# Shuffle relations for regularised integrals of symbols

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## Abstract

We prove shuffle relations which relate a product of regularised integrals of classical symbols $\int^{reg} \sigma_i \, d\xi_i, i = 1, \cdots, k$ to regularised nested iterated integrals:

$$\prod_{i=1}^{k} \int^{reg} \sigma_i \, d\xi_i = \sum_{\tau \in \Sigma_k} \int^{reg} d\xi_1 \int_{|\xi_2| \leq |\xi_1|} d\xi_2 \cdots \int_{|\xi_L| \leq |\xi_{k-1}|} d\xi_k \, \otimes_{i=1}^{k} \sigma_{\tau(i)}$$

where $\Sigma_k$ is the group of permutations over $k$ elements. We show that these shuffle relations hold if all the symbols $\sigma_i$ have vanishing residue; this is true of non integer order symbols on which the regularised integrals have all the expected properties such as Stokes' property [MMP]. In general the shuffle relations hold up to finite parts of corrective terms arising from a renormalisation on tensor products of classical symbols, a procedure adapted from renormalisation procedures on Feynman diagrams familiar to physicists.
We relate the shuffle relations for regularised integrals of symbols with shuffle relations for multiple zeta functions adapting the above constructions to the case of a symbol on the unit circle.

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# Introduction

Before describing the contents of the paper, let us give some general motivation. Starting from a function $f : \mathbb{N} \to \mathbb{C}$, one can build functions $P(f) : \mathbb{N} \to \mathbb{C}$ and $\tilde{P}(f) : \mathbb{N} \to \mathbb{C}$:

$$P(f)(n) = \sum_{n>m>0} f(m), \quad \tilde{P}(f)(n) = \sum_{n\geq m>0} f(m).$$

The operators $P$ and $\tilde{P}$ obey Rota-Baxter relations and define Rota-Baxter type operators of weight $-1$ and 1 respectively:

$$P(f)\,P(g) = P\left(f\,P(g)\right) + P\left(g\,P(f)\right) + P(fg)$$

and

$$\tilde{P}(f)\,\tilde{P}(g) = \tilde{P}\left(f\,\tilde{P}(g)\right) + \tilde{P}\left(g\,\tilde{P}(f)\right) - \tilde{P}(fg).$$

When applied to $f(n) = n^{-z_1}$, $g(n) = n^{-z_2}$, these relations lead to the "second shuffle relations" for zeta functions [ENR]:

$$\zeta(z_1)\,\zeta(z_2) = \zeta(z_1, z_2) + \zeta(z_2, z_1) + \zeta(z_1 + z_2)$$

where $\zeta(z) = \sum_{n>0} n^{-z}$ and $\zeta(z_1, z_2) = \sum_{n_1>n_2} n_1^{-z_1} n_2^{-z_2}$. Similarly,

$$\zeta(z_1)\,\zeta(z_2) = \tilde{\zeta}(z_1, z_2) + \tilde{\zeta}(z_2, z_1) - \tilde{\zeta}(z_1 + z_2)$$

where $\tilde{\zeta}(z_1, z_2) = \sum_{n_1\geq n_2} n_1^{-z_1} n_2^{-z_2}$.
Correspondingly, starting from $f \in L^1(\mathbb{R}, \mathbb{C})$, one can build $P(f) : R \to \mathbb{C}$:

$$P(f)(y) = \int_{y\geq x} f(x)\,dx.$$

Then the classical Rota-Baxter relation (of weight zero)

$$P(f)\,P(g) = P(f\,P(g)) + P(g\,P(f))$$

is an integration by parts in disguise. It leads to to shuffle relations for integrals:

$$\prod_{i=1}^{k} \int_{\mathbb{R}} f_i = \sum_{\tau\in\Sigma_k} \int_{\mathbb{R}} P\left(P\left(\cdots P(f_{\tau(k)})\,f_{\tau(k-1)}\cdots\right) f_{\tau(2)}\right) f_{\tau(1)} \quad \forall k \geq 2$$

under adequate integrability assumptions on the functions $f_i$.
Zeta functions generalize to zeta functions associated to elliptic classical pseudo-differential operators on a closed manifold $M$ defined by

$$\zeta_A(z) = \sum_{\lambda_n\in\mathrm{Spec}(A),\lambda_n\neq 0} \lambda_n^{-z}$$

modulo some extra under assumptions on the leading symbol of the operator $A$ to ensure the existence of its complex power $A^{-z}$. If $\sigma_A(z)$ denotes the symbol of this complex power then provided the order of $A$ is positive, for $\mathrm{Re}(z)$ large enough, $\zeta_A$ is actually an integral of the symbol on the cotangent bundle $T^*M$:

$$\zeta_A(z) = \int_M dx \int_{T_x^* M} \mathrm{tr}_x(\sigma_A(z))(x, \xi)\,đ\xi$$

with $d\!\!\!{}^-\xi := \frac{d\xi}{(2\pi)^n}$, $n$ being the dimension of $M$. It extends to a meromorphic function on the whole plane replacing the ordinary integral by a cut-off integral $-\!\!\!\!\!\!\int_{T^*M}$.

The main purpose of this paper is to establish shuffle relations for cut-off integrals of classical symbols $\sigma_i \in CS^{\alpha_i}(U_i)$ (see notations in the Preliminaries):

$$\prod_{i=1}^{k} -\!\!\!\!\!\!\int \sigma_i = \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int P\left(\cdots P\left(P(\sigma_{\tau(k)})\,\sigma_{\tau(k-1)}\right)\cdots \sigma_{\tau(2)}\right)\sigma_{\tau(1)} \quad \forall k \geq 2$$

and other regularised integrals built from cut-off integrals. We give sufficient assumptions on the symbols for such shuffle relations to hold, conditions which we shall specify below, once we have introduced the necessary technical tools. It turns out that on the class of non integer order classical symbols, on which these regularised integrals have the expected properties such as Stokes' property, translation invariance...(see [MMP]), these shuffle relations hold. Otherwise a renormalisation procedure is needed to take care of obstructions to these shuffle relations.

In order to make this statement precise, we first need to extend cut-off and other regularised integrals on classical symbols to cut-off and other regularised *iterated integrals* on tensor products of classical symbols; they are all continuous linear forms on spaces of symbols which naturally extend to continuous linear forms on the (closed) tensor product. The Wodzicki residue, which is also continuous on classical symbols of fixed order, extends in a similar way to a higher order residue density $\widetilde{\mathrm{res}}_{x,k}$ at point $x = (x_1, \ldots, x_k) \in U = U_1 \times \cdots \times U_k$ on the tensor product $\hat{\otimes}_{i=1}^k CS(U_i)$ and the well-known relation expressing the ordinary residue density $\mathrm{res}_x := \mathrm{res}_{x,0}$ as a complex residue:

$$\mathrm{Res}_{z=0} -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi)\, d\!\!\!{}^-\xi = -\frac{1}{\alpha'(0)}\mathrm{res}_x(\sigma(0)) \quad \forall \sigma \in CS(U)$$

extends to $\hat{\otimes}_{i=1}^k CS(U_i)$. Here $\sigma(z)$ is a holomorphic family of classical symbols with order $\alpha(z)$ such that $\alpha'(0) \neq 0$.
Indeed, the map $z \mapsto -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi)\, d\!\!\!{}^-\xi$ with $\sigma \in \hat{\otimes}_{i=1}^k CS^{\alpha_i}(U_i)$ is meromorphic with poles of order no larger than $k$ and we have (see Theorem 2)

$$\mathrm{Res}_{z=0}^k -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi)\, d\!\!\!{}^-\xi = \frac{(-1)^k}{\prod_{i=1}^k \alpha_i'(0)}\widetilde{\mathrm{res}}_{x,k}(\sigma(0)) \quad \forall \sigma \in \hat{\otimes}_{i=1}^k CS(U_i), \qquad (1)$$

which is independent of the choice of regularisation $\mathcal{R} : \sigma \mapsto \sigma(z)$ which sends the symbol $\sigma$ to a holomorphic family of symbols $\sigma(z)$ such that $\sigma(0) = \sigma$.

Another approach to regularised *iterated* integrals is to consider the operator $\sigma \mapsto P(\sigma)$

$$P(\sigma)(\eta) = \int_{|\xi| \leq |\eta|} \sigma(\xi)\, d\!\!\!{}^-\xi.$$

It maps $\sigma \in CS(U)$ to a symbol $P(\sigma)$ which is not anymore classical, since it raises the power of the logarithm entering the asymptotic expansion of the symbol by one. The fact that the algebra of classical symbols is not stable under the action of $P$ justifies the introduction of log-polyhomogeneous symbols in this context (see e.g. [L] for an extensive study of log-polyhomogeneous symbols and operators). Indeed, the operator $P$ satisfies a Rota-Baxter relation (of weight zero):

$$P(\sigma)\, P(\tau) = P(\sigma\, P(\tau)) + P(\tau\, P(\sigma))$$

and defines a Rota-Baxter operator on the algebra of logpolyhomogeneous symbols (see Proposition 3). In one dimension the Rota-Baxter relation is an integration by parts formula in disguise but for higher dimensions, this Rota Baxter formula does not merely reduce to an integration by parts formula. However, similarities are to be expected between the obstructions to shuffle relations for regularised integrals studied here and the obstructions to Stokes' formula for regularised integrals of symbol valued forms studied in [MMP]. In both cases the obstructions disappear under a non integrality assumption on the orders of the symbols involved. It is interesting to note that regularised integrals behave nicely specifically on symbols of non integer order, namely when they obey Stokes' property [MMP] and have good transformation properties [L], [MMP].
Unlike in the previous approach, we now take a fixed open subset $U \in \mathbb{R}^n$ so that $U_i = U$, $i = 1, \cdots, k$. From a tensor product $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ of classical symbols $\sigma_i \in CS(U)$ and operators

$$\begin{array}{rcl} \sigma & \mapsto & P_k(\sigma) \\ (\sigma)(x; \xi_1, \cdots, \xi_k) & := & P\left(\sigma(x; \xi_1, \cdots, \xi_k, \cdot)\right)(\xi_k), \end{array}$$

for fixed $x \in U$, one builds a map $(x, \xi) \mapsto (P_1 \circ \cdots \circ P_{k-1}(\sigma))\,(x, \xi)$ which is logpolyhomogeneous. The regularised cut-off iterated integral of $\sigma$ can then be seen as an ordinary regularised cut-off integral (extended by M. Lesch [L] to logpolyhomogeneous symbols) on the logpolyhomogeneous symbol $P_1 \circ \cdots \circ P_{k-1}(\sigma)$ in our case [1]:

$$-\!\!\!\!\!\!\int_{T_x^*U} \sigma = \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi_1\, P_1 \circ \cdots \circ P_{k-1}\,(\sigma_\tau)\,.$$

When $\sigma = \otimes \sigma_i$ and the (left) partial sums $\alpha_1 + \alpha_2 + \cdots + \alpha_j$, $j = 1, \cdots, k$ of the orders $\alpha_i$ of the symbols $\sigma_i \in CS(U)$ are *non integer*, the following shuffle relations hold (see Theorem 4)

$$\prod_{i=1}^{k} -\!\!\!\!\!\!\int_{T_x^*U} \sigma_i = \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi_1\, P_1 \circ \cdots \circ P_{k-1}\,(\sigma_\tau) \qquad (2)$$

where we have set $\sigma_\tau := \otimes_{i=1}^k \sigma_{\tau(i)}$.
A holomorphic regularisation procedure $\mathcal{R} : \sigma \mapsto \sigma(z)$ on $CS(U)$ (with some continuity assumption) induces a regularisation procedure $\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \sigma_1(z) \otimes \cdots \otimes \sigma_k(z)$ on $\hat{\otimes}^k CS(U)$. Using results by Lesch [L] on cut-off integrals of holomorphic families of logpolyhomogeneous symbols we build meromorphic maps $z \mapsto -\!\!\!\!\!\int_{T_x^*U} \sigma(z)$ with poles of order at most $k$ for any $\sigma \in \hat{\otimes}^k CS(U)$.

[1] Similar nested integrals arise in D.Kreimer's work [K1] in relation to a change of scale in the renormalisation procedure. His rooted trees describing nested integrations can be adapted to our context, decorating trees with symbols $\sigma_i$. We thank D. Kreimer for pointing this reference out to us, which we read after this article was completed.

When $\sigma(z)$ has order $q\,z+\alpha(0)$, then equation (2) implies the following equality of meromorphic functions

$$\prod_{i=1}^k \int\!\!\!\!\!\!-_{T_x^*U} \sigma_i(z) = \sum_{\tau\in\Sigma_k} \int\!\!\!\!\!\!-_{T_x^*U} d\xi\, P_1\circ\cdots\circ P_{k-1}\left(\otimes_{i=1}^k \sigma_{\tau(i)}(z)\right). \tag{3}$$

But in general, the constant term in the meromorphic expansion on the l.h.s does not coincide with the product of the regularised integrals $\int\!\!\!\!\!-_{T_x^*U}^{\mathcal{R}} \sigma_i := \mathrm{fp}_{z=0} \int\!\!\!\!\!- \sigma_i(z)$, namely in general

$$\mathrm{fp}_{z=0} \prod_{i=1}^k \int\!\!\!\!\!\!-_{T_x^*U} \sigma_i(z) \neq \prod_{i=1}^k \int\!\!\!\!\!\!-_{T_x^*U}^{\mathcal{R}} \sigma_i.$$

However, shuffle relations extend to these regularised integrals provided the symbols involved have *vanishing Wodzicki residue* (see Corollary 2):

$$\prod_{i=1}^k \int\!\!\!\!\!\!-_{T_x^*U}^{\mathcal{R}} \sigma_i = \sum_{\tau\in\Sigma_k} \int\!\!\!\!\!\!-_{T_x^*U}^{\mathcal{R}} d\xi_1\, P_1\circ\cdots\circ P_{k-1}\left(\sigma_\tau\right).$$

For general symbols, a renormalisation procedure borrowed from physicists keeps track of counterterms one needs to introduce in order to pick the “right” finite part thereby circumventing the problem that “taking finite parts” does not commute with ”taking products” of meromorphic functions.

The above constructions are adapted in section 5 to invariant classical pseudodifferential operators acting on sections over the unit circle $S^1$. Using the identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, one can relate the shuffle relations for integrals of the symbol of the modulus of the Dirac operator on the circle with “second shuffle relations” for multiple zeta-functions. The adaptation is not straightforward as the symbol is not a smooth function anymore; since it involves Dirac measures the integrals turn out to be discrete sums. The Euler-MacLaurin formula is the main tool which enables us to go from integrals of symbols to discrete sums of symbols.

These shuffle relations for regularised integrals of symbols and their link with shuffle relations for zeta functions are a hint towards deeper algebraic structures underlying cut-off multiple integrals on one hand and renormalisation procedures in quantum field theory on the other hand.
In particular, this leads to the following open questions. Combining tensor products $\sigma = \otimes_{i=1}^I \sigma_i$ considered previously with injective linear maps $B = B_0 \otimes I_n : \mathbb{R}^L \otimes \mathbb{R}^n \to \mathbb{R}^I \otimes \mathbb{R}^n$ with $L \leq I$, one can build a class of functions[2]

$$f(\xi_1 \cdots, \xi_L) = \sigma \circ B(\xi_1, \cdots, \xi_L)$$

in the momenta $\xi_1, \cdots, \xi_k$ which, for certain choices of $\sigma_i$'s are of Feynman type in the language of Etingof [E]. A regularisation procedure $\mathcal{R}$ on classical symbols gives rise to holomorphic families $z \mapsto \sigma_i(z)$ from which we can build a map $(z_1, \cdots, z_I) \mapsto \sigma_{z_1,\cdots,z_I} = \otimes_{i=1}^I \sigma_i(z_i)$. It is reasonable to expect the map

$$(z_1, \cdots, z_I) \mapsto \int\!\!\!\!\!\!- \sigma_{z_1,\cdots,z_I} \circ B(\xi_1, \cdots, \xi_L)\, d\xi_1 \cdots d\xi_L$$

[2]In the context of Feynman diagrams, $L$ stands for the number of loops and $I$ for the number of internal edges.

to give rise to a Laurent expansion in the $z_i$'s, on the grounds of work by Speer [S][3] who proves this fact when $\sigma_i(\xi) = (|\xi|^2 + m_i^2)^{-1} \quad \forall i \in \{1, \cdots, I\}$ and $\sigma(z) = \sigma^{1+z}$. Alternatively, following a dimensional regularisation type procedure, one can build maps

$$(z_1, \cdots, z_L) \mapsto \int \sigma \circ B(\xi_1, \cdots, \xi_L) \, |\xi_1|^{-z_1} \cdots |\xi_L|^{-z_L} \, d\xi_1 \cdots d\xi_L,$$

which again can be expected to give rise to Laurent expansions and hence to a meromorphic function at 0 when $z_1 = \cdots = z_L = z$. Etingof's results on dimensional regularisation [E] imply this meromorphicity property when $\sigma_i(\xi) = (|\xi|^2 + m_i^2)^{-1} \quad \forall i \in \{1, \cdots, I\}$ on the grounds of a theorem by Bernstein but further investigations are needed to prove the first part of the statement on the existence of a Laurent expansion in several variables.
In Theorem 6, we show that provided the class of functions under consideration has this expected Laurent expansion behaviour, the renormalisation procedure boils down to picking up the constant term in the Laurent expansion in $(z_1, \cdots, z_k)$. This fact was already proved by Speer [S] in the particular case we briefly described above in relation to his work.

