Darboux Transformations and Integrable Systems I

Vladimir B. Matveev

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

Laboratoire de Physique Mathématique
UFR Sciences et Techniques
Université de Bourgogne
BP 138
21004 Dijon Cedex

France

St. Petersburg Branch of
Steklov Mathematical Institut
Fontaka 27
191011 St. Petersburg

Russia

MPI 96-170

Darboux Transformations and Integrable Systems I

Vladimir B. Matveev*
Max-Planck-Institut für Mathematik ,
Göttfried-Claren-Strasse 26, 53228 Bonn
, e-mail matveev@mpim-bonn.mpg.de *
* On leave from Université de Bourgogne , Physique Mathématique
UFR Sciences et Techniques , BP 138 , F. 21004 Dijon Cedex
France
e-mail : matveev@satie.u-bourgogne.fr ,
and St. Petersburg Branch of Steklov Mathematical Institut
Fontanka 27 , 191011 , St. Petersburg , Russia

Abstract

We show how the construction of the integrable Sturm-Liouville equations proposed in 1882 by Gaston Darboux can be extended to the case of the linear and nonlinear integrable partial differential equations and their lattice versions. The power of the Darboux transformation method is illustrated by several examples including the simplest possible construction of the rational solutions of the KdV and KP equations. We mainly concentrate around the lattice equations since in the existing presentations of the Darboux transformation method many important points are missing. At the end of the paper we show how the method can be extended to the case of the multidimensional functional equations with generalized shift operators. More applications to concrete integrable systems and further generalisations will be reported in the second part.

1 Introduction

We wish to expose here some general features of the algebraic approach to completely integrable linear and nonlinear partial derivative or lattice equations invented in 1979 by the present author [1] as a natural extension of Darboux covariance theorem [2] concerning Sturm-Liouville equation. Later this method was extended to a larger class of systems mainly by Salle and present author and summarized in [5], where the more detailed bibliography and much more concrete examples can be found. Somehow the detailed explanation of space-time discretized or lattice version of the method , briefly outlined in [1, article 2] never was discribed in detailes. It was ommited in [5] since at the time there were no much social interest to such a systems. Somehow many of their remarkable properties were studied by Ablowitz and Ladik [6] , Hirota [7] , Miwa [8] and other researchers: [9] , [10] , [11] , [12] , [13], [14] .

In particular it was shown by Hirota that his bilinear functional-difference equation reproduces in certaine limits most part of the known integrable continuous or differential-difference systems. Last time the interest to the space time discretized systems grows quickly motivated by many parallels between quantum and classical integrable systems on a lattice , especially by recognizing the fact that the eigenvalues of the commuting quantum transfer matrices can be found as the solutions of the certain functional equations derived by means of the fusion rules [15] , [16] ,[17],[18], [19] , [20] , [21] . The associated equations can be identified with Hirota like equations with certain boundary conditions or with discrete Zakharov-Scabat equations. Some of these equations still are not well studied. Another source of appearence of the generalized Hirota equations involving some dilatation operators instead of lattice shifts is the theory of quantum group representations and quantum τ-functions relevant also to some aspects of the theory of matrix models [19]. Works of the last time mainly explore the Plukker relations as a tool for obtaining determinant representations for solutions of Hirota - like equations and also the algebrogeometric approach . Our approach is different and can be combined as in KP and KdV case with algebrogeometric method of integration.

By the pedagogical reasons somehow we start from the brief exposition of some basic features of Darboux transformation method as formulated in [1] for the case of the partial derivative equations , explaining how to construct some important classes of the solutions. Some formulas , for instance explicite expression of the Wronskian corresponding to 1-soliton-2positons solutions of the KdV equation ,were never published before. We give also a brief outline of main theorems concerning the differential difference case allowing systematic application (which can be found in [5], see also [26], [24], [25]) to the 1-dimensional and 2-dimensional Toda lattices and also to their nonabelian versions and higher flows.

For "continuous" systems our approach represents a far going generalisation of the factorisation method for ordinary differential equations coming back to the works of Frobenius and to the original work of Darboux [2] , developped further by Crum [22] . There are also deep parallels between the structure of Darboux-like dressing formulas explained below and algebraic Bethe anzatz method . In a case of stationary one dimensional Schrödinger operator (see [5]) the Dirac construction of the eigenfunctions and eigenvalues of quantum oscillator is easily recovered , quantum q-oscillators are also here [23]. Some how we have no intention to discuss stationary reductions leading to this kind of applications here.

We also omit the discussion of the hamiltonian aspects of the theory which are not still completely understood. More detailed applications to concrete functioinal equations will be exposed separately. We also omitted here the discussion of the binary Darboux transformation (see [5] and references there for differential-difference and continuous case) . Associated results also will be explained elshwere. The results connected with differential-difference systems were never explained in detailes before and are sufficiently precised and extended with respect to the brief remarks in [1, article 2] including the new concept of the mixed Darboux transformation (see subsection 3.3) . The results of the sections 4 , 5 are new and were never published before.

2 Darboux Transformation Approach to the matrix KP Hierarchie

2.1 Covariance Theorems and simplest applications

Let $f(x,t)$ be an $M \times M$ matrix valued solution of the following PDE :

$$\dot{f} = \sum_{m=0}^{n} u_m(x,t) f^{(m)}, \dot{f} \equiv \partial_t f, f^{(m)} \equiv \partial_x^m f \tag{1}$$

with $M \times M$ matrix coefficients. Let φ be some fixed invertible matrix solution of the same equation , $\sigma = \partial_x \varphi \cdot \varphi^{-1}$ where φ^{-1} means the inverse matrix. The matrix function ψ defined by the formula

$$\psi = (\partial_x - \sigma) f \equiv D(f) \tag{2}$$

is called the Darboux transformation of f . Obviously it is fixed by the choice of φ. Now we have the following statement (Generalized Darboux theorem):
Theorem 1 [1]
Equation (1) *is covariant with respect to the action of the Darboux transformation i.e.* ψ *is the solution of the following PDE*

$$\dot{\psi} = \sum_{m=1}^{n} \tilde{u}_m(x,t) \cdot \psi^{(m)} \tag{3}$$

The coefficients $\tilde{u}_m$ *are defined by the recursive relations:*

$$\tilde{u}_n = u_n, \tilde{u}_{n-1} = u_{n-1} + u_n^{(1)} + [u_n, \sigma], \tag{4}$$

$$\tilde{u}_{n-2} = u_{n-2} + u_{n-1}^{(1)} + n u_n \sigma^{(1)} + u_n^{(1)} \sigma + [u_{n-1}, \sigma] + [u_n, \sigma]\sigma, \tag{5}$$

$$\tilde{u}_{k-1} = u_{k-1} + u_k^{(1)} - \sigma u_k + \sum_{m \geq k}^{n} C_m^k \tilde{u}_m \sigma^{(m-k)}; k = n-1, \ldots, 1. \tag{6}$$

The $[\,,]$ above denotes the commutator of two matrices.

Th. 1 implies in particular the Darboux covariance property for the ordinary differential equations of any order . To get this result it is enough to assume that all the coefficients in (1) are t independent and reduce our consideration to the class of particular solutions of the form

$$f = \chi(x,\lambda) e^{\lambda t}, \quad \varphi = \phi(x,\lambda_1) e^{\lambda_1 t}.$$

Then we obviously get the following statement.
Theorem 2. *The equation*

$$\sum_{m=0}^{n} u_m \chi^{(n+m)} = \lambda \chi, \tag{7}$$

is covariant with respect to Darboux transformation (2) *i.e. the function*

$$\psi = \chi_x - \sigma \chi, \qquad \sigma = \phi_x \phi^{-1}$$

, *where* $\phi = \phi(x, \lambda_1)$ *is a fixed solution of (7) with* $\lambda = \lambda_1$, *gives the solution of the transformed equation*

$$\sum_{m=0}^{n} \tilde{u}_m \psi^{(n+m)} = \lambda \psi, \tag{8}$$

where the coefficients $\tilde{u}_m$ *are defined by the same formulas as in the Th.* 1 .

In scalar case the formulas for the coefficients become simpler since all commutators vanish. Original result of Darboux [2] is recovered just by taking $n = 2, u_2 = 1$ in the formulation of Th. 2 . Theorem 1 was proved in [1] where it was also proposed to call (2) Darboux transformation. Most important examples of scalar problems allowing the application of the Th. 1 are the second order evolution equations :

$$f_y = f_{xx} + uf, \tag{9}$$

$$f_t = f_{xxx} + \frac{3}{2} u_x f + vf. \tag{10}$$

According to the Th.1 for both of them the transfomation of the coefficient u with respect to the action of the Darboux transformation is given by the formula

$$\tilde{u} = u + 2\sigma^{(1)} = u + 2\partial_x^2 \log \varphi. \tag{11}$$

Now it is obvious from the Darboux theorem that having one solvable Sturm-Liouville equation with potential u we get another one with potential $\tilde{u}$ fixed by the choice of λ_1 and the solution $\phi(x, \lambda_1)$. Varying ϕ in the formula for Darboux transformation we recover all the solutions of the Sturm-Liouville equation with the potential $\tilde{u}$. Now we can take for this new equation one more point $\lambda = \lambda_2$, fix some solution at this point and generate a new solvable potential via the same theorem. This process can be continued and we obviously get this way an infinite number of the solvable potentials depending also of any desired number of parameters. Darboux himself used this process to construct the explicit solutions for the Sturm-Liouville equation with the potential $n(n+1)/\cosh^2 x$.

