge \widetilde{W}(n)_k \wedge (K(\mathbf{Z}_4, n-1) \times K(\mathbf{Z}_4, n-1)) =$$
$$= \widetilde{W}(n)_k \wedge S^1 \wedge (K(\mathbf{Z}_4, n-1) \times K(\mathbf{Z}_4, n-1)) \xrightarrow{id \wedge \kappa} \widetilde{W}(n)_k \wedge S^1 \wedge K(\mathbf{Z}_4, n-1),$$

where $\kappa(l) = l \otimes 1 + 1 \otimes l$ for the basic class $l \in H^{n-1}(K(\mathbf{Z}_4, n-1), \mathbf{Z}_4)$. Identifying $\widetilde{W}(n)_k \wedge S^1 \wedge (K(\mathbf{Z}_4, n-1) \times K(\mathbf{Z}_4, n-1))$ with

$$\{\widetilde{W}(n)_k \wedge S^1 \wedge K(\mathbf{Z}_4, n-1)\} \vee \{\widetilde{W}(n)_k \wedge S^1 \wedge (K(\mathbf{Z}_4, n-1)\} \vee$$

$$\vee \{\widetilde{W}(n)_k \wedge S^1 \wedge (K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1))\}.$$

It is readily to see that $f(x+y) = f(x) + f(y) + g$, here g is the composition

$S^{2n+k+1} \xrightarrow{id \wedge \Delta\alpha} S^1 \wedge T\xi \wedge M_+ \xrightarrow{id \wedge \mathbf{u} o w \wedge \Delta} S^1 \wedge \widetilde{W}(n)_k \wedge M_+ \wedge M_+ \xrightarrow{id \wedge x \wedge y} \widetilde{W}(n)_k \wedge S^1 \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1) \xrightarrow{id \wedge H(\kappa)} \widetilde{W}(n)_k \wedge S^1 \wedge K(\mathbf{Z}_4, n-1)$,
where $H(\kappa)$ the Hopf constuction of κ.

As $\widetilde{W}(n)_k/S^k$ is $(k+2)$-connected, it is easy to check

$$(i \wedge id)_* : \pi_{2n+k+1}(S^{k+1} \wedge K(\mathbf{Z}_4, n-1)) \to \pi_{2n+k+1}(S^1 \wedge \widetilde{W}(n)_k \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1))$$

is surjective. On the other hand, by Remark 2.4, the generator $\beta \in \pi^s_{2n}(K(\mathbf{Z}_4, n-1)) \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$ satisfies $\beta^*(l \otimes Sq^2 l) \neq 0$. Thus, for the inclusion map i, the composition $(i \wedge id) \circ \beta \in \pi_{2n+k+1}(S^1 \wedge \widetilde{W}(n)_k \wedge K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1))$ induces a nontrivial homomorphism on the $(2n+k)$-th homology and thus is not null homotopy. Thus $(i \wedge id)_*$ is an isomorphism. Moreover, the generator $g_0 \in \pi^s_{2n}(\widetilde{W}(n)_k \wedge K(\mathbf{Z}_4, n-1)) \wedge K(\mathbf{Z}_4, n-1))$ satisfies that $g_0^*(U_k \wedge Sq^2 l_{n-1} \wedge l_{n-1}) \neq 0$. Thus the composition $(id \wedge x \wedge y)(\mathbf{u} o w \wedge \Delta)(\Delta\alpha)$ is null homotopy if and only if $\langle x \cup Sq^2 y, [M]_2 \rangle = 0$. By Proposition 1.2, the proof now follows by the commutative diagram

$$\begin{array}{ccc}
S^k \wedge \sum K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1) & \xrightarrow{i \wedge id} & \widetilde{W(n)}_k \wedge \sum K(\mathbf{Z}_4, n-1) \wedge K(\mathbf{Z}_4, n-1) \\
\downarrow id \wedge H(\kappa) & & \downarrow id \wedge H(\kappa) \\
S^k \wedge \sum K(\mathbf{Z}_4, n-1) & \xrightarrow{i \wedge id} & \widetilde{W(n)}_k \wedge \sum K(\mathbf{Z}_4, n-1).
\end{array}$$

♣

The proof of 1.7 is obvious since the stable homotopy group $\pi_{2n}(K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ if $n = 2(mod 4)$ and the order of elements in $\pi^s_{2n}(K(\mathbf{Z}_4, n-1)$ is at most 2 if $n = 0, 1(mod 4)$.

To show 1.8, let us first begin with a lemma.

Lemma 3.1. *If $n = 2(mod 4)$ and $\alpha(n+3) \geq 3$, then the homomorphism*

$$i_* : \pi^s_{2n}(S^{n-1}) \to \pi^s_{2n}(K(\mathbf{Z}_4, n-1))$$

is zero, where i is a generator of $(n-1)$-th homotopy group.