It appears from the investigations carried out here, that iterated integrals of symbols seem to provide a stepping stone between Feynman type integrals in physics and the renormalisation procedures used to handle their divergences on one hand and multiple zeta functions and the regularised shuffle relations they obey, a line of thought we want to pursue further in a forthcoming paper.

[3] We thank Dirk Kreimer for drawing our attention to this reference. Speer's results are transposed here to the euclidean set up.

# 1 Preliminaries

For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$, the set $CS^{\alpha,k}(U)$ of scalar valued logpolyhomogeneous symbols of order $\alpha$ on an open subset $U$ of $\mathbb{R}^n$ can be equipped with a Fréchet structure. Such a symbol reads:

$$\sigma = \sum_{m=0}^{N-1} \psi\, \sigma_{\alpha-m} + \sigma_{(N)}, \tag{4}$$

where $\psi$ is a smooth function which vanishes at 0 and equals to one outside a compact, where $\sigma_{\alpha-m}(x,\xi) = \sum_{p=0}^{k} \sigma_{\alpha-m,p}(x,\xi)\, \log^p |\xi| \in C^\infty(S^*U)$ with $\sigma_{\alpha-m,p}(x,\xi)$ positively homogeneous in $\xi$ of order $\alpha - m$ and where $\sigma_{(N)} \in C^\infty(S^*U)$ is a symbol of order $\alpha - N$. The following semi-norms labelled by multiindices $\gamma, \beta$ and integers $m \geq 0, p \in \{1, \cdots, k\}$, $N$ give rise to a Fréchet topology on $CS^{\alpha,k}(U)$:

$$\begin{aligned}
&\sup_{x\in K, \xi\in \mathbb{R}^n} (1+|\xi|)^{-\alpha+|\beta|}\, |\partial_x^\gamma \partial_\xi^\beta \sigma(x,\xi)|;\\
&\sup_{x\in K, \xi\in \mathbb{R}^n} |\xi|^{-\alpha+N+|\beta|}\, |\partial_x^\gamma \partial_\xi^\beta \left(\sigma - \sum_{m=0}^{N-1} \psi(\xi)\, \sigma_{\alpha-m}\right)(x,\xi)|;\\
&\sup_{x\in K, |\xi|=1} |\partial_x^\gamma \partial_\xi^\beta \sigma_{\alpha-m,p}(x,\xi)|,
\end{aligned}$$

where $K$ ranges over compact sets in $U$.

**Remark 1** *Note that the first set of norms corresponds to the ordinary symbol topology, the second set of norms controls the rest term $\sigma_{(N)}$ whereas the last set of norms is the ordinary supremum norm on the homogeneous components of the symbol.*

Let us introduce some notations. The set $CS^{-\infty}(U) := \bigcap_{m\in \mathbb{R}} CS^m(U)$ corresponds to the algebra of smoothing symbols. The set

$$CS^{\mathbb{Z},*}(U) := \bigcup_{m\in\mathbb{Z}} \bigcup_{k\in \mathbb{N}} CS^{m,k}(U)$$

of integer order log-polyhomogeneous symbols, which is equipped with an inductive limit topology of Fréchet spaces is strictly contained in the algebra generated by log-polyhomogeneous symbols of any order

$$CS^{*,*}(U) := \langle \bigcup_{m\in \mathbb{R}} \bigcup_{k\in \mathbb{N}} CS^{m,k}(U)\rangle.$$

Following [KV] (see also [L]), we extend the continuity on symbols of fixed order to families of symbols with varying order as follows:

**Definition 1** *Let $k$ be a non negative integer. A map $b \mapsto \sigma(b) \in CS^{*,k}(U)$ of symbols parametrized by a topological space $B$ is continuous if the following assumptions hold:*

1. *the order $\alpha(b)$ of $\sigma(b)$ is continuous in $b$,*
2. *for any non negative integer $j$, the homogeneous components $\sigma_{\alpha(b)-j,l}(b)(x,\xi)$, $0 \leq l \leq k$ of the symbol $\sigma(b)(x,\xi)$ yield continuous maps $b \mapsto \sigma_{\alpha(b)-j}(b) := \sum_{l=0}^{k} \sigma_{\alpha(b)-j,l} \log^l |\xi|$ into $C^\infty(T^*U)$,*

*3. for any sufficiently large integer $N$, the truncated kernel*

$$K^{(N)}(b)(x,y) := \int_{T_x^*U} d\xi e^{i\xi\cdot(x-y)} \sigma_{(N)}(b)(x,\xi)$$

*where*

$$\sigma_{(N)}(b)(x,\xi) := \sigma(b)(x,\xi) - \sum_{j=0}^{N} \psi(\xi)\, \sigma_{\alpha(b)-j}(b)(x,\xi)$$

*yields a continuous map $b \mapsto \sigma_{(N)}(b)$ into some $C^{K(N)}(U\times U)$ where $\lim_{N\to\infty} K(N) = +\infty$.*

# 2 Regularised integrals of log-polyhomogeneous symbols

We recall for completeness, well-known results on regularisation techniques of integrals of ordinary log-polyhomogeneous symbols which lead to trace functionals on the corresponding pseudodifferential operators.

## 2.1 Cut-off integrals of log-polyhomogeneous symbols

We start by recalling the construction of cut-off integrals of log-polyhomogeneous symbols [L] which generalizes results previously established by Guillemin and Wodzicki in the case of classical symbols.

**Lemma 1** *Let $U$ be an open subset of $\mathbb{R}^n$ and for any non negative integer $k$, let $\sigma \in CS^{*,k}(U)$ be a log-polyhomogeneous symbol, then for any $x \in U$*

- *$\int_{B_x^*(0,R)} \sigma(x,\xi) d\xi$ has an asymptotic expansion in $R \to \infty$ of the form:*

$$\int_{B_x^*(0,R)} \sigma(x,\xi) d\xi \quad \sim_{R\to\infty} \quad C_x(\sigma) + \sum_{j=0, \alpha-j+n\neq 0}^{\infty} \sum_{l=0}^{k} P_l(\sigma_{\alpha-j,l})(\log R)\, R^{\alpha-j+n}$$
$$+ \quad \sum_{l=0}^{k} \frac{\mathrm{res}_{x,l}(\sigma)}{l+1} \log^{l+1} R \qquad (5)$$

*where $P_l(\sigma_{\alpha-j,l})(X)$ is a polynomial of degree $l$ with coefficients depending on $\sigma_{\alpha-j,l}$ and where $C_x(\sigma)$ is the constant term corresponding to the finite part:*

$$C_x(\sigma) \;:=\; \int_{T_x^*U} \sigma_{(N)}(x,\xi)\, d\xi + \int_{B_x^*(0,1)} \psi(\xi)\sigma(x,\xi)\, d\xi$$
$$+ \sum_{j=0, \alpha-j+n\neq 0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha-j+n)^{l+1}} \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi) d_S\xi$$

*which is independent of $N \geq \alpha + n - 1$.*

- *For any fixed $\mu > 0$,*

$$\mathrm{fp}_{R\to\infty} \int_{B_x^*(0,\mu\, R)} \sigma(x,\xi) d\xi = \mathrm{fp}_{R\to\infty} \int_{B_x^*(0,R)} \sigma(x,\xi)\, d\xi + \sum_{l=0}^{k} \frac{\log^{l+1}\mu}{l+1} \cdot \mathrm{res}_{l,x}(\sigma).$$

**Remark 2** *If $\sigma$ is a classical operator, setting $k = 0$ in the above formula yields*

$$\begin{aligned}\mathrm{fp}_{R\to\infty}\int_{B_x^*(0,R)}\sigma(x,\xi)d\xi \quad &:= \quad \int_{T_x^*U}\sigma_{(N)}(x,\xi)\,d\xi+\sum_{j=0}^N\int_{B_x^*(0,1)}\psi(\xi)\,\sigma_{\alpha-j}(x,\xi)\,d\xi \\ &- \quad \sum_{j=0,\alpha-j+n\neq 0}^N\frac{1}{\alpha-j+n}\int_{S_x^*U}\sigma_{\alpha-j}(x,\omega)d\omega.\end{aligned}$$

**Proof.** Given a log-polyhomogeneous symbol $\sigma\in CS^{\alpha,*}(U)$, for any $N\in\mathbb{N}$ we write:

$$\sigma(x,\xi)=\sum_{j=0}^N\psi(\xi)\sigma_{\alpha-j}(x,\xi)+\sigma_{(N)}(x,\xi)\quad\forall(x,\xi)\in T^*U, \tag{6}$$

where $\sigma_{(N)}\in S^{\alpha-N-1}(U)$.

- For some fixed $N\in\mathbb{N}$ chosen large enough such that $\alpha-N-1<-n$, we write $\sigma(x,\xi)=\sum_{j=0}^N\psi\sigma_{\alpha-j}(x,\xi)+\sigma_{(N)}(x,\xi)$ and split the integral accordingly:

  $$\int_{B_x^*(0,R)}\sigma(x,\xi)d\xi=\sum_{j=0}^N\int_{B_x^*(0,R)}\psi(\xi)\sigma_{\alpha-j}(x,\xi)d\xi+\int_{B_x^*(0,R)}\sigma_{(N)}(x,\xi)d\xi.$$

  Since $\alpha-N-1<-n$, $\sigma_{(N)}$ lies in $L^1(T_x^*U)$ and the integral $\int_{B_x^*(0,R)}\sigma_{(N)}(x,\xi)d\xi$ converges when $R\to\infty$ to $\int_{T_x^*U}\sigma_{(N)}(x,\xi)d\xi$. On the other hand, for any $j\leq N$

  $$\int_{B_x^*(0,R)}\psi(\xi)\sigma_{\alpha-j}(x,\xi)=\int_{B_x^*(0,1)}\psi(\xi)\sigma_{\alpha-j}(x,\xi)+\int_{D_x^*(1,R)}\sigma_{\alpha-j}(x,\xi) \tag{7}$$

  since $\psi$ is constant equal to 1 outside the unit ball. Here $D_x^*(1,R)=B_x^*(0,R)-B_x^*(0,1)$. The first integral on the r.h.s. converges and since

  $$\sigma_{\alpha-j}(x,\xi)=\sum_{l=0}^k\sigma_{\alpha-j,l}(x,\xi)\log^l|\xi|,$$

  the second integral reads:

  $$\int_{D_x^*(1,R)}\sigma_{\alpha-j}(x,\xi)d\xi=\sum_{l=0}^k\int_1^R r^{\alpha-j+n-1}\log^l r\,dr\cdot\int_{S_x^*U}\sigma_{\alpha-j,l}(x,\omega)d\omega.$$

  Hence the following asymptotic behaviours:

  $$\begin{aligned}&\int_{D_x^*(1,R)}d\xi\,\sigma_{\alpha-j}(x,\xi)\\ \sim_{R\to\infty}\quad&\sum_{l=0}^k\frac{\log^{l+1}R}{l+1}\cdot\int_{S_x^*U}\sigma_{\alpha-j,l}(x,\omega)d\omega=\sum_{l=0}^k\frac{\log^{l+1}R}{l+1}\mathrm{res}_{l,x}(\sigma)\quad\text{if }\alpha-j=-n\end{aligned}$$

  whereas:

$$\int_{D_x^*(1,R)} \sigma_{\alpha-j} \sim_{R\to\infty} \sum_{l=0}^{k}\left(\sum_{i=0}^{l} \frac{(-1)^{i+1}\frac{l!}{(l-i)!}\log^i R}{(\alpha-j+n)^i} \cdot R^{\alpha-j+n}\int_{S_x^*U}\sigma_{\alpha-j,l}(x,\omega)d\omega\right.$$
$$+ (-1)^l l! \frac{R^{\alpha-j+n}}{(\alpha-j+n)^{l+1}} \cdot \int_{S_x^*U}\sigma_{\alpha-j,l}(x,\omega)d\omega$$
$$\left. + \frac{(-1)^{l+1}l!}{(\alpha-j+n)^{l+1}} \cdot \int_{S_x^*U}\sigma_{\alpha-j,l}(x,\omega)d\omega\right) \text{ if } \quad \alpha-j\neq -n.$$

Putting together these asymptotic expansions yields the statement of the proposition with

$$C_x(\sigma) = \int_{T_x^*U}\sigma_{(N)} + \sum_{j=0}^{N}\int_{B_x^*(0,1)}\psi\sigma_{a_j} + \sum_{j=0, a_j+n\neq 0}^{N}\sum_{l=0}^{L}\frac{(-1)^{l+1}l!}{(a_j+n)^{l+1}}\int_{S_x^*U}\sigma_{a_j,l}.$$

- The $\mu$-dependence follows from

$$\log^{l+1}(\mu R) = \log^{l+1} R\left(1+\frac{\log\mu}{\log R}\right)^{l+1}$$
$$\sim_{R\to\infty} \log^{l+1} R\sum_{k=0}^{l+1} C_{l+1}^k\left(\frac{\log\mu}{\log R}\right)^k.$$

  The logarithmic terms $\sum_{l=0}^{k}\frac{\mathrm{res}_{l,x}(\sigma)}{l+1}\log^{l+1}(\mu R)$ therefore contribute to the finite part by $\sum_{l=0}^{k}\frac{\log^{l+1}\mu}{l+1}\cdot \mathrm{res}_{l,x}(\sigma)$ as claimed in the lemma. □

Discarding the divergences, we can therefore extract a finite part from the asymptotic expansion of $\int_{B(0,R)}\sigma(x,\xi)d\xi$ and set for $\sigma\in CS^{*,k}(\mathbb{R}^n)$:

**Definition 2** *Given an non negative integer $k$, an open subset $U\subset\mathbb{R}^n$ and a point $x\in U$, for any $\sigma\in CS^{\alpha,k}(U)$, the cut-off integral*

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*U}\sigma(x,\xi)d\xi &:= \mathrm{fp}_{R\to\infty}\int_{B_x^*(0,R)}\sigma(x,\xi)d\xi \\ &= \int_{T_x^*U}\sigma_{(N)}(x,\xi)\,d\xi + \sum_{j=0}^{N}\int_{B_x^*(0,1)}\psi(\xi)\sigma_{\alpha-j}(x,\xi)\,d\xi \\ &+ \sum_{j=0,\alpha-j+n\neq 0}^{N}\sum_{l=0}^{k}\frac{(-1)^{l+1}l!}{(\alpha-j+n)^{l+1}}\int_{S_x^*U}\sigma_{\alpha-j,l}(x,\xi)\,d_S\xi \end{aligned} \tag{8}$$

*is independent of $N>\alpha+n-1$.*
*It is independent of the parametrisation $R$ provided the higher Wodzicki residue*

$$\mathrm{res}_{x,l} := \int_{S_x^*U}\sigma_{-n,l}(x,\xi)d_S\xi$$

*vanishes for all integer $0\le l\le k$.*

This explicit description of the finite part leads to the following continuity result.