Later it was discovered by Crum [22] that the iterations of the Darboux transformation for the Sturm-Liouville case can be completely described by fixing finite number of points λ_j for starting equation and choosing in arbitrary way the solutions φ_j at these points. Than the resulting ψ-function corresponding to the n-times iterated Darboux transformation is expessed very simply (see next section) as the ratio of two Wronskian determinants. The Theorem 1 somehow gives much more allowing to perform the same kind of the constructions in the case of linear and nonlinear partial differential equations. To illustrate this point let us recall that the compatibility condition of the system (1-2) implies that the function $u(x,t)$ is the solution of the KP-I equation :

$$3u_{yy} + \partial_x(4u_t + 6uu_x + u_{xxx}). \tag{12}$$

Assuming that φ is a fixed solution of the system (1-2) we can conclude that the Darboux transformation ψ of f generated by φ solves the system of the same form with the coefficients $\tilde{u}$ and $\tilde{v}$, caculated from Th.1 . Hence for any fixed solution u of the KP equation and any solution φ of the associated linear system (9-10) we get the new solution of (12) defined by (11). Of course as in the case of the Sturm-Liouville equation we can iterate the process of construction of the new solvable systems of the form (9-10) and the associated solutions of the KP equation. Starting from the trivial solution $u = 0$ we already produce a lot of the explicit solutions including all the rational solutions decreasing when $x \to \pm\infty$ (see below for detailes). To apply the same technique to construct the explicit solutions of the KdV equation:

$$u_t = 6uu_x - u_{xxx} \tag{13}$$

we need only to apply Th.1 to the system

$$\begin{gathered} -f_{xx} + uf = \lambda f, \\ f_t = -4f_{xxx} + 6uf_x + 3u_x f \end{gathered} \tag{14}$$

and remark that in this case u transforms according to the formula

$$\tilde{u} = u - 2\sigma^{(1)}. \tag{15}$$

. For instance taking $\varphi = \cosh k_1(x - 4k_1^2 t + x_1), u = 0$, with real values of x_1, k_1 we recover immediatly 1-soliton of the KdV equation given by (15).In addition we get from (2) the solution of the linear system (14) associated with this solution. Again we can iterate this process of creation of the new solvable potentials an infinite number of times getting in particular all the nonsingular n-solitons solutions by applying n-times the Darboux transformation to the starting system (14) with $u = 0$. Again the result of the n-times application of the Darboux transformation can be described by simple formula containing the Wronskians of the solutions of the starting system as explained in the next subsection. It is obvious from the previous discussion that all the hierarchy of the higher KP flows (Zakharov-Shabat systems) also allows the application of the same Th. 1.

2.2 Multiple Iterations of the Darboux Transformations

For simplicity we discuss here only the scalar problems. Let $\varphi_1, \ldots, \varphi_N$ are different linearly independent solutions of (1) . Now let the function $\psi_{[N]}$ represents the result of action of the sequence of the N consecutive Darboux transformations on f:

$$\psi_{[N]} = D_N \cdot D_{N-1} \cdots D_{-1} f \tag{16}$$

Below we shall use the standart notation $W(\varphi_1, \ldots, \varphi_N)$
for the wronskian of N functions i.e. $W = \det A$, $A_{ik} = \partial_x^{i-1}\varphi_k$, $i, k = 1, \ldots, N$. In these notations the first order differential operator D_k can be written as

$$D_k = \partial_x - \sigma_k, \sigma_k = \partial_x \log \varphi_{[k]}, \tag{17}$$

where the function $\varphi_{[k]}$ is defined by the formula

$$\varphi_{[k]} = \frac{W(\varphi_1, \ldots, \varphi_k)}{W(\varphi_1, \ldots, \varphi_{k-1})}. \tag{18}$$

It is important to observe that $\sigma_{[k]}$ are the symmetric functions of φ_j . It will be clear that we get the same result as in (16) by replacing the operators

D_k by $\tilde{D}_k$ where the functions $\tilde{\sigma_k}$ are built from $\varphi_{i_1}, \ldots, \varphi_{i_k}$ in a same way as in (17) and $s_k = (i_1, \ldots, i_k)$ are any subsets of $(1, \ldots, N)$ formed from k different entries , $s_1 \in s_2, \ldots, \in s_n$.
Formula (16) constitutes the far going generalization of the Dirac construction of the eigenstates of the quantum harmonic oscillator in terms of the powers of creation operator applied to a vacuum vector.

Theorem 3

The function $\psi_{[N]}$ is a symmetric function of $\varphi_1, \ldots, \varphi_N$ given by the ratio of two Wronskians

$$\psi_{[N]} = \frac{W(\varphi_1, \ldots, \varphi_N, f)}{W(\varphi_1, \ldots, \varphi_N)} \tag{19}$$

For the particular case when (1) is replaced by the Sturm-Liouville equation this theorem was proved by Crum [22] . We shall give the proof in a general case of (1) with scalar coefficients. Originally the associated statement was first obtained in [1] . The hint of the proof was outlined also in [5].

First it is evident that the RHS of (19) vanishes if f coincide with one of φ_j . Next it is obvious that RHS of (19) represents the action of the linear differential operator with highest term ∂_x^N on f :

$$\psi_N = \sum_{m=0}^{N-1} a_m f^{(m)} + f^{(N)}. \tag{20}$$

The fact that this expression vanishes when when $f = \varphi_i, \quad i = 1, \ldots, N$ is equivalent to the system of N linear algebraic equations solvable via the Kramer rule i.e. fixes the coefficients a_m uniquely assuming that $W(\varphi_1, \ldots, \varphi_N) \neq 0$, since W is just the determinant of this system. Third step of the proof is given by induction. For N=1 the statement of the theorem is obtained by rewriting RHS of (2) as $W(\varphi, f)/\varphi$ and hence is trivial. Now to complete the induction we have to assume that the result of the action of the $N-1$ consecutive Darboux transformations applied to f is given by the formula

$$\psi_{[N-1]} = \frac{W(\varphi_1, \ldots, \varphi_{N-1}, f)}{W(\varphi_1, \ldots, \varphi_{N-1})}. \tag{21}$$

To fix some solution of the corresponding linear PDE satisfied by $\psi_{[N-1]}$ we have to take $f = \varphi_N$ in the RHS of (21) where φ_N is some additional solution of (1) linearly independent with $\varphi_1, \dots, \varphi_{N-1}$. Now the linear equation satisfied by (22) is covariant with respect to the action of the Darboux transformation $D_{[N]}$ on $\psi_{[N-1]}$:

$$D_{[N]}(\psi_{[N-1]}) = \psi^{(1)}_{[N-1]} - \sigma_N \psi_{[N-1]}, \qquad \sigma_N = \partial_x \log \psi_{[N-1]} \mid_{f=\varphi_N} . \tag{22}$$

Now it is evident that

$$D(\psi_{[N-1]}) \mid_{f=\varphi_k} = 0, \qquad k = 1, \dots, N, \tag{23}$$

and hence being differential operator acting on f (of the same structure as (20)) coincide with $\psi_{[N]}$. □

Remark It it easy to check that the coincidence of the RHS of (19) with (22) ,proved above,is equivalent to the Jacobi identity for Wronskians :

$$\begin{aligned} &W[\,W(\varphi_1, \dots \varphi_{N-1}, g), W(\varphi_1, \dots, \varphi_{N-1}, f)\,] = \\ &= W(\varphi_1, \dots, \varphi_{N-1}, g, f) \cdot W(\varphi_1, \dots, \varphi_{N-1}), \end{aligned} \tag{24}$$

with $g = \varphi_N$.

Every differential equation for f

$$W(\varphi_1, \dots, \varphi_N, f) = 0, \tag{25}$$

is of the form (7) with $\lambda = 0$ and hence is covariant with respect to the N-times application of the Darboux transformation where $\varphi_1, \dots, \varphi_N$ are arbitrary linearly independent N times differentiable functions.From thus we conclude that our proof of the Th. 5 provides the independent proof of the Jacobi identity . Crum [22] proved the particular case of the Th. 3 for the Sturm-Liouville equation by using the Jacobi identity. Our proof concerns much more general situation and we get the Jacobi identity for free. Using the Th. 3 it is easy to conclude that in scalar case we get simple expessions for the first coefficients of the N-times Darboux transformed equation (1) . Most simple case is to assume that $u_n = u(t), u_{n-1} = p(t)$ and hence are invariants with respect to the action of DT defined by (2). Now as a result of the repeated N-times action of the DT we get for $u_{[N],n-2}$ the following repesentation :

$$\begin{aligned} u_{[N],n-2} &= u_{n-2} + n u_n \cdot \sum_{m=1}^{N} \sigma^{(1)}_{[m]} = \\ &= u_{n-2} + n u_n \partial_x^2 \log W(\varphi_1, \dots, \varphi_N), \end{aligned} \tag{26}$$

$$\sigma_{[m]} := \partial_x \log \varphi_{[m]},\ \varphi_{[m]} := \frac{W(\varphi_1, \ldots, \varphi_{m-1}, \varphi_m)}{W(\varphi_1, \ldots, \varphi_{m-1})}. \tag{27}$$

Another way to compute the coefficient $u_{[N],n-2}$ is to substitute (20) into the differential equation

$$\dot{\psi}_{[N]} = \sum_{m=1}^{n} u_{[N],m} \cdot \psi_{[N]}^{(m)}, \tag{28}$$

for $\psi_{[N]}$. Than equating the coefficients of the higher order derivatives of f in the LHS and RHS of (28) it is easy to see that

$$u_{[N-2],n} = u_{n-2} + n u_n a_1^{(1)}, \tag{29}$$

The coefficient a_1 is given by the formula

$$a_1 = \frac{\begin{vmatrix} \varphi_1 & \ldots & \varphi_N & f \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1^{(N-2)} & \vdots & \varphi_N^{(N-2)} & f^{(N-2)} \\ \varphi_1^{(N)} & \ldots & \varphi_N^{(N)} & f^{(N)} \end{vmatrix}}{W(\varphi_1, \ldots, \varphi_N)}. \tag{30}$$

Taking the derivative of a_1 we get the same result as before.

The formulas enable one to get very easy all the rational solutions of the KP II equation decreasing when $x \to \infty$. To do this [1] it is enough to take $u = 0, v = 0$ and choose φ_j apropriately . For instance taking

$$\varphi_j = (\partial_k + g(k))\exp(kx + k^2 y + k^3 t)|_{k=k_j} \tag{31}$$

we get the whole family of the rational solutions of "general position" obtained in a different and much longer way by Krichever [37] . Solutions of general position correspond to asymptotically free movements of the particles of the associated Calogero-Moser system i.e. the poles $x = x_j(y, t)$ behave asymptotically linear as a function of y when $y \to \infty$. By contrast taking φ_j in a form of some linear combination of Schur polynomials [1, article 3] i.e.

$$\varphi_j = \sum_{l=1}^{n} c_{lj}\, \partial_k^l \exp(kx + k^2 y + k^3 t)|_{k=0}, \tag{32}$$

and substituing them into (26) with

$$u_n = 1, n = 2, u_{n-2} = u = 0$$

we get the family of "separatrix" (with respect to the trajectorys of associated Calogero-Moser system) rational solutions of the KP-I equation . See [1, article 3] for more details. We refere on [5] for construction of real valued nonsingular solutions of KP-II and KP-I equations depending on any number of the functional parameters. Briefly the idea is to take

$$\varphi_j = \int \rho_j(k) \exp(kx + k^2 y + k^3 t)\, dk \tag{33}$$

and next to impose the requirements on the densities $\rho_j(k)$ providing nonvanishing of the Wronskian $W(\varphi_1, \cdots, \varphi_n)$. To get the real valued nonsingular solutions including the nonsingula rational multi lumps solutions (first found in [40]) for KP-II case (which differs from (12) by the sign of the first term) the idea of the binary Darboux transformation invented in [27] and making use of conjugated linear system is extremely usefull .