Proof: By the Postnikov factorization of $S^q \wedge K(\mathbf{Z}_4, n-1)$ for q large it is immediately to see that:

(i) $\alpha \in \pi^s_{2n+q}(S^q \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_4$ is a generator if and only if the functional cohomology operator $Sq^{n+2}_\alpha(S^q \wedge l_{n-1}) \neq 0$. Thus the composition $S^{2n+q} \to S^{q+n-1} \to S^q \wedge K(\mathbf{Z}_4, n-1)$ is of order at most 2 if $n+2 \neq 2, 4$ or 8 by [1]. Therefore, under our assumption, every element in the image of i_* is of order at most 2.
(ii) An $\beta \in \pi^s_{2n+q}(S^q \wedge K(\mathbf{Z}_4, n-1))$ is of order 2 if and only if $\phi^{n+2}_\beta(S^q \wedge l_{n-1}) \neq 0$, where ϕ^{n+2}_β is the secondary functional cohomology operator.
If $\beta = i_*(z)$ for some $z \in \pi_{2n+q}(S^{q+n-1})$. By the naturality of the functional secondary cohomology operator ϕ_β it follows that $\phi^{n+2}_z(s_{n+q-1}) \neq 0$, where (s_{n+q-1}) is the $n+q-1$ dimensional cohomology generator of the sphere S^{n+q-1}. By [1] the proof of 4.3.2, we have a decomposition

$$\phi^{n+2}(s_{n+q-1}) = \sum_{i,j} a_{i,j}\phi_{i,j}(s_{n+q-1}) \text{ mod zero indeterminacy,}$$

where $\phi_{i,j}$ is defined by Adams in [1] for each $i \leq j$ and $j \neq i+1$, $\deg\phi_{i,j} = 2^i + 2^j - 1$, $a_{i,j} \in A_2$, the Steenrod algebra. Whenever $\alpha(n+3) \geq 3$, $a_{i,j}$ are not the unit in A_2 and so ϕ^{n+2} is zero when applied to the 2-cells complexes $S^{q+n-1} \cup e^{2n+q+1}$. Thus β is zero. This completes the proof. ♣

Proof of Theorem 1.8. Since $g \circ \beta \simeq *$ and $T(g') \circ T(\beta) : T(\beta^*\xi) \to T(\xi) \to MSpin_G\langle \bar{k}_2 \rangle$ factors as $j \circ V$, where $j : S^0 \to MSpin_G\langle \bar{k}_2 \rangle$ is the spectral inclusion map and $V : T(\beta^*\xi) \to S^0$. Note that $T(\beta^*\xi)$ is the S-dual of $T(\nu(\beta))$. We have the following immediate commutative diagram:

$$\begin{array}{ccc}
\{T(\nu(\beta)), K(\mathbf{Z}_4, n-1)\} & \overset{d}{\cong} & \{S^{2n+k}, T(\beta^*\xi) \wedge K(\mathbf{Z}_4, n-1)\} \\
\downarrow T^* & & \downarrow T(\beta)_* \\
\{M_+, K(\mathbf{Z}_4, n-1)\} & \overset{A_\alpha}{\cong} & \{S^{2n+k}, T\xi \wedge K(\mathbf{Z}_4, n-1)\}
\end{array}$$

where $T : M_+ \to T(\nu(\beta))$ is the Thom construction. $T(\beta^*\xi) \simeq S^k \vee S^{k+n+1}$ since $\beta^*\xi$ is stable trivial. We denote by U the Thom class of $\nu(\beta)$. Then $T^*U = x$ and $dU = \alpha_1 + \alpha_2$, where $\alpha_1 \in \{S^{2n+k}, S^k \wedge K(\mathbf{Z}_4, n-1)\}$ and $\alpha_2 \in \{S^{2n+k}, S^{k+n+1} \wedge K(\mathbf{Z}_4, n-1)\} \cong \mathbf{Z}_4$ is a generator. By the definition

$$\phi_{M,h}(x) = h(\mathbf{u} \circ T(g')_* \circ T(\beta)_*(\alpha_1)) + h(\mathbf{u} \circ T(g')_* \circ T(\beta)_*(\alpha_2)).$$

The second term is zero when $\alpha(n+3) \geq 3$ since $\mathbf{u} \circ T(g') \circ T(g')_*(\alpha_2)$ factors through $S^{2n+k} \to S^{n+k-1} \to S^k \wedge K(\mathbf{Z}_4, n-1)$ which is null in homotopy by lemma 3.1.

Notice that α_1 depends only on the bundle ν_β. Thus we may choose some special manifolds X of dimension $2n$ with a normal bundle $\theta \in \pi_n(SO(n)) \cong \mathbf{Z}_4$ of an embedded $(n+1)$ sphere in X in $X \times \mathbf{R}$ and to verify the Theorem. Consider first $X = S^{n-1} \times S^{n+1}$ and $\beta = pt \times S^{n+1}$. By the definition, $\phi_{X,h}(x) = 0$ since (X, x) is framed bordant zero, where x is the generator of $H^{n-1}(X)$. Consider $X = S^{n+1} \times_\theta S^{n-1}$, the sphere bundle over S^{n+1} with charateristic class $\theta \in \pi_n(SO(n)) \cong \mathbf{Z}_4$ and $x \in H^{n-1}(X, \mathbf{Z}_4)$ is a generator. These bundles are diffeomorphic to each other if and only if their charateristic classes coincide up to sign(c.f: theorem 1.11 or [10]). Moreover, by 1.11 we have that, these bundles are diffeomorphic to each other if and only if the corresponding quadratic functions $\phi_h : \mathbf{Z}_4 \to \mathbf{Q}/\mathbf{Z}$ are isometric to each other. Therefore there is a bijection between the charateristic classes in $\pi_n(SO(n))$ up to sign and the isometric classes of the quadratic functions. Note that these bundles have always a section(c.f: [11]) and let β be a section. The normal bundle of β in $X \times R$ is θ. We claim that θ is of order 2 if $\phi_{X,h}(x) = \frac{1}{2}$.

