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Abstract

The notion of block divisibility naturally leads one to introduce unitary cyclotomic polynomials. We formulate some basic properties of unitary cyclotomic polynomials and study how they are connected with cyclotomic, inclusion-exclusion and Kronecker polynomials. Further, we derive some related arithmetic function identities involving the unitary analog of the Dirichlet convolution.

Contents

1 Introduction

1.1 Unitary divisors

A divisor d of n ($d, n \in \mathbb{N}$) is called a *unitary divisor* (or block divisor) if $(d, n/d) = 1$, notation $d \,||\, n$ (note that this is in agreement with the standard notation $p^a \,||\, n$ used for prime powers p^a). If the prime power factorization of n is $n = p_1^{a_1} \cdots p_s^{a_s}$, then the set of its unitary divisors consists of the integers $d = p_1^{b_1} \cdots p_s^{b_s}$, where $b_i = 0$ or $b_i = a_i$ for any $1 \leqslant i \leqslant s$.

The study of *arithmetic functions* defined by unitary divisors goes back to Vaidyanathaswamy [29] and Cohen [4]. For example, the analogs of the sum-of-divisors function σ and Euler's totient function φ are $\sigma^*(n) = \sum_{d||n} d$, respectively $\varphi^*(n) = \#\{j : 1 \leqslant j \leqslant n, (j, n)_* = 1\}$, where

$$(j, n)_* = \max\{d : d \mid j,\, d \,||\, n\}.$$

Several properties of the unitary functions σ^* and φ^* run parallel to those of σ and φ, respectively. For example, both functions σ^* and φ^* are multiplicative, and $\sigma^*(p^a) = p^a + 1$, $\varphi^*(p^a) = p^a - 1$ for prime powers p^a ($a \geqslant 1$). The *unitary convolution* of the functions f and g is defined by

$$(f \times g)(n) = \sum_{d||n} f(d) g(n/d) \quad (n \in \mathbb{N}).$$

The set of arithmetic functions f such that $f(1) \neq 0$ forms a commutative group under the unitary convolution and the set of multiplicative functions is a subgroup. The identity is the function ϵ, given by $\epsilon(1) = 1$, $\epsilon(n) = 0$ ($n > 1$), similar to the case of Dirichlet convolution. The inverse of the constant 1 function under the unitary convolution is $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors of n. That is,

$$\sum_{d||n} \mu^*\left(\frac{n}{d}\right) = \sum_{d||n} \mu^*(d) = \epsilon(n) \quad (n \in \mathbb{N}). \tag{1}$$

See, e.g., the books by Apostol [1], McCarthy [14] and Sivaramakrishnan [21].

1.2 Unitary Ramanujan sums

The *unitary Ramanujan sums* $c_n^*(k)$ were defined by Cohen [4] as follows:

$$c_n^*(k) = \sum_{\substack{1 \leqslant j \leqslant n \\ (j,n)_* = 1}} \zeta_n^{jk} \quad (k, n \in \mathbb{N}),$$

where $\zeta_n := e^{2\pi i/n}$. (The classical Ramanujan sums are defined similarly, but with $(j, n)_* = 1$ replaced by $(j, n) = 1$.)

The identities

$$c_n^*(k) = \sum_{d||(k,n)_*} d\mu^*(n/d) \quad (n, k \in \mathbb{N}), \tag{2}$$

$$\sum_{d||n} c_d^*(k) = \varrho_n(k) = \begin{cases} n & \text{if } n \mid k; \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

can be compared to the corresponding ones concerning the classical Ramanujan sums $c_n(k)$. Note that $c_n^*(n) = \varphi^*(n)$, $c_n^*(1) = \mu^*(n)$ ($n \in \mathbb{N}$).

