Towards the intersection theory on Hurwitz spaces

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1 Introduction

1.1 The Hurwitz problem

In paper [8] A. Hurwitz posed the following problem: count the number of isomorphism classes of branched coverings of the 2-sphere by a surface of genus g, having prescribed ramification points in the target sphere and prescribed ramification type over each ramification point. (We give a precise statement of the problem below.) During the last century, this problem attracted the researcher's attention many times, however, this direction of investigations became a central one only recently. This explosion of interest is due, first of all, to the applications the Hurwitz problem found in modern mathematical physics, namely, in the string theory and in the theory of Gromov–Witten invariants.

Up to now, the Hurwitz problem does not have a satisfactory solution: the known formulas expressing the numbers of coverings in terms of the ramification data lead to transparent answers only in few partial cases. However, some hopes for finding explicit answers are revived due to the connection, found first in [1, 2], between the enumeration of ramified coverings of the sphere and the intersection theory on moduli spaces of meromorphic functions on algebraic curves. In these papers the Hurwitz problem is reduced, in fact, to a problem about the geometry of the moduli spaces. The latter is of its own interest. Because of the construction in [2], it is closely related to the now classical problem about the geometry of moduli spaces of algebraic curves.

The present paper is devoted to the analysis of the part of the geometry of Hurwitz spaces that can shed some light on the Hurwitz problem.

1.2 Hurwitz spaces and the strata cohomology classes

Consider ramified coverings of the 2-sphere by surfaces of genus g, and let n denote the degree of the coverings. Besides topological coverings we will also consider meromorphic functions of degree n on algebraic curves of genus g. Of course, each such function is a ramified covering from the topological point of view. Denote by $\mathcal{H}_{g,n}$ the space of such functions possessing the following properties:

- all the poles are simple; that is, a function has n poles of order one, and we assume that the poles are numbered;
- the sum of the critical values of a function is 0.

According to [2], this space is a smooth complex orbifold (or even a smooth variety, provided $g = 0$ or n is sufficiently large). It is fibered over the moduli space $\mathcal{M}_{g,n}$

of complex curves of genus g with n marked points: one can associate to each meromorphic function its curve of definition, with n marked points.

The space $\mathcal{H}_{g,n}$ admits a completion, denoted by $\overline{\mathcal{H}}_{g,n}$, which consists of *stable* meromorphic functions [2, 14]. The *boundary* $\overline{\mathcal{H}}_{g,n} \setminus \mathcal{H}_{g,n}$ consists of stable functions defined on singular curves, but the only singularities arising here are nodes (simple transversal intersections). The projection $\mathcal{H}_{g,n} \to \mathcal{M}_{g,n}$ extends to a projection $\overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ of the completed space to the moduli space of *stable* curves with marked points. The fibers of the latter projection are vector spaces since linear combinations of meromorphic functions having poles of order at most one at the marked points also possess the same property. Note, however, that generally speaking this projection is not a vector bundle, since the dimension of the fiber may vary from one point of the base to another. The fiberwise projectivization $P\overline{\mathcal{H}}_{g,n}$ of the completed Hurwitz space is a compact complex orbispace. It will serve as our main moduli space, while all other necessary spaces will be constructed by means of it. By some abuse of language, we will speak below about varieties and subvarieties having in mind that our moduli spaces are usually orbifolds and suborbifolds (that is, locally they look like quotients of a complex disc modulo a finite group action).

Being a projectivization, the space $\mathcal{P}\overline{\mathcal{H}}_{g,n}$ carries a natural 2-cohomology class, namely, the first Chern class of the tautological sheaf, which we denote by $\psi = \psi_{g,n} = c_1(\mathcal{O}(1)) \in H^2(P\overline{\mathcal{H}}_{g,n})$.

On the other hand, this space contains subvarieties consisting of functions with degenerate ramifications. By the Riemann–Hurwitz formula, a general meromorphic function of degree n on a curve of genus g has $2n+2g-2$ nondegenerate ramification points (both in the source and in the target). The functions having less ramification points in the target sphere form the *discriminant* in the space of functions. One can associate to each ramification point in the target a partition of n, which is the (unordered) set of multiplicities of the preimages of the given point. It will be more convenient for us, however, to make use of *reduced partitions*, that is, sets of multiplicities of the singular preimages of the degenerate ramification points, each diminished by 1. The closure in $P\overline{\mathcal{H}}_{g,n}$ of the set of functions having ramifications of prescribed types will be denoted by $\sigma_{\alpha_1,\dots,\alpha_c}$, where the index consists of the reduced partitions over the degenerate ramification points. We call these subvarieties the *strata* of the discriminant. Thus, $\sigma_{2^1} \subset P\overline{\mathcal{H}}_{g,n}$ denotes the *caustic*, i.e., the stratum consisting of functions with two ramification points in the source glued together, while σ_{1^2} is the *Maxwell stratum* whose open part consists of functions taking the same value at two distinct critical points. The caustic and the Maxwell stratum are the only two strata of (complex) codimension one. Similarly, the stratum $\sigma_{2^1,1^2}$, of codimension 2, is the closure of the set of functions having two degenerate critical values, with a ramification point of multiplicity three over one of them, and two ramification points of multiplicity 2 over the other one, and so on. The number of nondegenerate critical values (those corresponding to the reduced partition 1^1) for a generic function in a given stratum can be easily computed by means of the Riemann–Hurwitz formula. For the sake of notation simplicity, we do not include these nondegenerate reduced partitions, as well as the values g and n, in the notation for the strata.

Each stratum is a complex subvariety of pure dimension in $P\overline{\mathcal{H}}_{g,n}$ and determines, therefore, by the Poincaré duality, a homogeneous element of the cohomology ring $H^*(P\overline{\mathcal{H}}_{g,n}) = H^*(P\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$. (Everywhere below we study cohomology with rational coefficients; that is why we omit the reference to the ground ring in the notation for cohomology.) The degree of this element coincides with the (real) codimension of the stratum. Its product with the complementary degree of the class ψ is just a rational number, and — which is a result of the geometric approach suggested in [2] — this number is closely related to the number of ramified coverings we are interested in. Before describing this relationship let us first give a precise formulation of the Hurwitz problem.

Two ramified coverings $f_1 : C_1 \to S^2$, $f_2 : C_2 \to S^2$ of the 2-sphere are said to be *isomorphic* if there is a homeomorphism $h : C_1 \to C_2$ such that $f_1 = f_2 \circ h$. Clearly, isomorphic ramified coverings have the same ramification points in the target sphere, with the same ramification types over these points, and their degrees n_1, n_2 and the genera g_1, g_2 of the covering surfaces coincide, $n_1 = n_2 = n$, $g_1 = g_2 = g$. A ramification point in the target sphere is said to be *nondegenerate*, or *simple*, if it has $n-1$ geometrically distinct preimages, among them one ramification point of order 2 and $n-2$ smooth points of the covering. As it was already mentioned, we associate to each ramification point in the target sphere a reduced partition α, which consists of the multiplicities of its singular preimages, diminished by 1. The set of all partitions α, for a given ramified covering, form the *ramification data*. Denote by $h_{\alpha_1,\alpha_2,\dots}$ the *Hurwitz number*, which equals, by definition,

$$h_{\alpha_1,\alpha_2,\dots} = \sum_f \frac{1}{|\mathrm{Aut}(f)|},$$

where the summation is carried over all isomorphism classes f of ramified coverings of the sphere of degree n by the surface of genus g having reduced partitions $\alpha_1, \alpha_2, \dots$ over the degenerate ramification points, and $|\mathrm{Aut}(f)|$ denotes the number of elements in the automorphism group of such a ramified covering (this group is always finite). Of course, the Hurwitz numbers also depend on g and n, but we do not include these parameters into the notation.

The *Hurwitz problem* consists in computing the Hurwitz numbers, and it is related to the geometry of the Hurwitz spaces by the following theorem, first stated for the case $g = 0$ in [16] and following immediately from [2].

Theorem 1.1 *We have*

$$h_{\alpha_1,\alpha_2,\dots} = \frac{|\mathrm{Aut}(\alpha_1, \alpha_2, \dots)|}{n!} \langle \sigma_{\alpha_1,\alpha_2,\dots} \psi^d \rangle,$$

where $|\mathrm{Aut}(\alpha_1, \alpha_2, \dots)|$ *denotes the order of the automorphism group of the set of reduced partitions, that is, the product of factorials of the numbers of coinciding partitions, and the angle brackets denote the multiplication in the cohomology* $H^*(P\overline{\mathcal{H}}_{g,n})$ *of* $P\overline{\mathcal{H}}_{g,n}$.

This theorem reduces the Hurwitz problem to a question about the cohomology of Hurwitz spaces. In spite of the apparent complication of the problem caused by the replacement of a group combinatorial problem by an algebro-geometric one, the geometric interpretation not only clarifies the known computational results (see [8, 6, 15, 2]), but also leads to new ones (see, e.g. [16]).

Generally speaking, the cohomology rings of moduli spaces of stable curves and closely related Hurwitz spaces are very complicated, even in the case of rational functions ($g = 0$). Our experience shows, however, that all natural geometric classes related to the calculus of singularities (as well as multisingularitites and multimultisingularities, these are the classes playing the crucial role in the Hurwitz problem) can be expressed in terms of relatively small set of "basic" (tautological, in a sense) classes. Our aim is to identify these basic classes, to describe relations between them and to express the strata in terms of these classes.

1.3 A description of main results

Let us start with the results concerning the Hurwitz numbers.

Theorem 1.2 *The following formulas take place for rational* (*i.e., referring to the case* $g = 0$) *Hurwitz numbers:*

$$\begin{aligned} h_{2^1,2^1} &= \frac{3}{4}(27\,n^2 - 137\,n + 180)\,\frac{(2\,n-6)!}{(n-3)!}\,n^{n-6} \\ h_{2^1,1^2} &= 3\,(3\,n^2 - 15\,n + 20)\,\frac{(2\,n-6)!}{(n-4)!}\,n^{n-6} \\ h_{1^2,1^2} &= 2\,(2\,n^3 - 16\,n^2 + 43\,n - 40)\,\frac{(2\,n-6)!}{(n-4)!}\,n^{n-6}. \end{aligned}$$

These formulas refer to multimultisingularities, that is, the situation where the ramification over *several* points in the target sphere is degenerate. For multisingularities, where the ramification is degenerate over a single point, the rational Hurwitz numbers are given by the well-known Hurwitz formula. The first formula in Theorem 1.2 is obtained by D. Zvonkine in [20], the two others, as far as we know, are new.

The proof of these results is based on some relations in the cohomology of Hurwitz spaces on classes represented by discriminant strata and the class ψ. We shall also make an extensive use of the class $\delta \in H^2(P\overline{\mathcal{H}}_{g,n})$ represented by functions defined on singular curves. The class δ is the pullback of the boundary class in $\overline{\mathcal{M}}_{g,n}$ under the projection $\overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Its importance in the geometry of the moduli spaces of stable curves is well-known. For example, the index of maximal self-intersection of this class is the *Weil–Petersson volume* of the moduli space, see [17]. The principal innovation produced in the study of Hurwitz spaces, is the presence of another natural 2-cohomology class, namely, the class ψ.