If $\phi_{X,h}(x) = \frac{1}{2}$, $\theta \neq 0$. Otherwise X is diffeomorphic to the trivial bundle by 1.11. This is impossible. We now prove further that θ is of order 2. As $\phi_{X,h}(2x) = 0$, $(X, 2x)$ is therefore zero framed bordant. Let (V, y) be a framed manifold with boundary $(X, x)(y|_X = x)$. By use framed surgery we can modify V so that V is $(n-2)$ connected, $H_{n-1}(V) \cong \mathbf{Z}$ and $H_n(V) = 0$. By duality one can check $H_{n+1}(V) \cong \mathbf{Z}_2$. It is elementary to show that the exponent of $\pi_{n+1}(V)$ is 2. Thus the charateristic class of the normal bundle of each embedded $(n+1)$-sphere in V is of order at most 2. Notice that the normal bundle of β in V is the same as that in $X \times \mathbf{R}$. This proves that $2\theta = 0$ and so $j(\theta) = \frac{1}{2}$. Consequently, if $\phi_{X,h}(x) = \frac{1}{4}$, θ is of order 4. Otherwise, it contradicts with 1.11. This completes the proof. ♣

Let $\{Y_k\}_{k \in \mathbf{Z}_+}$ be an ϕ-orientable Thom spectrum. For k large, let W_i, $i = 1, 2$, are maps $Y_k \to \widetilde{W}_k(n)$ which lift the Thom class U_k, where $\widetilde{W}(n)$ is the universal Ω-spectrum defined in §1. Note that $\Pi_2 W_1$ and $\Pi_2 W_2$ are differed by a map to the fibre $K_{k+1} \times K_{k+n+1}$ of the fibration of $\Pi_1 : W_k(n) \to K(\mathbf{Z}, k)$. Let $d_1(W_1, W_2)$ denote this difference. Of course W_1 and W_2 are homotopy if $d_1(W_1, W_2) = 0$ and a secondary obstruction vanishes. The following theorem says that this secondary obstruction does not affect our quadratic function ϕ.

Theorem 3.3. *Let (M, ξ, α) be a Poincaré triple where ξ is an ϕ-orientable k plane bundle. Let $(W_i, h)(i = 1, 2)$ be ϕ-orientations of $T\xi$ and let ϕ_i be the quadratic functions associated with (W_i, h). Suppose that $d_1(W_1, W_2) = 0$. Then $\phi_1(x) = \phi_2(x)$ for all $x \in H^{n-1}(M, \mathbf{Z}_4)$.*

Proof: Let $\mu : K_{n+k+1} \times \widetilde{W}_k(n) \to \widetilde{W}_k(n)$ denote the fibre multiplication. By the assumption $d_1(W_1, W_2) = 0$, W_2 is the composition

$$T\xi \xrightarrow{\Delta} T\xi \times T\xi \xrightarrow{W_1 \times vU_k} \widetilde{W}_k(n) \times K_{n+k+1} \xrightarrow{\mu} \widetilde{W}_k(n),$$

where $vU_k \in H^{k+n+1}(T\xi, \mathbf{Z}_2)$ is the second difference of W_i, $i = 1, 2$, i.e, an obstruction between W_1 and W_2. We have a commutative diagram:

$$\begin{array}{ccccccc}
S^{2n+k} & \xrightarrow{\alpha'} & (T\xi \wedge M_+) \vee (T\xi \wedge M_+) & \xrightarrow{a} & \widetilde{W}_k(n) \wedge K(\mathbf{Z}_4, n-1) \\
\| & & \cap & & \| \\
S^{2n+k} & \xrightarrow{\Delta\alpha} & (T\xi \times T\xi) \wedge M_+ & \xrightarrow{b} & \widetilde{W}_k(n) \wedge K(\mathbf{Z}_4, n-1)
\end{array}$$

where α' is a lifting of $\Delta\alpha$, $b = \mu(W_1 \times vU_k) \wedge x$, $a = (W_1 \wedge x) \vee c$, and $c = i(vU_k) \wedge x$, $i : K_{n+k+1} \to \widetilde{W}_k(n)$ the inclusion of the fibre. Write $\alpha' = \alpha_1 + \alpha_2$, here α_1 and α_2 are the factors of the wedge. Note that $\phi_2(x) = h(b \circ \Delta\alpha) = h(a\alpha_1) + h(a\alpha_2) = \phi_1(x) + h(a\alpha_2)$. We are going to show $h(a\alpha_2) = 0$.

As $a\alpha_2$ factors through the map $i \wedge id : K_{n+k+1} \wedge K(\mathbf{Z}_4, n-1) \to \widetilde{W}_k(n) \wedge K(\mathbf{Z}_4, n-1)$. To show $a\alpha_2$ is null homotopy, it suffices to prove that

$$(i \wedge id)_* : \pi_{2n+k}(K_{n+k+1} \wedge K(\mathbf{Z}_4, n-1)) \to \pi_{2n+k}(\widetilde{W}_k(n) \wedge K(\mathbf{Z}_4, n-1))$$

is zero. Note the homomorphism

$$(Sq^1 \wedge id)_* : \pi_{2n+k}(K_{n+k} \wedge K(\mathbf{Z}_4, n-1)) \to \pi_{2n+k}(K_{n+k+1} \wedge K(\mathbf{Z}_4, n-1)) \cong \mathbf{Z}_2$$

is an isomorphism as it induces an ismomorphism on the $(2n+k)$-th homology groups. The composition $K_{n+k} \xrightarrow{Sq^1} K_{n+k+1} \xrightarrow{i} \widetilde{W}_k(n)$ is null homotopy. Thus $(i \wedge id)_* = 0$. This completes the proof.♣

Proof of Theorem 1.9. Consider the Poincaré triple $(M_1, f^*\xi_2, \alpha_3)$, where $\alpha_3 = T(\tilde{f})_*^{-1}\alpha_2$, $\tilde{f} : f^*\xi_2 \to \xi_2$ is a bundle map over f which is a bundle homotopy equivalence. $T(\tilde{f})$ is the Thom construction of $\tilde{f}$. Let ϕ_3 denote the quadratic functions assoiciated with the Poincaré triple $(M_1, f^*\xi_2, \alpha_3)$ and the orientation $\mathbf{u} \circ w_2 \circ T(f')$, where f' is a bundle map over f. By the defintion it is clear that $\phi_3(f^*x) = \phi_2(x)$ for all $x \in H^{n-1}(M_2, \mathbf{Z}_4)$.