1.3 Unitary cyclotomic polynomials

The *cyclotomic polynomials* $\Phi_n(x)$ are defined by

$$\Phi_n(x) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} \left(x - \zeta_n^j\right). \tag{4}$$

They arise as irreducible factors (see Weintraub [30]) on factorizing $x^n - 1$ over the rationals:

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{5}$$

By Möbius inversion it follows from (5) that

$$\Phi_n(x) = \prod_{d|n} \left(x^{n/d} - 1\right)^{\mu(d)} = \prod_{d|n} \left(x^d - 1\right)^{\mu(n/d)}, \tag{6}$$

where μ denotes the Möbius function.

The *unitary cyclotomic polynomial* $\Phi_n^*(x)$ is defined by

$$\Phi_n^*(x) = \prod_{\substack{j=1 \\ (j,n)_*=1}}^{n} \left(x - \zeta_n^j\right), \tag{7}$$

see [21, Ch. X]. It is monic, has integer coefficients and is of degree $\varphi^*(n)$. Furthermore, for any natural number n we have

$$x^n - 1 = \prod_{d||n} \Phi_d^*(x) \tag{8}$$

and

$$\Phi_n^*(x) = \prod_{d||n} \left(x^{n/d} - 1\right)^{\mu^*(d)} = \prod_{d||n} \left(x^d - 1\right)^{\mu^*(n/d)}. \tag{9}$$

See Section 2 for short direct proofs of these properties and further basic properties of unitary cyclotomic polynomials.

If n is squarefree, then the unitary divisors of n coincide with the divisors of n and hence comparing (6) with (9) yields $\Phi_n^*(x) = \Phi_n(x)$. In this case $\Phi_n^*(x)$ is irreducible over the rationals. However, a quick check shows, that for certain non-squarefree values of n, the polynomial $\Phi_n^*(x)$ is reducible over the rationals. For example, $\Phi_{12}^*(x) = \Phi_6(x)\Phi_{12}(x)$ and $\Phi_{40}^*(x) = \Phi_{10}(x)\Phi_{20}(x)\Phi_{40}(x)$. Indeed, we will show that Φ_n^* is reducible for *every* non-squarefree integer n. This is a corollary of the fact that each polynomial $\Phi_n^*(x)$ can be written as the product of the cyclotomic polynomials $\Phi_d(x)$, where d runs over the divisors of n such that $\kappa(d) = \kappa(n)$, with $\kappa(n)$ the *squarefree kernel* of n (Theorem 2). In fact, this is a consequence of a more general result (Theorem 6) involving unitary divisors.

One can introduce the *bi-unitary cyclotomic polynomials* $\Phi_n^{**}(x)$ defined by

$$\Phi_n^{**}(x) = \prod_{\substack{j=1 \\ (j,n)_{**}=1}}^{n} \left(x - \zeta_n^j\right),$$

where $(j,n)_{**}$ stands for the greatest common unitary divisor of j and n. The degree of the polynomial $\Phi_n^{**}(x)$ equals $\varphi^{**}(n)$, the bi-unitary Euler function, which is defined as $\varphi^{**}(n) = \#\{j : 1 \leqslant j \leqslant n, (j,n)_{**} = 1\}$, see the paper [26]. Although these definitions seem to be more natural than the previous ones, the properties of $\Phi_n^{**}(x)$ and $\varphi^{**}(n)$ are not similar to their unitary analogs. For example, the function $\varphi^{**}(n)$ is not multiplicative and the coefficients of the polynomials $\Phi_n^{**}(x)$ are in general not integers (we have, e.g., $\Phi_6^{**}(x) = x^3 - \overline{\eta}x^2 + \overline{\eta}x + \eta$, where $\eta = (1+i\sqrt{3})/2$ and $\overline{\eta} = (1-i\sqrt{3})/2$).