Note that all the results below are valid not only in the cohomology ring, but also in the Chow ring of $P\overline{\mathcal{H}}_{g,n}$. The only reason for speaking about cohomology is that cohomology calculations are sufficient for the prospective computational applications.

As far as we know, the first result concerning the relations for strata in cohomology rings in Hurwitz spaces was obtained in [16]. It expresses the caustic and the Maxwell stratum in spaces of rational functions ($g = 0$) in terms of the boundary stratum and the class ψ:

$$\begin{aligned} \sigma_{2^1} &= 6(n-1)\psi - 3\delta \\ \sigma_{1^2} &= 2(n-1)(n-6)\psi + 4\delta. \end{aligned} \tag{1}$$

Below we give this fact a new derivation. Classes of complex codimension greater than one also are subject to numerous relations, which are described in more detail in the main body of the paper since their formulation requires additional definitions.

1.4 The degree

Theorem 1.1 reduces the computation of a Hurwitz number to the calculation of the intersection index of a stratum with the class ψ, taken in the complementary degree. Such a class ψ naturally arises on any space of orbits of the group $\mathbb{C}^*$ action, and it behaves functorially under equivariant maps of varieties endowed with actions of this group. That is why the intersection of a subvariety with the complementary degree of this class deserves the name of "degree".

Let A be a complex variety, and suppose the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers acts on it without fixed points. Let B be the variety of orbits of the $\mathbb{C}^*$-action on A and suppose B is compact. Denote by $\Psi = c_1(O(1)) \in H^2(B)$ the tautological class of the $\mathbb{C}^*$-action. For any class $\beta \in H^*(B)$ we call the *degree* of β the result of the pairing

$$\deg\,\beta = \int_B \frac{\beta}{1-\Psi} = \int_B \beta(1 + \Psi + \Psi^2 + \Psi^3 + \dots).$$

In other words, the degree of a homogeneous class β is its intersection index with the class Ψ taken in the degree equal to the codimension of β.

Example 1.3 A weighted projective space $\mathbb{C}P_{w_0,\dots,w_n}$ is the quotient space of the complement to the origin in the vector space $\mathbb{C}^{n+1}$ endowed with the following action of $\mathbb{C}^*$:

$$\lambda : (x_0, \dots, x_n) \mapsto (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n), \quad \lambda \in \mathbb{C}^*.$$

The integer exponents w_i are called the *weights* of the action. Each linear representation of $\mathbb{C}^*$ has such a form. The degree of a weighted projective space $\mathbb{C}P_{w_0,\dots,w_n}$ with weights $w_0, \dots, w_n$ is $(\prod w_i)^{-1}$.

The following functoriality property of the degree follows immediately from the definitions.

Proposition 1.4 *Suppose $f : B_1 \to B_2$ is a finite morphism of two compact orbispaces induced by an equivariant morphism $g : A_1 \to A_2$ of two complex varieties of the same dimension. Then* $\deg\,B_1 = \deg\,f \cdot \deg\,B_2$.

Theorem 1.1 is an immediate consequence of this proposition if we consider the map $L_{g,n} : P\overline{\mathcal{H}}_{g,n} \to \mathbb{C}P_{2,3,\dots,2n+2g-2}$ taking a meromorphic function to the unordered set of its critical values (both the function and the set of its critical values are considered up to multiplication by a constant).

1.5 Singularities, relative Chern classes, and universal polynomials

Equations (1) are obtained in [16] by counting intersection indices of the classes σ_{2^1}, σ_{1^2} with a basic set of classes of complementary dimension (that is, of complex dimension 1, curves). However, as the genus or the codimension of the strata grow, this method becomes cumbersome: both the choice of a basic set of classes of complementary dimension and the computation of the intersection indices with such a class are complicated problems, without any effective solutions known. In the present paper, we suggest another approach to the search for relations among the cohomology classes. This approach is based on Thom's theory of universal polynomials of singularities, extended to the case of multisingularities in [13, 12, 11, 10]. Let us explain briefly the basics of this theory.

Let $F : M \to N$ be a general holomorphic map of smooth complex varieties (we assume, for definiteness, that the varieties are of the same dimension: although the theory works for other cases as well, we will apply it only in the situation of coinciding dimensions). The *total Chern class* of F is the fraction

$$\begin{aligned} c(F) &= \frac{c(F^*TN)}{c(TM)} = 1 + c_1(F^*(TN) - c_1(TM)) \\ &\quad + (c_2(F^*TN) - c_1(F^*TN)c_1(TM) + c_1^2(TM)) + \cdots \in H^*(M); \end{aligned}$$

here $c(\cdot)$, for a vector bundle argument, denotes, as usual, the total Chern class of the vector bundle. The homogeneous components of the total Chern class are called the *Chern classes* of F and are denoted by $c_i(F) \in H^{2i}(M)$:

$$\begin{aligned} c_1(F) &= c_1(F^*TN) - c_1(TM) \\ c_2(F) &= c_2(F^*TN) - c_1(F^*TN)c_1(TM) - c_2(TM) + c_1^2(TM) \end{aligned}$$

and so on. A map F has a singularity at a given point if its differential dF is degenerate at this point. Singularities are classified according to their types. The closure of the locus of points where F has a singularity of a given type determines, by the Poincaré duality, a cohomology class in $H^*(M)$. Thom's theory states that this class can be expressed as a polynomial in the Chern classes of F, and the coefficients of this polynomial depend only on the type of the singularity, but not on the varieties M and N or the map F.

The notion of both a general map and the type of a singularity should be defined more exactly, see, e.g. [13]. Here we give only the simplest example, that of the singularity A_1: a map has a singularity A_1 at a given point if the rank of its differential at this point is one less than the maximal one and the restriction of the map to the kernel of the differential is nondegenerate. For a general map, the locus of points with the A_1-singularity is a subvariety of complex codimension 1. The cohomology class of its closure simply coincides with $c_1(F) = c_1(F^*TN) - c_1(TM)$, which can be easily seen by considering the pullback of a differential 1-form on N to M. Already in the simplest cases this equation for the class A_1 produces nontrivial results. Suppose, for example, that the dimension of both varieties is 1, i.e., we deal with a ramified covering, say of degree n, of a complex curve by a complex curve. Then we obtain the Riemann–Hurwitz formula for the number of ramification points of the ramified covering:

$$c_1(F) = n(2 - 2g_N) - (2 - 2g_M) = 2(n-1) - 2(ng_N - g_M),$$

where g_M and g_N are the genus of the covering and the covered surface, respectively.

A similar statement for *multisingularities* states that the closure in N of the locus of points such that F has singularities of prescribed types at these points is described as a universal polynomial in the pushforwards $F_*((c_1(F))^{k_1}(c_2(F))^{k_2}\dots) \in H^*(N)$ of monomials in the Chern classes of F. We also extend this approach to the case of *multimultisingularities*, that is, subvarieties in moduli spaces of functions consisting of functions with prescribed multisingularities.

We apply the universal polynomials approach to the universal map over the Hurwitz space, which we are now going to define. Denote by $\overline{\mathcal{U}}_{g,n}$ the *universal curve* over $\overline{\mathcal{H}}_{g,n}$, i.e., a complex orbifold together with a map $\overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{H}}_{g,n}$ whose fiber over a point $f \in \overline{\mathcal{H}}_{g,n}$ is the curve of definition of f (or, to be more precise, the quotient of this curve modulo the action of the automorphism group of f). Another variety is simply the direct product $\mathbb{C}P^1 \times \overline{\mathcal{H}}_{g,n}$, with a distinguished coordinate along the first factor. The *universal map* over the Hurwitz space takes a point of the universal curve $\overline{\mathcal{U}}_{g,n}$ to the value of the corresponding function at this point. It is fibered over the Hurwitz space.

Remark 1.5 It is well known that the universal curve over the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ coincides with the moduli space $\overline{\mathcal{M}}_{g,n+1}$ of stable curves with $n+1$ marked points. Indeed, a point in a fiber of the universal curve can be considered as the $(n+1)$ st marked point along the fiber. Hurwitz spaces admit a similar construction. Namely, the space $\overline{\mathcal{H}}_{g,n+1}$ can be identified with the total space of the relative tangent bundle over the universal curve $\overline{\mathcal{U}}_{g,n}$. Indeed, a point of a fiber determines the $(n+1)$ st marked point, while a tangent vector at this point determines the principal part of a function with a pole of order one at this point.

The multiplicative group $\mathbb{C}^*$ of nonzero complex numbers acts on each of the three spaces $\overline{\mathcal{H}}_{g,n}$, $\overline{\mathcal{U}}_{g,n}$, and $\mathbb{C}P^1 \times \overline{\mathcal{H}}_{g,n}$ by multiplying a function by a constant. On $\overline{\mathcal{U}}_{g,n}$, this action is fiberwise trivial, while $\mathbb{C}^*$ acts as multiplication by a constant on a fiber $\mathbb{C}P^1$ of the direct product (the latter action is well defined since we have chosen a coordinate along the fiber). The universal map is equivariant with respect to these actions, and the strata in the Hurwitz space are invariant with respect to them. After restricting the universal curve and the direct product to the complement to the locus of fixed points of the action on $\overline{\mathcal{H}}_{g,n}$ and taking the quotient of the resulting spaces modulo the action we obtain the triangle of spaces and maps which will be the main subject of our study. Note that since the action of $\mathbb{C}^*$ on the fibers of $\mathbb{C}P^1 \times \overline{\mathcal{H}}_{g,n}$ is nontrivial, the quotient of the direct product does not coincide with the direct product $\mathbb{C}P^1 \times P\overline{\mathcal{H}}_{g,n}$; instead, it is isomorphic to the projectivization of the rank two vector bundle $\mathbb{C} \oplus \mathcal{O}(1)$ over $P\overline{\mathcal{H}}_{g,n}$.

Introduce the simplified notation $B_{g,n}$ for the projectivized Hurwitz space $P\overline{\mathcal{H}}_{g,n}$, $X_{g,n}$ for the quotient of the universal curve (which is, in its own turn, the universal curve over the projectivized Hurwitz space), $Y_{g,n}$ for the quotient of the direct product, $p_{g,n}$ (respectively, $q_{g,n}$) for the projection of $X_{g,n}$ (respectively, $Y_{g,n}$) to $B_{g,n}$, and $f_{g,n}$ for the factorized universal map, $f_{g,n} : X_{g,n} \to Y_{g,n}$. In other words, we consider the commutative triangle

$$\begin{tikzcd} X_{g,n} \arrow[rr, "f_{g,n}"] \arrow[dr, "p_{g,n}"'] & & Y_{g,n} \arrow[dl, "q_{g,n}"] \\ & B_{g,n} & \end{tikzcd} \tag{2}$$

Our first goal is to show that the relative Chern classes of the universal map can be expressed in terms of few 'basic' classes in $H^*(X_{g,n})$ and then describe the strata in the space $B_{g,n}$ in terms of the pushforwards of polynomials in these basic classes. We are able to do this because the types of singularities of $f_{g,n}$ are few in number and they are independent of g and n, the fact that leads to universal formulas.