Note that $f^*\xi_2$ and ξ_1 are stably equivalent as spherical fibration since f is a homotopy equivalence. Without lossing of generality we can assume that $f^*\xi_2$ and ξ_1 are the same. Thus we have two orientations for ξ_1, $(\mathbf{u} \circ w_1, h)$ and $(\mathbf{u} \circ w_2 \circ T(f'), h)$, where $w_i = T(f_i')$ are the Thom maps of $(f_i', f_i) : (\xi_i, M_i) \to (\pi^*\gamma, BSpin_G\langle \bar{k}_2\rangle)$, f_i are the classifying maps of ξ_i, $i = 1, 2$. Note that f preserves the spin structures/Wu orientations. Thus $p \circ f_2 \circ f \simeq p \circ f_1$, where $p : BSpin_G\langle \bar{k}_2\rangle \to BSpin_G/BSpin_G\langle v_{n+2}\rangle$ is the principal fibration defined in Example (ii) and (iii) of §1.

It is not hard to show that there exists a fibre automorphism $g' \in Aut(\xi_1)$ over the identity such that $T(p' \circ f_2' \circ f') \simeq T(p' \circ f_1') \circ T(g')$. Notice that g' gives a unique element in $[M_1, G_k]$, namely g_0. By the formula in Brown [4], the $(n+1)$-dimensional component of the difference $d_1(\mathbf{u} \circ w_1 \circ T(g'), \mathbf{u} \circ w_1)$ is $\sum v_{n+2-2^i} \cup g_0^* u_{2^i-1} = 0$ under our assumption, where u_{2^i-1} is the transgression of $w_{2^i} \in H^{2^i}(BG_k, \mathbf{Z}_2)$. The 1-dimensional component of the difference is exactly determined by the spin structures and so it is zero for the fixed spin structure. Thus the first difference of $\mathbf{u} \circ w_2 \circ T(f')$ and $\mathbf{u} \circ w_1$ is zero. By Theorem 3.3, ϕ_1 is the same as the quadratic function associated with (X_1, ξ_1, α_1) and the orientation $\mathbf{u} \circ w_2 \circ T(f')$.

Recall that ϕ_3 is a quadratic function associated with the Poincaré triple (M_1, ξ_1, α_3) and $\mathbf{u} \circ w_2 \circ T(f')$. By the proof of 1.18 in [4], there is an $g_1 \in [X, G_k]$, k large, such that it gives an automorphism $g \in Aut(\xi_1)$ so that $T(g)_*(\alpha_3) = \alpha_1$. For the same reasoning as above it follows that ϕ_1 is the same as a function associated with the Poincaré triple (M_1, ξ_1, α_3) and the orientation $(\mathbf{u} \circ T(g) \circ w_2 \circ T(f'), h)$. Notice now again that the first difference of $\mathbf{u} \circ T(g) \circ w_2 \circ T(f')$ and $\mathbf{u} \circ w_2 \circ T(f')$ is zero. Thus theorem 3.3 implies that ϕ_1 and ϕ_3 are the same and so $\phi_1(f^*x) = \phi_2(x)$ for all x. This completes the proof. ♣

§4. Classification of almost parallelizable $2n$-manifolds

Recall that a manifold X is called almost parallelizable if the restriction of the tangent bundle τX on $X - pt$ is trivial. The immediate examples of such manifolds are the S^{n-1}-bundles over S^{n+1} with charateristic classes in $ker\{\pi_n(SO(n)) \to \pi_n(SO)\}$. Recall that the homotopy groups of $\pi_n(SO(n))$ are as the following table (c.f: [11]):

$\pi_n(SO(n))$, $n \geq 3, \neq 6$

$n \geq 3, \neq 6$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
$\pi_n(SO(n))$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}$

and $\pi_6(SO(6)) = 0$.

Follows the notations in [10], let A_α and B_β denote the S^{n-1}-bundle over S^{n+1} with charateristic number $\alpha, \beta \in \pi_n(SO(n))$ respectively so that $\pi(\alpha) = 0, \pi(\beta) = 1$ for $\pi : \pi_n(SO(n)) \to \pi_n(S^{n-1}) \cong \mathbf{Z}_2$. $(n \geq 4)$. Obviously $Sq^2 : H^{n-1}(A_\alpha) \to H^{n+1}(A_\alpha)$ is zero, and $Sq^2 : H^{n-1}(B_\beta) \to H^{n+1}(B_\beta)$ is an isomorphism.

Proof of 1.10.. For $x \in H^{n-1}(M, \mathbf{Z}_2)$, consider the bordism class $[M, x] \in \tilde{\Omega}^{fr}_{2n}(K_{n-1}) \cong \mathbf{Z}_2$. First note that $x \cup Sq^2x[M]$ is a bordism invariant. To show this, it suffices to prove $x \cup Sq^2x = 0$ if (M, x) is framed bordant zero. Let (V, y) is a framed manifold with boundary (M, x), we may assume that V is simply connected by using framed surgery. Thus $y \cup Sq^2y \in H^{2n}(V, \mathbf{Z}_2) \cong H_1(V, M, \mathbf{Z}_2) = 0$ and thereby $x \cup Sq^2x = 0$. Moreover, it is directly to see that

$$\begin{gathered}\tilde{\Omega}^{fr}_{2n}(K_{n-1}) \to \mathbf{Z}_2 \\ [M, x] \to x \cup Sq^2x[M]\end{gathered}$$

is a homomorphism.