2.3 Generalized wronskian formulas and applications to KdV

Here we explain the extended version of Wronskian formulas obtaiened in [3] allows one to construct all the rational solutions of KdV equation (much easier than in other approaches : compare with [39] , [37] ,) . We also get from these generalized wronskian formulas some long range oscillating solutions introduced in [3] . The author proposed to call them positons since in spectral sense they are connected with Wigner- von Neumann resonances : spectral singularities embedded in the positive continuous spectrum of the Schrödinger operator (for more detailed study of their properties and interactions with solitons see [4] or [28] .

Let $u(x,t)$ be some fixed solution of the KdV equation We denote by ϕ_1, ... ϕ_n different solutions of system () taken at some values $\lambda = \lambda_1$, ..., $\lambda = \lambda_n$ respectively. With these functions we construct two Wronskian determinants W_1 and W_2 as follows:

$$W_1 := W(\phi_1, \ldots, \phi_1^{(m_1)}, \phi_2, \ldots, \phi_2^{(m_2)}, \ldots, \phi_n, \ldots, \phi_n^{(m_n)}), \tag{34}$$

$$W_2 := W(\phi_1, \ldots, \phi_1^{(m_1)}, \phi_2, \ldots, \phi_2^{(m_2)}, \ldots, \phi_n, \ldots, \phi_n^{(m_n)}, f), \tag{35}$$

where $m_i \geq 0$ are some given numbers and $\phi_j^{(n)} := \partial_\lambda^n \phi_j(x, \lambda)|_{\lambda=\lambda_j}$.

Moreover, we define

$$\tilde{u} := u - 2\partial_x^2 \log W_1, \tag{36}$$

$$\tilde{f} := W_2/W_1. \tag{37}$$

Equations (35-36) represent a natural generalization of the result corresponding to the case m_1, $m_2 \ldots$, $m_n = 0$. The same arguments as in [3] lead to the following theorem:

The system (14) is covariant with respect to the generalized Darboux transformation $f \to \tilde{f}$, i.e. $\tilde{f}$ satisfies the system obtained from (14) by the change $u \to \tilde{u}$, $f \to \tilde{f}$. The function $\tilde{u}$ represents a new solution of the KdV equation.

Note that we can replace the parameter λ by any parameter k being locally in analytic one to one correspondence with λ, replacing the derivatives with respect to λ by derivatives with respect to k. In particular we can use $k = \sqrt{\lambda}$ in the neighborhoud of the points $\lambda = \lambda_1, \ldots, \lambda_n$, assuming that the dependence of ψ_j on k is also of analytical character.

Here we dicuss briefly the application of (35-36) to the case $u = 0$. Therefore we will write u instead of $\tilde{u}$ when it can not create a confusion.

Now to construct the rational solutions of KdV equation we take $m_1, \ldots, m_n$ to be arbitrary entries . The functions ϕ_j are defined by the formula

$$\phi_j = \exp(kx - 4k^3 t + P_j(k))|_{k=k_j}, Im\, k = 0. \tag{38}$$

The $P_j(k)$ are arbitrary real polynomials of order $\leq m_j$ and To obtain the rational solutions we substitute into the generalized Wronskian formula the functions $\phi_j^{(m)}$:

$$\phi_j^{(m)} = \partial_k^m \exp(kx - 4k^3 t + P_j(k))|_{k=k_j}. \tag{39}$$

The solution obviously comprises $n + \sum_{j=1}^n m_j$ arbitrary real parameters. The obtained formula containes more particular family of solutions calculated by Krichever which is obvious from the simple comparison with results [37] .

2.4 Positon and Soliton-Positon solutions

Positon solution of the KdV equation is defined by the formula

$$u = -2\partial_x^2 \log W(p_1, p_{1_{k_1}}) = -2\partial_x^2 \log(2k_1 g_1 - \sin 2p_1) =$$

$$= \frac{32k_1^2(\sin T - k_1 g \cos T)\sin T}{(\sin 2T - 2k_1 g)^2}, \tag{40}$$

$$p = \sin T, T = k_1(x + 4k_1^2 t + x_1(k_1)),$$

$$g = \partial_{k_1} T = x + 12k_1^2 t + y_1; k_1, x_1, y_1 \ni R, p_{k_1} = \partial_{k_1} p = g \cos T. \tag{41}$$

The unique pole $x_0(t)$ of (2) is determined by the formula

$$x_0 = -12k_1^2 t + y_1 + \frac{\delta(t)}{2k_1}, \delta = \sin(\delta - 16k_1^3 t + 2k_1(x_1 - y_1)), \tag{42}$$

$$\delta(t + \frac{2\pi}{2k_1}) = \delta(t). \tag{43}$$

The associated solution is slowly decaying at infinity and essentialy differs from well known soliton solutions.

Soliton-positon solution of the KdV equation is defined by the formula

$$u = -2\partial_x^2 \log W(p_1, p_{1k_1}, s), s = \cosh Y,$$

$$Y = b(x - 4b^2 t + r), r \in R. \tag{44}$$

The Wronskian in this formula can be computed explicitely :

$$W = 2kb \sin^2 T \sinh Y + ((b^2 - k^2)2^{-1} \sin 2T - k(k^2 + b^2)g) \cosh Y. \tag{45}$$

For the plot and asymptotic properties see [4]. It is remarkable that for all real values of parameters the expession (45) has only one zero on the real axis as a function of x.

The **2- positons solution** is determined by the formulas

$$u = -2\partial_x^2 \log W(p_1, p_{1k_1}, p_2, p_{2k_2}),$$
$$p_1 = \sin T_1, T_1 = k_1(x + x_1 + 4k_1^2 t), \; p_{1k_1} = g_1 \cos T_1$$
$$g_1 = x + y_1 + 12k_1^2 t;$$
$$p_2 = \sin T_2, \; T_2 = k_2(x + x_2 + 4k_2^2 t),$$
$$p_{2k_2} = g_2 \cos T_2,$$
$$g_2 = x + y_2 + 12k_2^2 t,$$
$$Im\, x_1 = Im\, y_1 = Im\, x_2 = Im\, y_2 = 0; \; k_{1,2} > 0$$

Remarkably the associated τ-function almost factorises into the product of the two *tau*-functions τ_1, τ_2 corresponding to individual positons : It has to be mentioned that the same wronskian can be written in a much more compacte forme in terms of the τ-fonctions $\tau_{1,2}$ of the individual positons ($\tau_1 = \frac{\sin 2T_1}{2} - k_1 g_1$, $\tau_2 = \frac{\sin 2T_2}{2} - k_2 g_2$) :

$$W \ = (k_1^2 - k_2^2)^2 \tau_1 \tau_2 - 4k_1 k_2 (k_1 \cos T_1 \sin T_2 - k_2 \sin T_1 \cos T_2)^2. \qquad (46)$$

Using (46) is evident that in the area where $0 < c_1 \leq \tau_1 \leq\ll c_2$ and $t \to \infty$ (implying $g_2 \to \infty$), we have the following asymptotic estimate for the solution

$$u = -2\partial_x^2 \log \tau_1 [1 + 0(g_2^{-1})]. \qquad (47)$$

In the area where τ_2 is fixed and $t \to \infty$ (i.e. $g_1 \to \infty$), we have similar asymptotics up to the change $\tau_1 \to \tau_2, g_1 \to g_2$. Thus we have shown that asymptotically two positons are going from their mutual collision without any change. Even additional phases appearing e.g. in the description of a collision of two solitons are absent.

For all real values of parameters with $k_1 \neq k_2$ the Wronskian (46) has exactly two real zeros which never coincide.

2-positon-1-soliton solution.

This solution being also of the form (37) is produced by the wronskian $W(p_1, p_{1k_1}, p_2, p_{2k_2}, s)$.This wronskian also can be computed explicitely with help of Maple. Initial form of the answer produced by Maple takes about 3 pages of printed output but it can be simplified ordering the answer by isolating terms proportional $g_1 g_2, g_1, g_2$, and terms free of these factors. Below we write g instead of g_1 and d instead of g_2 to simplify the notations. The final result is

$$\begin{aligned}
&W = gdk_1k_2(k_1^2 - k_2^2)^2(b^2 + k_1^2)(b^2 + k_2^2)\cosh Y - \\
&-2dk_1k_2b(b^2 + k_2^2)(k_1^2 - k_2^2)^2 \sin^2 T_1 \sinh Y + \\
&+\tfrac{d}{2}k_2(k_1^2 - k_2^2)^2(k_1^2 - b^2)(k_2^2 + b^2)\sin 2T_1 \cosh Y + \\
&+\tfrac{g}{2}k_1(k_2^2 - k_1^2)^2(k_2^2 - b^2)(k_1^2 - b^2)\sin 2T_2 \cosh Y - \\
&-2gk_1k_2b(b^2 + k_1^2)(k_2^2 - k_1^2)^2 \sin^2 T_2 \sinh Y + \\
&+\cosh Y\{-4k_1^3k_2(k_2^2 + b^2)^2 \cos^2 T_1 - 4k_2^3k_1(k_1^2 + b^2)^2 \cos^2 T_2 + \\
&+4k_1k_2[k_1^2(b^2 + k_2^2)^2 + k_2^2(b^2 + k_1^2)^2]\cos^2 T_1 \cos^2 T_2 + \\
&+\tfrac{1}{4}[b^2(9k_1^2k_2^4 + 9k_1^4 - k_1^6 - k_2^6) + (b^4 + k_1^2k_2^2)(k_1^4 + k_2^4 + 6k_1^2k_2^2)]\sin 2T_1 \sin 2T_2\} + \\
&+[k_2b(k_2^4b^2 - 3k_1^4b^2 + 3k_1^2k_2^4 - k_1^6 - 2k_1^4k_2^2 + 2k_1^2k_2^2b^2)\sin 2T_1 \sin^2 T_2 + \\
&+k_1b(k_1^4b^2 - 3k_2^4b^2 + 3k_1^4k_2^2 - k_2^6 - 2k_2^4k_1^2 + 2k_1^2k_2^2b^2)\sin 2T_2 \sin^2 T_1]\sinh Y
\end{aligned}$$

When $k_1 = k_2 = k$ the expression for W in (10) simplifies:

$$W = -4k^4(k^2 + b^2)^2 \sin^2(x_1 - x_2) \cosh Y \tag{48}$$

Again Wronskian has exactly two real zeros as a function of x , for $k_1 \neq k_2$ which never coincide.