Note that since the Hurwitz space $B_{g,n}$ is a moduli space of stable meromorphic functions of degree n on curves of genus g, and $X_{g,n}$ is the universal curve over $B_{g,n}$, any family of stable meromorphic functions of degree n on curves of genus g is induced from the projection $p_{g,n} : X_{g,n} \to B_{g,n}$. In particular, linear relations among cohomology classes that are valid in $X_{g,n}$ and $B_{g,n}$ are preserved in arbitrary families.

1.6 The structure of the paper

In the next section we analyze local singularities that the universal map can acquire, and introduce basic cohomology classes. Then we show that the relative Chern classes of the universal map indeed can be expressed in terms of the basic classes. Next we produce expressions for some strata in $B_{g,n}$ in terms of pushforwards of polynomials in the basic classes. The last section is devoted to calculations of the degrees of the strata.

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2 Singularities of fibered maps with one-dimensional fibers

The relative Chern classes of the universal map $f_{g,n}$ are closely related to the singularities of this map. Since the map $f_{g,n}$ is fibered over the base $B_{g,n}$ and the fibers are one-dimensional, the singularities are not too complicated. The classification of simple isolated singularities of fibered maps with one-dimensional fibers [5] includes singularities of types A_k and $I_{k,l}$. The map $f_{g,n}$ acquires singularities of these types. However, besides isolated singularities this map also has (with the exception of the simplest cases $g = 0$, $n = 2$, or $n = 3$) nonisolated ones. The latter arise if the underlying fiber possesses an irreducible component where the function is constant. Note that being constant on an irreducible component does not necessarily make the function unstable: it remains stable if either the genus of the component is at least 1, or, for a rational component, if it has at least three nodes. Of course, a complete study of Hurwitz spaces following our approach requires an analysis of nonisolated singularities as well, which we do not present here, and we obtain only partial results. We plan to discuss nonisolated singularities in a separate paper.

2.1 Local singularities

Consider the commutative triangular diagram of spaces and holomorphic maps of the form

$$\begin{tikzcd} \mathbb{C}^{m+1} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{C}\times\mathbb{C}^m \arrow[dl, "q"] \\ & \mathbb{C}^m & \end{tikzcd} \tag{3}$$

where q is the projection of the direct product to the second factor. We suppose that the fibers of p are one-dimensional (although not necessarily smooth). Restricting f to the fibers of p we obtain a family of functions on (noncompact) curves. The natural 'right' equivalence of such diagrams allows arbitrary coordinate changes in the base $\mathbb{C}^m$, as well as fiberwise coordinate changes in the source space $\mathbb{C}^{m+1}$ fibered over the chosen coordinate change in the base. We add to this equivalence group the possibility to multiply the function by an arbitrary constant holomorphically depending on the point of the base. Such extension does not affect the classification of singularities, but leads to a more rich theory of characteristic classes.

Since the map q is a given projection, and p is the composition of f and q, the singularities of the diagram are totally determined by that of f. A point $x \in \mathbb{C}^{m+1}$ is a *local singularity* of f if the rank of f at x is smaller than $m + 1$. We make the following two assumptions:

- all singularities f acquires are generic, i.e., the complement to the set of such singularities forms an open dense subset in the space of the diagram jets;

- the projection $p : \mathbb{C}^{m+1} \to \mathbb{C}^m$ has no singularities more complicated than the Morse fold $(x, y) \mapsto xy$.

For the universal map we are interested in, the second assumption is satisfied, while the first one is not: the restriction of a map to a fiber may prove to be constant on some irreducible component of the fiber. As a result, the formulas we obtain are true for the universal mapping ‘up to some classes supported on the subvariety of nonisolated singularities’. However, in the case of rational functions the subvariety of nonisolated singularities has codimension 3, which allows us to obtain exact formulas for up to codimension 2 strata.

Under the above assumptions a map f has singularities of the following types:

- singularities of type A_k, $k = 1, 2, \ldots$, which are realized at those smooth points of fibers of p where the restriction of f to the fiber has the form $x \mapsto x^{k+1}$ in a suitable coordinate x;

- singularities of type $I_{k,l}$, $k \geq l \geq 1$, which are realized at those singular points of fibers of p where the restriction of f to the plane with the coordinates x, y such that p looks like $(x, y) \mapsto xy$ in these coordinates has the form $f : (x, y) \mapsto (x^k + y^l, xy)$.

At these points, one can choose coordinates on the base $\mathbb{C}^m$ such that f is a direct sum of an identity map and the following miniversal unfoldings, for A_k and $I_{k,l}$, respectively:

- A_k:$(x, b_2, \ldots, b_{k-1}) \mapsto (x^{k+1} + b_2x^{k-1} + \cdots + b_kx, b_2, \ldots, b_k)$
- $I_{k,l}$: $(x, y, a_1, \ldots, a_{k-1}, b_1, \ldots, b_{l-1}) \mapsto (x^k + a_1x^{k-1} + \cdots + a_{k-1}x + y^l + b_1y^{l-1} + \cdots + b_{l-1}y, xy, a_1, \ldots, a_{k-1}, b_1, \ldots, b_{l-1})$.

The action of the group $\mathbb{C}^*$ in these coordinates looks as follows:

- for the singularity A_k:

$$\lambda : (x, b_2, \ldots, b_k) \mapsto (\lambda x, \lambda^2 b_2, \ldots, \lambda^k b_k), \qquad \lambda \in \mathbb{C}^*;$$

- for the singularity $I_{k,l}$:

$$\begin{aligned}\lambda :&(x, y, a_1, \ldots, a_{k-1}, b_1, \ldots, b_{l-1}) \\ &\mapsto (\lambda^l x, \lambda^k y, \lambda^l a_1, \ldots, \lambda^{l(k-1)} a_{k-1}, \lambda^k b_1, \ldots, \lambda^{k(l-1)} b_{l-1}), \qquad \lambda \in \mathbb{C}^*.\end{aligned}$$

2.2 Characteristic classes of singularities

Suppose that diagram (3) has a singularity of type A_k at the origin. Without loss of generality we may suppose that $k = m + 1$ and f is simply the miniversal unfolding of A_k, as above. The space $\mathbb{C}^k$ at the upper left corner of the diagram contains subvarieties corresponding to all singularities A_i with $i \leq k$. These subvarieties are invariant with respect to the action of $\mathbb{C}^*$, therefore, to the singularity type A_i a subvariety $[A_i]$ in the weighted projective space $\mathbb{C}P^{(k)} = \mathbb{C}P_{1,2,\ldots,k}$ obtained by factorization of $\mathbb{C}^k$ modulo the action of $\mathbb{C}^*$ is associated, whence a cohomology class $[A_i] \in H^{2i}(\mathbb{C}P^{(k)})$. Denote by $\Sigma \in H^2(\mathbb{C}P^{(k)})$ the characteristic class of the singularity A_1, that is, the first Chern class of f, and denote by $\Psi \in H^2(\mathbb{C}P^{(k)})$ the characteristic class $c_1(\mathcal{O}(1))$.

Theorem 2.1 *The characteristic classes of the singularities $[A_i]$ in the unfoldings of the singularities A_k are given by the formula*

$$[A_i] = P_i(\Sigma, \Psi) = \Sigma(2\Sigma - \Psi)(3\Sigma - 2\Psi)\dots(i\Sigma - (i-1)\Psi).$$

In particular, the expressions for these classes in terms of Ψ, Σ are independent of the order k of the deformed singularity A_k.

Proof. The cohomology of the space $\mathbb{C}P^{(k)}$ are induced from the cohomology of the classifying space BG of the group $G = \mathbb{C}^*$. This classifying space is $\mathbb{C}P^\infty$, and its cohomology ring is isomorphic to the ring of polynomials $\mathbf{Z}[t]$ under the natural choice of the generator t. Therefore, all elements of the cohomology ring $H^*(\mathbb{C}P^{(k)})$ can be expressed as polynomials in the variable t. In particular, $\Psi = (k+1)t$, $\Sigma = kt$. The expression for $[A_i]$ in terms of t must be

$$[A_i] = c_i \cdot (k)_i t^i$$

for some constant c_i. Here the Pohgammer symbol $(k)_i$ denotes the polynomial of degree i in k equal to

$$(k)_i = k(k-1)(k-2)\dots(k-i+1).$$

Indeed, $[A_i]$, being homogeneous of degree i, is proportional to t^i and the coefficient of proportionality is a polynomial in k of degree i which vanishes for all $k < i$. Now, in the case $i = k$ we have $[A_i] = i!t^i$, which means that $c_i = 1$. The expression for $A[i]$ in terms of Ψ and Σ is obtained by substituting the values $t = \Psi - \Sigma$, $k = \Sigma/(\Psi - \Sigma)$. The theorem is proved.

The miniversal unfolding of the singularity $I_{k,l}$ contains strata corresponding to singularities of type A_i, for $i < k+l$ as well as strata corresponding to the singularities of type $I_{i,j}$, for $i \le k$, $j \le l$. These strata are also invariant with respect to the $\mathbb{C}^*$ action. After taking the quotient modulo this action we obtain the weighted projective space $\mathbb{C}P^{(k,l)} = \mathbb{C}P_{l,l,2l,\dots,(k-1)l,k,k,2k,\dots,(l-1)k}$, and all the cohomology classes considered below in this section belong to the cohomology of this space.

The singularities of types $I_{i,j}$ have support on the subvariety Δ of singular points of the projection p. In the above local coordinates, this subvariety is given by the equation $x = y = 0$. Let us add to the classes $\Psi, \Sigma \in H^2(\mathbb{C}P^{(k,l)})$ the class $\Delta \in H^4(\mathbb{C}P^{(k,l)})$. Consider also the normal bundle to Δ and denote by N the first Chern class of this normal bundle. The class Δ itself can be understood as the second Chern class of this normal bundle, and these two classes can be treated as (commuting) operators acting on Δ. Even less formally, we may treat N as the cohomology class $N \in H^2(\mathbb{C}P^{(k,l)})$ (which makes sense only when applied to Δ). We are interested in expressing the classes of singularities of type A_i in the unfoldings of $I_{k,l}$ in terms of the classes Σ, Ψ, Δ, N.

Theorem 2.2 *The classes $[A_i]$ in the unfoldings of the singularities $I_{k,l}$ can be represented as weighted homogeneous polynomials of degree i in the variables Σ, Ψ, Δ, N in such a way that*

- *these polynomials are universal, that is, they depend only on i and not on k and l;*
- *the polynomial corresponding to the class $[A_i]$ can be represented as the sum of two polynomials, the first one coinciding with the polynomial $P_i(\Sigma, \Psi)$ defined above, the second one having support on the stratum Δ of double points;*
- *any polynomial having support on Δ can be expressed as a polynomial in Δ, N, and Ψ.*

The last statement of the theorem follows from the obvious relation $\Sigma\Delta = \Psi\Delta$.