**Proposition 1** *For any fixed $\alpha \in \mathbb{R}$ and any non negative integer $k$, and given an open subset $U \in \mathbb{R}^n$, a point $x \in U$, the map*

$$\begin{array}{rcl} CS^{\alpha,k}(U) & \to & C^\infty(U, \mathbb{C}) \\ \sigma & \mapsto & \left(x \mapsto -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)\, d\xi\right) \end{array}$$

*is continuous in the Fréchet topology of $CS^{\alpha,k}(U)$ and the natural topology of $C^\infty(U, \mathbb{C})$.*

**Remark 3** *The assumption that $\alpha$ be fixed is essential here.*

**Proof:** From formula (8) and the fact that symbols are smooth functions on $U \times \mathbb{R}^n$, it follows that the cut-off integral is $C^\infty(U, \mathbb{C})$-valued.
The maps $\sigma \mapsto \left(x \mapsto \int_{B_x^*(0,1)} \psi(\xi)\sigma_{\alpha-j}(x,\xi)\, d\xi\right)$ and $\sigma \mapsto \left(x \mapsto \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi)\, d_S\xi\right)$ are clearly continous as integrals over compact sets of continuous maps. On the other hand the map $\sigma \mapsto \left(x \mapsto \int_{T_x^*U} \sigma_{(N)}(x,\xi)\, d\xi\right)$ is continuous since $\sigma \mapsto \sigma_{(N)}$ is continuous and $\sigma_{(N)}(x,\xi) \leq C(1+|\xi|)^{-N}$ can be uniformly bounded by an $L^1$ function. $\square$

As well as the higher order residue density function $\mathrm{res}_{x,k}$, one can define on $CS^{*,k}(U)$ an extension of the ordinary residue density function $\mathrm{res}_x$ as follows:

$$\mathrm{res}_x(\sigma) := \int_{S_x^*U} (\sigma(x,\xi))_{-n}\, d_S\xi$$

where $d_S\xi$ is the volume measure on the unit cotangent sphere $S_x^*U$ induced by the canonical volume measure on $T_x^*U$. Even though it certainly does not induce a graded trace on the algebra of log-polyhomogeneous operators on a closed manifold as the higher order residue does [L], it is a useful tool for what follows since we have the following continuity result:

**Lemma 2** *Given any non negative integer $k$, and given any $\alpha \in \mathbb{R}$, the map:*

$$\begin{array}{rcl} CS^{\alpha,k}(U) & \to & C^\infty(U, \mathbb{C}) \\ \sigma & \mapsto & (x \mapsto \mathrm{res}_x(\sigma)) \end{array}$$

*is continuous for the Fréchet topology on $CS^{\alpha,k}(U)$.*

## 2.2 Integrals of holomorphic families of log-polyhomogenous symbols

Following [KV] (see also [L]), we define a holomorphic family of logpolyhomogeneous symbols in $CS^{*,k}(U)$ in the same way as in definition 1 replacing continuous by holomorphic.
We quote from [PS] the following theorem which extends results of [L] relating the Wodzicki residue of holomorphic families of log-polyhomogeneous symbols with higher Wodzicki residues. For simplicity, we restrict ourselves to holomorphic families with order $\alpha(z)$ given by an affine function of $z$, a case which covers natural applications.

**Theorem 1** *Let $U$ be an open subset of $\mathbb{R}^n$ and let $k$ be a non negative integer. For any holomorphic family $z \mapsto \sigma(z) \in CS^{\alpha(z),k}(U)$ of symbols parametrised by a domain $W \subset \mathbb{C}$ such that $z \mapsto \alpha(z) = \alpha'(0)\, z + \alpha(0)$ is an affine function with $\alpha'(z) = \alpha'(0) \neq 0$, then for any $x \in U$, there is a Laurent expansion in a neighborhood of any $z_0 \in P$*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) d\xi &= \mathrm{fp}_{z=z_0} -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) d\xi \\
&+ \sum_{j=1}^{k+1} \frac{r_j(\sigma)(z_0)(x)}{(z-z_0)^j} \\
&+ \sum_{j=1}^{K} s_j(\sigma)(z_0)(x)\, (z-z_0)^j \\
&+ o\left((z-z_0)^K\right),
\end{aligned}$$

*where for $1 \leq j \leq k+1$, $R_j(\sigma)(z_0)(x)$ is locally explicitly determined by a local expression (see [L] for the case $\alpha'(0) = 1$)*

$$\begin{aligned}
& r_j(\sigma)(z_0)(x) \\
:= & \sum_{l=j-1}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \frac{l!}{(l+1-j)!} \, \mathrm{res}_x \left( \left(\sigma_{(l)}\right)^{(l+1-j)} \right)(z_0). \qquad (9)
\end{aligned}$$

*Here $\sigma_{(l)}(z)$ is the local symbol given by the coefficient of $\log^l |\xi|$ of $\sigma$ i.e.*

$$\sigma(z) = \sum_{l=0}^{k} \sigma_{(l)}(z) \log^l |\xi|.$$

*On the other hand, the finite part $\mathrm{fp}_{z=z_0} -\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) d\xi$ consists of a global piece $-\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(z_0)(x,\xi)\, d\xi$ and a local piece:*

$$\begin{aligned}
\mathrm{fp}_{z=z_0} -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) d\xi &= -\!\!\!\!\!\!\int_{T_x * U} \sigma(z_0)(x,\xi)\, d\xi \\
&+ \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \frac{1}{l+1} \, \mathrm{res}_x \left( \left(\sigma_{(l)}\right)^{(l+1)} \right)(z_0). \qquad (10)
\end{aligned}$$

*Finally, for $1 \leq j \leq K$, $S_j(\sigma)(z_0)(x)$ reads*

$$\begin{aligned}
s_j(\sigma)(z_0) &:= -\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(j)}(z_0)\, d\xi \\
&+ \sum_{l=0}^{k} \frac{(-1)^{l+1} l! j!}{(\alpha'(z_0))^{l+1}\, (j+l+1)!} \, \mathrm{res}_x \left( \left(\sigma_{(l)}\right)^{(j+l+1)}(z_0) \right). \qquad (11)
\end{aligned}$$

As a consequence, the finite part $\mathrm{fp}_{z=z_0} -\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) d\xi$ is entirely determined by the derivative $\alpha'(z_0)$ of the order and by the derivatives of the symbol $\sigma^{(l)}(z_0), \quad l \leq k+1$ via the cut-off integral and the Wodzicki residue density.

## 2.3 Regularised integrals of log-polyhomogeneous symbols

Let us briefly recall the notion of holomorphic regularisation taken from [KV] (see also [PS]).

**Definition 3** *A holomorphic regularisation procedure on $CS^{*,k}(U)$ for any fixed non negative integer $k$ is a map*

$$\begin{array}{rcl} \mathcal{R}: CS^{*,k}(U) & \to & \mathrm{Hol}\left(CS^{*,k}(U)\right) \\ \sigma & \mapsto & \sigma(z) \end{array}$$

*where $\mathrm{Hol}\left(CS^{*,k}(U)\right)$ is the algebra of holomorphic maps with values in $CS^{*,k}(U)$, such that*

1. *$\sigma(0) = \sigma$,*
2. *$\sigma(z)$ has holomorphic order $\alpha(z)$ (in particular, $\alpha(0)$ is equal to the order of $\sigma$) such that $\alpha'(0) \neq 0$.*

*We call a regularisation procedure $\mathcal{R}$ continuous whenever the map*

$$\begin{array}{rcl} \mathcal{R}: CS^{*,k}(U) & \to & \mathrm{Hol}\left(CS^{*,k}(U)\right) \\ \sigma & \mapsto & (z \mapsto \sigma(z)) \end{array}$$

*is continuous.*

**Remark 4** *It is easy to check [PS] that if $z \to \sigma(z) \in CS^{\alpha(z),k}(U)$ then $\sigma^{(j)}(z_0) \in CS^{\alpha(z_0),k+j}(U)$.*

Examples of holomorphic regularisations are the well known Riesz regularisation $\sigma \mapsto \sigma(z)(x,\xi) := \sigma(x,\xi)\cdot|\xi|^{-z}$ and generalisations of the type $\sigma \mapsto \sigma(z)(x,\xi) := H(z)\cdot\sigma(x,\xi)\cdot|\xi|^{-z}$ where $H$ is a holomorphic function such that $H(0) = 1$. The latter include dimensional regularisation (see [P]). These regularisation procedures are clearly continuous.
As a consequence of the results of the previous paragraph, given a holomorphic regularisation procedure $\mathcal{R}: \sigma \mapsto \sigma(z)$ on $CS^{*,k}(U)$ and a symbol $\sigma \in CS^{*,k}(U)$, for every point $x \in U$, the map $z \mapsto ⨍_{T_x^*U} \sigma(z)(x,\xi)\,d\xi$ is meromorphic with poles of order at most $k+1$ at points in $\alpha^{-1}([-n,+\infty[\cap\mathbb{Z})$ where $\alpha$ is the order of $\sigma(z)$ so that we can define the finite part when $z \to 0$ as follows.

**Definition 4** *Given a holomorphic regularisation procedure $\mathcal{R}: \sigma \mapsto \sigma(z)$ on $CS^{*,k}(U)$, a symbol $\sigma \in CS^{*,k}(U)$ and any point $x \in U$, we define the regularised integral*

$$\begin{aligned} \int_{T_x^*U}^{\mathcal{R}} \sigma(x,\xi)\,d\xi &:= \mathrm{fp}_{z=0} ⨍_{T_x^*U} \sigma(z)(x,\xi)\,d\xi \\ &:= \lim_{z\to 0}\left( ⨍_{T_x^*U} d\xi\,\sigma(z)(x,\xi) - \sum_{j=1}^{k+1} \frac{1}{z^j}\mathrm{Res}_{z=0}^j ⨍_{T_x^*U} d\xi\,\sigma(z)(x,\xi)\right). \end{aligned}$$

We have the following continuity result.

**Proposition 2** *Given a continuous holomorphic regularisation procedure $\mathcal{R} : \sigma \mapsto \sigma(z)$ on $CS^{*,k}(U)$ where $k$ is a non negative integer, for any fixed $\alpha \in \mathbb{R}$, there is a discrete set $P_\alpha \subset \mathbb{C}$ such that the map*

$$\begin{array}{rcl} CS^{\alpha,k}(U) & \to & C^\infty(U, \mathrm{Hol}(\mathbb{C} - P_\alpha)) \\ \sigma & \mapsto & -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)(z)\, d\xi \end{array}$$

*is continuous on $C^\infty\left(U, \mathrm{Hol}(\mathbb{C} - P_\alpha)\right)$. Moreover the map*

$$\begin{array}{rcl} CS^{\alpha,k}(U) & \to & C^\infty(U, \mathbb{C}) \\ \sigma & \mapsto & -\!\!\!\!\!\!\int_{T_x^*U}^{\mathcal{R}} \sigma(x,\xi)\, d\xi \end{array}$$

*is continuous on $CS^{\alpha,k}(U)$.*

**Remark 5** *The assumption that $\alpha$ be constant is essential here.*

**Proof:** From Theorem 1 we know that the map $z \mapsto -\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\cdot)$ is meromorphic with simple poles in some discrete set $P_\alpha$. From Proposition 1 we know that the map $\sigma \mapsto -\!\!\!\!\!\int \sigma$ is continuous. Combining these two results gives the continuity of the map $\sigma \mapsto \left(z \mapsto -\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)(z)\, d\xi\right)$ where the r.h.s is understood as a holomorphic map on $\mathbb{C} - P_\alpha$.
We now prove the second part of the proposition. By theorem 1 applied to $z_0 = 0$, it is sufficient to check that the maps $\sigma \mapsto -\!\!\!\!\!\int_{T_x^*U} \sigma(0)(x,\xi)\, d\xi$ and the maps $\sigma \mapsto \mathrm{res}_x\left(\sigma^{(j)}(0)\right)$ are $C^\infty(U, \mathbb{C})$ valued and continuous for any $1 \leq j \leq k+1$ for the Fréchet topology on log-polyhomogenous symbols and the Fréchet topology on smooth functions.
From the continuity assumption on the regularisation $\mathcal{R}$ combined with Proposition 1 and Lemma 2 it follows that for a log-polyhomogeneous symbol $\tau$, both $x \mapsto -\!\!\!\!\!\int_{T_x^*U} \tau(x,\xi)\, d\xi$ and $x \mapsto \mathrm{res}_x(\tau)$ are smooth functions. Applying this to $\tau = \sigma^{(j)}(0)$ (which is log-polyhomogeneous by the above remark) with $0 \leq j \leq k+1$ yields the result. □

# 3 Regularised integrals on tensor products of classical symbols

## 3.1 Tensor products of symbols

Let $U_1, \cdots, U_L$ be open subsets of $\mathbb{R}^n$. Since the spaces $CS^{m_i}(U_i)$ and $CS^{m_i,k_i}(U_i)$ are Fréchet spaces, we can form their closed tensor products, where the closed tensor product of two Fréchet spaces $E$ and $F$ is the Fréchet space $E\hat{\otimes}F$ built as the closure of $E \otimes F$ for the finest topology for which $\otimes : E \times F \to E \otimes F$ is continuous.

**Definition 5** *For any multiindices $(m_1, \cdots, m_L) \in R^L$, $(k_1, \cdots, k_L) \in \mathbb{N}^L$ we set*

$$CS_w^{(m_1,\cdots,m_L)}(U_1 \times \cdots \times U_L) := \hat{\otimes}_{i=1}^L CS^{m_i}(U_i)$$

*and*

$$CS_w^{(m_1,\cdots,m_L),(k_1,\cdots,k_L)}(U_1 \times \cdots \times U_L) := \hat{\otimes}_{i=1}^L CS^{m_i,k_i}(U_i).$$

*The multiindex $(m_1, \cdots, m_L)$ is called the multiple order of $\sigma$ and $m_1 + \cdots + m_L$ its total order.*

There are at least two ways of continuously extending regularised integrals to tensor products of symbols.

### 3.2 A first extension of regularised integrals to tensor products

**Definition 6** *Let $U = U_1 \times \cdots \times U_L$ with $x = (x_1, \cdots, x_L)$, $x_i \in U_i, i = 1, \cdots, L$ open subsets in $\mathbb{R}^n$. Let $(\alpha_1, \cdots, \alpha_L) \in \mathbb{C}^l$ and let $(k_1, \cdots, k_L)$ be a multiindex of non negative integers.*
*The continuous maps*

$$\begin{aligned} CS^{\alpha_i,k_i}(U_i) &\to C^\infty(U_i, \mathbb{C}) \\ \sigma_i &\mapsto \left(x_i \mapsto ⨍_{T^*_{x_i}U_i} \sigma_i(x_i, \xi_i)\, d\xi_i\right), \quad i = 1, \cdots, L \end{aligned}$$

*induce a uniquely defined map:*

$$\begin{aligned} CS_w^{(\alpha_1,\cdots,\alpha_L),(k_1,\cdots,k_L)}(U) &\to C^\infty(U, \mathbb{C}) \\ \sigma &\mapsto \left(x \mapsto ⨍_{T^*_x U} \sigma(x, \xi)\, d\xi_1 \cdots d\xi_L\right) \end{aligned}$$

*which gives rise to a linear map on $\hat{\otimes}^k CS(U_i)$ called the multiple regularised cut-off integral of $\sigma(x, \cdot)$.*

Clearly, if $\sigma(x, \cdot) = \otimes_{i=1}^k \sigma_i \sigma_i(x_i, \cdot)$ we have:

$$⨍_{T^*_x U} \sigma(x, \xi)\, d\xi_1 \cdots d\xi_L = \prod_{i=1}^L ⨍_{T^*_{x_i} U} \sigma(x_i, \xi_i)\, d\xi_i.$$

The following extends holomorphic regularisations to tensor products of symbol spaces.