Remark The last statement is confirmed by the asymptotic analysis and numerical experiments but still the complete proof is missing. Most remarkable in the behaviour of the multi positon-soliton solutions studied in [4] that asymptotically they behave almost as simple as multisoliton solutions :it is possible to find explicit formulas describing the associated phase shifts given in terms of the spectral data.

Darboux transformations can be applied as well to the starting finite-gap solutions expressed by means of the Riemann theta-functions of compact Riemann surfaces since associated solutions of the linear problems (Baker-Akhieser functions) are known [32] ,[34], [34] ,[36], [38] . Associated results are most interesting in the case of $2+1$ dimensional systems but we have no intention to discuss these applications here and pass to the lattice version of Darboux transformations.

3 Lattice Darboux Transformations

Now we start to describe the lattice versions of the Darboux transformations which can be applied along the same lines as above to the difference , differential difference or difference difference linear and nonlinear systems. First we consider the class of the differential difference equations of the form

$$\dot{f}_n = \sum_{m=-M}^{N} b_m(n,t) f_{n+m}, n = 0, \pm 1, \pm 2, \dots \tag{49}$$

for the matrix function $f_n(t)$ with the coefficients $b_m(n,t)$ of the same matrix dimension. Let $\varphi_n(t)$ be the fixed solution of the same system , and $\sigma_n^\pm = \varphi_n \varphi_{n\pm1}^{-1}$. Right and left Darboux transformations $\psi_n^\pm$ of f_n are defined by the formula

$$\psi_n^\pm = f_n - \sigma_n^\pm f_{n\pm1}. \tag{50}$$

The following statement represents the natural lattice version of the Th.1

Theorem 4

The function $\psi_n^\pm$ defined above satisfy to the following system of differential-difference equations

$$\dot{\psi}_n^\pm = \sum_{m=-M}^{N} d_m(n,t) f_{n+m}, \tag{51}$$

where the coefficients $d_m(n,t)$ are given by the formulas
a. The case $(+)$ i.e. for the right Darboux transformation :

$$d_{-M} = b_{-M},$$
$$d_j = \left(\sum_{m=j}^{M-1}\left[b_{-m} - \sigma_n b_{-m-1}(n+1)\right]\varphi_{n-m} + b_{-M}\varphi_{n-M}\right)\varphi_{n-j}^{-1},$$
$$j = 0,\dots,M-1;$$

$$d_k = \left(\sum_{m=k}^{N-1}\left[\sigma_n b_m(n+1) - b_{m+1}\right]\varphi_{n+m+1} + \sigma_n b_N(n+1)\varphi_{N+n+1}\right)\varphi_{n+k}^{-1},$$

$$k = 1,\dots,N-1;$$

$$d_N = \sigma_n b_N(n+1)\varphi_{n+N+1}\varphi_{n+N}^{-1}.$$

b.*case $(-)$ i.e. for the left Darboux transformation :*

$$d_N = b_N,$$
$$d_k(n,t) = \left[\sum_{m=k}^{N-1}\left(b_m(n) - \sigma_n^- b_{m+1}(n-1)\right)\varphi_{n+m} + b_N(n)\varphi_{n+N}\right]\varphi_{n+k}^{-1},$$
$$k = 0,\dots,N-1;$$

$$d_{-k}(n,t) = \left[\sum_{m=-M}^{-k}\sigma_n^- b_m(n-1)\varphi_{n+m-1} - \sum_{m=-M}^{-k-1} b_m(n)\varphi_{n+m}\right]\varphi_{n-k}^{-1},$$

$$k = 1,\dots,M-1;$$

$$d_{-M} = \sigma_n^- b_{-M}(n-1)\cdot(\sigma_{n-M}^-)^{-1}.$$

The proof of the Th. 4 for the case b can be found in [1, article 2] . In the case of the right DT the same idea as their works without any troubles so we omit the proof here. It is important to observe that the first coefficient b_{-M} remains invariant under the action of the right DT and the last coefficient b_N is stable with respect to the action of the left DT.

Another important remark is that for the coefficient d_N in the case a. and respectively for the coefficient d_{-M} for the case b. we get from Th. 6 two different representations . Their origin follows naturaly from the proof which we omitted here since the same effect take place for the difference difference equation. This explains also the presence of the two types of Darboux dressing formulas for the Toda lattice and its two-dimensional and nonabelian versions first discovered in [24], [26]. For further applications to nonlinear systems and explanation of important concept of the binary lattice DT see [5] . For the multiple iterations of left and right DT we obtain the compact expessions similar to (19) but involving instead of Wronskians their difference versions : Casorati determinants . Since the corresponding formulas are the same for the case of the difference difference equations their proof follows after the next subsection.

3.1 DT Covariance of the Difference-difference equations

Let us replace now continuous time evolution by the discrete time evolution. This leads to the following difference diffrence equation representing the time discretized version of (49).

$$f_n(j+1) = \sum_{m=-M}^{N} b_m(n,j) f_{n+m}(j);\; n = 0, \pm 1, \pm 2 \ldots; j = 0, \pm 1, \pm 2, \ldots . \tag{52}$$

Let $\varphi_n(j)$ be a fixed solution of (52) and $\sigma_n(j) = \varphi_n(j)\varphi_{n+1}^{-1}(j)$. If the discrete time argument is omitted below this means that it takes the value j. Once again all the functions and coefficients in (52) are square matrices of the same fixed order. The definition (50) of the Darboux transformation used in the differential difference case remains valid without any change. The covariance properties of (52) with respect to its action can be summarized as follows.
Theorem 5.
The equation () is covariant with respect to the action of the Darboux transformation

$$D := f_n \to \psi_n = f_n - \sigma_n f_{n+1}, \tag{53}$$

i.e. the function ψ_n satisfy to the following lattice equation.

$$\psi_n(j+1) = \sum_{m=-M}^{N} d_m(n,j)\psi_{m+n}(j) \tag{54}$$

where the coefficients are defined by the formulas

$$d_{-M} = b_{-M},$$
$$d_{-M+1} = -\sigma_{n-M}b_{-M} + b_{-M+1} - \sigma_n(j+1)b_{-M}(n+1,j),$$
$$d_m = \sum_{l=0}^{m+M}[b_{-M+l} - \sigma_n(j+1)b_{-M+l-1}(n+1,j)]\varphi_{n-M+l} \cdot \varphi_{n+m}^{-1} -$$
$$-b_{-M} \cdot \varphi_{n-M} \cdot \varphi_{n+m}^{-1},$$
$$m = -M+1, \dots, N.$$

Proof. According to the definition (50) of ψ_n and taking into account (52) $\psi_n(j+1)$ can be written as follows

$$\psi_n(j+1) = f_n(j+1) - \sigma_n(j+1) \cdot f_{n+1}(j+1) =$$
$$= \sum_{m=-M}^{N} b_m(n,j)f_{n+m}(j) - \sigma_n(j+1)\sum_{m=-M}^{N} b_m(n+1,j) \cdot f_{n+m+1}(j). \tag{55}$$

Substitutig (50) into RHS of (54) we can also represent $\psi_n(j+1)$ in the form

$$\psi_n(j+1) = \sum_{m=-M}^{N} d_m(n,j)[f_{n+m}(j) - \sigma_{n+m}(j)f_{n+m+1}(j)]. \tag{56}$$

Now the equation (54) certainly holds if the coeficients of $f_{n+m}(j)$ in the RHS of (55) and (56) are the same. Equating these coefficients we get the following system of the matrix recursive relations

$$d_M = b_M,$$
$$d_m(n,j) - d_{m-1}(n,j)\sigma_{n+m-1}(j) = b_m(n,j) - \sigma_n(j+1)b_{m-1}(n+1,j),$$
$$m = -M+1, \dots, N,$$
$$-d_N(n,j)\sigma_{n+N}(j) = -\sigma_n(j+1)b_N(n+1,j).$$

This system is overdetermined. We can consider $M+N+1$ first relations as the system of linear algebraic equations for the apriory unknown coefficients d_m, $m=-M,\ldots,N$. The last relation also defines the coefficient d_N uniquely and the obtained result should be compatible with that obtained from solving the system of the first $M+N+1$ equations. In abelian case the matrix M of this system is given by the formula

$$M=\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -\sigma_{n-M} & 1 & 0 & \ldots & 0 & 0 \\ 0 & -\sigma_{n-M+1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -\sigma_{n+N-2} & 1 & 0 \\ 0 & 0 & \ldots & 0 & -\sigma_{n+N-1} & 1 \end{pmatrix}.$$

The inverse matrix can be easily computed :

$$M^{-1}=\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \sigma_{n-M} & 1 & 0 & \ldots & 0 & 0 \\ \sigma_{n-M}\sigma_{n-M+1} & \sigma_{n-M+1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \prod_{k=-M}^{N-2}\sigma_{n+k} & \prod_{k=-M+1}^{N-2}\sigma_{n+k} & \ldots & \sigma_{n+N-2} & 1 & 0 \\ \prod_{k=-M}^{N-1}\sigma_{n+k} & \prod_{k=-M+1}^{N-1}\sigma_{n+k} & \ldots & \sigma_{n+N-2}\sigma_{n+N-1} & \sigma_{n+N-1} & 1 \end{pmatrix}.$$

The products of σ terms simplifies being written in terms of φ-functions:

$$\prod_{k=m}^{m+n}\sigma_k(j)=\varphi_m(j)\cdot\varphi_{m+n+1}^{-1}(j) \tag{57}$$

This means that instead of sigmas every matrix element of M^{-1} different from zero can be written as a ratio of the two φ-functions . Now for the solutions of the system under consideration we obviously obtain the formulas provided by the Th.4

The straightforward calculation shows that the same formulas gives the solution of the same system in nonabelian case if we keep the order of all matrix products as written above. Of course the inversion of M was not neccessary for the proof of Th.4 . It was included only for explain how the solutions of recursive relations were found. To complete the proof we have to check that the last of formulas in the overdetermined system defining the coefficient d_N as

$$d_N = \sum_{l=1}^{M+N} b_{-M+l}\varphi_{n-M+l}\varphi^{-1}_{n+N} -$$
$$-\sigma_n(j+1)\sum_{l=0}^{M+N} b_{-M+l-1}\varphi_{n-M+1}\cdot\varphi^{-1}_{n+N}, \tag{58}$$

is compatible with the last of relations produced by equating the coefficients of f_{n+N+1}. Changing the summation index by taking $k=-M+l$ we can in abelian case identify the first sum in the RHS of (58) with $\varphi_n(j+1)\cdot\varphi^{-1}_{n+N}(j)$. The second sum in the RHS of (58) can be easily computed taking into account that by definition of φ,

$$\varphi_{n+1}(j+1) = \sum_{m=-M}^{N} b_m(n+1,j)\varphi_{n+m+1}(j), \tag{59}$$

This formula can be rewritten as follows

$$\sum_{-M}^{N-1} b_m(n+1,j)\varphi_{n+m+1}(j) = \varphi_{n+1}(j+1) - b_N(n+1,j)\varphi_{N+n+1}. \tag{60}$$

Changing in the second sum of the RHS of (58) the summation index from l to $m=-M+l-1$ we easily conclude with help of (60) that the second term in RHS of (58) is equal to

$$-\sigma_n(j+1)\left[\varphi_{n+1}(j+1) - b_N(n+1,j)\varphi_{N+n+1}\right]\varphi^{-1}_{N+n}(j).$$

Now it becomes obvious that the RHS of (58) coincides with $\sigma_n(j+1)\cdot b_N(n=1,j\,)\sigma^{-1}_{n+N}(j)$, which completes the proof.□

In full analogy with the consideration above for the left DT we have the following statement.