Denote the polynomial corresponding to the class $[A_i]$ by R_i. It has the form $R_i = P_i + Q_{i-2}\Delta$, where the polynomial P_i is already known, and $Q_{i-2} = Q_{i-2}(\Delta, N, \Psi)$ is a weighted homogeneous polynomial of degree $i-2$. We did not manage to find an explicit formula for these polynomials, but they can be computed using the indeterminate coefficients method. Let us show how this method works for small i.

Similarly to the situation above, the cohomology ring of the weighted projective space $\mathbb{C}P^{(k,l)}$ is induced from the cohomology ring of the classifying space $BG = \mathbb{C}P^{\infty}$ of the group $G = \mathbb{C}^*$. Therefore, each element of $H^*(\mathbb{C}P^{(k,l)})$ can be expressed as a polynomial in t. In particular, we have

$$\Psi = \Sigma = klt, \qquad N = (k+l)t, \qquad \Delta = klt^2.$$

Let us find, using these values, the coefficients of the polynomials Q_0 and Q_1. We know that $Q_0 = a$ for some constant a and that the stratum A_2 in the unfolding of $I_{1,1}$ is empty. Here $k = l = 1$ and we have $\Psi = \Sigma = t$, $\Delta = t^2$, whence

$$R_2 = \Sigma(2\Sigma - \Psi) + a\Delta = t^2 + at^2 = 0.$$

Therefore, $a = -1$, i.e.,

$$R_2 = \Sigma(2\Sigma - \Psi) - \Delta.$$

Similarly, $Q_1 = aN + b\Psi$ for some constants a and b. Using the fact that the stratum A_3 is empty in the unfoldings of each of the two singularities $I_{1,1}$ and $I_{2,1}$, we obtain the following system of two linear equations:

$$\begin{cases} 1 + 2a + b = 0 \\ 8 + 6a + 4b = 0 \end{cases}$$

(we divided both equations by t^3 thus making use of their homogeneity), whence $a = 2$, $b = -5$, i.e.,

$$Q_1 = 2N - 5\Psi.$$

Similar calculations give

$$\begin{aligned} Q_2 &= -(6N^2 - 15N\Psi + 15\Psi^2 - 8\Delta) \\ Q_3 &= 24N^3 - 62N^2\Psi + 63N\Psi^2 - 35\Psi^3 - 60N\Delta + 84\Psi\Delta \\ Q_4 &= -(120N^4 - 322N^3\Psi + 343N^2\Psi^2 - 196N\Psi^3 + 70\Psi^4 \\ &\qquad - 432N^2\Delta + 812N\Psi\Delta - 469\Psi^2\Delta + 180\Delta^2). \end{aligned}$$

Note that for A_i, the number of indeterminate coefficients is $[i^2/4]$, where the square brackets denote the floor, and this is exactly the number of $I_{k,l}$-singularities not containing the stratum A_i in their miniversal unfoldings.

2.3 Residual polynomials for multisingularities

Fix a partition $\alpha = 1^{m_1}2^{m_2}\dots$ (here only finitely many exponents are nonzero). Consider the set of points y in the space $\mathbb{C} \times \mathbb{C}^m$ in the upper right vertex of the commutative triangle (3) possessing the following properties:

- the fiber $p^{-1}(q(y))$ of the projection p is nonsingular;
- the restriction of f to this fiber has, among the preimages of y, exactly m_1 copies of the singularity A_1, exactly m_2 copies of the singularity A_2, and so on. (Besides singularities of prescribed types, the point y can also have nonsingular preimages, whose number is determined by the degree of f.)

Denote the closure of this set by $A_\alpha \subset \mathbb{C} \times \mathbb{C}^m$.

Introduce the exponential generating function in infinitely many variables $t_1, t_2, \dots$ having formal coefficients R_α, where α runs over nonzero partitions:

$$\mathcal{R}(t_1, t_2, t_3, \dots) = \sum R_{1^{m_1}2^{m_2}} \frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \dots .$$

We are going to prove that *there exist weighted homogeneous polynomials R_α in the classes Σ, Ψ, Δ, N such that after substituting them as the coefficients into the generating series $\mathcal{R}$ the cohomology classes of the strata $A_{1^{k_1}2^{k_2}\dots}$ coincide with the coefficients of the monomials $t_1^{k_1}t_2^{k_2}\dots$ in the exponent* $\exp(f_*\mathcal{R})$.

Here f_* denotes the *Gysin homomorphism*, i.e., the pushforward homomorphism in the cohomology of the quotient spaces modulo the $\mathbb{C}^*$-action. This map is only additive, not multiplicative, structure homomorphism. The polynomials R_α introduced in this way are called the *residual polynomials for multisingularities.* Of course, there is nothing strange in the fact that the polynomials R_{k^1} coincide with the polynomials R_k in the previous section (indeed, in this case the points under consideration have only one singular preimage).

In order to compute the residual polynomials we must know how the homomorphism f_* acts on the cohomology classes we are interested in. Since the target space of f also is a weighted projective space, its cohomology ring also is induced from $H^*(BG)$ being thus a ring of polynomials in t. Therefore, f_* looks like $f_* : t^i \mapsto ct^i$ for some constant c. This constant cannot be other than the constant f_*1, that is, the degree of f. If f is the unfolding of A_k, then its degree is $k+1$.

We start with computing the polynomials R_α for the standard unfoldings of singularities of type A_k (that is, in the presence of only singularities of type A). These polynomials depend only on the classes Σ and Ψ. They can be computed as follows.

After substituting $\Sigma = kt$, $\Psi = (k+1)t$ into the generating function $\mathcal{R}$ we must obtain

$$\exp((k+1)\mathcal{R}(t_1,\dots)) = 1 + (k+1)_2\frac{t_1}{1!}t + \left((k+1)_3\frac{t_2}{1!} + (k+1)_4\frac{t_1^2}{2!}\right)t^2$$
$$+ \left((k+1)_4\frac{t_3}{1!} + (k+1)_5\frac{t_1}{1!}\frac{t_2}{2!} + (k+1)_6\frac{t_1^3}{3!}\right)t^3 + \dots \qquad (4)$$

Indeed, the coefficient of $\frac{t_1^{i_1}}{i_1!}\frac{t_2^{i_2}}{i_2!}\dots$ in this exponent must be $t^{i_1+2i_2+3i_3+\dots}$ times a polynomial in k which vanishes for all $k = -1, 0, 1, \dots, 2i_1 + 3i_2 + \dots - 2$, since this is the codimension of the corresponding stratum.

Note that the right-hand side of Eq. (4) can be conveniently rewritten in the form[1]

$$\exp((k+1)\mathcal{R}(t_1,\dots)) = e^{\tau_1\tau+\tau_2\tau^2+\tau_3\tau^3+\dots}s^{k+1}|_{s=1}.$$

Here s is an auxiliary variable commuting with t_i and t, and we use the notation

$$\tau = t\frac{d}{ds}, \qquad \tau_i = t_i\frac{d}{ds}.$$

Now, the residual polynomials R_α are easily calculated by taking the logarithm of the right-hand side of Eq. (4), dividing it by $(k+1)$ and substituting $t = \Psi - \Sigma$, $k = \Sigma/(\Psi - \Sigma)$. For small codimensions we thus obtain

$$\begin{aligned}
R_{1^2} &= -\Sigma(3\Sigma - \Psi)\\
R_{1^12^1} &= -6\Sigma(2\Sigma - \Psi)^2\\
R_{1^3} &= \frac{4}{3}\Sigma(15\Sigma^2 - 13\Sigma\Psi + 3\Psi^2)\\
R_{1^13^1} &= -4\Sigma(5\Sigma - 3\Psi)(3\Sigma - 2\Psi)(2\Sigma - \Psi)\\
R_{2^2} &= -\frac{3}{2}\Sigma(2\Sigma - \Psi)(20\Sigma^2 - 25\Sigma\Psi + 8\Psi^2)\\
R_{1^22^1} &= -\frac{3}{2}\Sigma(2\Sigma - \Psi)(15\Sigma^2 - 17\Sigma\Psi + 5\Psi^2)\\
R_{1^4} &= -2\Sigma(105\Sigma^3 - 160\Sigma^2\Psi + 84\Sigma\Psi^2 - 15\Psi^3).
\end{aligned}$$

The Δ-part of the residual polynomials in the unfoldings of the singularities $I_{k,l}$ can be computed recursively, using the indeterminate coefficients method as above.

Let us compute, for example, the polynomial R_{1^2}. Because of weighted homogeneity, it has the form

$$R_{1^2} = \Sigma\Psi - 3\Sigma^2 + a\Delta$$

for some constant a (the Δ-free part is already known), while the stratum A_{1^2} can be expressed as the unknown coefficient of $t_1^2/2!$ in $\exp(f_*\mathcal{R})$:

$$\begin{aligned}
[A_{1^2}] &= \frac{1}{2}(f_*R_{1^1})^2 + f_*R_{1^2}\\
&= \frac{1}{2}(f_*\Sigma)^2 + f_*(\Sigma\Psi - 3\Sigma^2 + a\Delta).
\end{aligned}$$

[1]We are obliged for this remark to D. Zagier.

For the unfolding of $I_{k,l}$, the degree is $k+l$. Substituting the expressions for the classes Ψ, Δ, Σ in terms of the class t for the unfolding of the singularity $I_{1,1}$ and using the fact that the stratum A_{1^2} in this unfolding is empty, we obtain the linear equation

$$\frac{1}{2} \cdot 2^2 + 2 - 6 + 2a = 0$$

which yields $a = 1$, i.e.,

$$R_{1^2} = \Sigma\Psi - 3\Sigma^2 + \Delta = -\Sigma(3\Sigma - \Psi) + \Delta.$$

Similar calculations for residual polynomials of higher codimensional multisingularities produce the following answers:

$$\begin{aligned}
R_{1^1 2^1} &= -6\Sigma(2\Sigma - \Psi)^2 + 6\Delta(3\Psi - N_1) \\
R_{1^3} &= \frac{4}{3}\Sigma(15\Sigma^2 - 13\Sigma\Psi + 3\Psi^2) - \frac{4}{3}\Delta(10\Psi - 3N) \\
R_{1^1 3^1} &= -4\Sigma(5\Sigma - 3\Psi)(3\Sigma - 2\Psi)(2\Sigma - \Psi) \\
&\quad +4\Delta(20\Psi^2 - 17N\Psi + 6N^2 - 8\Delta) \\
R_{2^2} &= -\frac{3}{2}\Sigma(2\Sigma - \Psi)(20\Sigma^2 - 25\Sigma\Psi + 8\Psi^2) \\
&\quad +\frac{3}{2}\Delta(25\Psi^2 - 21N\Psi + 8N^2 - 12\Delta) \\
R_{1^2 2^1} &= -\frac{3}{2}\Sigma(2\Sigma - \Psi)(15\Sigma^2 - 17\Sigma\Psi + 5\Psi^2) \\
&\quad -12\Delta(20\Psi^2 - 15N\Psi + 5N^2 - 7\Delta) \\
R_{1^4} &= -2\Sigma(105\Sigma^3 - 160\Sigma^2\Psi + 84\Sigma\Psi^2 - 15\Psi^3) \\
&\quad +2\Delta(70\Psi^2 - 48N\Psi + 15N^2 - 21\Delta).
\end{aligned}$$

2.4 Residual polynomials for multimultisingularities

Now we are interested in the subvarieties in the lower vertex of the triangle (3) consisting of points b such that the restriction of f to the fiber $p^{-1}(b)$ has several degenerate multisingularities with prescribed reduced partitions $\alpha_1^{m_1}, \ldots, \alpha_c^{m_c}$ (and the necessary complementary number of nondegenerate singularities). Here the exponent marks the number of occurences of the corresponding partition. Denote by $A_{\alpha_1^{m_1}, \ldots, \alpha_c^{m_c}} \subset \mathbb{C}^m$ the closure of the set of these points.