1.4 Inclusion-exclusion and Kronecker polynomials

Let $\rho = \{r_1, r_2, \ldots, r_s\}$ be a set of increasing natural numbers satisfying $r_i > 1$ and $(r_i, r_j) = 1$ for $i \neq j$, and put

$$n_0 = \prod_i r_i, \;\; n_i = \frac{n_0}{r_i}, \;\; n_{ij} = \frac{n_0}{r_i r_j} \;\; [i \neq j], \ldots$$

For each such ρ we define a function Q_ρ by

$$Q_\rho(x) = \frac{(x^{n_0} - 1) \cdot \prod_{i<j}(x^{n_{ij}} - 1) \cdots}{\prod_i (x^{n_i} - 1) \cdot \prod_{i<j<k}(x^{n_{ijk}} - 1) \cdots}. \tag{10}$$

It can be shown that $Q_\rho(x)$ is a polynomial of degree $n_0 \prod_{r_i | n_0}(1 - 1/r_i)$ having integer coefficients. This class of polynomials was introduced by Bachman [2], who named them *inclusion-exclusion polynomials*.

A *Kronecker polynomial* $f \in \mathbb{Z}[x]$ is a monic polynomial having all its roots inside or on the unit circle. It was proved by Kronecker, cf. [6], that such a polynomial is a product of a monomial and cyclotomics and so we can write

$$f(x) = x^s \prod_d \Phi_d(x)^{e_d}, \tag{11}$$

with $s, e_d \geqslant 0$ and $e_d \geqslant 1$ for only finitely many d.

We will show how a unitary cyclotomic can be realized as an inclusion-exclusion cyclotomic. As $Q_\rho(x)$ is monic and in $\mathbb{Z}[x]$, it follows from (10) that it is Kronecker. Thus we have the following inclusions:

$$\{\text{unitary cyclotomics}\} \subset \{\text{inclusion-exclusion polynomials}\} \subset \{\text{Kronecker polynomials}\}. \tag{12}$$

The inclusion-exclusion polynomials that are unitary can be precisely identified (for the proof see Section 6.1).

Theorem 1. *The set of unitary polynomials $\Phi_n^*(x)$ with $n \geqslant 2$ equals the set of inclusion-exclusion polynomials $Q_\rho(x)$ with ρ having prime power entries, with no base prime repeated. More precisely there is a one-to-one map between these sets that sends n to $\rho = \{p_1^{e_1}, \ldots, p_s^{e_s}\}$, where $p_1^{e_1} \cdots p_s^{e_s}$ with $p_1^{e_1} < \ldots < p_s^{e_s}$ is the prime factorization of n, resulting in*

$$\Phi_n^*(x) = Q_{\{p_1^{e_1}, \ldots, p_s^{e_s}\}}(x).$$

This theorem shows that the first inclusion in (12) is strict, e.g., $Q_{\{5,6\}}(x)$ is not a unitary cyclotomic. By Theorem 2 (or Theorem 26) any Kronecker polynomial divisible by $\Phi_d(x)^2$ for some $d \geqslant 1$ cannot be an inclusion-exclusion polynomial, and so also the second inclusion is strict. Even more, it is easy to see that for both inclusions the set theoretic differences are infinite.

We would like to point out that in this paper, with the exception of Theorem 22, the nomination "theorem" is not used to indicate a deep result, but rather a key fact.

2 Elementary properties of unitary cyclotomic polynomials

The polynomials $\Phi_n^*(x)$ have integer coefficients. This follows by induction on n by taking into account identity (8), similar to the case of classical cyclotomic polynomials. Indeed, various of our arguments in this section closely mirror those for cyclotomic polynomials and can, in somewhat more detail than we provided, be found in Thangadurai [24].

By the definition (7) and (1),

$$\log \Phi_n^*(x) = \sum_{\substack{j=1 \\ (j,n)_*=1}}^{n} \log\left(x - \zeta_n^j\right) = \sum_{j=1}^{n} \log\left(x - \zeta_n^j\right) \sum_{d||(j,n)_*} \mu^*(d).$$

Note that $d \,||\, (j,n)_*$ holds if and only if $d \mid j$ and $d \,||\, n$. Hence

$$\log \Phi_n^*(x) = \sum_{d||n} \mu^*(d) \sum_{k=1}^{n/d} \log\left(x - \zeta_{n/d}^k\right) = \sum_{d||n} \mu^*(d) \log\left(x^{n/d} - 1\right),$$

giving (9), which by unitary Möbius inversion is equivalent to (8).