By the proof of Theorem 2.1, the reduction homomorphism

$$\pi^s_{2n}(K(\mathbf{Z}, n-1)) \to \pi^s_{2n}(K_{n-1})$$

is surjective if n is even. Under the Thom-Pontryagin map, this corresponds to the reduction

$$\tilde{\Omega}^{fr}_{2n}(K(\mathbf{Z}, n-1)) \to \tilde{\Omega}^{fr}_{2n}(K_{n-1}).$$

By Theorem 1.8 it follows that the generator of $\tilde{\Omega}^{fr}_{2n}(K_{n-1})$ can be represented by $(S^{n+1} \times_\theta S^{n-1}, z)$, where $S^{n+1} \times_\theta S^{n-1}$ is a sphere bundle and $z \in H^{n-1}(S^{n+1} \times_\theta S^{n-1}, \mathbf{Z}_2)$. By the tables (I)(II) of [11], it is immediate to see $z \cup Sq^2z = 0$ if $n = 2(mod4)$, and the bundle B_β for $\beta \in kerS_* : \pi_n(SO(n)) \to \pi_n(SO)$ is an example so that $z \cup Sq^2 z = 0$ if $n = 0(mod 4)$, here z is the generator. This proves (i) and (iii).

When n is odd, by 2.7

$$q(Sq^1) : \tilde{\Omega}^{fr}_{2n}(K_{n-1}) \to \mathbf{Z}_2$$
$$[M, x] \to q_M(Sq^1 x)$$

is an isomorphism. Thus there is an $\delta \in \mathbf{Z}_2$ so that $\delta q_M(Sq^1 x) = x \cup Sq^2 x[M]$ for all $[M, x]$. In particular, if x can be lifted to the $\mathbf{Z}_4$-coefficients, $Sq^1 x = 0$ and so $x \cup Sq^2 x = 0$. This completes the proof of (ii) and so 1.10. ♣

To set up the classification of $(n-2)$-connected $2n$-dimensional almost parallelizable manifolds, we are going to decompose the manifolds as the connected sum of two simpler pieces.

Lemma 4.1. Let M be a $(n-2)$-connected $2n$ dimensional manifold. If $n \geq 4$, then M is homeomorphic to $K \# N$, where K is $(n-1)$-connected and N satisfies $\beta_n(N, \mathbf{Q}) = 0$.
Proof: Since the Hurewicz homomorphism $\pi_n(M) \to H_n(M)$ is surjective, we may represent each n-dimensional homology class by an embedded n-sphere by using Whitney trick. Let $I = (a_{i,j})_{\beta \times \beta}$ $(\beta = \beta_n(M))$ denote the intersection matrix of M with respect to a basis $\alpha_1, \cdots, \alpha_\beta$ of $H_n(M)$ represented by β embedded spheres satisfying
(i): $\alpha_i \cap \alpha_j \cap \alpha_k = \phi$(empty) if i, j and k are pairwise different.
(ii): α_i and $\alpha_j (i \neq j)$ intersects in $a_{i,j}$ points transversally.
I is unimodular.

Let K_0 denote the closed regular neighborhood of $\alpha_1 \cup \cdots \cup \alpha_\beta$ in M. K_0 is a smooth manifold with boundary. $H_i(\partial K_0) = H_i(K_0) = 0$ for $2 \leq i \leq n-1$, $\pi_1(\partial K_0) \to \pi_1(K_0)$ is a free group with finite letters. Let C be the closure of $M - K_0$. Notice that $\pi_1(\partial K_0) \to \pi_1(C)$ is a zero homomorphism. Representing a generator set of $\pi_1(K_0)$ by embedded $S^1 \times D^{2n-2}$'s and extend to $D^2 \times D^{2n-2}$'s in C. Here we have to change the framing of the embedded $S^1 \times D^{2n-2}$'s if it is not compatible with the induced framings. Add these 2-handles to K_0 , we get a smooth manifold K_0' with boundary a homotopy sphere and so homeomorphic to S^{2n-1}. $K_0' \subset M$. Let $K = K_0' \cup D^{2n}$ and $N = (M - intK_0') \cup D^{2n}$. It is now easy to see that $M \approx K \# N$, K is $(n-1)$-connected and $H_n(K_0') \to H_n(M)$ is a rational isomorphism with cokernel the torsion subgroup. This completes the proof. ♣

Notice the classification of $(n-1)$-connected $2n$-manifolds have been done by Wall [22] completely. To consider the classification of $(n-2)$-connected $2n$-manifolds, by lemma 4.1, it suffices to handle the case of N with $b_n(N, \mathbf{Q}) = 0$. Thus $signN = 0$ and N is therefore stably parallelizable if M is almost parallelizable. As we mentioned in §1, for the standard spin structure or Wu orientation on N, the equivalence class of the quadratic function ϕ_h is a homotopy invariant.

Proof of Theorem 1.11. By 1.7 and 1.10 we see that ϕ and the bilinear form μ has the property described in the Theorem. Now we prove every data in the Theorem can be realized by an almost parallelizable manifold.