Introduce the exponential generating function, in infinitely many indeterminates $s_{1^2}, s_{2^1}, \ldots$ indexed by partitions α, with formal coefficients $T_{\alpha_1^{m_1}, \ldots, \alpha_c^{m_c}}$ indexed by multipartitions:

$$\mathfrak{R}(s_{1^2}, s_{2^1}, \ldots) = \sum T_{\alpha_1^{m_1}, \ldots, \alpha_c^{m_c}} \frac{s_{\alpha_1}^{m_1}}{m_1!} \cdots \frac{s_{\alpha_c}^{m_c}}{m_c!}.$$

Then the following statement takes place.

Theorem 2.3 *After choosing for T_α the polynomials expressing the classes A_α, there exist weighted homogeneous polynomials $R_{\alpha_1^{m_1}, \ldots, \alpha_c^{m_c}}$, for $c \geq 2$, in the classes Σ, Ψ, Δ, N*

such that after substituting their Gysin pushforwards $f_*R_{\alpha_1,\ldots,\alpha_c}$ for $T_{\alpha_1^{m_1},\ldots,\alpha_c^{m_c}}$ in the above generating function $\mathfrak{R}$ the cohomology classes of the strata $A_{\alpha_1^{m_1},\ldots,\alpha_c^{m_c}}$ coincide with the coefficients of the monomial $\frac{s_{\alpha_1}^{m_1}}{m_1!}\ldots\frac{s_{\alpha_c}^{m_c}}{m_c!}$ in the exponent $\exp(q_*\mathfrak{R})$.

We formulate the last statement as a theorem, although it does not deserve this name yet, and it would be better termed a 'conjecture'. However, numerous calculations confirming it, as well as the similarity of this statement to that in the cases of singularities and multisingularities, ensure its validity.

Once again, the polynomials $R_{\alpha_1,\ldots,\alpha_c}$ can be found inductively using the indeterminate coefficients method. The computations make use of the fact that the map p is proper, when restricted to Σ, and all the cohomology classes we are interested in have support on Σ. Hence the action of the Gysin homomorphism p_* reduces, when expressed in terms of the basic class t, to multiplication by the degree of $p|_\Sigma$, which is the number of critical points of f on a fiber, that is, $k+1$ for the singularity A_k and $k+l$ for the singularity $I_{k,l}$. Here are the results of sample calculations:

$$\begin{aligned}
R_{2^1,2^1} &= -\Sigma(2\Sigma-\Psi)(5\Sigma-3\Psi)+\frac{1}{2}\Delta(14\Psi-5N)\\
R_{2^1,1^2} &= 3\Sigma(2\Sigma-\Psi)(5\Sigma-2\Psi)-3\Delta(7\Psi-2N)\\
R_{1^2,1^2} &= -\frac{3}{4}\Sigma(30\Sigma^2-21\Psi\Sigma+4\Psi^2)+\frac{1}{4}\Delta(57\Psi-14N).
\end{aligned}$$

3 Relative Chern classes of the universal map

In this section we are going to show that the relative Chern classes of the universal map indeed belong to the subring in $H^*(X_{g,n})$ generated by the basic classes. This will confirm that the classes $[A_k]$, as well as the residual polynomials for multisingularities and multimultisingularities also belong to this subring. The main tool in the computation of the relative Chern classes of $f_{g,n}$ is the Grothendieck–Riemann–Roch theorem. It also produces nontrivial relations for the Gysin pushforwards of polynomials in the basic classes. We start with recalling this theorem in the specific situation we are interested in. Our exposition follows [3, 18].

3.1 The Grothendieck–Riemann–Roch theorem

The Grothendieck–Riemann–Roch (below, simply GRR) formula compares the Chern character of a coherent sheaf over a variety M with the Chern character of the image of this sheaf under a map $P : M \to A$. We suppose that both M and A are compact and smooth.

In one of the forms (which is probably the most useful one) the GRR formula reads

$$\mathrm{ch}(P_!\beta) = P_*(\mathrm{ch}(\beta)\mathrm{td}(P)^{-1}). \tag{5}$$

Here β is a coherent sheaf on M, ch denotes the Chern character, and td is the Todd class of the morphism P.

Let us give definitions of the notions participating in the statement of the GRR formula.

3.1.1 The Chern character

The *Chern character* of the sheaf of sections of a vector bundle or, simply, the Chern character of a vector bundle, is defined as the sum of exponents of the *Chern roots*

$$\mathrm{ch}(E) = \exp(e_1) + \cdots + \exp(e_r),$$

where r is the rank of the vector bundle E. In particular, for a line bundle E we have $\mathrm{ch}(E) = \exp(c_1(E))$.

Being a symmetric function in the Chern roots, the Chern character of a vector bundle can be expressed in terms of its Chern classes:

$$\begin{aligned}\mathrm{ch}(E) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &+ \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \\ &\frac{1}{120}(c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5) + \dots,\end{aligned}$$

where each c_i is the i th elementary symmetric function in the Chern roots.

Due to the properties of Chern roots, for an exact sequence of vector bundles

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0,$$

we have the *Whitney formula*

$$\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'').$$

The Whitney formula allows one to extend the notion of Chern character to arbitrary coherent sheaves. A *coherent sheaf* on M is either the sheaf of sections of some vector bundle, or, more generally, a sheaf that can be inserted as a last nontrivial term into an exact sequence of sheaves all whose elements but the last one are sheaves of sections of vector bundles (such an exact sequence is called a *resolvent* for the given sheaf). The Chern character of a coherent sheaf is defined as the alternating sum of the Chern characters of the other terms of the exact sequence.

If $\Sigma \subset M$ is a subvariety of codimension 1 and $i : \Sigma \hookrightarrow M$ is the inclusion, then the corresponding resolvent simply looks like

$$0 \longrightarrow N_\Sigma^\vee \longrightarrow \mathcal{O}_M \longrightarrow i_*(\mathcal{O}_\Sigma) \longrightarrow 0,$$

where N_Σ is a line bundle over M such that Σ coincides with the zero locus of its section. In a tubular neighborhood of Σ the bundle N_Σ simply coincides with the natural extension of the normal bundle to Σ in M. Now the Whitney formula yields

$$\begin{aligned}\mathrm{ch}(i_*(\mathcal{O}_\Sigma)) &= \mathrm{ch}(\mathcal{O}_M) - \mathrm{ch}(N_\Sigma^\vee) = 1 - \exp(-\Sigma) \\ &= \frac{(1 - \exp(-\Sigma))}{\Sigma}\Sigma = \left(\frac{\Sigma}{1 - \exp(-\Sigma)}\right)^{-1}\Sigma,\end{aligned}$$

where we are able to replace the operator of intersection with the first Chern class of the normal line bundle by Σ because, by assumption, Σ coincides with the zero locus of some holomorphic section of the normal bundle.

If Z is a subvariety of codimension 2, then the resolvent becomes more complicated. Namely, it has the form

$$0 \longrightarrow \Lambda^2 N_Z^\vee \longrightarrow N_Z^\vee \longrightarrow \mathcal{O}_M \longrightarrow i_*(\mathcal{O}_Z) \longrightarrow 0, \tag{6}$$

where Λ^2 denotes the skew-symmetric square of the vector bundle. This sequence is written in the assumption that Z is the zero set of a globally defined section of some vector bundle N_Z over M. We shall see that the final expression for $\mathrm{ch}(i_*(\mathcal{O}_Z))$ is determined by the restriction of N_Z to Z, which is the normal bundle of Z. Moreover, one can show (see [3]) that the resulting expression is universal and is valid without the assumption that the normal bundle to Z extends to the whole M. The Whitney formula yields

$$\mathrm{ch}(i_*\mathcal{O}_Z) = \mathrm{ch}(\Lambda^2 N_Z^\vee) + \mathrm{ch}(\mathcal{O}_M) - \mathrm{ch}(N_Z^\vee).$$

The fictitious vector bundle $N_Z^\vee$ is of rank 2 and its total Chern class is $c(N_Z^\vee) = 1 - N_1 + N_2$, where $N_i = c_i(N_Z)$, $i = 1, 2$. Hence, its Chern character is

$$\begin{aligned} \mathrm{ch}(N_Z^\vee) &= 2 - N_1 + \frac{1}{2}(N_1^2 - 2N_2) + \frac{1}{6}(3N_1N_2 - N_1^3) + \frac{1}{24}(N_1^4 - 4N_1^2N_2 + 2N_2^2) \\ &\quad - \frac{1}{120}(N_1^5 - 5N_1^3N_2 + 5N_1N_2^2) + \dots . \end{aligned}$$

The vector bundle $\Lambda^2 N_Z^\vee$ is linear and

$$c(\Lambda^2 N_Z^\vee) = 1 - N_1,$$

whence

$$\mathrm{ch}(\Lambda^2 N_Z^\vee) = \exp(-N_1).$$

Now, combining these results we obtain

$$\mathrm{ch}(i_*\mathcal{O}_Z) = N_2 - \frac{1}{2}N_1N_2 + \dots = \left(1 - \frac{1}{2}N_1 + \frac{1}{12}(N_1^2 - N_2) - \frac{1}{24}(N_1^3 - N_1N_2)\dots\right) Z,$$

since both Z and N_2 are identified with the class of the zero section of the normal bundle N_Z.