Theorem 5.
The equation (52) is covariant with respect to the action of the left Darboux transformation D^-

$$D^- := f_n \to \psi^-_n = f_n - \sigma^-_n f_{n-1},$$
$$\sigma^-_n(j) = \varphi_n(j)\cdot\varphi^{-1}_{n-1}(j). \tag{61}$$

i.e. the function ψ_n^- satisfy to the following lattice equation.

$$\psi_n(j+1) = \sum_{m=-M}^{N} d_m(n,j)\psi_{m+n}(j) \tag{62}$$

where the coefficients are defined by the formulas

$$d_N(n,j) = b_N(n,j),$$
$$d_k(n,j) = \left[\sum_{m=k}^{N} b_m(n,j) - \sigma_n(j+1)\sum_{m=k}^{N-1} b_{m+1}(n-1,j)\varphi_{n+m}\right]\varphi_{n+k}^{-1},$$

$$k = -M, \ldots, N-1;$$

$$d_{-M} = \sigma_n(j+1)\cdot b_{-M}(n-1,j)\sigma_{n-M}^{-1},$$
$$\sigma(n,j+1) = \left[\sum_{m=-M}^{N} b_m(n,j)\varphi_{n+m}(j)\right]\cdot\left[\sum_{m=-M}^{N} b_m(n-1,j)\varphi_{n+m-1}(j)\right]^{-1}$$

3.2 Iterations of the lattice Darboux transformations and Casorati determinants

Here we show that the n-times iterated lattice Darboux transformation is described by the simple formula containing the ratio of the two Casorati determinants. We start from the case (+) i.e. Darboux transformation of $f_n(j)$ is given by the formula

$$D_1 f_n = \psi_n = f_n - \sigma_n[1]f_{n+1} = \frac{Cas[f_n, \varphi_n(1)]}{\varphi_{n+1}(1)}, \tag{63}$$

$$Cas[f_n, \varphi_n(1)] \stackrel{\text{def}}{=} \begin{vmatrix} f_n & \varphi_n(1) \\ f_{n+1} & \varphi_{n+1}(1) \end{vmatrix}. \tag{64}$$

More generaly we define Casorati determinant of m functions $f_n(1), \ldots, f_n(m)$ by the formula

$$Cas[f_n(1), \ldots, f_n(m)] = \det A, A_{jk} = f_{n+j-1}(k); j,k = 1, \ldots, m. \tag{65}$$

Let $\varphi_n(1), \ldots, \varphi_n(m)$ be m different solutions of (49) or (52) with $Cas[\varphi_n(1), \ldots, \varphi_n(m)] \neq 0$.
Theorem 6

The result $\psi_{[m],n}$ of the m-times application of the Darboux transformation to any given solution f_n of (49) *or* (52) *is given by the following formula*

$$\psi_{[m],n} = \frac{Cas[f_n, \varphi_n(1), \ldots, \varphi_n(m)]}{Cas[\varphi_n(1), \ldots, \varphi_n(m)]}. \tag{66}$$

Defining the "left" lattice DT as

$$\psi_n^{(-)} = f_n - \sigma_n^{(-)} f_{n-1}, \sigma_n^{(-)} = \varphi_n \cdot \varphi_{n-1}^{-1} \tag{67}$$

we can describe the result $\psi_{[m],n}^{(-)}$ of the m-times action of DT on f_n by the formula

$$\psi_{[m],n}^{(-)} = \frac{Cas^{(-)}[\, f_n, \varphi_n(1), \ldots, \varphi_n(m)\,]}{Cas^{(-)}[\, \varphi_{n-1}(1), \ldots, \varphi_{n-1}(m)\,]}, \tag{68}$$

where

$$Cas^{(-)}[\, \varphi_n(1), \ldots, \varphi_n(m)] := \det A, A_{jk} = \varphi_{n-j+1}(k),\ j, k = 1, \ldots, m. \tag{69}$$

Proof We give the proof only for case $(+)$ since case $(-)$ can be treated in a same way. Below in the proof sign $(+)$ will be omitted for brevity. First it is obvious that the result of the m-times iterative application of the lattice Darboux transformation to f_n can be written as follows

$$\psi_{[m],n} = D_m D_{m-1} \cdots D_1 f_n = \tag{70}$$

$$= f_n + a_1 f_{n+1} + \ldots + a_m f_{n+m}, \tag{71}$$

where D_j is defined by specifying some solution of the equation obtained after $j - 1$ application of the DT to the starting equation. Expanding the nominator of (66) by the elements of its first column we obviously get the same structure. Now the proof can be easily achieved by induction. We observe that RHS of (66) vanish when $f_n = \varphi_n(j)$, $j = 1, \ldots, m$. If we show that RHS of (71) has the same property this fixes the coefficients $a_1, \ldots, a_m$ uniquely since we can interpret the relations

$$f_n + \sum_{k=1}^{m} a_k f_{n+k}|_{f_n = \varphi_n(j)} = 0;\ j = 1, \ldots, m, \tag{72}$$

as a system of linear algebraic equations for unknown coefficients a_k. The determinant of this system is obviously coincide with denominator of the

RHS of (66). Now to complete the induction ,since the formula ( ) is true for $m = 1$, it is enough to demonstrate that the validity of the formula

$$\psi_{[m-1],n} = \frac{Cas[f_n, \varphi_n(1), \dots, \varphi_n(m-1)]}{Cas[\varphi_n(1), \dots, \varphi_n(m-1)]}, \tag{73}$$

for the result of $m - 1$ application of the DT to f_n implies that its own Darboux transformation

$$\psi_{[m-1],n} - \sigma_{[m-1],n}\psi_{[m-1],n+1}, \tag{74}$$

$$\sigma_{[m-1],n} = \psi_{[m-1],n}\psi^{-1}_{[m-1],n+1}|_{f_n=\varphi_n(m)}, \tag{75}$$

vanish if $f_n = \varphi_n(j)$, which is completely evident. Now we can identify RHS of (71) with RHS of (66) taking into account the uniqueness of the solution of (72).□

Remark. In whole analogy with continuos case the coincidence of the RHS of (74) with (66) proved above is equivalent to the following identity (we take $g_n = \varphi_n(m)$. relating four Casorati determinants (below we write C instead of Cas in order to compactify the notations):

$$\begin{aligned} C[\varphi_n(1), \dots, \varphi_n(m-1), g_n, f_n] \cdot C[\varphi_{n+1}(1), \dots, \varphi_{n+1}(m-1)] = \\ = C[C[\varphi_n(1), \dots, \varphi_n(m-1), g_n], C[\varphi_n(1), \dots, \varphi_n(m), f_n]]. \end{aligned} \tag{76}$$

In our approach this identity (known in the literature as Silvester formula) is obtained as a free biproduct. The existing proofs of this identity using Laplace decomposition of the determinants are a bit longer. For the case $(-)$ we have of course quite similar identity obtained by replacing C by $C^{(-)}$ and $n+1$ by $n-1$.

3.3 Mixed lattice DT

The simple observation concerning the subsequtive application of the left and right DT to initial linear equation is that they do not commute i.e.

$$D^+ \cdot D^- f_n = c_1 f_n + c_2 f_{n+1} + c_3 f_{n-1} \neq \tag{77}$$

$$D^- \cdot D^+ = g_1 f_n + g_2 f_{n+1} + g_3 f_{n-1}. \tag{78}$$

Here we mean that $D^- f$ in (77) was defined by fixing some solution $\varphi_n(1)$ of (52) or (49) . Next taking $f = \varphi_n(2)$ we get a fixed solution of the transformed equation thus defining the action of D^+ on $D^- \cdot f_n$. In (78) we first fix $\varphi_n(2)$ or $\varphi_n(1)$ for define the action of D^+ on f_n and take the notused of these two functions for define the action of D^- on $D^+ f_n$. For any order of use $\varphi_n, \varphi_n(2)$ the RHS of (77) and (78) are different and $c_1, g_1 \neq 1$. So it is obvious that for a mixed compositions of finite number of DT there is no chances to get the formulas as simple as (66) or (73).