**Definition 7** *Let $U = U_1 \times \cdots \times U_L$ be a product of open subsets of $\mathbb{R}^n$. For a given multiindex $(k_1, \cdots, k_L)$ with $k_i$ non negative integers, a regularisation procedure $\mathcal{R}$ on $CS_w^{*,(k_1,\cdots,k_L)}(U)$ is a map:*

$$\begin{aligned} \mathcal{R} : CS_w^{*,(k_1,\cdots,k_L)}(U) &\to \mathrm{Hol}\left(CS_w^{*,(k_1,\cdots,k_L)}(U)\right) \\ \sigma &\mapsto \mathcal{R}(\sigma) : z \mapsto \sigma(z) \end{aligned}$$

*such that*

1. $\sigma(0)=\sigma$,

2. $\sigma(z)$ *has holomorphic (multiple) order* $\alpha(z)=(\alpha_1(z),\cdots,\alpha_L(z))\in\mathbb{R}^L$ *(in particular,* $\alpha(0)$ *is equal to the (multiple) order of* $\sigma$*) such that* $\mathrm{Re}\,(\alpha_i'(0))>0$ *for all* $i\in\{1,\cdots,L\}$.

*Here* $\mathrm{Hol}\left(CS_w^{*,(k_1,\cdots,k_L)}(U)\right)$ *is the algebra of holomorphic maps with values in* $CS^{*,k}(U)$.

Clearly, regularisation procedures $\mathcal{R}_1,\cdots,\mathcal{R}_L$ on $CS^{*,k_1}(U_1),\cdots,CS^{*,k_L}(U_L)$ induce a regularisation procedure $\mathcal{R}=\hat{\otimes}_{i=1}^L\mathcal{R}_i$ on $CS_w^{*,(k_1,\cdots,k_L)}(U)$, which we refer to as a *product* regularisation procedure.

**Definition 8** *Let* $U=U_1\times\cdots\times U_L$ *with* $U_i, i=1,\cdots,L$ *open subsets in* $\mathbb{R}^n$ *and let* $(k_1,\cdots,k_L)$ *be a multiindex of non negative integers.*
*Given a product regularisation procedure*

$$\mathcal{R}=\hat{\otimes}_{i=1}^L\mathcal{R}_i:\sigma=\otimes_{i=1}^k\sigma_i\mapsto\sigma(z)=\otimes_{i=1}^k\sigma_i(z)$$

*on* $\hat{\otimes}_{i=0}^kCS(U_i)$ *of continuous regularisations* $\mathcal{R}_i, i=1,\cdots,L$, *the continuous maps*

$$\begin{array}{rcl} CS^{\alpha_i}(U_i) & \to & C^\infty(U_i,\mathrm{Hol}(\mathbb{C}-P_i)) \\ \sigma_i & \mapsto & \left(x_i\mapsto\int_{T^*_{x_i}U_i}\mathcal{R}_i(\sigma_i)(z)(x_i,\xi_i)\,d\xi_i\right), \quad i=1,\cdots,L \end{array}$$

*induce a uniquely defined map:*

$$\begin{array}{rcl} \hat{\otimes}_{i=0}^kCS^{\alpha_i}(U_i) & \to & C^\infty(U,\mathrm{Hol}(\mathbb{C}-\cup_{i=1}^kP_i)) \\ \sigma & \mapsto & \left(x\mapsto{\int\!\!\!\!\!\!-}_{T^*_xU}\mathcal{R}(\sigma)(z)(x,\xi)\,d\xi_1\cdots d\xi_L\right). \end{array}$$

*Similarly the continuous maps*

$$\begin{array}{rcl} CS^{\alpha_i}(U_i) & \to & C^\infty(U_i,\mathbb{C}) \\ \sigma_i & \mapsto & \left(x_i\mapsto{\int\!\!\!\!\!\!-}_{T^*_{x_i}U_i}^{\mathcal{R}_i}\sigma_i(x_i,\xi_i)\,d\xi_i\right), \quad i=1,\cdots,L \end{array}$$

*induce a uniquely defined map:*

$$\begin{array}{rcl} \hat{\otimes}_{i=0}^kCS^{\alpha_i}(U_i) & \to & C^\infty(U,\mathbb{C}) \\ \sigma & \mapsto & \left(x\mapsto{\int\!\!\!\!\!\!-}_{T^*_xU}^{\mathcal{R}}\sigma(x,\xi)\,d\xi_i\right) \end{array}$$

*which induces a linear map on* $\hat{\otimes}_{i=0}^kCS(U_i)$ *called the multiple regularised integral associated with the product regularisation* $\mathcal{R}$.

The Wodzicki residue density $\mathrm{res}_{x_i}$ on $CS(U_i)$ similary give rise by continuity to $\widetilde{\mathrm{res}}_{x,k}$ on $\hat{\otimes}_{i=1}^k CS(U_i)$ in such a way that for any $x=(x_1,\cdots,x_L)\in U_1\times\cdots\times U_L$:

$$\widetilde{\mathrm{res}}_{x,k}(\otimes\sigma_i(x_i,\cdot))=\prod_{i=1}^k \mathrm{res}_{x_i}(\sigma_i(x_i,\cdot)).$$

**Theorem 2** *Let $U=U_1\times\cdots\times U_L$ with $U_i, i=1,\cdots,L$ open subsets in $\mathbb{R}^n$ and let $\sigma\in\hat{\otimes}_{i=1}^k CS(U_i)$.*
*Given a product regularisation procedure*

$$\mathcal{R}=\hat{\otimes}_{i=1}^L\mathcal{R}_i:\otimes_{i=1}^L\sigma_i\mapsto\otimes_{i=1}^L\sigma_i(z)$$

*on $CS_w(U)$ of continuous regularisations $\mathcal{R}_i, i=1,\cdots,L$ such that $\mathcal{R}_i(\sigma)(z)$ has order $\alpha_i(z)$, the map*

$$z\mapsto ⨏_{T_x^*U}\mathcal{R}(\sigma)(z)(x,\xi)\,d\xi_1\cdots d\xi_L$$

*is meromorphic with poles at most of order $L$ and:*

$$\mathrm{Res}_{z=0}^L ⨏_{T_x^*U}\mathcal{R}(\sigma)(z)(x,\xi)\,d\xi_1\cdots d\xi_L=\frac{(-1)^L}{\prod_{i=1}^L\alpha_i'(0)}\widetilde{\mathrm{res}}_{x,L}(\sigma).$$

*In particular, when $\alpha_i'(0)=\alpha'(0)$ is constant this yields*

$$\mathrm{Res}_{z=0}^L ⨏_{T_x^*U}\mathcal{R}(\sigma)(z)(x,\xi)\,d\xi_1\cdots d\xi_L=\frac{(-1)^L}{(\alpha'(0))^L}\widetilde{\mathrm{res}}_{x,L}(\sigma).$$

**Proof:** By a continuity argument, this follows from the fact that this same relation holds on products $\sigma=\otimes_{i=1}^L\sigma_i$:

$$\begin{aligned}\mathrm{Res}_{z=0}^L ⨏_{T_x^*U}\prod_{i=1}^L\mathcal{R}_i(\sigma_i)(z)(x_i,\xi_i)\,d\xi_1\cdots d\xi_L &= \prod_{i=1}^L\mathrm{Res}_{z=0}⨏_{T_{x_i}^*U_i}\mathcal{R}_i(\sigma_i)(z)(x_i,\xi_i)\,d\xi_i\\ &= \prod_{i=1}^L\frac{-1}{\alpha_i'(0)}\mathrm{res}_{x_i}(\sigma_i)\\ &= \frac{(-1)^L}{\prod_{i=1}^L\alpha_i'(0)}\widetilde{\mathrm{res}}_{x,L}(\sigma).\end{aligned}$$

□

On the grounds of this theorem, taking finite parts we set:

**Definition 9** *Given a product regularisation $\mathcal{R}=\hat{\otimes}_{i=1}^L\mathcal{R}_i$ on $CS_w(U)$, for any $\sigma\in CS_w(U)$ we call*

$$⨏_{T_x^*U}^{\mathcal{R}}\sigma(x,\xi)\ ::=\mathrm{fp}_{z=0}⨏_{T_x^*U}\sigma(z)(x,\xi)\,d\xi$$

*with $\mathcal{R}:\sigma\mapsto\sigma(z)$, the $\mathcal{R}$-regularised iterated integral of $\sigma$.*

**Remark 6** *With these notations we have:*

$$⨏_{T_{x_i}^*U}^{\mathcal{R}}d\xi\ \otimes_{i=1}^L\sigma_i(x_i,\xi_i)=\prod_{i=1}^L⨏_{T_{x_i}^*U_i}^{\mathcal{R}_i}d\xi_i\,\sigma_i(x_i,\xi_i).$$

# 4 An alternative extension of regularised integrals to tensor products of classical symbols

We now give an alternative extension of regularised integrals to tensor products of classical symbols which we then compare with the one previously defined. For this purpose we consider a map similar to the map $\sigma \mapsto \int_{|\xi|\leq R} \sigma(x,\xi) d\xi$ underlying the construction of cut-off integrals. We will henceforth work under the assumption $U_1 = \cdots = U_k = U$ an open subset of $\mathbb{R}^n$.

## 4.1 Rota-Baxter relations

**Proposition 3** *1. The map $\sigma \mapsto P(\sigma)$ defined by*

$$P(\sigma)(x,\eta) := \int_{|\xi|\leq|\eta|} \sigma(x,\xi)\, d\xi$$

*maps $CS^{*,k-1}(U)$ to $CS^{*,k}(U)$. Given $\sigma \in CS^{*,k-1}(U)$, $P(\sigma) = C + \tau$ for some constant $C$ and with $\tau \in CS^{\alpha+n,k}$. In particular, when $\alpha \in \mathbb{R}$, it has order* $\max(0, \alpha + n)$.

*2. For any $\sigma \in CS^{*,k-1}(U)$*

$$P(\sigma)(x,\eta) - \frac{\mathrm{res}_{x,k-1}(\sigma)}{k} \log^k|\eta| \quad \in CS^{*,k-1}(U) \tag{12}$$

*so that if $\sigma$ has vanishing residue of order $k-1$ then $P(\sigma)$ also lies in $CS^{*,k-1}(U)$.*

*3. $P$ obeys the following Rota-Baxter relation [EGK]:*

$$P(\sigma)\, P(\tau) = P(\sigma\, P(\tau)) + P(\tau\, P(\sigma)). \tag{13}$$

**Proof:** Replacing $R$ by $|\eta|$ in the asymptotic expansion (5) yields:

$$\begin{aligned} P(\sigma)(x,\eta) \quad &\sim \quad C_x(\sigma) + \sum_{j=0,\alpha-j+n\neq 0}^{\infty} \sum_{l=0}^{k-1} P_l(\sigma_{\alpha-j,l})(\log|\eta|)\, |\eta|^{\alpha-j+n} \\ &+ \quad \sum_{l=0}^{k-1} \frac{\mathrm{res}_{x,l}(\sigma)}{l+1} \log^{l+1}|\eta| \end{aligned} \tag{14}$$

where $P_l(\sigma_{\alpha-j,l})(X)$ is a polynomial of degree $l$ with coefficients depending on $\sigma_{\alpha-j,l}$ and where $C_x(\sigma)$ is the constant term corresponding to the finite part.
$P(\sigma)$ is therefore the sum of a symbol of order zero (the constant $C_x(\sigma)$) and a symbol $\tau$ of order $\alpha + n$ so that when $\alpha \in \mathbb{R}$, its order is $\max(0, \alpha + n)$. Furthermore, it lies in $CS^{\alpha,k}(U)$ and the coefficient of $\log^k|\eta|$ is $\frac{\mathrm{res}_{x,k-1}(\sigma)}{k}$.
The Rota-Baxter relation then follows from:

$$P(\sigma)(\eta)\, P(\tau)(\eta) \quad = \quad \int_{|\xi|\leq|\eta|} \sigma(\xi)\, d\xi \int_{|\xi|\leq|\eta|} \tau(\xi)\, d\xi$$

$$
\begin{aligned}
&= \int_{|\xi|\leq|\eta|}\sigma(\xi)\,d\xi\int_{|\tilde\xi|\leq|\xi|}\tau(\tilde\xi)\,d\tilde\xi+\int_{|\xi|\leq|\eta|}\tau(\xi)\,d\xi\int_{|\tilde\xi|\leq|\xi|}\sigma(\tilde\xi)\,d\tilde\xi\\
&= P(\sigma\,P(\tau))(\eta)+P(\tau\,P(\sigma))(\eta).
\end{aligned}
$$

□

Let $\mathcal{C}_k := \hat\otimes_{i=1}^{k+1} CS^{*,*}(U_i)$ be the space of $k$-chains built from the $CS^{*,*}(U_i)$'s. Using the Rota-Baxter map we define a map

$$P_\bullet : \mathcal{C}_{\bullet+1} \to \mathcal{C}_\bullet$$

by

$$
\begin{aligned}
P_k : \hat\otimes_{i=1}^{k+1} CS^{*,*}(U_i) &\to \hat\otimes_{i=1}^{k} CS^{*,*}(U_i)\\
P_k(\sigma)(\xi_1,\cdots,\xi_k) &:= P\left(\sigma(\xi_1,\cdots,\xi_k,\cdot)\right)(\xi_k).
\end{aligned}
$$

In particular we have:

$$P_k\left(\sigma_1\otimes\cdots\otimes\sigma_{k+1}\right)(\xi_1,\cdots,\xi_k)=\sigma_1(\xi_1)\cdots\sigma_k(\xi_k)\,P(\sigma_{k+1})(\xi_k).$$

**Theorem 3** *Let $U$ be an open subset of $\mathbb{R}^n$. For any integer $k>1$,*

1. *the composition $P_1\circ\cdots\circ P_{k-1}$ maps $\hat\otimes_{i=1}^{k} CS^{\alpha_i}(U)$ to $CS^{*,\,k-1}(U)$.*
   *For $\sigma_i\in CS(U)$,*

   $$P_1\circ P_2\circ\cdots\circ P_{k-1}(\sigma_1\otimes\cdots\otimes\sigma_k)=P\left(\cdots P(\sigma_k)\sigma_{k-1}\cdots\right)\sigma_2)\ \sigma_1 \qquad (15)$$

   *is a finite sum of log-polyhomogeneous symbols of order given by the partial sum $\alpha_1+\alpha_2+\cdots+\alpha_j+(j-1)n$ with $j=1,\cdots,k$.*
   *In particular, when $\alpha_1,\cdots,\alpha_k\in\mathbb{R}$, then $P_1\circ P_2\circ\cdots\circ P_{k-1}(\sigma)$ has order given by*

   $$
   \begin{aligned}
   & o\left(P_1\circ P_2\circ\cdots\circ P_{k-1}(\sigma)\right)\\
   =\ & \max\left(0,\cdots,\max(0,\max(0,\alpha_k+n)+\alpha_{k-1}+n),\cdots\right)+\alpha_2+n)+\alpha_1.
   \end{aligned}
   $$

2. *Furthermore,*

   $$P_1\circ\cdots\circ P_{k-1}(\sigma_1\otimes\cdots\otimes\sigma_k)(\xi_1)-\frac{\mathrm{res}_x(\sigma_k)}{(k-1)!}\,\log^{k-1}|\xi_1|\quad\in CS^{*,k-2}(U). \ (16)$$

3. *The following shuffle (or iterated Rota-Baxter) relations hold:*

   $$
   \begin{aligned}
   \prod_{i=1}^{k} P(\sigma_i) &= \sum_{\tau\in\Sigma_k} P\circ P_1\circ\cdots\circ P_{k-1}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})\\
   &= \sum_{\tau\in\Sigma_k} P\left(P\left(\cdots P(\sigma_{\tau(k)})\sigma_{\tau(k-1)}\cdots\right)\sigma_{\tau(2)}\right)\sigma_{\tau(1)}\big). \qquad (17)
   \end{aligned}
   $$

**Remark 7** *For $k=2$ equation (17) yields back equation (13).*

**Proof:**

1. By a continuity argument, it suffices to show that $P_1 \circ P_2 \circ \cdots \circ P_{k-1}(\sigma) \in CS^{*,k-1}(U)$ for any $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$. This follows from the first point in Proposition 3 by induction on $k$. Indeed, appplying it to $k = 2$, we first check that $P_1(\sigma_1) \in CS^{*,1}(U)$; then assuming that the statement holds for $k$ we can apply Proposition 3 to $P_2 \circ P_3 \circ \cdots \circ P_k(\sigma_2 \otimes \cdots \otimes \sigma_{k+1}) \in CS^{*,k-1}(U)$ from which we infer that
$$\begin{aligned} & P_1 \circ P_2 \circ \cdots \circ P_k(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{k+1}) \\ = \quad & P\left(P_2 \circ P_3 \circ \cdots \circ P_k\left(\sigma_1 \otimes \sigma_2 \otimes \cdots \sigma_{k+1}\right)\right) \in CS^{*,k}(U). \end{aligned}$$
This formula combined with Proposition 3 also yields in a similar manner that $P_1 \circ P_2 \circ P_3 \circ \cdots \circ P_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)$ is a finite sum of log-polyhomogeneous symbols of order $\alpha_1 + \cdots + \alpha_j + (j-1)n$ with $j = 1, \cdots, k$. From there we easily derive the formula for degree of $P_1 \circ P_2 \circ P_3 \circ \cdots \circ P_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)$ when the $\alpha_i$'s are real.