3.1.2 The Todd class

We start with defining the Todd class of a vector bundle, and then turn to the Todd class of a mapping. For a vector bundle E over M, its *Todd class* is another symmetric function in the Chern roots, namely,

$$\begin{aligned} \mathrm{td}(E) &= \frac{e_1}{1 - \exp(-e_1)} \cdots \frac{e_r}{1 - \exp(-e_r)} \\ &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \dots . \end{aligned}$$

For an exact sequence of vector bundles like above, we have

$$\mathrm{td}(E) = \mathrm{td}(E')\mathrm{td}(E'').$$

The following formula is true for the top Chern class of a vector bundle E of rank r:

$$\sum_{i=0}^{r}(-1)^i \mathrm{ch}(\Lambda^i E^\vee) = c_r(E)\mathrm{td}(E)^{-1}.$$

The *Todd class of a variety* M is the Todd class of its tangent bundle TM. The *Todd class* of a proper mapping $F : M \to N$ is the Todd class of the difference $F^*TN - TM$, that is,

$$\mathrm{td}(F) = \mathrm{td}(F^*TN - TM) = \frac{\mathrm{td}\ F^*TN}{\mathrm{td}\ TM}.$$

3.1.3 The push-forward of a coherent sheaf

For a coherent sheaf E over M, the push-forward $P_!(E)$ of E with respect to a map $P : M \to A$ is the most complicated object among those defined in the present section. Methods of computing it depend on the relative dimensions of A and M. In particular, if the dimension of A coincides with that of M, then the image $P_!(E)$ coincides with the direct image $P_*(E)$. Recall that, by definition, the *direct image* P_*E of a sheaf E over M with respect to a map $P : M \to A$ is the sheaf whose group of global sections over a sufficiently small open set $U \subset A$ coincides with the restriction of E to $P^{-1}(U)$. If dim M is dim $A + 1$ (this is exactly the situation to which we will apply the GRR theorem), then

$$P_!(E) = R^0(P_*(E)) - R^1(P_*(E)),$$

where $R^0(P_*(E)) = P_*(E)$ is the direct image of E, and $R^1(P_*(E))$ is the first Grothendieck higher direct image. The latter can be computed by applying the Serre duality:

$$R^1(P_*(E)) = (R^0(P_*(E^\vee \otimes \omega)))^\vee = (P_*(E^\vee \otimes \omega))^\vee,$$

where ω is the relative dualizing sheaf.

3.2 Relative Chern classes of the universal map

Now let us return to the commutative triangle (2) of spaces and maps. From now on, we simplify the notation by omitting the indices g, n, i.e., $X_{g,n}$ is denoted by X, $f_{g,n}$ is denoted by f, and so on. Introduce the following cohomology classes in the space X, the universal curve over the projectivized Hurwitz space, which we call *basic* classes. Denote by ω the relative dualizing sheaf of the map $p : X \to B$, that is, the only holomorphic line bundle over X whose fiber at a smooth point of a fiber of p coincides with the cotangent line to the fiber. The first Chern class $c_1(\omega)$ is an element of the second cohomology group $H^2(X)$. Let Π denote the divisor of poles of the universal map $f_{g,n}$, that is, the set of points in X such that the restriction of f

to the fiber of p has a pole at this point. Recall that, being a quotient modulo an action of the group $\mathbb{C}^*$, each of the spaces X, Y and B carries the cohomology class $c_1(\mathcal{O}(1)) \in H^2$, which we denote by Ψ_X, Ψ_Y, and ψ, respectively. We also introduce notation $\Sigma = c_1(\omega) + \Psi_X + 2\Pi$ for the first relative Chern class of f.

Besides the subvarieties and classes of codimension one, we will also make use of the class $\Delta \subset X$ represented by the set of singular points of the fibers of p. The normal bundle to Δ in X has rank two. Locally it admits a splitting into a direct sum of two line bundles. Indeed, each point of Δ is a point of transversal intersection of the two sheets of the fiber curve at this point, and the tangent lines to the sheets provide a splitting of the normal bundle. (In the case of rational curves, $g = 0$, this splitting can be extended to a global splitting, since each double point cuts a rational curve into two connected components, but we will not use this property below.) Denote by N the first Chern class of the normal bundle to $\Delta \subset X$. This Chern class can be understood as an operator that can be applied to Δ. Its powers N^i also are such operators. The second Chern class of the normal bundle can be simply identified with Δ itself.

Lemma 3.1 *The basic classes satisfy the following relations (in $H^*(X)$):*

$$\Pi^2 = -\Psi_X \, \Pi, \quad \Sigma \, \Pi = 0, \quad \Pi \, \Delta = 0, \quad \Sigma \, \Delta = \Psi_X \, \Delta.$$

Proof. The equality $\Pi \, \Delta = 0$ means that the poles of any function in the Hurwitz space belong to the smooth part of the curve so that the varieties Δ and Π do not intersect. Similarly, the equality $\Sigma \, \Pi = 0$ is the algebraic expression of the fact that all the poles are simple and there is no ramification over infinity.

Now, the dualizing sheaf ω is locally trivial over Δ and it is isomorphic to the conormal line bundle of the hypersurface Π at the points of this hypersurface. This implies the equalities $c_1(\omega) \, \Delta = 0$ and $\Pi^2 = -c_1(\omega) \, \Pi$. Substituting $c_1(\omega) = \Sigma - \Psi_X - 2\Pi$ we get

$$0 = (\Sigma - \Psi_X - 2\Pi) \, \Delta = \Sigma \, \Delta - \Psi_X \, \Delta$$

and

$$\Pi^2 = -(\Sigma - \Psi_X - 2\Pi) \, \Pi = \Psi_X \, \Pi + 2\Pi^2,$$

respectively. This proves the equalities $\Sigma \, \Delta = \Psi_X \, \Delta$ and $\Pi^2 = -\Psi_X \, \Pi$ of the lemma. Note that the last one is also the pullback of the equality $\Pi_Y^2 = -\Psi_Y \, \Pi_Y$ which takes place in Y, see the next section.

Corollary 3.2 *Every polynomial in the basic classes can be represented in the form*

$$P_1(\Psi_X) \, \Pi + P_2(\Psi_X, \Sigma) + P_3(\Psi_X, N, \Delta) \, \Delta,$$

where P_1, P_2, and P_3 are certain polynomials.

The main goal of the present section is to prove the following statement and to deduce from it some relations on the pushforwards of the basic classes.

Theorem 3.3 *The total relative Chern class of f is*

$$c(f) \; = \; 1+\frac{\Sigma}{1-\Sigma+\Psi_X}-\frac{\Delta}{1+N+\Delta}. \tag{7}$$

As an immediate corollary we obtain the following statement.

Corollary 3.4 *If the only singularities of the map f, when restricted to the fibers of p, are of type A_k or $I_{k,l}$, then the residual polynomials $R_{\alpha_1,\dots,\alpha_c}$ for multimultisingularities belong to the subring in $H^*(X)$ generated by the basic classes $\Sigma, \Psi_X, \Delta, \mathbf{N}$.*

Since p is the composition of the two maps f and q, $p = f \circ q$, its total relative Chern class is the ratio of the total relative Chern class of p and the pullback of the total relative Chern class of q:

$$c(f) = \frac{f^*c(q)}{c(p)}.$$

Therefore, in order to compute it it suffices to compute the two classes $c(p)$, $c(q)$ as well as the pullback of the latter class to X. This is exactly what we are going to do now.

3.2.1 The class $c(q)$ and its pullback $f^*c(q)$

The total space Y of the fibration q is the projectivization of the rank two bundle $E = \mathbb{C} \oplus \mathcal{O}(1)$ over B. Denote by $\mathcal{T}$ the tautological line bundle over Y and set $\Pi_Y = c_1(\mathcal{T}^\vee) = -c_1(\mathcal{T})$. This class is subject to the relation $c_2(q^*(E)/\mathcal{T}) = 0$, that is

$$\Pi_Y^2 + \Pi_Y\,\Psi_Y = 0, \tag{8}$$

where $\Psi_Y = q^*(\psi) = c_1(\mathcal{O}(1))$ is the corresponding characteristic class of the $\mathbb{C}^*$-action. This relation determines the cohomology ring of Y as an algebra over $H^*(B)$ via q^*.

The lines $\mathbb{C} \oplus \{0\}$ and $\{0\} \oplus \mathcal{O}(1)$ in the fibers of E correspond to the points 'zero' and 'infinity' in the fibers of q, respectively. These points form hypersurfaces with the Poincaré dual classes

$$c_1(\mathrm{Hom}(\mathcal{T}, \mathcal{O}(1))) = \Pi_Y + \Psi_Y, \quad c_1(\mathrm{Hom}(\mathcal{T}, \mathbb{C})) = \Pi_Y,$$

respectively. Thus, Π_Y can be regarded as the class dual to the 'infinity locus' in Y, and Eq. (8) geometrically means that the zero and infinity loci have no common points.

The total Chern class of q is inverse to the total Chern class of the relative tangent bundle $\ker(q_*)$. This bundle is determined by the isomorphism $\ker(q_*) \simeq \mathrm{Hom}(\mathcal{T}, q^*(E)/\mathcal{T})$. Therefore,

$$c(q) = \frac{1}{c(\mathrm{Hom}(\mathcal{T}, q^*(E)/\mathcal{T}))} = \frac{1}{1+\Psi_Y+2\Pi_Y}.$$

Applying f^* we get $f^*(\Psi_Y) = \Psi_X$ and $f^*(\Pi_Y) = \Pi$. Therefore, we get finally

$$f^*(c(q)) = \frac{1}{1+\Psi_X+2\Pi}.$$

3.2.2 The total relative Chern class of p

For the map p, the computation of the relative Chern class is slightly more complicated. The relative Chern class of p can be computed via the exact sequence

$$0 \longrightarrow p^* T^\vee B \xrightarrow{p^\vee} T^\vee X \longrightarrow \omega \longrightarrow \mathcal{O}_\Delta \longrightarrow 0. \tag{9}$$

where, recall, ω denotes the relative dualizing sheaf of p. The subvariety $\Delta \subset X$ is exactly the locus of points, where the map p^* fails to be of corank 1 (surjective) and is of corank 2. The structure sheaf $\mathcal{O}_\Delta$ of Δ measures the difference between the relative dualizing sheaf ω and the sheaf of relative differentials.

Applying to the above exact sequence the Whitney formula we obtain

$$c(p^\vee) = 1 - c_1(p) + c_2(p) - \cdots = c(p^* T^\vee B - T^\vee X) = \frac{c(\mathcal{O}_\Delta)}{c(\omega)} = \frac{c(\mathcal{O}_\Delta)}{1 + c_1(\omega)}.$$

In order to compute the Chern class $c(\mathcal{O}_\Delta))$ we make use of the Coszul projective resolvent (6). Another application of the Whitney formula yields

$$c(\mathcal{O}_\Delta) = \frac{1 - N}{1 - N + \Delta} = 1 - \frac{\Delta}{1 - N + \Delta}.$$

Another way to obtain the same formula is to use the GRR-formula for the embedding $\Delta \hookrightarrow X$. Combining with the above expression for $c(p^\vee)$ we get

$$c(p) = \frac{1}{1 - c_1(\omega)} \left(1 - \frac{\Delta}{1 + N + \Delta}\right).$$

Now we are able to compute the total Chern class of f. Applying relations of Lemma 3.1, we get

$$\begin{aligned} c(f) &= \frac{c(p)}{f^*(c(q))} = \frac{1 + \Psi_X + 2\Pi}{1 - c_1(\omega)} \left(1 - \frac{\Delta}{1 + N + \Delta}\right) \\ &= \frac{1 + \Psi_X + 2\Pi}{1 - \Sigma + \Psi_X + 2\Pi} - \frac{1 + \Psi_X}{1 + N + \Delta}\,\Delta \\ &= \left(\frac{1 + \Psi_X}{1 - \Sigma + \Psi_X} - \frac{1 + \Psi_X}{1 + N + \Delta}\,\Delta\right) + \left(\frac{1 + \Psi_X + 2\Pi}{1 - \Sigma + \Psi_X + 2\Pi} - \frac{1 + \Psi_X}{1 - \Sigma + \Psi_X}\right) \end{aligned}$$

The first summand is equal to the expression in the formula of Theorem 7; the second summand, after reducing to the common denominator, attains the numerator

$$(1 + \Psi_X + 2\Pi)(1 - \Sigma + \Psi_X) - (1 + \Psi_X)(1 - \Sigma + \Psi_X + 2\Pi) = -2\Pi\,\Sigma = 0.$$

Theorem 7 is proved.