It is easy to prove somehow that RHS of both equations (77-78) vanishe if we take $f_n = \varphi_n(1), \varphi_n(2)$. This means that multiplying $D^+ \cdot D^- f_n$ on c_1^{-1} and $D^- \cdot D^+ f_n$ on g_1^{-1} we get the same resulting function $\Psi_n(1,2)$ of the form

$$\psi_n(1,2) = f_n + a_1 \cdot f_{n-1} + b_1 \cdot f_{n+1} \tag{79}$$

vanishing when $f = \varphi_1, \varphi_2$. This vanishing properties as before define f uniquely and it is obvious that (52) is covariant with respect to the action of the mixed DT defined by the formula

$$D(1,2) f_n = \psi_n(1,2). \tag{80}$$

The covariance follows from the left and right DT covariance established above and from the fact that (52) is covariant with respect to the action of the trivial gauge transformations $f \to G(x,j) \cdot f$ where $G \in GL(N,C)$, reducing in abelian case to multiplication on nonvanishing functions. It is easy to obtain the general formulas decribing the transformation of the coefficients of (52) under the action of the mixed DT . We give here for brevity only the associated result for the simplest difference - difference equation

$$f_n(j, r-1) = p \cdot f_n + u \cdot f_{n-1}, p = p(j,r), u = u(j,r). \tag{81}$$

According to the covariance of (81) with respect to the action of (80) we have (writing ψ_n instead of $\psi_n(1,2,j,r)$ and specifing only the shifted variables)

$$\psi_n = \tilde{p}\psi_n + \tilde{u}\psi_{n-1}. \tag{82}$$

The coefficients $\tilde{p}$, $\tilde{u}$ are decribed by the following formulas

$$\begin{aligned}
&\tilde{p} + \tilde{u} \cdot b_1(n-1) = p + b_1(r-1) \cdot u(n+1), \\
&\tilde{p} \cdot a_1 + \tilde{u} = u + a_1(r-1) \cdot p(n-1), \\
&\tilde{u} a_1(n-1) = a_1(r-1) u(n-1), \\
&\tilde{p} \cdot b_1 = b_1(r-1) \cdot p(n+1).
\end{aligned}$$

More generally normalising the same way result of N-times application of the right Darboux transformation to $\psi_{[M]}$ obtained after M -times application of the left Darboux transformation to f_n, we get

$$\psi_n(N,M) = g(r,j)\cdot D_N^+ \dots D_1^+ \cdot D_M^- \dots D_1^- f_n = \tag{83}$$

$$= f_n + \sum_{l=1}^{N} a_l f_{n+l} + \sum_{m=1}^{M} b_m f_{n-m}. \tag{84}$$

The coefficients in (84) are fixed by the following vanishing requirement

$$\psi_n(N,M)|_{f_n=\varphi_n(k)} = 0, k = 1,\dots,M+N. \tag{85}$$

where $\varphi_n(i)$ are $N+M$ fixed different linearly independent solutions of (52) and g is the normalisation factor. In whole analogy with previous considerations we can prove the following statement

Theorem 7

The function $\psi_n(N,M)$ defined above is given by the formula

$$\begin{aligned}
&\psi_n(N,M) = \tfrac{\det A}{\det B},\\
&A_{ik} = \varphi_{n+N+1-i}(k),\\
&i = 1,\dots,M+N+1, k = 1,\dots,N,N+2,\dots,N+M+1;\\
&A_{i,N+1} = f_{n+N+1-i}.\\
&B_{ik} = \varphi_{n+N+1-i}(k), i,k = 1,\dots,N+M.
\end{aligned}$$

Remark.$\psi_{N,M}$ *is a symmetric function of* $\varphi_n(j)$.

The proof of the theorem literally repeats the proof of the Th.6 and can be easily completed by the reader. Again it is obvious from previous considerations that (52) is covariant with respect to the action of the mixed lattice DT : $f_n \to \psi_n(N,M)$

The theorems proved above do not exhaust the list of useful covariance properties leading to the interesting applications to integrable systems. One usefull concept which we also missed to discuss for continous systems is the lattice binary DT . The interested reader can find some preliminary results about it in [5] , [25]. Its more detailed presentation applicable to the hierarchies of the lattice equations of any order will be reported elsewhere. It is also not difficult to obtaine the law of transformation of the coefficients of (52) under the action of the mixed DT of any order defined above. We omit here the assosiated resuts for brevity. The necessary type of calculation will be illustrated in the last section devoted to the functional equations with generalized shift operators. Also we have no intention to reproduce all the results concerning the particular classes of solutions for continous systems for the lattice systems (i.e. the construction of the rational solutions , multisoliton solutions , discussion of lattice lumps , positons etc.) Here we will only illustrate the main ideas of applications of the lattice DT to the simplest lattice Zakharov-Schabat equations.

4 Construction of the solutions of the lattice Zakharov-Schabat equations

We consider here for concretness only the simplest case corresponding (see below) to the Hirota bilinear difference equation. Our appproach somehow works without any modifications for the higher Zakharov Scabat equations and their nonabelian versions. Asssociated applications of general theorems proved above will be reported elsewhere. The "minimal" Zakharov Shabat equation which we will call also Hirota equation can be obtained as the compatibility condition of the following linear system.

$$f_n(j, r-1) = f_n(j, r) + u(n, j, r) \cdot f_{n-1}(j, r), \tag{86}$$

$$f_n(j-1, r) = f_{n+1}(j, r) + v(n, j, r) \cdot f_n(j, r), \tag{87}$$

$$n, j, r \in Z.$$

More precisely we can represent $f_n(j-1, r-1)$ as a linear combination of $f_k(j, r)$ first replacing j by $j-1$ in (86) and than transforming the RHS of the obtained equation with help of (87). Next we can compute the same quantity first replacing r by $r-1$ in (87) and than transforming the RHS of the produced equation with help of (86). Equating the coefficients of $f_n(j, r)$ and $f_{n-1}(j, r)$ we get the following system

$$v(n, j, r-1)u(n, j, r) = u(n, j-1, r)v(n-1, j, r), \tag{88}$$

$$v(n, j, r) + u(n, j-1, r) = u(n+1, j, r) + v(n, j, r-1). \tag{89}$$

This system represents the simplest possible of the " Zakharov-Schabat " difference-difference equations. The system (88-89) can be also rewritten as follows

$$u(n,j-1,r)-u(n+1,j,r)=v(n,j,r-1)-v(n,j,r), \tag{90}$$

$$\frac{u(n,j,r)}{u(n,j-1,r)}=\frac{v(n-1,j,r)}{v(n,j,r-1)}. \tag{91}$$

It is connected with original Hirota bilinear form by the following formulas

$$u(n,j,r)=\frac{\tau_{n-1}(j,r)\cdot\tau_{n+1}(j,r-1)}{\tau_n(j,r-1)\cdot\tau_n(j,r)}, \tag{92}$$

$$v(n,j,r)=\frac{\tau_n(j,r)\cdot\tau_{n+1}(j-1,r)}{\tau_n(j-1,r)\cdot\tau_{n+1}(j,r)}. \tag{93}$$

The function *tau* is the solution to original Hirota bilinear equation first written explicitly in [7]. Hirota also found some kind of 3-solitons solution and conjectured the formula for N-solitons solutions. This formula was proved and generalized by Miwa who also found some four terms bilinear integrable equation [8] and solved it using the free fermions formalism of Kyoto school. The form of the solution obtained by Miwa somehow is rather complicated and has some advantages and disadvantages from the point of view of concrete applications. Hirota equation in bilinear form reads :

$$\tau_n(j+1,r)\tau_n(j,r+1)-\tau_n(j,r)\tau_n(j+1,r+1)+\tau_{n+1}(j+1,r)\tau_{n-1}(j,r+1)=0. \tag{94}$$

Both equations (86-87) are covariant with respect to the action of the left lattice DT (50). With respect to the action of the right DT the system (86-87) is not covariant . From the Th. 5 we get immediatly the following formulas for the coefficients of the DT transformed system

$$u_1(n,j,r)=\sigma_n(j,r-1)\cdot u(n-1,j,r)\cdot\sigma_{n-1}^{-1}(j,r)= \tag{95}$$

$$\frac{\varphi_n(j,r-1)\varphi_{n-2}(j,r)}{\varphi_{n-1}(j,r-1)\varphi_{n-1}(j,r)}= \tag{96}$$

$$=u(n,j,r)+\sigma_n(j,r)-\sigma_n(j,r-1); \tag{97}$$

$$v_1(n,j,r)=\sigma_n(j-1,r)\cdot v(n-1,j,r)\cdot\sigma_n^{-1}(j,r)= \tag{98}$$

$$=v(n-1,j,r)\cdot\frac{\varphi_n(j-1,r)\varphi_{n-1}(j,r)}{\varphi_{n-1}(j-1,r)\varphi_n(j,r)}= \tag{99}$$

$$=v(n,j,r)+\sigma_{n+1}(j,r)-\sigma_n(j-1,r). \tag{100}$$

It has to be mentioned that according to the Th.5 the formulas (95 ,97,98,100) remain valid in the nonabelian case.

It is easy to prove (see the last section for the derivation in more general context) that the coefficients of the system

$$\psi_{[N],n}(j,r-1) = \psi_{[N],n}(j,r) + u_N(n,j,r)\psi_{[N],n-1}(j,r), \tag{101}$$
$$\psi_{[N],n}(j-1,r) = \psi_{[N],n+1}(j,r) + v_N(n,j,r)\psi_{[N],n}(j,r) \tag{102}$$

obtained after application of the N-fold left DT are described by the following formulas

$$u_N(n,j,r) = \frac{C_N^-(n,j,r-1)\cdot C_N^-(n-2,j,r)}{C_N^-(n-1,j,r-1)\cdot C_N^-(n-1,j,r)}\cdot u(n-N,j,r), \tag{103}$$
$$v_N(n,j,r) = \frac{C_N^-(n,j-1,r)\cdot C_n^-(n-1,j,r)}{C_N^-(n-1,j-1,r)\cdot C_N^-(n,j,r)}\cdot v(n-N,j,r). \tag{104}$$

Here as before we put

$$C_N^-(n,j,r) = \det A,\, A_{ik} = \varphi_{n-i+1}(k); i,k = 1,\ldots,N.$$

Now if we put in particular $u = v = 1$ we can obviously identify the tau functions in (92-93) with Casorati determinants

$$\tau_n(j,r) = C_N^-(n-1,j,r). \tag{105}$$

In the last case we also can easily construct the solutions to the starting system (86-87), depending on arbitrary number of functional parameters in whole analogy with the similar construction for the KP equation [1, article 1]. First we find the particular solutions by separation of variables :

$$f_n(j,r,k) = k^{r-j}\cdot(k-1)^{n-r}, \tag{106}$$

where $k = k_1 + ik_2$ is an arbitrary complex parameter. For any real valued continuous density function $\rho_l(k_1,k_2)$ with compact support not including the points $k=0, k=1$ the integral

$$\varphi_n(l) = \int\int \rho_l(k_1,k_2)\cdot k^{r-j}(k-1)^{n-r}\,dk_1\cdot dk_2 \tag{107}$$

gives the solution to the system (86-87). Its real part and imaginary part represent the real valued solutions to the same system. In particular taking k

real and choosing N different densities $\rho_l(k)$ we get the real valued solutions of (86-87) with $u = 1, v = 1$ in a following form :

$$\varphi_n(l) = \int \rho_l(k) \cdot k^{r-j}(k-1)^{n-r} dk. \tag{108}$$

Substituting the obtained solutions with $l = 1, \ldots, N$ into determinant (105) we obviously get the solution of the Hirota equation depending on N functional parameters. In whole analogy with the similar consideration in [5] we can isolate nonsingular globaly bounded solutions. It is important to observe that the "starting solutions" (107) are the functions of the form $f_l(r-j, n-r)$. Performing one step Darboux transformation with one of them we already get the nontrivial τ-function solving the Hirota equation :

$$\tau_n(j,r) = \varphi_{n-1}(j,r,l) = \int \rho(k) \cdot k^{r-j}(k-1)^{n-1-r}\, dk \tag{109}$$

The τ-functions (109) corresponding different densities $\rho_l(k)$ form an infinite dimensional linear subspace in the space of the solutions of the Hirota equation.