2. Similarly, an induction using equation (12) implies equation (16).

3. Equation (17) follows from equation (13) in a similar manner.

□

## 4.2 Iterated cut-off integrals of classical symbols

By the results of the previous paragraph, the operator $P_1 \circ \cdots \circ P_{k-1}$ sends $\hat{\otimes}_{i=1}^k CS(U)$ to $CS^{*,k-1}(U)$, a space on which we can apply cut-off regularisation described in section 2.

**Definition 10** *Let $U \subset \mathbb{R}^n$ be an open subset. For $\sigma \in \hat{\otimes}_{i=1}^k CS(U)$ and given a point $x \in U$ we set*
$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^* U} \sigma(\xi) d\xi \quad &:= \quad \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_x^* U} d\xi P_1 \circ \cdots \circ P_{k-1}(\sigma \circ \tau)(\xi) \\ &= \quad \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_{x_1}^* U_1} d\xi_1 \int_{|\xi_2| \le |\xi_1|} d\xi_2 \cdots \int_{|\xi_k| \le |\xi_{k-1}|} d\xi_k \, \sigma(\xi_{\tau(1)}, \cdots, \xi_{\tau(k)}). \end{aligned}$$

**Lemma 3** *Let $U \subset \mathbb{R}^n$ be an open subset. For $\sigma_1, \cdots, \sigma_k \in CS(U)$ such all the (left) partial sums of the orders $\alpha_1 + \alpha_2 + \cdots + \alpha_j$, $j = 1, \cdots, k$ are non integer valued, then*
$$\prod_{i=1}^k -\!\!\!\!\!\!\int_{T_x^* U_i} \sigma_i(x_i, \xi_i) \, d\xi_i = \mathrm{fp}_{R \to \infty} \prod_{i=1}^k \int_{|\xi_i| \le R} \sigma_i(x_i, \xi_i) \, d\xi_i.$$

**Proof:** We need to show that
$$\prod_{i=1}^k \mathrm{fp}_{R_i \to \infty} \int_{|\xi_i \le |R_i} \sigma_i(x_i, \xi_i) \, d\xi_i = \mathrm{fp}_{R \to \infty} \prod_{i=1}^k \int_{|\xi_i| \le R} \sigma_i(x_i, \xi_i) \, d\xi_i.$$

For each $i \in \{1, \cdots, k\}$ we have the following asymptotic expansion (see equation (5)):

$$\begin{aligned} \int_{|\xi_i| \le R_i} \sigma_i(x_i, \xi_i) d\xi_i \quad &\sim_{R_i \to \infty} \quad C_x(\sigma_i) + \sum_{m=0, \alpha_i - m + n \neq 0}^{\infty} \sum_{p=0}^{k_i} P_p(\sigma_{\alpha_i - m, p})(\log R_i) R_i^{\alpha_i - m + n} \\ &+ \quad \sum_{p=0}^{k_i} \frac{\mathrm{res}_{p, x_i}(\sigma_i)}{p+1} \log^{p+1} R_i. \end{aligned}$$

Multiplying these asymptotic expansions and setting $R_i = R$ can give rise to new finite parts other than $\prod_{i=1}^k \mathrm{fp}_{R_i \to \infty} \int_{|\xi_i| \le R_i} \sigma_i(x_i, \xi_i)\, d\xi_i = \prod_{i=1}^k C_x(\sigma_i)$. Indeed, when setting $R_i = R_j = R$, positive powers of $R_i$ arising from the asymptotic expansion of $\int_{|\xi_i| \le R_i} \sigma_i(x_i, \xi_i)\, d\xi_i$ might compensate negative powers of $R_j$ arising from the asymptotic expansion of $\int_{|\xi_j| \le R_j} \sigma_i(x_j, \xi_j)\, d\xi_j$ thus leading to a new constant term. But since such powers arise in the form $R^{\alpha_1 + \alpha_2 + \cdots + \alpha_j - m + j\, n}$ such a compensation can only happen if $\alpha_1 + \alpha_2 + \cdots + \alpha_j$ takes integer values. One therefore avoids such compensations assuming that non of all the (left) partial sums of the orders $\alpha_1 + \alpha_2 + \cdots + \alpha_j$ are non integers. $\square$

We deduce from the definition and the above lemma that cut-off regularisation "commutes" with products of symbols in certain special cases: the cut-off iterated integral of a product of symbols coincides with the product of the cut-off integrals of the symbols provided these have orders whose (left) partial sums are non integer valued.

**Proposition 4** *Let $\sigma_i \in CS^{\alpha_i}(U)$, $i = 1, \cdots, k$ such that all the (left) partial sums of the orders $\alpha_1 + \alpha_2 + \cdots + \alpha_j, j = 1, \cdots, k$ are non integer valued. Then*

$$-\!\!\!\!\!\!\int_{T_x^* U} \prod_{i=1}^k \sigma_i(x, \xi_i)\, d\xi_i = \prod_{i=1}^k -\!\!\!\!\!\!\int_{T_{x_i} U_i} \sigma_i(x, \xi_i)\, d\xi_i. \tag{18}$$

**Proof:** From the above lemma it follows that

$$\begin{aligned} & \prod_{i=1}^k -\!\!\!\!\!\!\int_{T_x U} \sigma_i(x, \xi_i)\, d\xi_i \\ = \quad & \mathrm{fp}_{R \to \infty} \prod_{i=1}^k \int_{|\xi_i| \le R} \sigma_i(x, \xi_i)\, d\xi_i \\ = \quad & \mathrm{fp}_{R \to \infty} \sum_{\tau \in \Sigma_k} \int_{|\xi_1| \le R} d\xi_1 \int_{|\xi_2| \le |\xi_1|} \cdots \int_{|\xi_k| \le |\xi_{L-1}|} d\xi_k \prod_{i=1}^k \sigma_{\tau(i)} \left(x, \xi_{\tau(i)}\right) \\ = \quad & -\!\!\!\!\!\!\int_{T_x^* U} \prod_{i=1}^k \sigma_i(x, \xi_i)\, d\xi_i. \end{aligned}$$

$\square$

**Theorem 4** *Let $\sigma_i \in CS^{\alpha_i}(U)$, $i = 1, \cdots, k$ be such that all the (left) partial sums $\alpha_1 + \alpha_2 + \cdots + \alpha_j, \quad j = 1, \cdots, k$ are non integer valued. Then the following shuffle* *relations hold:*

$$\begin{aligned}
&\prod_{i=1}^{k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi_i\, \sigma_i \\
= \quad & \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi\, P_1\circ\cdots\circ P_{k-1}\left(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)}\right)(\xi)\, d\xi \qquad (19)\\
= \quad & \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi_1 \int_{|\xi_2|\le|\xi_1|} d\xi_2 \cdots \int_{|\xi_{k-1}|\le|\xi_k|} d\xi_{k-1}\, \sigma_{\tau(k)}(\xi_k)\cdots\sigma_{\tau(1)}(\xi_1).
\end{aligned}$$

**Proof:** Recall that $P(\sigma_i)(\eta_i) = \int_{|\xi|\le|\eta_i|}\sigma_i(\xi)\, d\xi$. Applying equation (17) to $\eta_i = R$ for $i = 1, \cdots k$ and then taking the finite part when $R\to\infty$ yields the result:

$$\begin{aligned}
\prod_{i=1}^{k} -\!\!\!\!\!\!\int_{T_x^*U} \sigma_i &= \prod_{i=1}^{k} \mathrm{fp}_{R\to\infty} \int_{B_x(0,R)} \sigma_i \\
&= \mathrm{fp}_{R\to\infty}\left(\sum_{\tau\in\Sigma_k}\int_{B_x(0,R)} P_1\circ\cdots\circ P_{k-1}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})\right)\\
&= \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} P\left(\cdots P(\sigma_{\tau(k)})\sigma_{\tau(k-1)}\cdots)\sigma_{\tau(2)}\right)\sigma_{\tau(1)}.
\end{aligned}$$

The above lemma then yields the result under the assumption that all partial orders are non integer. □

## 4.3 Iterated integrals of holomorphic families of classical symbols

When the symbols have integer order, neither does the iterated cut-off integral of the tensor product of the symbols coincide with the product of their cut-off integrals (see equation (18)), nor do the shuffle relations (19) hold for cut-off integrals. However holomorphic perturbation of these symbols will have holomorphic orders, the (left) partial sums of which will be non integer outside a discrete set and both equation (18) and the shuffle relations (19) hold for these perturbed symbols.

**Proposition 5** *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $CS^{*,*}(U)$ such that $\sigma(z)$ has order $\alpha(z) = q\,z+\alpha(0)$ with $q\neq 0$.*
*For any $\sigma_i\in CS^{*,k_i}(U), i = 1,2,$ with $\sigma_i(z)$ of order $\alpha_i(z) = q\,z+\alpha_i(0)$*

1. *the map*

$$z\mapsto -\!\!\!\!\!\!\int_{T_x^*U} P(\sigma_2(z))(\xi)\,\sigma_1(z)(\xi)\, d\xi$$

*is meromorphic with at most poles of order $k_1+k_2+2$ in the discrete set*

$$P_2 := q^{-1}\left(\mathbb{Z}-\alpha_1(0)\right)\,\cup\,(2\,q)^{-1}\left(\mathbb{Z}-\alpha_1(0)-\alpha_2(0)\right).$$

*2. We have the following identity of meromorphic functions:*

$$\begin{aligned} & -\!\!\!\!\!\!\int_{T_x^*U} d\xi_1\, \sigma_1(z) -\!\!\!\!\!\!\int_{T_x^*U} d\xi_2\, \sigma_2(z) \\ = \; & -\!\!\!\!\!\!\int_{T_x^*U} P\left(\sigma_1(z)\right)(\xi)\, \sigma_2(z)(\xi)\, d\xi + -\!\!\!\!\!\!\int_{T_x^*U} P\left(\sigma_2(z)\right)(\xi)\, \sigma_1(z)(\xi)\, d\xi. \end{aligned} \tag{20}$$

**Proof:**

1. We first observe that $P(\sigma_2(z))\,\sigma_1(z)$ is the sum of a symbol $\tau_1(z) \in CS^{\alpha_1(z),k_1}(U)$ proportional to $\sigma_1(z)$ and a symbol $\tau_2(z)\,\sigma_1(z) \in CS^{\alpha_1(z)+\alpha_2(z)+n,k_1+k_2+1}(U)$ with $\tau_2(z) \in CS^{\alpha_2(z)+n,k_2+1}(U)$ (see Proposition 3). By Theorem 1 and using the linearity of the cut-off integral, we find that the cut-off integral

$$-\!\!\!\!\!\!\int_{T_x^*U} P(\sigma_2(z))(\xi)\, \sigma_1(z)(\xi)\, d\xi = -\!\!\!\!\!\!\int_{T_x^*U} \tau_1(z)(x,\xi)\, d\xi + -\!\!\!\!\!\!\int_{T_x^*U} \tau_2(z)(\xi)\, \sigma_1(z)(\xi)\, d\xi$$

is meromorphic with poles of order at most $k_1 + k_2$ at points in $P_2$ defined as in the proposition since $\alpha_1(z) = q\,z + \alpha_1(0)$ and $\alpha_1(z) + \alpha_2(z) + n = 2q\,z + \alpha_1(0) + \alpha_2(0) + n$.

2. Equation (20) then follows from applying (19) to $\sigma_i := \sigma_i(z)$ (with $k = 2$) outside the discrete set of poles.

$\square$

This generalises to the tensor product of $k$ symbols.

**Theorem 5** *Let $U$ be an open subset of $R^n$ and let $\mathcal{R}$ be a holomorphic regularisation procedure $\sigma \mapsto \sigma(z)$ on $CS(U)$ such that $\sigma(z)$ has order $\alpha(z) = q\,z + \alpha(0)$ with $q \neq 0$. For any $\sigma_i \in CS(U)$ with $\sigma_i(z)$ of order $\alpha_i(z) = q\,z + \alpha_i(0)$*

*1. the map $z \mapsto -\!\!\!\!\!\int_{T_x^*U} d\xi\, P_1 \circ \cdots \circ P_{k-1}\left(\sigma_1(z) \otimes \cdots \otimes \sigma_k(z)\right)(\xi)$ is meromorphic with poles of order at most $k$ in*

$$\mathcal{P}_k := \bigcup_{j=1}^{k} (j\,q)^{-1}\left(\mathbb{Z} - \alpha_1(0) - \alpha_2(0) - \cdots - \alpha_j(0)\right).$$

*2. The map*

$$z \mapsto -\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^{k} \sigma_i(z)\, d\xi$$

*is meromorphic with poles of order at most $k$ and we have the following equality of meromorphic functions:*

$$\begin{aligned} & \prod_{i=1}^{k} -\!\!\!\!\!\!\int_{T_x^*U} d\xi_i\, \sigma_i(z) \\ = \; & \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} P_1 \circ \cdots \circ P_k\left(\sigma_{\tau(1)}(z) \otimes \cdots \otimes \sigma_{\tau(k)}(z)\right)(\xi)\, d\xi, \end{aligned} \tag{21}$$

*where $\Sigma_k$ denotes the group of permutations on $k$ elements.*

**Proof:** Statements 1 and 2 in the theorem follow by induction on k from statements 1 and 2 of Proposition 5. Indeed, Proposition 5 with $k_1 = k_2 = 0$ yields the theorem for $k = 1$. Replacing $\sigma_2$ in Proposition 5 by $P_2 \circ \cdots \circ P_k(\sigma_2 \otimes \cdots \sigma_{k+1}) \in CS^{*,k-1}(U)$ (so that $k_2 = k - 1$ here) then yields the induction step $k \to k + 1$ since

$$\begin{aligned} & P_1 \circ P_2 \circ \cdots \circ P_k(\sigma_1(z) \otimes \sigma_2(z) \otimes \cdots \otimes \sigma_{k+1}(z)) \\ = \quad & P\left(P_2 \circ P_3 \circ \cdots \circ P_k\left(\sigma_1(z) \otimes \sigma_2(z) \otimes \cdots \sigma_{k+1}(z)\right)\right). \end{aligned}$$

□

**Corollary 1** *Under the same assumptions and using the same notations as in Theorem 5, we have the following equality of meromorphic maps:*

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi \quad &= \quad \not{-}\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi \\ &= \quad \prod_{i=1}^k -\!\!\!\!\!\!\int_{T_x^*U_i} \sigma_i(z)(x_i, \xi_i)\, d\xi_i. \end{aligned} \tag{22}$$

*The highest order pole is given by:*

$$\mathrm{Res}_{z=0}^k -\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi = \frac{(-1)^k}{\prod_{i=1}^k \alpha_i'(0)} \widetilde{\mathrm{res}}_{x,k}(\otimes_{i=1}^k \sigma_i) = \prod_{i=1}^k \frac{-1}{\alpha_i'(0)} \mathrm{res}_x(\sigma_i).$$

**Proof:** As a consequence of the shuffle relations (21), we have the following equality of meromorphic functions

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi \quad &= \quad \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{T_x^*U} P_1 \circ \cdots \circ P_k\left(\sigma_{\tau(1)}(z) \otimes \cdots \otimes \sigma_{\tau(k)}(z)\right)(\xi)\, d\xi \\ &= \quad \prod_{i=1}^k -\!\!\!\!\!\!\int_{T_x^*U_i} \sigma_i(z)(x_i, \xi_i)\, d\xi_i. \end{aligned}$$

On the other hand, by the results of section 3 we have a further equality of meromorphic functions:

$$\not{-}\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi = -\!\!\!\!\!\!\int_{T_x^*U} \otimes_{i=1}^k \sigma_i(z)\, d\xi,$$

which shows that the two regularised integrals $-\!\!\!\!\!\!\int$ and $\not{-}\!\!\!\!\!\!\int$ both coincide on tensor products of holomorphic symbols with the product of the regularised integral of each of the symbols. The Wodzicki residue formula then follows from Theorem 2. □

## 4.4 Obstructions to shuffle relations for regularised integrals of general classical symbols

The finite part of a product of meromorphic functions with poles generally does not coincide with the product of the finite parts. As a result, when the symbols have non vanishing residues, taking finite parts of the above shuffle relations on the level of meromorphic functions does not yield the expected shuffle equations for the corresponding finite parts. However, in that case a renormalisation procedure familiar to

physicists provides the obstruction in terms of counterterms arising in the renormalisation.
Let $\mathcal{M}(\mathbb{C})$ denote the algebra of meromorphic functions on $\mathbb{C}$, and let $\mathcal{M}^k(\mathbb{C})$ denote the space of meromorphic functions on $\mathbb{C}$ with poles of order at most $k$ at $z = 0$. Clearly, if $f_1, \cdots, f_k \in \mathcal{M}^1(\mathbb{C})$ then $\prod_{i=1}^k f_i \in \mathcal{M}^k(\mathbb{C})$. Let as before $\mathrm{f.p.}_{z=0} f = \lim_{z\to 0} z\, f(z)$ denote the finite part at $z = 0$ of a function $f \in \mathcal{M}^1(\mathbb{C})$. Then, in general

$$\prod_{i=1}^k \mathrm{f.p.}_{z=0} f_i(z) \neq \mathrm{f.p.}_{z=0} \prod_{i=1}^k f_i(z).$$

A renormalisation procedure taken from physics provides a recursive procedure to compute the obstruction to the equality; when the products $\prod_{i=1}^k f_i(z)$ arise from applying dimensional regularisation to Feynman type functions in the language of Etingof [E], this comes down to applying the renormalisation procedure used by physicists for connected Feynman graphs to a concatenation of disjoint one loop diagrams.