3.3 The pushforward homomorphisms

In order to apply the universal formulas for the classes of multisingularities, we need also to compute the pushforward homomorphisms f_*, p_*, q_*. The following theorem asserts that the homomorphism p_* determines the two others.

Theorem 3.5 *For any collection of cohomology classes $h_1, \dots, h_s$ on X satisfying the condition $h_i\,\Pi = 0$, $i = 1, \dots, s$ we have*

$$q_*(f_*(h_1)\dots f_*(h_s)) = \psi^{s-1} p_*(h_1)\dots p_*(h_s).$$

The condition $h_i\Pi = 0$ geometrically means that h_i can be represented by a cycle having no common points with the divisor Π of poles. This condition is satisfied for residual polynomials of multisingularities.

Proof. Since q is the projectivization of a vector bundle, the direct image $f_*(h)$ of an arbitrary class $h \in H^*(X)$ can be represented in the form

$$f_*(h) = q^*(a) + q^*(b)\,\Pi_Y$$

with certain classes a and b on B. The class b can be obtained by applying q_* to both sides of this equation. From the equation $q_*\Pi_Y = 1$ and the projection formula we have

$$b = q_*(f_*(h)) = p_*(h).$$

The class a is obtained by a similar argument. Namely, multiplying both sides of the above equation by $\Pi_Y + \Psi_Y$, we get

$$f_*((\Psi_X + \Pi)\,h) = q^*(a)\,(\Psi_Y + \Pi_Y) + q^*(b)\,(\Psi_Y + \Pi_Y)\,\Pi_Y = q^*(\psi\,a) + q^*(a)\,\Pi_Y,$$

and applying q_* we find $a = p_*((\Psi_X + \Pi)\,h)$. If the class h satisfies the assumption $h\,\Pi = 0$ of the theorem, then the answer simplifies:

$$f_*(h) = q^*(p_*(a))\,(\Psi_Y + \Pi_Y).$$

Therefore, under the assumptions of the theorem we have

$$f_*(h_1)\dots f_*(h_s) = (\Psi_Y + \Pi_Y)^s \prod_{i=1}^{s} q^*(p_*(h_i)) = (\Psi_Y^s + \Psi_Y^{s-1}\Pi_Y) \prod_{i=1}^{s} q^*(p_*(h_i)).$$

Applying q_*, we get from the projection formula the desired relation.

Thus to complete the computation of the cohomology classes of strata in the Hurwitz space $P\overline{\mathcal{H}}_{g,n}$ we should be able to compute the pushforward class $p_*(R)$ for any residual polynomial represented in the form $R = P_1(\Psi_X, \Sigma) + P_2(\Psi_X, N, \Delta)\,\Delta$. Set

$$\xi_k = p_*(\Sigma^{k+1}), \qquad \delta_{k,\ell} = p_*(N^k \Delta^{\ell+1}).$$

There are no universal relations between the classes ξ_k and $\delta_{k,\ell}$ which would be valid for any g and n. Of course, for any particular g and n the cohomology space $H^*(P\overline{\mathcal{H}}_{g,n})$ is of finite dimension, so that for fixed g and n such relations do exist. Some of them are discussed in the next section.

3.4 Relations for the cohomology classes in B

Let us apply the GRR formula to the projection $p : X \to B$ and to the relative dualizing sheaf ω. Our approach follows that of [18], where the same thing is done for the universal curve projection onto the moduli space of curves. We have

$$\mathrm{ch}(p_!\omega) = p_*(\mathrm{ch}(\omega) \cdot \mathrm{td}(p)^{-1}).$$

The sheaf on the left-hand side $p_!\omega = R^0p_*\omega - R^1p_*\omega$ is expressed in terms of the Hodge bundle $\Lambda = \Lambda_{g,n}$ over $B = P\overline{\mathcal{H}}_{g,n}$ (recall that the fiber of the Hodge bundle at a point $(C; f)$ is the space of holomorphic 1-forms over C, so that the rank of the bundle is g): $R^0p_*\omega = p_*\omega = \Lambda$ and, by the Serre duality, $R^1p_*\omega = p_*\mathbb{C} = \mathbb{C}$. The classes entering the right-hand side of the formula were computed in the previous sections: using the equality $c_1(\omega)\Delta = 0$, we have

$$\begin{aligned}\mathrm{ch}(\omega)\mathrm{td}^{-1}(p) = \mathrm{ch}(\omega)\mathrm{td}(\omega^\vee - \mathcal{O}_\Delta^\vee) = e^{c_1(\omega)}\frac{-c_1(\omega)}{1-e^{c_1(\omega)}}\mathrm{td}(-\mathcal{O}_\Delta^\vee)&\\ = \frac{c_1(\omega)}{1-e^{-c_1(\omega)}} + \mathrm{td}(-\mathcal{O}_\Delta^\vee),&\end{aligned}$$

where the Todd class of the (fictitious) sheaf $-\mathcal{O}_\Delta^\vee$ is determined by its Chern class

$$c(-\mathcal{O}_\Delta^\vee) = \frac{1+N+\Delta}{1+N} = 1 + \frac{\Delta}{1+N}.$$

Substituting to the GRR formula, we get

$$\begin{aligned}&(g-1) + \mathrm{ch}_1(\Lambda) + \mathrm{ch}_2(\Lambda) + \mathrm{ch}_3(\Lambda) + \mathrm{ch}_4(\Lambda) + \mathrm{ch}_5(\Lambda) + \dots\\ &\quad = \frac{1}{2}p_*(c_1(\omega)) + \frac{1}{12}p_*(c_1^2(\omega)+\Delta) - \frac{1}{720}p_*(c_1^4(\omega) + (N^2-3\Delta)\Delta)\\ &\qquad + \frac{1}{30240}p_*(c_1^6(\omega) + (N^4 - 5N^2\Delta + 5\Delta^2)\Delta) + \dots .\end{aligned} \tag{10}$$

This relation says that the homogeneous components of the Chern character (and hence the Chern classes) of the Hodge bundle are determined by the pushforwards of the basic classes.

Note that the right-hand side of (10) has no terms of even positive degree. It follows that the even part of $\mathrm{ch}(\Lambda)$ is trivial, that is $c(\Lambda + \Lambda^\vee) = 1$. In the case $g = 0$ the Hodge bundle is trivial so that the left-hand side of (10) is trivial in positive degrees. It provides non-trivial identities between the pushforwards of the basic classes:

$$p_*(c_1^2(\omega)+\Delta) = 0, \quad p_*(c_1^4(\omega) + (N^2-3\Delta)\Delta) = 0, \quad \dots .$$

4 Applying universal polynomials to the study of stratification of Hurwitz spaces

The knowledge of the residual polynomials for singularities, multisingularities, and multimultisingularities allows one to obtain cohomological expressions for the strata in the Hurwitz spaces. Unfortunately only complete formulas, including the contribution of nonisolated singularities, are required. That's why we present universal expressions for strata and compute the degrees of the corresponding strata only in the case of rational functions and only for small codimensions.

4.1 Universal expressions for strata

In the table below we present the expressions for the strata of codimension one and two moduli classes supported on the strata of nonisolated singularities via pushforwards of the basic classes, from the computed residual classes. In this table, we use notation

$$\xi_k = p_*(\Sigma^{k+1}), \quad \xi_0 = 2n - 2 + 2g, \quad \delta_{k,l} = p_*(N^k \Delta^{l+1}).$$

$$\begin{aligned}
\sigma_{2^1} &= -\psi\,\xi_0 + 2\,\xi_1 - \delta_{0,0}\\
\sigma_{1^2} &= \frac{1}{2}\,\psi\,\xi_0\,(\xi_0+2) - 3\,\xi_1 + \delta_{0,0}\\
\sigma_{3^1} &= 2\,\xi_0\,\psi^2 - 7\,\xi_1\,\psi - 5\,\delta_{0,0}\,\psi + 6\,\xi_2 + 2\,\delta_{1,0}\\
\sigma_{1^1 2^1} &= -\,\xi_0\,(\xi_0+6)\,\psi^2 + 2\,(\xi_0+12)\,\xi_1\,\psi + (18-\xi_0)\,\delta_{0,0}\,\psi - 24\,\xi_2 - 6\,\delta_{1,0}\\
\sigma_{1^3} &= \frac{1}{6}\,\xi_0\,(\xi_0^2 + 6\,\xi_0 + 24)\,\psi^2\\
&\quad - \frac{1}{3}\,(9\,\xi_0 + 52)\,\xi_1\,\psi + \frac{1}{3}\,(3\,\xi_0 - 40)\,\delta_{0,0}\,\psi + 20\,\xi_2 + 4\,\delta_{1,0}\\
\sigma_{2^1,2^1} &= \frac{1}{2}\,(\xi_0 - 6)\,\xi_0\,\psi^2 - (2\,\xi_0 - 11)\,\xi_1\,\psi + \frac{1}{2}\,\delta_{0,0}^2 + 2\,\xi_1^2 - 10\,\xi_2\\
&\quad + (\xi_0 + 7)\,\delta_{0,0}\,\psi - 2\,\xi_1\,\delta_{0,0} - \frac{5}{2}\,\delta_{1,0}\\
\sigma_{2^1,1^2} &= -\frac{1}{2}\,\xi_0\,(\xi_0^2 - 2\,\xi_0 - 12)\,\psi^2 + (\xi_0^2 + \xi_0 - 27)\,\xi_1\,\psi - \delta_{0,0}^2 - 6\,\xi_1^2 + 30\,\xi_2\\
&\quad - \frac{1}{2}\,(\xi_0^2 + 42)\delta_{0,0}\,\psi + 5\,\xi_1\,\delta_{0,0} + 6\,\delta_{1,0}\\
\sigma_{1^2,1^2} &= \frac{1}{8}\,(\xi_0 - 4)\,\xi_0\,(\xi_0^2 + 4\,\xi_0 + 6)\,\psi^2 - \frac{3}{4}\,(2\,\xi_0^2 - 4\,\xi_0 - 21)\,\xi_1\,\psi + \frac{1}{2}\,\delta_{0,0}^2\\
&\quad + \frac{9}{2}\,\xi_1^2 - \frac{45}{2}\,\xi_2 + \frac{1}{4}\,(2\,\xi_0^2 - 4\,\xi_0 + 57)\,\delta_{0,0}\,\psi - 3\,\xi_1\,\delta_{0,0} - \frac{7}{2}\,\delta_{1,0}.
\end{aligned}$$