4.1 More general lattice Zakharov-Shabat equation

Next particular example deals with an auxilary linear system slightly more general than that of the previous subsection .

This equation can be obtained as the compatibility condition of the following linear system.

$$f_n(j, r-1) = p \cdot f_n + u \cdot f_{n-1}, \tag{110}$$

$$f_n(j-1, r) = q \cdot f_{n+1} + v(n,j,r) \cdot f_{n-1}, \tag{111}$$

$$n, j, r \in Z.$$

Here and below to shorten the notations we shall often omit arguments of the coefficients if they coincide with (n, j, r) specifying only the shifted variables so that $p(r-1) = p(n, j, r-1)$, $f_{n+1} = f_{n+1}(j, r)$ and so on. We can represent $f_n(j-1, r-1)$ as a linear combination of $f_k(j, r)$ first replacing j by $j-1$ in (110) and than transforming the RHS of the obtained equation with help of (111). Next we can compute the same quantity first replacing r by $r-1$ in (111) and than transforming the RHS of the produced equation with help of

(110). Equating the coefficients of $f_{n+1}(j,r)$, $f_n(j,r)$ and $f_{n-1}(j,r)$ we get the following system of 3 nonlinear equations:

$$p(j-1)\cdot q = q(r-1)\cdot p(n+1), \tag{112}$$

$$q(r-1)\cdot u(n+1) + v(r-1)\cdot p =$$
$$= p(j-1)\cdot v + u(j-1)\cdot q(n-1), \tag{113}$$

$$u(j-1)\cdot v(n-1) = v(r-1)\cdot u. \tag{114}$$

This system represents the slightly more complicated case with respect to that of the previous section. Of course we can reduce it to the previous case performing appropriated gauge transformation but we prefere not fixing the gauge and consider the system (110-111) in its original more symmetric form. First the linear system (110) is covariant with respect to the action of the left and right Darboux transformations (50). The left DT leaves the coefficents p and q invariant and the transformation law for u and v under the action of the M-fold left DT is given by the formulas :

$$u_M = u(n-M)\cdot a_M(r-1)\cdot a_M^{-1}(n-1) = \tag{115}$$
$$= u(n,j,r) + a_1(r-1)\cdot p(n-1) - p\cdot a_1. \tag{116}$$
$$v_M = v(n-M)\cdot a_M(j-1)\cdot a_M^{-1} = \tag{117}$$
$$= v(n,j,r) + a_1(j-1)\cdot q(n-1) - q\cdot a_1(n+1). \tag{118}$$

The coefficients $a_1(n,j,r)$, $a_M(n,j,r)$ can be easily computed from decomposition of $\psi^+_{[M],n}$

$$\psi^+_{[M,n]} = f_n + \sum_{m=1}^{M} a_m f_{n-m}. \tag{119}$$

. From (73) we obviously get the following expessions for a_1, a_M

$$a_1(n,j,r) = \frac{\det B}{\det A}, \tag{120}$$

$$B_{ik} = \varphi_{n-i+1}(k); i = 1,3,4,\ldots,M+1; k = 1,\ldots,M. \tag{121}$$

$$A_{ik} = \varphi_{n-i}(k); i,k = 1,\ldots,M. \tag{122}$$

$$a_M = (-1)^M \frac{\det C}{\det A}, \tag{123}$$

$$C_{ik} = \varphi_{n-i+1}(k); i,k = 1,\ldots,M. \tag{124}$$

Here as before $\varphi_n(k) = \varphi_n(j,r,k)$ are different fixed solutions of the system (110-111). In particular (117) and (117) mean that the multiplicative dressing formulas for u_M, v_M can be written in a following explicit form :

$$u_N(n,j,r) = \frac{C_N^-(n,j+1,r)\cdot C_N^-(n-2,j,r)}{C_N^-(n-1,j+1,r)\cdot C_N^-(n-1,j,r)}\cdot u(n-N,j,r), \quad (125)$$

$$v_N(n,j,r) = \frac{C_N^-(n,j,r+1)\cdot C_n^-(n-2,j,r)}{C_N^-(n-1,j,r+1)\cdot C_N^-(n-1,j,r)}\cdot v(n-N,j,r). \quad (126)$$

Here as before we put

$$C_N^-(n,j,r) = \det A, A_{ik} = \varphi_{n-i+1}(k); i,k = 1,\ldots,N.$$

Now using the covariance of the same system with respect to the action of the right DT applying it N-times we keep the coefficients u and v invariant and we get for the coefficients p and q of the transformed system the following formulas :

$$p_M = p(n+M)\cdot\frac{C_N^+(n,j,r-1)\cdot C_N^+(n+1,j,r)}{C_N^+(n+1,j,r-1)\cdot C_N^+(n,j,r)}, \quad (127)$$

$$q_N = q(n+M)\frac{C_N^+(n,j-1,r)\cdot C_N^+(n+2,j,r)}{C_N^+(n+1,j-1,r)\cdot C_N^+(n+1,j,r)}. \quad (128)$$

Now if we put in particular $u = v = p = q = 1$ we obviously get the same starting solutions of (110-111) as given by the formulas (107-108) and we get again the family of the solutions described by Casorarty determinants formed from the functions (107-108). Of course the system (110-111) allows also the application of the mixed DT. But we omitted these formulas here for brevity.

5 Equations with generalized shift operators

Let us consider the following class of the functional-differential equations

$$f_t(x,t) = \sum_{m=-M}^{N} u_m(x,t)\cdot T^m(f), x\in R^n, \quad (129)$$

or functional difference equations of the form

$$f(x, j+1) = \sum_{m=-M}^{N} u_m(x, j) \cdot T^m(f). \tag{130}$$

In (130-131) T is assumed to be an invertible linear operator commuting with ∂_t in the case (130) and with the shift operator $\Delta : \ \Delta(f)(x, j) = f(x, j+1)$ in case of (131) which is of course more general and contains continuous time evolution in appropriated limit. So below we formulate all the statments for (130). T is also supposed to satisfy the following additional requirement

$$T(f_1 \cdot f_2) = T(f_1) \cdot T(f_2), \; T(I) = I. \tag{131}$$

For many particular choices of T listed above x instead of be in R^n might be the point of some Lie group or more generally the point of some smooth manifold : most part of the theorems listed below are valid for this more general case also .

Again we can consider the general case where the coefficients u_m are matrix valued functions , f is a matrix valued solution of the same matrix dimension as the coefficients , and $T(f)$ is a matrix valued function of the same matrix dimension as f . I is a unit matrix. Of course there is a lot of operators satisfying to these requirements.

Let us list the some important examples of T operators.

$$\begin{aligned}
&T(f)(x,t) = f(x+\delta); x, \delta \in R, \\
&T^{-1}f(x,t) = f(x-\delta, t) \\
&T(f)(x,t) = f(qx,t), q \in C, q \neq 0, \\
&T^{-1}f(x,t) = f(q^{-1}x, t) \\
&T(f)(x,t) = f(Ux,t), T^{-1}(f)(x,t) = f(U^{-1}x, t), \\
&U \in GL(n,R), x \in R^n \\
&T(f)(x,t) = f(g(x),t), T^{-1}f(x,t) = f(g^{-1}(x),t), \\
&g \in Diff(R^n), \\
&T(f) = U \cdot f \cdot U^{-1}, \; U = U(x) \in GL(n,C).
\end{aligned}$$

Let $\varphi(x,t)$ be a fixed solution of (131) and (in nonabelian case) φ^{-1} means the result of its matrix inversion. In whole analogy with previous sections we define left and right Darboux transformations $D^{\pm}$ by the formulas

$$D^+ f = f - \sigma^+ \cdot T(f), \; \sigma^+ = \varphi \cdot [T(\varphi)]^{-1}, \tag{132}$$

$$D^- f = f - \sigma^- \cdot T^{-1}(f), \; \sigma^- = \varphi \cdot [T^{-1}(\varphi)]^{-1}, \tag{133}$$

It is instructive to observe that under the first two choices of T in the list above in a limit $\delta \to 0$ and $q \to 1$ respectively we recover from $D^{\pm}$ the classical Darboux transformation . More precisely we have the following formulas

$$\sigma f - \partial_x \cdot f = \lim_{\delta\to 0} \delta_{-1} D^{\pm} f = \lim_{q\to 1} (1-q)^{-1} D^{\pm} \tag{134}$$

Of course we can reproduce the formulas for the iterated classical Darboux transformation coming to the same kind of limits in the formulas listed above.

We can prove that (131) is again covariant with respect to the action of D^+ and D^-. The transformation law of the coefficients remains the same as described by the formulas of the previous section if we replace there f_{n+m} by $T^m(f)(x,j)$, $\sigma_n^+(j)$ by σ^+ defined by (133) etc . More precisely for the coefficients of the D^+ transformed equation

$$\psi^+(x, j+1) = \sum_{k=-M}^{N} \tilde{u}_k(x,j) \cdot T^k(\psi)(x,j), \tag{135}$$

we get the following formulas :

$$\tilde{u}_{-M} = u_{-M},$$
$$\tilde{u}_{-M+1} = u_{-M+1} + T^{-M}(\sigma^+) u_{-M} - \sigma^+(j+1) T(u_M),$$
$$\tilde{u}_m = \sum_{l=0}^{m+M} [u_{-M+l} - \sigma^+(j+1) T(u_{-M+l-1})] T^{-M+l}(\varphi) \cdot [T^m(\varphi)]^{-1} -$$
$$-u_{-M} \cdot T^{-M}(\varphi) \cdot [T^m(\varphi)]^{-1}, \qquad m = -M+1, \ldots, N;$$
$$\tilde{u}_N = \sigma^+(j+1) T(u_n) [T^N(\sigma)]^{-1}$$

There is no any difference in the proof providing that the general requirements imposed on T are satisfied.