By Theorem 2.1 the algebraic data in the theorem where $b = 0$ and so $sign = 0$ if n is even, determines a reduced framed bordism class in $\tilde{\Omega}_{2n}^{fr}(K(H, n-1))$. Let $[M_0, f]$ denote such a bordism class. Without lossing of generality we can assume that $f_* : H_{n-1}(M_0) \to H$ is an epimorphism. Otherwise we may sum some $S^{n-1} \times S^{n+1}$ to M_0 and modify f_* to fulfill this property. Using framed surgery to kill the kernel of f_* we get a framed manifold, namely M with $H_{n-1}(M) \cong H$. The connected sum of M with some $\pm|E_8|$ and $S^n \times S^n$ will realize the data, where $|E_8|$ is a $(n-1)$-connected $2n$-dimensional almost parallelizable manifold with intersection form E_8.

Now we are going to prove that these algebraic data determine the homemorphism types of the manifolds. As we mentioned in §1, these invariants set are homotopy invariants of the manifolds. Thus the homotopy and homeomorphism classification of such manifolds are the same.

Suppose that X_i, $i = 1, 2$, are two smooth manifolds with the same data(for TOP manifold, the similar argument works identically). Without lossing of generality we assume that $b_n(X_i, \mathbf{Q}) = 0$ since lemma 4.1 and the classification theorem of Wall applies here. Thus X_i are stable parallelizable. Put framings on X_i. By the assumption there are maps $f_i : X_i \to K(H, n-1)$, $i = 1, 2$, so that (X_1, f_1) and (X_2, f_2) are reduced framed bordant, where f_i induces an isomorphism at the $(n-1)$-th homology groups. If $n \neq 1, 3, 7$, X_i are both framed bordant to a framed homotopy sphere, if n is even or n odd and the Kervaire invariants of X_i vanish. In this case, we can assume X_1 and $X_2 \# \Sigma$ are framed bordant, here Σ is a homotopy sphere. If $n = 7$(we have assumed that $n \geq 4$), we can change the framing on X_1 if necessary, so that X_1 and X_2 are framed bordant. Therefore (X_1, f_1) and $(X_2 \# \Sigma, f_2)$ are framed bordant. By Freedman [8] or Kreck [14] it follows that X_1 and X_2 are diffeomorphic since $b_n(X_i, \mathbf{Q}) = 0$. If X_i both have Kervaire invariants 1. Up to connected sum with a framed homotopy sphere, X_1 and X_2 are framed cobordant. The same argument above applies to show that X_1 and X_2 are homeomorphic to each other. This completes the proof. ♣

Proof of 1.14. By Theorem 1.11 and 1.12 (vi), $\sigma : \Delta_{2n}(G) \to \mathbf{Z}_3$ is a injective if n is even. By Theorem 5.5, every value in $\mathbf{Z}_3(\mathbf{Z}_2)$ can be realized as the Arf invariant of a $\mathbf{Q}/\mathbf{Z}$-valued quadratic function (factoring through $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$). Combining this with Theorem 1.11 (i) and (ii) follows.

To prove (iii), note that $q(Sq_i^1) : G \to \mathbf{Z}_2$ is a linear function. Let $e_1, \cdots, e_m$ be a symplectic basis of G/G_0. If $q(Sq_i^1)(e_1) = 1$ and $q(Sq_i^1)(e_i) = 1$ for some i. Let $T \in Aut(G/G_0)$ be the automorphism such that $T(e_1) = e_1$, $T(e_i) = e_i + e_1$ and $T(e_j) = e_j$ if $j \neq 1, i$. T preserves the inner product μ and $q(Sq_i^1)(T(e_1)) = 1$, $q(Sq_i^1)(T(e_i)) = 0$.

Thus we can assume $q(Sq_i^1)(e_1) = 1$ and $q(Sq_i^1)(e_l) = 0$ if $l \geq 2$. Therefore there is only a nonzero function up to equivalence. By Theorem 1.11 the proof of (iii) follows. Similar argument applies to show (iv). This completes the proof. ♣

Proof of 1.15. By Theorem 1.11 it suffices to verify that $M\#K$ and $M'\#K$ have the same algebraic data. It is clear to see that, as an inner product module, $H_{n-1}(M\#K)$ and $H_{n-1}(M'\#K)$ are isometric. When $n = 2(mod4)$, by §5 it is easy to check that the quadratic functions for $M\#K$ and $M'\#K$ are equivalent.
When $n = 3(mod4)$, notice $q_K(Sq^1)(x) = 1$ since $q_{S^n \times S^n}(D(\Delta)) = 1$, where Δ is the diagonal of $S^n \times S^n$, D is the Poincaré duality isomorphism and q is the Kervaire form. It is easy to see that $q_{M\#K}(Sq_i^1)$ and $q_{M'\#K}(Sq_i^1)$ are both equivalent to $\omega = 0 \oplus (1)$: $Hom(tor(H)\otimes Z_{2^i}, \mathbf{Z}_2)\oplus\mathbf{Z}_2 \to \mathbf{Z}_2$, where $H = H_{n-1}(M) \cong H_{n-1}(M')$. Applying Theorem 1.11 the proof follows. ♣

§5. Appendix: Numeral invariants of $\mathbf{Q}/\mathbf{Z}$-valued quadratic functions

Throughout this section we let V denote a $\mathbf{Z}_4$ module with a $\mathbf{Z}_2$-valued inner product "$\cdot$". Assuming $x \cdot y = 0$ if either x or y is of order 2. We say a function $\phi : V \to \mathbf{Z}_4$ is a quadratic if

$$\phi(x+y) = \phi(x) + \phi(y) + j(x \cdot y);\ \forall x, y \in V$$

where $j : \mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$ is the inclusion.