In the next table we present simplified expressions for the same strata in the case $g = 0$ with the identities

$$\xi_0 = 2n - 2, \quad \xi_1 = 4\,(n-1)\,\psi - \delta_{0,0}$$

of Sec. 3.4 taken into account. Note that in the case $g = 0$ the nonisolated singularities have codimension 3 in B, therefore, they give no contribution to the strata of the table.

$$
\begin{aligned}
\sigma_{2^1} &= 6\,(n-1)\,\psi - 3\,\delta_{0,0}\\
\sigma_{1^2} &= 2\,(n-6)\,(n-1)\,\psi + 4\,\delta_{0,0}\\
\sigma_{3^1} &= -24\,(n-1)\,\psi^2 + 2\,\delta_{0,0}\,\psi + 6\,\xi_2 + 2\,\delta_{1,0}\\
\sigma_{1^1 2^1} &= 12\,(n-1)\,(n+6)\,\psi^2 - 6\,n\,\delta_{0,0}\,\psi - 24\,\xi_2 - 6\,\delta_{1,0}\\
\sigma_{1^3} &= \frac{4}{3}\,(n-1)\,(n^2-17\,n-30)\,\psi^2 + 4\,(2\,n-1)\,\delta_{0,0}\,\psi + 20\,\xi_2 + 4\,\delta_{1,0}\\
\sigma_{2^1,2^1} &= 2\,(n-1)\,(9\,n+10)\,\psi^2 - 2\,(9\,n-7)\,\delta_{0,0}\,\psi + \frac{9}{2}\,\delta_{0,0}^2 - 10\,\xi_2 - \frac{5}{2}\,\delta_{1,0}\\
\sigma_{2^1,1^2} &= 12\,(n-9)\,(n-1)\,n\,\psi^2 - 6\,(n^2-13\,n+11)\,\delta_{0,0}\,\psi - 12\,\delta_{0,0}^2 + 30\,\xi_2 + 6\,\delta_{1,0}\\
\sigma_{1^2,1^2} &= (n-1)\,(2\,n^3-30\,n^2+145\,n-60)\,\psi^2 + \frac{1}{2}\,(16\,n^2-144\,n+125)\,\delta_{0,0}\,\psi\\
&\qquad + 8\,\delta_{0,0}^2 - \frac{45}{2}\,\xi_2 - \frac{7}{2}\,\delta_{1,0}.
\end{aligned}
$$

4.2 Degrees of the strata in genus 0

In the case $g = 0$, the compactified Hurwitz space $P\overline{\mathcal{H}}_{0,n}$ is fibred over the compactified moduli space $\overline{\mathcal{M}}_{0,n}$ of rational curves with marked points. The fiber is the projective space PE, where $E = L_1^\vee \oplus \cdots \oplus L_n^\vee$ is the Whitney sum of the tangent lines to the curve at the marked points. It follows that the cohomology of $\overline{\mathcal{M}}_{0,n}$ considered as an algebra over $H^*(\overline{\mathcal{M}}_{0,n})$ is generated by the class $\psi = c_1(\mathcal{O}(1))$ which is subject to the relation

$$\psi^n + c_1(E)\,\psi^{n-1} + \cdots + c_n(E) = 0$$

(cf. [16]). In particular, any class α in $H^{2d}(P\overline{\mathcal{H}}_{0,n})$ can be represented in the form

$$\alpha = \pi^*(\eta_d) + \pi^*(\eta_{d-1})\psi + \pi^*(\eta_{d-2})\psi^2 + \dots,$$

where $\pi : P\overline{\mathcal{H}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ is the natural projection and the classes η_i are defined on $\overline{\mathcal{M}}_{0,n}$. It follows from the well-known relation $\pi_*\psi^s = c_{s-n+1}(-E)$ (see, e.g., [3]) that the degree of any class represented in such form can be computed from the relation

$$\deg\alpha = \int\limits_{P\overline{\mathcal{H}}_{0,n}} \frac{\pi^*(\eta_d) + \pi^*(\eta_{d-1})\psi + \dots}{1-\psi} = \int\limits_{\overline{\mathcal{M}}_{0,n}} \frac{\eta_d + \eta_{d-1} + \eta_{d-2} + \dots}{c(E)}.$$

Thus the computation of the degree is reduced to the computation of certain integrals over the moduli spaces of rational curves with marked points. The machinery of computing such integrals is well developed. Below we present the results of computation of the degrees of the pushforwards of the basic classes of codimension at most two:

$$
\begin{aligned}
\deg(1) &= n^{n-3} \\
\deg(\delta_{0,0}) &= \frac{1}{2}\,(n-1)\,(n+6)\,n^{n-4} \\
\deg(\xi_2) &= \frac{1}{3}\,(n-1)\,(17\,n^2-28\,n+12)\,n^{n-5} \\
\deg(\delta_{1,0}) &= -\frac{1}{6}\,(n-1)\,(n^2+10\,n-120)\,n^{n-5} \\
\deg(\delta_{0,0}^2) &= \frac{1}{12}\,(n-1)\,(3\,n^3+31\,n^2+82\,n-120)\,n^{n-5}.
\end{aligned}
$$

Substituting these equalities to the computed expressions for the singularity strata we obtain the following formulas for the degrees of the these strata:

$$
\begin{aligned}
\deg(\sigma_{2^1}) &= \frac{9}{2}\,(n-2)\,(n-1)\,n^{n-4} \\
\deg(\sigma_{1^2}) &= 2\,(n-3)\,(n-2)\,(n-1)\,n^{n-4} \\
\deg(\sigma_{3^1}) &= \frac{32}{3}\,(n-3)\,(n-2)\,(n-1)\,n^{n-5} \\
\deg(\sigma_{1^1 2^1}) &= 9\,(n-4)\,(n-3)\,(n-2)\,(n-1)\,n^{n-5} \\
\deg(\sigma_{1^3}) &= \frac{4}{3}\,(n-5)\,(n-4)\,(n-3)\,(n-2)\,(n-1)\,n^{n-5} \\
\deg(\sigma_{2^1,2^1}) &= \frac{3}{8}\,(n-2)\,(n-1)\,(27\,n^2-137\,n+180)\,n^{n-5} \\
\deg(\sigma_{2^1,1^2}) &= 3\,(n-3)\,(n-2)\,(n-1)\,(3\,n^2-15\,n+20)\,n^{n-5} \\
\deg(\sigma_{1^2,1^2}) &= (n-3)\,(n-2)\,(n-1)\,(2\,n^3-16\,n^2+43\,n-40)\,n^{n-5}.
\end{aligned}
$$

By Theorem 1.1, this leads to the following formulas for the Hurwitz numbers:

$$
\begin{aligned}
h_{2^1} &= \frac{9}{2}\,\frac{(2\,n-4)!}{(n-3)!}\,n^{n-5}\\
h_{1^2} &= 2\,\frac{(2\,n-4)!}{(n-4)!}\,n^{n-5}\\
h_{3^1} &= \frac{32}{3}\,\frac{(2\,n-5)!}{(n-4)!}\,n^{n-6}\\
h_{1^1 2^1} &= 9\,\frac{(2\,n-5)!}{(n-5)!}\,n^{n-6}\\
h_{1^3} &= \frac{4}{3}\,\frac{(2\,n-5)!}{(n-6)!}\,n^{n-6}\\
h_{2^1,2^1} &= \frac{3}{4}\,(27\,n^2-137\,n+180)\,\frac{(2\,n-6)!}{(n-3)!}\,n^{n-6}\\
h_{2^1,1^2} &= 3\,(3\,n^2-15\,n+20)\,\frac{(2\,n-6)!}{(n-4)!}\,n^{n-6}\\
h_{1^2,1^2} &= 2\,(2\,n^3-16\,n^2+43\,n-40)\,\frac{(2\,n-6)!}{(n-4)!}\,n^{n-6}
\end{aligned}
$$

of which those corresponding to a single degeneration are known by the Hurwitz formula.

The alternative approach to the compution of the degrees of the strata consists in a direct application of the Hurwitz formula (or, for arbitrary genus, of the formula in [1], which is, however, less explicit) for the Hurwitz numbers of multisingularities. Via Theorem 1.1, it gives us the degree of each stratum of the form σ_α. In the rational case $g = 0$, knowing these degrees and the expressions of these strata in terms of the pushforwards of the basic ones, we are able to compute the degrees of at least some of these pushforwards. In our case, it is easy to compute the degrees $\deg(1)$, $\deg(\delta_{0,0})$, $\deg(\delta_{1,0})$, $\deg(\xi_2)$. Note, however, that these degrees are not sufficient for the computation of the degrees of the multisingular strata: their decomposition contains the class $\delta_{0,0}^2$. The degree of the latter can be deduced form that of the stratum $\sigma_{2^1,2^1}$, which we know due to the work of Zvonkine [21].

4.3 On nonisolated singularities

Although we do not yet have a complete picture of what happens when nonisolated singularities are included, some words are in order. Consider the first case where the nonisolated singularities give nontrivial contribution, namely the case of codimension 3 classes for $g = 0$. Denote by $\sigma^{\exp}_{\alpha_1,\alpha_2,\dots}$ the expected expression for the corresponding class obtained by the universal formulas. It differs from the actual class $\sigma_{\alpha_1,\alpha_2,\dots}$ by some terms supported on the locus $I_\infty \subset B$ of functions with non-isolated singularities. Since the correction term has the same codimension as the subvariety I_∞, it should be proportional to the fundamental class of I_∞ with some coefficient. It is natural to assume that this coefficient is universal, it depends neither on the irreducible component of I_∞, nor on n. The numerical manifestation of this assumption is the 'unexpected' observation that the degree of the difference

$$\deg(\sigma_{\alpha,\alpha_2,\dots} - \sigma^{\exp}_{\alpha,\alpha_2,\dots})$$

is proportional to the degree of the stratum I_∞,

$$\deg(I_\infty) = \frac{1}{8}(n-1)(n^3+11n^2+34n-120)n^{n-5},$$

with integer coefficient. The degrees of $\sigma^{\exp}_{\alpha,\alpha_2,\dots}$ are computed by the methods of this paper, and the actual degree of the class $\sigma_{\alpha,\alpha_2,\dots}$ can be derived from the known Hurwitz numbers. In the following table we present the conjectural contribution of nonisolated singularities for some codimension 3 strata:

$$\begin{aligned}
\sigma_{4^1} &= \sigma^{\exp}_{4^1} + 5\,I_\infty \\
\sigma_{3^1 1^1} &= \sigma^{\exp}_{3^1 1^1} - 16\,I_\infty \\
\sigma_{2^2} &= \sigma^{\exp}_{2^2} - 9\,I_\infty \\
\sigma_{2^1 1^2} &= \sigma^{\exp}_{2^1 1^2} + 36\,I_\infty \\
\sigma_{1^4} &= \sigma^{\exp}_{1^4} - 16\,I_\infty.
\end{aligned}$$

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