Now the result ψ_N^+ of the action of L-fold right Darboux transformation on f reads

$$\psi_N = D_L D_{L-1} \cdots D_1 = f + \sum_{l=1}^{L} a_l \cdot T^l(f) = \frac{\det A}{\det B}, \tag{136}$$

where

$$A_{i1} = T^{i-1}(f),\ A_{ij} = T^{i-1}(\varphi(j-1));$$
$$i = 1, \ldots, L+1;\ j = 2, \ldots, L+1,$$
$$B_{ij} = T^i(\varphi(j)\,); \quad i,j = 1, \ldots, L,$$

Above $\varphi(j), j = 1, \ldots, L$ are L different fixed solutions of (131). The proof of the last statement contains nothing new with respect to similar statements of the previous sections and explores the fact that the RHS of (136) obviously vanishes if we take $f = \varphi(j), j = 1, ..., L$ As before expanding the $\det A$ by the elements of the first column we easily compute all the coefficients a_l in (136). In particular for a_1 and a_L we get the formulas

$$a_1 = -\tfrac{\det Q}{\det B},$$
$$Q_{1j} = \varphi_j;\ j = 1, \ldots L; \qquad Q_{ij} = T^i(\varphi_j)\,;\ i = 2, \ldots, L.$$

$$a_L = (-1)^L \cdot \tfrac{\det P}{\det B},$$

$$P_{ij} = T^{i-1}(\varphi_j)\,; \qquad i, j = 1, \ldots, L.$$

Remark. Despite the fact that all the formulas here are looking quite similar to the lattice situation of previous section we have much more flexibility. For instance assuming that operator T is cyclic i.e . $T^L = I$ we see that $\psi_{[L]}$ vanishes as well as the coefficients $a_l, l \leq L-1$ and $a_L = -1$.

Substiting the decomposition given by the middle equation (136) into the LHS and RHS of the equation satisfied by $\psi_L(x, j)$

$$\psi_L(x, j+1) = \sum_{m=M}^{N} \tilde{u}_m \cdot T^m(\psi_L) \tag{137}$$

it is easy to show that the coefficients $\tilde{u}_j$ can be found from the following recursive relations

$$\tilde{u}_{-M} = u_{-M},$$
$$\tilde{u}_{-M+1} = u_{-M+1} + a_1(x, j+1) \cdot T(u_{-M}) - u_{-M} \cdot T^{-M}(a_1),$$

$$\tilde{u}_{-M+2} + \tilde{u}_{-M+1} \cdot T^{-M+1}(a_1) + u_{-M}T^{-M}(a_2) =$$

$$u_{-M+2} + a_1(x, j+1)T(u_{-M+1}) + a_2(x, j+1)T^2(u_{-M}),$$
$$\tilde{u}_p + \textstyle\sum_{m+l=p} \tilde{u}_m \cdot T^{m+l}(a_l) = u_p + \sum_{l+m=p} a_l(x, j+1) \cdot T^l(u_m),$$

$$p = -M+1, -M+2, \ldots, L;$$

$$\tilde{u}_N = a_L(x, j+1)T^L(u_N) \cdot [T^N(a_L)]^{-1}.$$

From this structure it is obvious that *in the case of the cyclic operator T satisfying $T^L = I$ we have a periodic closure of the chain of the Darboux transformations : the equations (137) and (130) coincide.* The same conclusion concerns of course the functional equations

$$\sum_{m=-M}^{N} u_m(x) \cdot T^m(f) = \lambda \cdot f, \tag{138}$$

obtained as stationary reduction of (1-2).

Remark Important particular case of (138) with scalar coefficients and $M = -1, N = 1$ with general T satisfying the requirements of this section was first considered by Salle [23] , who considered in particular the applications of the special case $T(f)(x) = f(q \cdot x)$ to construct explicitely the spectrum and eigenfunctions of the certain q-oscillator reproducing in a limit $q \to 1$ the spectrum and eigenfunctions of quantum harmonic oscillator. Somehow in [23] it was not mentioned that algebraically this is the same construction as in the case of the lattice Schrödinger equation and hence it is in one to one correspondence with stationary reduction of the simplest case of [1, article 2]

6 Concluding Remarks

So far we have proposed here enough general concept of the generalized Darboux transformation allowing application to a broad class of linear and nonlinear patial differential , differential difference or functional difference equations with matrix valued coefficients. The main tool to introduce them was to explore certain vanishing properties of the assosiated ψ-functions defining N-fold or mixed (N,M) DT. It is instructive to compare the results of the last section concerning periodic closures of the sequence of Darboux transformations with the discussion of the same problem in [29] , [30]. There the same question was studuied for the case of the stationary 1-dimensional Shrödinger operator allowing application of the original result by Darboux [2] and Crum [22]. It was shown that the possibility of the periodic closure depends on the choice of parameters (starting solutions) and its period depends also on the structure of the associated background potentials. This was used to give some new characterisation to the finite gap periodic potentials (first explicitly constructed together with the associated eigenfunctions in terms of the Riemann theta functions of the hyperelliptic curves in [34] ,[33], [32] see also

[35] , [36] for detailed review) and also to some new trancsendents generalizing Penleve transcendents. Our situation in the case of cyclic operator T is essentialy different : we have the periodic closure for any initial variety of "potentials". Looking at the sequence of the DT as on some discrete time dynamics we can hence generate the strictly periodic dynamics in a huge variety of situations. Additional comment is that in all known situations the Darboux transformations in contrast with general dressing transformations are canonical i.e. they preserve certain symplectic structures or Poisson brackets (see the end of [5] and references there. This observation and systematic hamiltonian interpretation of DT are not yet developped enough. Some additional results in this direction can be found in [31].

7 Aknowledgements

The author is deeply grateful to Max-Planck-Institut für Mathematik in Bonn for financical support and very pleasant atmosphere of work during my stay in Bonn in September-December 1996 where this work was written. I aknowledge also the stimulating discusssions with M.A. Salle, A. Zabrodin , A. Volkov and Y.Suris on different topics connected with the contents of this article.

References

[1] V.B.Matveev , Lett. Math. Phys. **3** (1979) , 213-216 , 217-222 , 503-512

[2] G. Darboux , Compt. Rend. **94** (1882) , 1456

[3] V. B. Matveev, Phys. Lett. A **166** (1992) 209-213

[4] V.B. Matveev , J. of Math. Phys. **35** n.6 (1994) 2955-2960

[5] V.B. Matveev , M.A. Salle : "Darboux Transformations and Solitons" Spinger Series in Nonlinear Dynamics , Springer Verlag (1991) , 1- 120

[6] M.J. Ablowitz , F. J.Ladik . Stud. Appl Math. **55** (1976) 213-229 , ibid. **57** (1977) 1-12

[7] R. Hirota , Journ. Phys. Soc. Japan **50** (1981) 3875-3791

[8] T. Miwa , Proc. Japan. Acad., ser. A **58** (1982) 9-12

[9] Y.B. Suris J. Phys. A :Math. Gen. **29** (1996) ,451-465

[10] Y.B.Suris , Leningrad Math. G. **2** (1991) 339-352

[11] J. Moser , A.P. Veselov , Comm. Math. Phys. **139** (1991) 217-43

[12] A.P. Veselov Funct. Anal. Appl. **22** (1988) 1-13

[13] L.D. Faddeev , A. Volkov , Lett. Math. Phys. **32** (1994) 125-130

[14] F. Nijhoff , H. Capel, Acta Appl. Math. **39** (1995) 133 - 157

[15] A.Kuniba , T.Nakanishi and J.Suzuki , Int. Journ. of Mod. Phys . **A9** (1994) 5215 - 5312

[16] Z. Tsuboi , A. Kuniba "Solutions of a discretized Toda field equation" hep-th Oct. 22 (1996) 1-14

[17] Z. Tsuboi "Discretised affine Toda field equations" hep-th Nov. 12 (1996) 1-14 hep-th (1986)

[18] I. Krichever , O. Lipan , P. Wiegmann , A. Zabrodin , hep-th 96

[19] Morozov et al , hep-th 9405011

[20] H. Au-Yang , J.H.H. Perk, Physica **18D** , (1986) 365-366 ,

[21] B. McCoy , J.H.H. Perk , T.T.Wu , Nucl. Phys. **B180** (1981) , 89-115

[22] M.Krum , Quat. J. Math. **6** (1954) 121-128

[23] M.A. Salle , Darboux transformations : solitons , special functions , integrable maps , supersymmetry , Preprint CSMAE n. 9511-1 (1995) 1-17 (St. Petersburg)

[24] V.B. Matveev , M.A. Salle , Lett. Math. Phys. **3** (1979) 425-429

[25] M.V. Babich , V.B. Matveev , M.A. Salle , Zap. Semin. LOMI **145** (1985) , 34-45

[26] V.B. Matveev , M.A Salle : Zap. Semin. LOMI, **101** (1981) 111-117

[27] V.B. Matveev , M.A. Salle , "Some Topics on Inverse Problems" , ed. by P.Sabatier , (World scientific , Singapore) (1988) 182-212

[28] V.B. Matveev : Algebraic and Geometric Methods in Mathematical Physics (A. Boutet de Monvel and V.A. Marchenko eds.) Kluver Acad. Publ. (1996) 343-355

[29] A.B. Schabat , R.I. Yamilov , Lening. Math. Journ. **2** (1991) 377-400

[30] A.P. Veselov , A.B. Schabat , Functional Anal. and apl. **27** (1993) 1-21

[31] A.P. Fordy , A. Schabat , A.P. Veselov Factoristion and Poisson Correspondences Preprint (1994) 1-26 (To appear in the volume dedicated to the memory of Ira Dorfman)

[32] A.R. Its , V.B. Matveev,Sov. J. Theoret and Math. Phys.**23** (1975) 51-67

[33] A.R.Its , V.B. Matveev , Funct. Anal. Pril. **9** (1975) 69-70

[34] V.B. Matveev , Uspekhi Math. Nauk **30**:6 (1975) 201-203

[35] B.A. Dubrovin , S.P. Novikov , V.B. Matveev , Usp. Math. Nauk **31** (1976) 55-136 (Reprinted in London Math. Soc. Lecture Notes Ser. **60** Cambridge Univ. Press , Cambridge (1981).

[36] E.D. Belokolos , A.I. Bobenko , A.R. Its , V.Z.Enolskij , V.B. Matveev : Algebro-Geometric Approach to Nonlinear Integrable Equations , Springer series in Nonlinear Dynamics , Springer Verlag (1994) 1-330

[37] I.M. Krichever , Funct. Anal. Appl. **12** (1978) 76-78

[38] I.M. Krichever , Dokl. Acad. Nauk. SSSR **227** (1976) 291-294

[39] M.Adler, J.Moser , Comm. Math. Phys **61** (1978) 1-30

[40] L.Bordag , A.Its , S.Manakov, V.Matveev , V.Zakharov , Phys. Lett. A **63** (1977) 205-208