Definition 5.1. *If V is finitely generated, and $\phi : V \to \mathbf{Z}_4$ is a quadratic function. We denote by $\lambda(\phi)$ the Gauss sum:*

$$\lambda(\phi) = \sum_{x \in V} e^{2\pi\phi(x)i} \in \mathbf{C} \ \textit{where}\ i^4 = 1$$

It is obvious that $\lambda(\phi)$ is well defined and is an invariant of the Witt class(isometry class) of ϕ. If (V, ϕ_1) and (V, ϕ_2) are two quadratic functions, we denote by $(V, \phi_1 \oplus \phi_2)$ and $(V, -\phi_i)$ the direct sum of (V, ϕ_i) and the multiplication by -1, $i = 1, 2$.

Proposition 5.2. *(i) $\lambda(\phi)$ is a real number.*
(ii) $\lambda(\phi_1 \oplus \phi_2) = \lambda(\phi_1)\lambda(\phi_2)$.
(iii) $\lambda(-\phi) = \bar{\lambda}(\phi)$.
(iv) If $\phi : V \to \mathbf{Q}/\mathbf{Z}$ is linear, then $\lambda(\phi) = |V|$ or 0 by $\phi = 0$ or not.
Proof: (ii), (iii) and (iv) are obvious. As $-\phi$ is equivalent to ϕ, thus (i) is a consequence of (iii).

We say an inner module V is *hyperbolic* if $x \cdot x = 0$ for all $x \in V$(In §1 we said that it is of diagonal zero). V is *nonsingular* if for each $x \in V$, there exists an $y \in V$ such that $x \cdot y \neq 0$. Let H be a free $\mathbf{Z}_4$-module of dimension 2 with nonsingular hyperbolic inner product. We denote by e_1, e_2 a basis of H with inner products $e_1 \cdot e_2 = e_2 \cdot e_1 = 1$,

$e_i \cdot e_i = 0$, $i = 1, 2$. There are the following three quadratic functions on H:

$$\begin{array}{ll} \phi_0 : & \phi_0(e_1) = 0, \quad \phi_0(e_2) = 0 \\ \phi_1 : & \phi_1(e_1) = \frac{1}{4}, \quad \phi_1(e_2) = \frac{1}{4} \\ \phi_2 : & \phi_2(e_1) = \frac{2}{4}, \quad \phi_2(e_2) = \frac{2}{4} \end{array}$$

The Gauss sum $\lambda(\phi)$ is:

$$\lambda(\phi_1) = 0 \ ; \ \lambda(\phi_2) = -8 \ ; \ \lambda(\phi_0) = 8 \cdots (5.3)$$

Proposition 5.3. *There are only the above three* $\mathbf{Q}/\mathbf{Z}$*-valued quadratic forms on* H *up to isomorphism.*

Proof: (1) Let $\phi : H \to \mathbf{Q}/\mathbf{Z}$ be a function with $\phi(e_1) = \pm\frac{1}{4}$, we claim that ϕ is equivalent to ϕ_1. If $\phi(e_1) = \frac{1}{4}$ and $\phi(e_2) = 0$, we set $T : H \to H$ for the following isometry

$$\begin{array}{l} T(e_1) = e_1; \\ T(e_2) = 3e_1 + e_2 \end{array}$$

Thus $\phi(3e_1 + e_2) = \frac{1}{4}$ and so $\phi \circ T = \phi_1$. Other cases in the claim can be checked similarly.
(2) Let $\phi : H \to \mathbf{Q}/\mathbf{Z}$ be a form such that $2\phi(e_i) = 0$ for $i = 1, 2$, then ϕ is either isomorphic to ϕ_0 or ϕ_2.
This completes the proof. ♣

Lemma 5.4.

$$\begin{array}{l} \phi_2 \oplus \phi_2 \cong \phi_0 \oplus \phi_0; \\ \phi_1 \oplus \phi_1 \cong \phi_1 \oplus \phi_0 \cong \phi_1 \oplus \phi_2. \end{array}$$

Proof: Let $e_1, e_2; e_1', e_2'$ denote a basis of $H \oplus H$; and set $T : H \oplus H \to H \oplus H$ for the following isometry

$$\begin{array}{ll} T(e_1) = e_1 + e_1' + e_2'; & T(e_2) = e_2 + e_1' + e_2' \\ T(e_1') = e_1' + e_1 + e_2; & T(e_2') = e_2' + e_1 + e_2 \end{array}$$

Note that

$$\begin{array}{ll} (\phi_0 \oplus \phi_0)(e_1 + e_1' + e_2') = \frac{1}{2}; & (\phi_0 \oplus \phi_0)(e_2 + e_1' + e_2') = \frac{1}{2} \\ (\phi_0 \oplus \phi_0)(e_1' + e_1 + e_2) = \frac{1}{2}; & (\phi_0 \oplus \phi_0)(e_2 + e_1' + e_2') = \frac{1}{2} \end{array}$$

Hence $\phi_2 \oplus \phi_2 \cong \phi_0 \oplus \phi_0$.
Similarly, the isometries

$$\begin{array}{ll} e_1 \to e_1 + e_1'; & e_2 \to e_2 + e_1' \\ e_1' \to e_1' & e_2' \to e_2' + e_1 + e_2 \end{array}$$

and

$$\begin{array}{ll} e_1 \to e_1; & e_2 \to e_2 \\ e_1' \to e_1' + 2e_1; & e_2' \to e_2' + 2e_2 \end{array}$$

will set up the other two isomorphisms in the lemma.
This completes the proof. ♣

Obviously every quadratic functions on $H \oplus \cdots \oplus H$ is isomorphic to the direct sum of quadratic functions on H. To use Lemma 5.4 iteratively we obtain