The underlying Hopf algebra ([K2],[CK]) in the situation considered here is the symmetric algebra $\mathcal{H} := \oplus_{k=0}^{\infty} \bigodot^k CS(U)$ [4] built on the vector space $CS(U)$. It is in particular commutative and cocommutative. Although very simple, this toy model is instructive. The (deconcatenation) coproduct on $\sigma = \sigma_1 \odot \cdots \odot \sigma_k$ reads:

$$\Delta\sigma = \sigma \otimes 1 + 1 \otimes \sigma + \sum_{J \subsetneq \{1,\ldots,k\}, J \neq \phi} \bigodot_{j \in J} \sigma_j \odot \bigodot_{i \notin J} \sigma_i.$$

A regularisation procedure $\mathcal{R} : \sigma \mapsto \sigma(z)$ induces a map $\phi : CS(U) \to \mathcal{M}^1(\mathbb{C})$ defined by

$$\phi(\sigma)(z) = -\!\!\!\!\!\!\int_{T_x^* U} \sigma(z)(x, \xi)\, d\xi.$$

Our previous constructions show it extends to an algebra morphism

$$\begin{array}{rcl} \Phi : \mathcal{H} & \to & \mathcal{M}(\mathbb{C}) \\ \sigma = \sigma_1 \otimes \cdots \otimes \sigma_k & \mapsto & -\!\!\!\!\!\!\int_{T^*U \times \cdots \times T^*U} \sigma_1(z)(x, \xi_1) \cdots \sigma_k(z)(x, \xi_k) d\xi_1 \cdots d\xi_k. \end{array}$$

The Hopf structure on $\mathcal{H}$ provides a recursive procedure to get a Birkhoff decomposition of the corresponding loop $\Phi(\sigma)$ for any $\sigma \in \mathcal{H}$ i.e. a factorisation of the form

$$\Phi(\sigma) = \Phi_-(\sigma)^{-1}\, \Phi_+(\sigma),$$

where $\Phi_+(\sigma)$ is holomorphic at 0. Namely, with Sweedler's notations $\Delta x = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$

$$\Phi_-(\sigma) := -T\left(\Phi(\sigma) + \sum \Phi_-(\sigma')\Phi(\sigma'')\right),$$

$$\Phi_+(\sigma) := \Phi(\sigma) + \Phi_-(\sigma) + \sum \Phi_-(\sigma')\Phi(\sigma''),$$

where $T$ is the projection on the pole part. This corresponds to Bogolioubov's prescription by which one first "prepares" [5] the symbol $\sigma$.

[4] $\odot$ denotes the symmetrised tensor product.

[5] We borrow this expression and the notations that follow from [CM] but we refer the reader to Kreimer [K2], see also [CK] for the Hopf algebra that underlies this renormalisation procedure.

There is another way of describing this renormalisation procedure via a renormalisation operator $R$ on the space of Laurent series $(z_1, \cdots, z_k) \mapsto f(z_1, \cdots, z_k)$ in several variables. For this, instead of

$$\Phi(\sigma) : z \mapsto -\!\!\!\!\!\!\int_{T_x^*U\times\cdots\times T_x^*U} \sigma_1(z)(x,\xi_1)\cdots\sigma_k(z)(x,\xi_k)\, d\xi_1\cdots d\xi_k,$$

let us consider the map

$$(z_1, \cdots, z_k) \mapsto -\!\!\!\!\!\!\int_{T_x^*U\times\cdots\times T_x^*U} \sigma_1(z_1)(x,\xi_1)\cdots\sigma_k(z_k)(x,\xi_k)\, d\xi_1\cdots d\xi_k$$

which defines a Laurent series in $(z_1, \cdots, z_k)$; setting $z_1 = z_2 = \cdots = z_k = z$ gives back the meromorphic function $\Phi(\sigma)$. Given a subset $J = \{i_1, \cdots, i_{|J|}\} \subsetneq \{1, \cdots, k\}$, setting $\bar{J} := \{i_{|J|+1}, \cdots, i_{|K|}\}$ to be its complement in $\{1, \cdots, k\}$, from such a Laurent series $f$ we build the map

$$f_J : (z, z_{i_{|J|+1}}, \cdots z_{i_{|\overline{J}|}}) \mapsto f(z_1, \cdots, z_{i_{|K|}})_{|_{z_i=z, \forall i\in J}}.$$

When $f = \Phi(\sigma_1\otimes\cdots\otimes\sigma_k) = f_1\otimes\cdots\otimes f_k$ with $f_i = \phi(\sigma_i)$ then $f_J(z, z_{i_{|J|+1}}, \cdots z_{i_{|\bar{J}|}}) = \prod_{j\in J} f_j(z)\cdot\prod_{j\in\overline{J}} f_j(z_j)$. Let us set

$$\bar{R}(f)(z) := f(z_1, \cdots, z_k)_{|_{z_i=z, 1\le i\le k}} + \sum_{\phi\neq J\subsetneq\{1,\ldots,k\}} C\left(f_J(z, z_{i_{|J|+1}}, \cdots, z_{i_{|\bar{J}|}})\right)_{z_{i_j}=z}$$

which, in the case $f = \otimes_{i=1}^k f_i$ considered above reads

$$\bar{R}(\otimes_{i=1}^k f_i)(z) := \prod_{i=1}^k f_i(z) + \sum_{\phi\neq J\subsetneq\{1,\ldots,k\}} C(\otimes_{j\in J} f_j)(z) \prod_{i\notin J} f_i(z).$$

The counterterm $C$ is defined inductively on $|J|$ by

$$C\left(f_J(z, z_{i_{|J|+1}}, \cdots, z_{i_{|\bar{J}|}})\right) := -T\left(\bar{R}\big(f_J(z, z_{i_{|J|+1}}, \cdots, z_{i_{|\bar{J}|}})\big)\right)$$

where $T$ is the projection onto the pole part of the Laurent series in $z$.
The renormalisation operator $R$ is then defined by

$$\begin{aligned} R(f) &:= \bar{R}(f) + C(f) \\ &= (1-T)(f) + (1-T)(\sum_{\phi\neq J\subsetneq\{1,\ldots,k\}} T\left(C(f_J)\right)), \end{aligned}$$

which for $f = \otimes_{i=1}^k f_i$ reads:

$$\begin{aligned} R(\otimes_{i=1}^k f_i) &:= \bar{R}(\otimes_{i=1}^k f_i) + C(\otimes_{i=1}^k f_i) \\ &= (1-T)(\prod_{i=1}^k f_i) + (1-T)(\sum_{J\subsetneq\{1,\ldots,k\}, J\neq\phi} T\left(C(\otimes_{j\in J} f_j)\right) \prod_{i\notin J} f_i. \end{aligned}$$

To illustrate this construction, let us take $k=2$ and compute $R(f)$ with $f$ a Laurent series in $z_1, z_2$ in each variable $z_i$. There are only two subsets $J \subset \{1,2\}$ to consider in the renormalisation procedure $J_1 = \{1\}$ and $J_2 = \{2\}$ and we set $f_i := f_{J_i}$ so that

$$R(f) = (1-T)(f) - (1-T)\left(T(f_1) + T(f_2)\right).$$

Writing

$$f(z_1, z_2) = \sum_{-I \leq i \leq 1; -J \leq j \leq 1} a_{i\,j} z_1^i\, z_2^j + o(\sup(|z_1|, |z_2|)),$$

where $I$, resp. $J$ is the largest order of the poles at 0 of $f_1$, resp. $f_2$ respectively, we get

$$\begin{aligned} R(f)(z) &= (1-T)\left(\sum_{-I \leq i \leq 1; -J \leq j \leq 1} a_{i\,j} z^{i+j} + o(z)\right) \\ &- (1-T)\left(\left(\sum_{i>0} a_{i\,j} z^i\, z_2^j\right)_{|z_2=z} + \left(\sum_{j>0} a_{i\,j} z_1^i\, z^j\right)_{|z_1=z}\right) \\ &= \sum_{0 \leq i+j} a_{i\,j} z^{i+j} + o(z) \\ &- \left(\sum_{i>0, i+j\geq 0} a_{i\,j} z^{i+j} + \sum_{j>0, i+j \geq 0} a_{i\,j} z^{i+j}\right) \\ &= a_{0\,0} + o(1). \end{aligned}$$

In particular, for two meromorphic functions $f_1$ and $f_2$ with simple poles:

$$\mathrm{fp}_{z=0}\left(R(f_1 \otimes f_2)\right)(z) = R(f_1\, f_2)(0) = \mathrm{fp}_{z=0} f_1(z)\, \mathrm{fp}_{z=0} f_2(z).$$

More generally, an induction procedure yields:

**Theorem 6** *Let $(z_1, \cdots, z_k) \mapsto f(z_1, \cdots, z_k)$ have a Laurent expansion in each of the variables $z_i$. The map $z \mapsto R(f)(z)$ is holomorphic at $z=0$ and its value at $z=0$ coincides with the constant term in the Laurent expansion in $(z_1, \cdots, z_k)$.*
*In particular, when $f = \otimes_{i=1}^k f_i$ where the functions $f_i, i = 1, \cdots, k$ are meromorphic at $z=0$, then $R(f)(0)$ coincides with the product of the finite parts of the $f_i$'s:*

$$\mathrm{fp}_{z\to 0}\left(R(f_1 \otimes \cdots \otimes f_k)(z)\right) = R(f_1 \otimes \cdots \otimes f_k)(0) = \prod_{i=1}^k \mathrm{fp}_{z=0} f_i(z).$$

**Proof**: The operator $R$ yields an algebra morphism on the algebra of Laurent series and takes values in meromorphic functions which are holomorphic at $z=0$ [CK]. As $f \mapsto R(f)(z)$ restricted to $\mathcal{M}(\mathbb{C})$ takes $f$ to a holomorphic function at 0 with value $R(f)(0)$ given by the finite part of $f$ at $z=0$, on a tensor product $f_1 \otimes \cdots \otimes f_k \mapsto R(f_1 \otimes \cdots \otimes f_k)(0)$ picks up the product of the finite parts of the $f_i$'s at $z=0$. By a closure argument, we conclude that the map $z \mapsto R(f)(z)$ is holomorphic at $z=0$ on the whole algebra of Laurent series and that its value at $z=0$ coincides with the constant term in the Laurent expansion in $(z_1, \cdots, z_k)$. The second assertion is straightforward. $\square$

**Remark 8** *As a consequence, if instead of using one complex parameter $z$, we regularise each $\sigma_i$ by $\sigma_i \mapsto \sigma_i(z_i)$ using a different complex parameter $z_i$ we can avoid this renormalisation procedure:*

$$\mathrm{fp}_{z_1,\cdots,z_k\to 0}\left(\otimes_{i=1}^k -\!\!\!\!\!\!\int \sigma_i(z_i)\right) = \mathrm{fp}_{z_1,\cdots,z_k\to 0}(\otimes_{i=1}^k f_i)(z_1,\cdots,z_k) = \prod_{i=1}^k \mathrm{fp}_{z=0} f_i(z).$$

Applying the above theorem to $f_i : z \mapsto -\!\!\!\!\!\!\int_{T^*_{x_i}U_i} \sigma_i(z)$ we get the following description of the obstructions to shuffle relations for general classical symbols:

**Corollary 2** *Given a regularisation procedure $\mathcal{R}$ on $CS(U)$ for any $i = 1, \cdots, k$, for any $\sigma_i \in CS(U)$,*

$$\begin{aligned}
&\prod_{i=1}^k -\!\!\!\!\!\!\int_{T^*_xU}^{\mathcal{R}} \sigma_i - \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T^*_{x_k}U}^{\mathcal{R}} d\xi_k\, P_1 \circ \cdots \circ P_{k-1}\,(\sigma_\tau)\\
= \;& \mathrm{fp}_{z=0} -\!\!\!\!\!\!\int_{T^*_xU\times\cdots T^*_xU} (f_1(z)\cdots f_k(z) - R(f_1\otimes\cdots\otimes f_k)(z))\\
= \;& \sum_{i_1+\cdots+i_k=0,(i_1,\cdots,i_k)\neq 0} a^1_{i_1}\cdots a^k_{i_k},
\end{aligned}$$

*where as before, $\sigma_\tau(i) := \sigma_{\tau(i)}$ and where the $a_i$'s correspond to the coefficients in the meromorphic expansion at $z=0$ of the cut-off integrals $-\!\!\!\!\!\!\int_{T^*_xU} \sigma_i(z) = \frac{a^i_{-1}}{z} + a^i_0 + a^i_1\, z + o(z)$.*
*In particular, the shuffle relations therefore hold if all the $\sigma_i$'s have vanishing residue.*

**Proof:** As in the proof of Corollary 1 we have

$$\begin{aligned}
&\mathrm{fp}_{z=0}\left[\prod_{i=1}^k -\!\!\!\!\!\!\int_{T^*_xU} \sigma_i(z)\, d\xi_i\right] =\\
= \;& \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T^*_xU}^{\mathcal{R}} d\xi_1 P\left(\cdots P(\sigma_{\tau(k)})\sigma_{\tau(k-1)}\cdots)\sigma_{\tau(2)}\right)(\xi_1)\sigma_{\tau(1)}(\xi_1)\\
= \;& \sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{T^*_xU}^{\mathcal{R}} d\xi_1 \int_{|\xi_2|\leq|\xi_1|} d\xi_2\cdots \int_{|\xi_k|\leq|\xi_{k-1}|} d\xi_k\, \sigma_{\tau(1)}(\xi_1)\cdots\sigma_{\tau(k)}(\xi_k).
\end{aligned}$$

On the other hand, Theorem 6 applied to $f_i : z \mapsto -\!\!\!\!\!\!\int_{T^*U} \sigma_i(z)$ yields

$$\begin{aligned}
&\mathrm{fp}_{z=0}\left(\prod_{i=1}^k -\!\!\!\!\!\!\int_{T^*_{x_i}U_i} \sigma_i(z) - R\left(\bigotimes_{i=1}^k -\!\!\!\!\!\!\int_{T^*_xU} \sigma_i(z)\right)\right)\\
= \;& \mathrm{fp}_{z=0}\left[\prod_{i=1}^k -\!\!\!\!\!\!\int_{T^*_xU} \sigma_i(z)\right] - \prod_{i=1}^k -\!\!\!\!\!\!\int_{T^*_xU}^{\mathcal{R}} \sigma_i(z)\\
= \;& \sum_{i_1+\cdots+i_k=0,(i_1,\cdots,i_k)\neq 0} a^1_{i_1}\cdots a^k_{i_k}
\end{aligned}$$

which in turn yields the result of the theorem. $\square$

# 5 Relation to multiple zeta functions

We want to adapt the previous results to symbols of operators on the unit circle. But instead of using an atlas on $S^1$ and expressing the symbol of the operators in local charts (e.g. using stereographic projections), we view $S^1$ as the Lie group $U(1)$ seen as the range of $(\mathbb{R}, +)$ under the group morphism:

$$\begin{array}{rcl} \Phi : \mathbb{R} & \to & S^1 \\ x & \mapsto & e^{i\,x} \end{array}$$

which has kernel $2\pi\mathbb{Z} \simeq \pi_1(S^1)$. This amounts to identifying $S^1$ with the quotient $\mathbb{R}/2\pi\mathbb{Z}$. In this picture, the additive group structure on $\mathbb{R}/2\pi\mathbb{Z}$ is identified with the multiplicative group structure on $S^1$:

$$\Phi(x + y + 2\pi n) = \Phi(x + 2\pi k)\Phi(y + 2\pi l) \quad \forall k, l, n \in \mathbb{Z},$$

an important fact for what follows.