Theorem 5.5. *Suppose that $V \cong nH$, then there are exactly three isomorphism classes of the quadratic functions on V*

$$\phi_1 \oplus (n-1)\phi_0, \phi_2 \oplus (n-1)\phi_0, n\phi_0.$$

Their Gauss sums are

$$\lambda(\phi_1 \oplus (n-1)\phi_0) = 0, \lambda(\phi_2 \oplus (n-1)\phi_0) = -8^n, \lambda(n\phi_0) = 8^n.$$

Notice that, any $\mathbf{Z}_4$-module V with a hyperbolic $\mathbf{Z}_2$-valued inner product can be written as the direct sum of a nonsingular module and V_0, where V_0 is a submodule with trivial inner product. It is obvious that a nonsingular inner module over $\mathbf{Z}_4$ must be free and so isomorphic to nH for some integer n.

Definition 5.6. Let $V \cong nH \oplus V_0$ be an orthogonal decomposition of V, the inner product on V_0 vanishes. For each quadratic function ϕ on V, we denote by

$$\sigma(\phi) = \frac{\lambda(\phi)}{8^n \cdot |V_0|} \in \{0, \pm 1\} = \mathbf{Z}_3$$

the Arf invariant of ϕ.
We say that ϕ is of type $I(II)$ if $2\phi \neq 0 (2\phi = 0)$.
If ϕ is of type I, denote by $\epsilon(\phi) \in \mathbf{Z}_2$:

$$\begin{aligned}&\epsilon(\phi) = 1 \text{ if there is an } x \in V \text{ s.t. } \phi(x) = \tfrac{1}{4} \text{ and } x \cdot V = 0\\ &\epsilon(\phi) = 0 \text{ otherwise}\end{aligned}$$

Obviously a form on V_0 is a homomorphism. Up to isomorphism there are only the following types($V_0 \neq 0$):

$$\begin{array}{ll} 0 : V_0 \to \mathbf{Q}/\mathbf{Z} & \text{the zero homomorphism,} \\ h : V_0 \to \mathbf{Q}/\mathbf{Z} & 2h \neq 0, \\ h' : V_0 \to \mathbf{Q}/\mathbf{Z} & h \neq 0 \text{ and } 2h = 0 \end{array}$$

It is easy to check:

$$\begin{array}{ll} (1): & \phi_1 \oplus 0 \cong \phi_1 \oplus h' \\ (2): & \phi_1 \oplus h \cong \phi_2 \oplus h \cong \phi_0 \oplus h \\ (3): & \phi_2 \oplus h' \cong \phi_0 \oplus h' \neq \phi_1 \oplus h' \\ (4): & \phi_1 \oplus 0 \neq \phi_1 \oplus h \end{array}$$

(4) holds as both sides have different ϵ invariants. The second inequality of (3) holds since they have different types. Consequently we have a complete list of all possible quadratic functions on V, where $V = nH \oplus \mathbf{Z}_4^{m_1} \oplus \mathbf{Z}_2^{m_2}$, $m_1 \geq 1$.

$$\begin{aligned}
&\phi_1 \oplus (n-1)\phi_0 \oplus (h) \cong \phi_2 \oplus (n-1)\phi_0 \oplus (h) \cong n\phi_0 \oplus (h),\\
&\phi_2 \oplus (n-1)\phi_0 \oplus (h') \cong n\phi_0 \oplus (h'),\\
&n\phi_0 \oplus (0),\\
&\phi_2 \oplus (n-1)\phi_0 \oplus (0),\\
&\phi_1 \oplus (n-1)\phi_0 \oplus (0) \cong \phi_1 \oplus (n-1)\phi_0 \oplus (h'),
\end{aligned}$$

Therefore we obtain that

Proposition 5.7. *Let V be a hyperbolic $\mathbf{Z}_4$-module. Then $\mathbf{Q}/\mathbf{Z}$-valued quadratic forms on V are isometric if and only if they have the same type, Arf invariant σ and the ϵ invariant.*

Let us consider the classification of the quadratic functions on V which factors through the inclusion $\mathbf{Z}_2 \to \mathbf{Q}/\mathbf{Z}$. Note that such quadratic functions are the composition of quadratic functions over $\mathbf{Z}_2$-vector space $V \otimes \mathbf{Z}_2$ and the reduction homomorphism $\rho : V \to V \otimes \mathbf{Z}_2$. By the classical result of Arf-invariant it follows that:

Proposition 5.8. *Let V be a nonsingular diagonal zero inner module over $\mathbf{Z}_4$. Then there are exactly two $\mathbf{Z}_2$-valued quadratic functions detected by the classical Arf invariant of the factors on $V \otimes \mathbf{Z}_2$.*

Suppose that $j(x \cdot y) = 0$ if x or y is of order 2. Note that the bilinear form μ defined in §1 satisfies this property. Therefore the nonsingular part of V is isomorphic to $\mathbf{Z}_4^m$ for some m.

Proposition 5.9. *Let $V = nH \oplus V_0$ be an inner module over $\mathbf{Z}_4$, where $V_0 \neq 0$. Then there are exactly three $\mathbf{Z}_2$-valued quadratic functions on V as the follows:*

$$\phi_0 \oplus 0, \phi_1 \oplus 0, \phi_1 \oplus \phi \cong \phi_1 \oplus \phi,$$

where ϕ_0, ϕ_1 have Arf invariants $0, 1$ respectively. ϕ is a nonzero linear function on V_0.

Consequently the proof of Theorem 1.12 follows readily.

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