## 5.1 The symbol of invariant operators on the unit circle

We then identify $S^1$ with $\mathbb{R}/2\pi\mathbb{Z}$ and note the group law additively. The kernel $K(x, y)$ of an invariant operator $P$ depends only on the difference $x - y$. It lifts to a $2\pi$-periodic function $\tilde{K}$ on $\mathbb{R}$. The Fourier transform of $\tilde{K}$ is a linear combination of Dirac masses at the integers, and can reasonably be taken as a symbol for the operator $P$. It defines then a $S^1$-invariant distribution on the cotangent $T^*S^1$. The trace of $P$, when it exists, will be given by the integral of the symbol on $T^*S^1$.

We will illustrate this principle on complex powers of the laplacian. The Laplacian

$$\Delta = -\partial_t^2$$

on $S^1$ has discrete spectrum $\{n^2, n \in \mathbb{Z}\}$. The operator $\Delta' := \Delta_{|_{\mathrm{Ker}\Delta^\perp}}$ where $\mathrm{Ker}\Delta^\perp$ denotes the orthogonal space to the kernel, has spectrum $\{n^2; n \in \mathbb{Z} - \{0\}\}$ and its square root $\sqrt{\Delta'}$ has spectrum

$$\{|n|, n \in \mathbb{Z} - \{0\}\}$$

as a consequence of which its zeta function is given by:

$$\begin{array}{rcl} \zeta_{\sqrt{\Delta'}}(z) & := & \displaystyle\sum_{n \in \mathbb{Z} - \{0\}} |n|^{-z} \\ & = & 2\displaystyle\sum_{n=1}^{\infty} n^{-z} = 2\zeta(z) \end{array}$$

where $\zeta$ is the Riemann zeta function.
$\zeta_{\sqrt{\Delta'}}(z)$ can also be seen as the canonical trace of the operator $\sqrt{\Delta'}^{-z}$ so that:

$$\begin{array}{rcl} \zeta_{\sqrt{\Delta'}}(z) & = & \mathrm{TR}\left(\sqrt{\Delta'}^{-z}\right) \\ & = & \displaystyle\int_{T^*S^1} \sigma_z(x, \xi) dx\, d\xi \end{array}$$

where $\sigma_z$ is the symbol of $\sqrt{\Delta'}$ (still to be defined). We use the Mellin transform to express $\sqrt{\Delta'}^{-z}$ in terms of the heat-kernel of $\Delta$ on $S^1$:

$$\sqrt{\Delta'}^{-z} = \frac{1}{\Gamma(\frac{z}{2})} \int_0^\infty t^{\frac{z}{2}-1} e^{-t\Delta'} \, dt.$$

We want to compute its symbol.

**Proposition 6** *The symbol of $\sqrt{\Delta'}^{-z}$ where $\Delta$ is the Laplacian on $S^1$ reads for $\xi \in \mathbb{R}$:*

$$\sigma_z(x,\xi) = \sum_{k \in \mathbb{Z} - \{0\}} |k|^{-z} \delta_k(\xi)$$

**Proof :** If $H_t(x,y) = h_t(x-y)$ denotes the heat-kernel of $\Delta$ on $S^1$ we have for every $f \in C^\infty(S^1, \mathbb{R}) \cap \mathrm{Ker}\Delta^\perp$:

$$\left(\sqrt{\Delta'}\right)^{-z} f = \frac{1}{\Gamma\left(\frac{z}{2}\right)} \int_0^\infty t^{\frac{z}{2}-1} h_t \star f \, dt.$$

Taking Fourier transforms we get

$$\sigma_z = \frac{1}{\Gamma\left(\frac{z}{2}\right)} \int_0^\infty t^{\frac{z}{2}-1} \widehat{h_t} dt$$

since $\widehat{h_t \star f} = \widehat{h_t} \cdot \hat{f}$. We therefore need to compute the Fourier transform of $h_t$ and hence an explicit expression for the heat-kernel of the Laplace operator on $S^1$.
The heat kernel of the corresponding Laplace operator on $\mathbb{R}$ at time $t$ is given by $K_t(x,y) = k_t(x-y)$ with:

$$k_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and when identifying $S^1$ with $\mathbb{R}/2\pi\mathbb{Z}$, the heat-kernel of the Laplacian on $S^1$ is given by

$$H_t(x,y) = \sum_{n \in \mathbb{Z}} k_t(x - y + 2\pi n).$$

The fact that it is "translation invariant modulo $2\pi$" enables us to define the symbol using an ordinary Fourier transform. Setting $H_t(x,y) = h_t(x-y)$ we have:

$$e^{-t\Delta} f = h_t * f \Rightarrow \widehat{e^{-t\Delta} f} = \hat{h}_t \, \hat{f}$$

so that the Fourier transform of $h_t$ can be intepreted as the symbol of $e^{-t\Delta}$. We first derive $h_t$ using the Poisson summation formula:

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{k \in \mathbb{Z}} e^{2i\pi kx} \int_{-\infty}^{+\infty} f(y) e^{-2i\pi ky} dy.$$

Hence

$$\begin{aligned} h_t(x) &= \sum_{n \in \mathbb{Z}} \tilde{k}_t\left(\frac{x}{2\pi} + n\right) \quad (\text{with} \quad \tilde{k}_t(y) := k_t(2\pi y)) \\ &= \sum_{k \in \mathbb{Z}} e^{ikx} \int_{-\infty}^{+\infty} k_t(2\pi y) e^{-2i\pi ky} dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ikx}\int_{-\infty}^{+\infty} k_t(y)e^{-iky}dy \\
&= \frac{1}{2\pi\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}} e^{ikx}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{4t}}e^{-iky}dy \\
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ikx}e^{-tk^2}
\end{aligned}
$$

since for any $\lambda > 0$ we have $\int_{-\infty}^{+\infty} e^{-iy\xi}e^{-\frac{\lambda y^2}{2}}dy = \frac{\sqrt{\pi}}{\sqrt{\lambda}}e^{-\frac{1}{2\lambda}\xi^2}$. Considering any test function $\varphi \in C_c^\infty(\mathbb{R})$ and taking Fourier transforms we find:

$$
\begin{aligned}
< \hat{h}_t, \varphi > &= < h_t, \check{\varphi} > \\
&= \int_{-\infty}^{+\infty}\sum_{k\in\mathbb{Z}} e^{-iky}e^{-tk^2}\check{\varphi}(y)\,dy \\
&= \sum_{k\in\mathbb{Z}} e^{-tk^2}\int_{-\infty}^{+\infty} e^{-iky}\check{\varphi}(y)\,dy \text{ (by Fubini's theorem)} \\
&= \sum_{k\in\mathbb{Z}} \varphi(k)e^{-tk^2}.
\end{aligned}
$$

On the other hand the orthogonal projection $p$ on Ker $\Delta$ (i.e. the constant functions) is given by:

$$
p(f)(x) = \int_{S^1} f(y)\,dy.
$$

Its kernel $K_p$ is then the constant function on $S^1 \times S^1$ equal to 1. The associated function $\tilde{K}_p$ is the constant function 1 on $\mathbb{R}$, so the symbol of $p$ is the Dirac mass at 0. From that we deduce that the symbol of $e^{-t\Delta'}$ is given by:

$$
\sum_{k\in\mathbb{Z}-\{0\}} e^{-tk^2}\delta_k.
$$

Applying the Mellin transform we finally get:

$$
\begin{aligned}
\sigma_z(x,\,\xi) &= \frac{1}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z}{2}-1}\sum_{k\in\mathbb{Z}-\{0\}} e^{-tk^2}\delta_k(\xi)\,dt \\
&= \sum_{k\in\mathbb{Z}-\{0\}} |k|^{-z}\delta_k(\xi).
\end{aligned}
$$

□

## 5.2 Discrete sums of symbols and the Euler-MacLaurin formula

The symbol $\sigma_z$ just described involves Dirac measures so that we cannot directly apply the results of sections 2, 3 and 4 derived for smooth symbols to define its truncated and regularised integrals. The presence of Dirac measures leads to discrete sums which

we need to truncate and regularise all the same; we therefore focus in this paragraph on truncated and regularised discrete sums of symbols.
As we shall see, the Euler-MacLaurin formula ([Ha] Chap. 13) builds a bridge between discrete sums on one hand and continuous integrals of symbols on the other hand. It enables to transpose the properties derived previously for regularised integrals and iterated nested integrals to regularised sums and iterated nested sums. Let us consider symbols $(x,\xi) \mapsto \sigma(x,\xi)$ of log-polyhomogeneous symbols on $\mathbb{R}$ in the class $CS^{*,k}$ (see section 1 and subsection 2.2) "with constant coefficients", i.e. independent of the first variable $x$. They clearly define symbols on the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ which we also call $\sigma$. We drop the first variable $x \in S^1$ and consider $\sigma$ as a function of a single variable $\xi \in \mathbb{R}$ (here identified with $T_x^* S^1$ for any $x \in S^1$). Let us denote by $CS^{*,k}(\mathbb{R})$ the class of such symbols and $CS^{*,*}(\mathbb{R})$ the algebra generated by the union over $l \in \mathbb{N}$ of these sets.
There is a discretised version $\mathcal{P}$ of the Rota-Baxter $P(\sigma)(\eta) = \int_{|\xi|\leq|\eta|} \sigma(\xi)\, d\xi$ of section 4:

$$\mathcal{P}(\sigma)(n) = \sum_{|k|\leq|n|} \sigma(k) \quad \forall \sigma \in CS^{*,*}(\mathbb{R}), \tag{23}$$

which has properties similar to those of $P$ as the following lemma shows.

**Lemma 4** *For any $\sigma \in CS^{*,k}(\mathbb{R})$, there is a symbol $\overline{\mathcal{P}(\sigma)} \in CS^{*,k+1}(\mathbb{R})$ with same order max$(0,\alpha+1)$ (where $\alpha$ is the order of $\sigma$) as $P(\sigma)$, which interpolates $\mathcal{P}(\sigma)$. More precisely, $\mathcal{P}(\sigma)(n) = \overline{\mathcal{P}(\sigma)}(n) \quad \forall n \in \mathbb{N}$ and for any $\sigma \in CS^{*,k}(\mathbb{R})$, $P(\sigma) - \overline{\mathcal{P}(\sigma)}$ lies in $CS^{*,k}(\mathbb{R})$.*

**Remark 9** *Let $\sigma_1$ and $\sigma_2$ be two classical symbols of order $\alpha_1$, $\alpha_2$ respectively. It follows from the above lemma and Proposition 3 that $\sigma_1\, \overline{\mathcal{P}(\sigma_2)}$ has order $\alpha_1 + \max(0, \alpha_2 + 1)$ so that if $\alpha_1 < -1$ and $\alpha_2 \leq -1$ it lies in $L^1(\mathbb{R}) \cap CS^{*,1}(\mathbb{R})$.*

**Proof:** The results of subsection 2.1 and the Euler-MacLaurin formula are the essential ingredients. We set $\tau(t) := \sigma(t) + \sigma(-t)$, so that we have:

$$\mathcal{P}(\sigma)(m) = \sum_{k=0}^{m} \tau(k).$$

Let us first recall the Euler-MacLaurin formula (formula (13.6.3) in G.H. Hardy's monograph [Ha], with adapted notations): Consider the Bernoulli numbers, defined by:

$$\frac{t}{e^t - 1} = \sum_k \frac{B_k}{k!} t^k,$$

so that

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = \frac{1}{30}, \ldots$$

and $B_{2k+1} = 0$ for $k \geq 1$. Define for any $n$ the function $\phi_n$ by the equation:

$$t\frac{e^{xt} - 1}{e^t - 1} = \sum_{n\geq 1} \phi_n(x) \frac{t^n}{n!}, \tag{24}$$

and define $\psi_n$ as the 1-periodic function equal to $\psi_n$ on the interval $[0, 1[$. We then have for $N \in \mathbb{N}$:

$$\mathcal{P}(\sigma)(N) - P(\sigma)(N) \quad = \quad \sum_{m=0}^{N} \tau(m) - \int_0^N \tau(t)\, dt$$

$$= \frac{1}{2}\tau(N) + \sum_{r=1}^{k}(-1)^{r-1}\frac{B_{2r}}{(2r)!}\tau^{(2r-1)}(N) + C_k + T_{k,N} \quad (25)$$

with:

$$\begin{aligned} C_k &= \int_0^1 \tau(t)\,dt + \frac{1}{2}\tau(1) \\ &- \sum_{r=1}^{k}(-1)^{r-1}\frac{B_{2r}}{(2r)!}\tau^{(2r-1)}(1) \\ &- \frac{1}{(2k+2)!}\int_1^{+\infty}\psi_{2k+2}(t)\tau^{(2k+2)}(t)\,dt \end{aligned} \quad (26)$$

and:

$$T_{k,N} = \frac{1}{(2k+2)!}\int_N^{+\infty}\psi_{2k+2}(t)\tau^{(2k+2)}(t)\,dt.$$

Setting

$$\overline{\mathcal{P}(\sigma)}(\xi) := P(\sigma)(\xi) + \frac{1}{2}\tau(\xi) + \sum_{r=1}^{k}(-1)^{r-1}\frac{B_{2r}}{(2r)!}\tau^{(2r-1)}(\xi) + C_k + T_{k,|\xi|}$$

then yields a symbol $\overline{\mathcal{P}(\sigma)}$ in $CS^{*,k+1}(\mathbb{R})$. Indeed, we know by Proposition 3 in section 4 that $P(\sigma)$ lies in $CS^{*,k+1}(\mathbb{R})$ and has order $\max(0,\alpha+1)$ where $\alpha$ is the order of $\sigma$. The other terms on the r.h.s lie in $CS^{*,k}(\mathbb{R})$ as a result of the fact that $\sigma$ itself lies in $CS^{*,k}(\mathbb{R})$ and have order $\leq \alpha$. Indeed, since $\tau$ lies in $CS^{*,k}(\mathbb{R})$, $\tau^{(2k+2)}$ also lies in $CS^{*,k}(\mathbb{R})$ and the remainder term $\xi \mapsto T_{k,|\xi|}$ is arbitrarily smoothing.
In particular, we see that $\overline{\mathcal{P}(\sigma)} - P(\sigma)$ lies in $CS^{*,k}(\mathbb{R})$ and has order $\leq \max(0,\alpha)$ (0 is due to the presence of the constant $C_k$) so that $\overline{\mathcal{P}(\sigma)}$ and $P(\sigma)$ have same order. $\square$

**Remark 10** *Formula (25) applied for $k$ and $k+1$ respectively shows $C_{k+1} = C_k$ so that $C_k$ stabilises at a constant $C$ for large $k$.*

On the grounds of this result we set the following definition.

**Definition 11** *For any $\sigma \in CS^{*,k}(\mathbb{R})$ the expression:*

$$\sum\!\!\!\!\!\!-_{k\in\mathbb{Z}} \sigma := \mathrm{fp}_{N\to+\infty}\sum_{k=-N}^{N}\sigma(k) := \mathrm{fp}_{R\to+\infty}\overline{\mathcal{P}(\sigma)}(R)$$

*defines the* cut-off sum *of $\sigma$ on the integers.*

**Remark 11** *Since $\overline{\mathcal{P}(\sigma)}$ has same order as $P(\sigma)$, the sum $\sum_{k=-N}^{N}\sigma(k)$ converges when the corresponding integral $\int_{-N}^{N}\sigma(\xi)\,d\xi$ converges, namely when $\sigma$ has order $< -1$ in which case we have:*

$$\sum\!\!\!\!\!\!-_{k\in\mathbb{Z}}\sigma(k) = \sum_{k\in\mathbb{Z}}\sigma(k).$$

Let us now consider holomorphic perturbations of a symbol $\sigma \in CS^{*,k}(\mathbb{R})$ (these are closely related to the "gauged symbols" of [G2]).

**Proposition 7** *Let $z \mapsto \sigma_z$ be a holomorphic family of log-polyhomogeneous symbols on $\mathbb{R}$ of order $\alpha(z) = -qz + \alpha(0)$ with $q > 0$ that lie in the class $CS^{*,k}(\mathbb{R})$.*

1. *The cut-off sum:*

$$\sum_{k \in \mathbb{Z}} \sigma_z(k) := \mathrm{fp}_{R \to +\infty} \sum_{k=-R}^{R} \sigma_z(k) \tag{27}$$

*is a meromorphic function of $z$ which coincides with $\sum_{k=-\infty}^{+\infty} \sigma_z(k)$ on the half-plane $\mathrm{Re}\ z > \frac{\mathrm{Re}\ \alpha(0)+1}{q}$, with poles in $\{\frac{\mathrm{Re}\ \alpha(0)+1-j}{q},\ q \in \mathbb{N}\}$ of order $\leq l+1$.*

2. *The difference:*

$$-\!\!\!\!\!\!\int_{\mathbb{R}} \sigma_z(\xi)\, d\xi - \sum_{k \in \mathbb{Z}} \sigma_z(k)$$

*is a holomorphic function of $z$.*

**Proof:** As can be seen from the expression of $\overline{\mathcal{P}(\sigma)}$, a holomorphic perturbation of $\sigma$ in $CS^{*,k}(\mathbb{R})$ induces a holomorphic perturbation $\overline{\mathcal{P}(\sigma)}(z) := \overline{\mathcal{P}(\sigma_z)}$ of $\overline{\mathcal{P}(\sigma)}$ in $CS^{*,k+1}(\mathbb{R})$ which reads:

$$\overline{\mathcal{P}(\sigma)}(z)(\xi) = \int_{-|\xi|}^{|\xi|} \sigma_z(t)\, dt + \frac{1}{2}\tau_z(\xi) + \sum_{r=1}^{k} (-1)^{r-1} \frac{B_{2r}}{(2r)!} \tau_z^{(2r-1)}(\xi) + C_k(z) + T_{k,|\xi|}(z)$$

where the various terms are obtained by substituting $\sigma_z$ to $\sigma$ in the r.h.s. of (25). By Theorem 1 the integral term $\int_{-|\xi|}^{|\xi|} \sigma_z$ shares all properties listed in Proposition 7. The term $\frac{1}{2}\tau_z(\xi)$ and each term inside the sum yields a holomorphic family in the symbol class $CS^{*,k}(\mathbb{R})$. The remainder term $\xi \mapsto T_{k,|\xi|}(z)$ yields a holomorphic family of smoothing symbols. Finally, formula (26) shows that $C(z)$ is holomorphic in the half-plane:

$$H_k := \{z \in \mathbb{C},\ \mathrm{Re}\ z > \frac{1 + \alpha(0) - 2k - 2}{q}\}.$$

As this holds for any $k$, the function $C(z)$ is holomorphic in the whole complex plane, and Proposition 7 is proven. $\square$

As a fundamental example, consider the holomorphic family:

$$\sigma_z(\xi) = \chi(\xi)|\xi|^{-z},$$

where $\chi$ is a cut-off function which vanishes around 0 and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$. One gets the expected relation between the cut-off sum of the symbol $\sigma_z$ and the zeta function:

**Corollary 3** *We have the following equality of meromorphic functions with simple poles at integer numbers:*

$$\sum_{k \in \mathbb{Z}} \sigma_z(k) = 2\,\zeta(z).$$

**Proof:** Since the cut-off sum coincides with the ordinary sum of the series when it converges absolutely, the equality holds for $z$ in the half-plane $\{\mathrm{Re}\ z > 1\}$. By item 1. of Proposition 7 the cut-off sum is a meromorphic function of $z$, which therefore coincides with the well-known meromorphic continuation of $2\zeta$. $\square$

**Remark 12** *A simple computation shows that the cut-off integral of $\sigma_z$ reads:*

$$-\!\!\!\!\!\!\int \sigma_z(\xi)\, d\xi = \frac{2}{z-1} + h(z),$$

*where $h$ is holomorphic. We then recover from item 3. of Proposition 7 that $\zeta(z) - \frac{1}{z-1}$ is holomorphic in the whole complex plane.*

## 5.3 Discrete Chen sums of symbols

Similarly to the operator $P$, the operator $\mathcal{P}$ satisfies relations reminiscent of Rota-Baxter relations of weight $-1$:

$$\mathcal{P}(\sigma)(n)\, \mathcal{P}(\tau)(n) = \mathcal{P}(\sigma\, \overline{\mathcal{P}(\tau)})(n) + \mathcal{P}(\tau\, \overline{\mathcal{P}(\sigma)})(n) + \mathcal{P}(\sigma\, \tau)(n) \quad \forall n \in \mathbb{N}$$

with an extra term $\mathcal{P}(\sigma\,\tau)$ that did not arise in the weight zero Rota-Baxter relations for integrals we considered previously. We want to build from $\mathcal{P}$ discrete Chen sums of symbols inductively in a similar manner to the way we built continuous Chen integrals of symbols from $P$. We first define from $\mathcal{P}$ the operators

$$\begin{array}{rcl} \mathcal{P}_j : \hat{\otimes}_{i=1}^{j+1} CS(\mathbb{R}) & \to & \hat{\otimes}_{i=1}^{j} \mathrm{Map}(\mathbb{N}, \mathbb{C}) \\ \mathcal{P}_j(\sigma)(n_1, \cdots, n_j) & := & \mathcal{P}\left(\sigma(n_1, \cdots, n_j, \cdot)\right)(n_j) \end{array}$$

On the grounds of Lemma 4 we derive the following result.

**Lemma 5** *Let $\sigma \in \hat{\otimes}_{i=1}^{j+1} CS(\mathbb{R})$, then*

1. *$\overline{\mathcal{P}(\sigma)}_j$ defined by*

$$\overline{\mathcal{P}(\sigma)}_j(\xi_1, \cdots, \xi_j) := \overline{\mathcal{P}\left(\sigma(\xi_1, \cdots, \xi_j, \cdot)\right)}(\xi_j)$$

   *lies in $\hat{\otimes}_{i=1}^{j-1} CS(\mathbb{R}) \otimes CS^{*,1}(\mathbb{R})$.*

2. *Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k \in \hat{\otimes}_{i=1}^{k} CS(\mathbb{R})$, then $\overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)}$ defined inductively by*

$$\overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)} := \overline{\mathcal{P}\left(\overline{\mathcal{P}_2 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)}\right)}$$

   *lies in $CS^{*,k-1}(\mathbb{R})$ and has the same order as $P_1 \circ \cdots \circ P_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)$, given by $\max(0, \cdots, \max(0, \max(0, \alpha_k + 1) + \alpha_{k-1} + 1), \cdots) + \alpha_2 + n) + \alpha_1$ where $\alpha_i$ is the order of $\sigma_i$.*

**Proof:** The first assertion is a direct consequence of Lemma 4. The second assertion then follows from an induction procedure on $j$ to check that $\overline{\mathcal{P}_{k-j} \circ \cdots \circ \mathcal{P}_{k-1} = (\sigma_1 \otimes \cdots \otimes \sigma_k)}$ maps $\hat{\otimes}^k CS(\mathbb{R})$ to $\hat{\otimes}^{k-j-1} CS(\mathbb{R}) \otimes CS^{*,j}(\mathbb{R})$. The computation of the order also follows by induction using the fact that by Lemma 4, $\overline{P(\sigma)}$ and $P(\sigma)$ have the same order derived in Theorem 3. □

We are now ready to define discrete Chen sums of symbols. Combining Lemma 5 with Lemma 4 shows that the cut-off sum of the symbol $\overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)}$ is well defined so that we can set the following definition.

**Definition 12** *For $\sigma_1, \cdots, \sigma_k \in CS(\mathbb{R})$, we call*

$$\sum^{Chen} \sigma_1 \otimes \cdots \otimes \sigma_k := \sum \overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)}$$

*the cut-off Chen sum of $\sigma := \sigma_1 \otimes \cdots \otimes \sigma_k$.*

**Remark 13** *Given the expression of the order of $\overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{k-1}(\sigma_1 \otimes \cdots \otimes \sigma_k)}$ explicited in the above lemma, it converges whenever $\alpha_1 < -1$ and $\alpha_i \leq -1$ for all $i \neq 1$ in which case we have that*

$$\sum^{Chen} \sigma_1 \otimes \cdots \otimes \sigma_k = \sum^{Chen} \sigma_1 \otimes \cdots \otimes \sigma_k$$

*is an ordinary discrete Chen sum.*

## 5.4 Multiple zeta functions

We now apply the above results to

$$\sigma_i := \sigma_{s_i} := \chi(\xi)\, |\xi|^{-s_i},$$

where $s_1, \cdots, s_k$ are real numbers and $\chi$ is a cut-off function which vanishes around 0 and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$. We want to generalise Corollary 3 to integrals of tensor products $\otimes_{i=1}^k \sigma_i(s_i)$ relating them to multiple zeta functions (investigated in [H] and [Z], see also [C] or [Wa] for a review on the subject). Applying the results of the previous paragraph to the $\sigma_i$'s of order $-s_i$ leads to the following result which gives back a known domain of convergence for multiple zeta functions.

**Theorem 7** *If $s_1 > 1$ and $s_i \geq 1$ for $i = 2, \cdots, k$ the discrete Chen sum $\sum^{Chen} \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k}$ converges and is proportional to the multiple zeta function:*

$$\sum^{Chen} \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k} = 2^k \tilde{\zeta}(s_1, \cdots, s_k) := 2^k \sum_{1 \leq n_k \leq n_{k-1} \leq \cdots \leq n_1} n_k^{-s_k} \cdots n_1^{-s_1}.$$

*It extends to all $s_i \in \mathbb{R}$ by a cut-off Chen integral of the type defined above:*

$$\tilde{\zeta}(s_1, \cdots, s_k) := 2^{-k} \sum^{Chen} \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k},$$

*where we have used the same symbol for the extended mutiple $\zeta$-function.*

**Proof:** It follows immediately from applying the results of the previous paragraph to $\sigma_i = \sigma_{s_i}$ of order $-s_i$. $\square$

As a consequence we can also write:

$$\tilde{\zeta}(s_1, \cdots, s_k) = \sum_{n=1}^{\infty} \tilde{P}_1 \circ \cdots \circ \tilde{P}_{k-1}(\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k})(n)$$

where

$$\tilde{P}(f)(m) := \sum_{1 \leq n \leq m} f(m), \quad \forall f \in \mathcal{F}(\mathbb{N}, \mathbb{C})$$

and

$$\begin{array}{rcl} \tilde{P}_j : \hat{\otimes}_{i=1}^{j+1}\mathrm{Map}(\mathbb{N},\mathbb{C}) & \to & \hat{\otimes}_{i=1}^{j}\mathrm{Map}(\mathbb{N},\mathbb{C}) \\ \tilde{P}_j(f)(n_1,\cdots,n_j) & := & \tilde{P}\left(f(n_1,\cdots,n_j,\cdot)\right)(n_j). \end{array}$$

If $s_1 > 1$ and $s_i \geq 1$ for $i \neq 1$ then clearly, we have ordinary sums:

$$\tilde{\zeta}(s_1,\cdots,s_k) = \sum_{n=1}^{\infty} \tilde{P}_1 \circ \cdots \circ \tilde{P}_{k-1}(\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k})(n).$$

**Remark 14** • *One can check that the same type of results holds with the usual multiple zeta functions*

$$\zeta(s_1,\ldots,s_k) := \sum_{1\leq n-1<n_2<\cdots<n_k} n_k^{-s_k}\cdots n_1^{-s_1}$$

*instead of* $\tilde{\zeta}(s_1,\ldots,s_k)$ *provided the large inequalities between the* $|\xi_j|$*'s and* $|n_j|$*'s are replaced by strict ones.*

- *The above results can be extended*[6] *to complex numbers* $z_i$ *instead of real numbers* $s_i$ *replacing* $s_1 \geq 1$ *and* $s_i > 1, i \neq 1$ *in the convergence assumptions by* $\mathrm{Re}(z_1) \geq 1$ *and* $\mathrm{Re}(z_i) > 1, i \neq 1$.

The well known "second shuffle relations" for multiple zeta functions [ENR] come from the natural partition of the domain:

$$P_{k,l} := \{x_1 > \cdots > x_k > 0\} \times \{x_{k+1} > \cdots > x_{k+l} > 0\} \subset ]0,+\infty[^{k+l}$$

into:

$$P_{k,l} = \coprod_{\sigma\in \mathrm{mix\ sh}(k,l)} P_\sigma,$$

where mix sh$(k,l)$ stands for the *mixable shuffles*, i.e. the surjective maps $\sigma$ from $\{1,\ldots k+l\}$ onto $\{1,\ldots m(\sigma)\}$ (for some $m(\sigma) \leq k+l$) such that $\sigma_1 < \cdots < \sigma_k$ and $\sigma_{k+1} < \cdots < \sigma_{k+l}$. The domain $P_\sigma$ is defined by:

$$P_\sigma = \{(x_1,\ldots,x_{k+l})\,/\,x_{\sigma_r} > x_{\sigma_{r+1}} \text{ if } \sigma_r \neq \sigma_{r+1} \text{ and } x_r = x_{r+1} \text{ if } \sigma_r = \sigma_{r+1}\}.$$

The second shuffle relations are:

$$\zeta_k(z_1,\cdots,z_k)\,\zeta_l(z_{k+1},\cdots,z_{k+l}) = \sum_{\sigma\in\mathrm{mix\ sh}(k,l)} \zeta_{m(\sigma)}(Z_\sigma), \tag{28}$$

where $Z_\sigma$ is the $m(\sigma)$-uple defined by:

$$Z(\sigma)_j = \sum_{i\in\{1,\ldots,k+l\},\,\sigma(i)=j} z_i.$$

[6] via an extra statement on Chen sums of holomorphic families which we omit here, but which can be established along the same lines as was the meromorphicity result on Chen integrals of holomorphic families.

For $k = l = 1$ they read:

$$\zeta(z_1)\zeta(z_2) = \zeta(z_1, z_2) + \zeta(z_2, z_1) + \zeta(z_1 + z_2).$$

Using the identification $⨍_{\mathbb{R}} \sigma_z(\xi)d\xi = 2\zeta(z)$ derived previously we can indeed compute:

$$\begin{aligned}
4\zeta(z_1)\zeta(z_2) &= \prod_{i=1}^{2} ⨍_{\mathbb{R}} D(\sigma_{z_i}) \\
&= ⨍_{\mathbb{R}} D(\sigma_{z_2}) \int_{|\xi_1|<|\xi_2|} D(\sigma_{z_1}) + ⨍_{\mathbb{R}} D(\sigma_{z_1}) \int_{|\xi_2|<|\xi_1|} D(\sigma_{z_2}) \\
&+ ⨍_{|\xi_1|=|\xi_2|} D(\sigma_{z_1}) \otimes D(\sigma_{z_2}) \\
&= 4\zeta(z_1, z_2) + 4\zeta(z_2, z_1) + 4\zeta(z_1 + z_2).
\end{aligned}$$

The verification of the general formula (28) goes along the same lines.

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