The unitary divisors of prime powers p^a $(a \geqslant 1)$ are 1 and p^a. We deduce by (8) that

$$\Phi_{p^a}^*(x) = \frac{x^{p^a} - 1}{x - 1} = \prod_{j=1}^{a} \Phi_{p^j}(x). \tag{13}$$

¿From formula (9) we immediately see that the Taylor series of $\Phi_n^*(x)$ around $x = 0$ has integer coefficients, showing again that the coefficients of $\Phi_n^*(x)$ have to be integers.

Using (1), we see that, for $n > 1$, we can rewrite (9) as

$$\Phi_n^*(x) = \prod_{d||n} \left(1 - x^d\right)^{\mu^*(n/d)}. \tag{14}$$

¿From (14) and (9) it follows that for $n > 1$

$$\Phi_n^*(x) = x^{\varphi^*(n)} \Phi_n^*(1/x), \tag{15}$$

in other words, unitary cyclotomics are *self-reciprocal*.

For odd $n > 1$, we have

$$\Phi_{2n}^*(x) = \Phi_n^*(-x). \tag{16}$$

In order to prove this we invoke (14) and group the even and odd unitary divisors together. This leads to

$$\begin{aligned}
\Phi^*_{2n}(x) &= \prod_{2d||2n}(1-x^{2d})^{\mu^*(n/d)}\prod_{d||n}(1-x^d)^{\mu^*(2n/d)};\\
&= \prod_{d||n}(1-x^{2d})^{\mu^*(n/d)}\prod_{d||n}(1-x^d)^{-\mu^*(n/d)};\\
&= \prod_{d||n}(1+x^d)^{\mu^*(n/d)} = \Phi^*_n(-x).
\end{aligned}$$

Let $k \geqslant 1$ be an integer and $p \nmid n$ a prime. The unitary divisors of $p^k n$ come in two flavors: those of the form $p^k d$ with $d \mid\mid n$, and those of the form $d \mid\mid n$. On grouping these together we obtain from (14) that

$$\Phi^*_{p^k n}(x) = \frac{\Phi^*_n(x^{p^k})}{\Phi^*_n(x)}. \tag{17}$$

Also,

$$\Phi^*_{p^k n}(x) = \prod_{j=0}^{k-1}\Phi^*_{pn}(x^{p^j}). \tag{18}$$

To see this we write each of the terms appearing in right hand side as a quotient of two unitaries given by (17). We so obtain a quotient of two unitaries, which equals the left hand side of (18) by (17) again.

Let $\Phi^*_n(x) = x^{\varphi^*(n)} + b_1 x^{\varphi^*(n)-1} + \cdots + b_{\varphi^*(n)}$. It follows, similar to the classical case, that $b_1 = -c^*_n(1) = -\mu^*(n)$ for every $n \in \mathbb{N}$.

3 Unitary cyclotomic polynomials as products of cyclotomic polynomials

Recall that $\kappa(n) = \prod_{p|n} p$ is the square-free kernel of n.

Theorem 2. *For any natural number n we have*

$$\Phi^*_n(x) = \prod_{\substack{d|n\\ \kappa(d)=\kappa(n)}}\Phi_d(x). \tag{19}$$

Proof. Combination of (9) with (5) yields

$$\Phi^*_n(x) = \prod_{d||n}\left(x^d-1\right)^{\mu^*(n/d)} = \prod_{d||n}\Big(\prod_{\delta|d}\Phi_\delta(x)\Big)^{\mu^*(n/d)}. \tag{20}$$

We thus find that $\Phi^*_n(x) = \prod_{\delta|n}\Phi_\delta(x)^{e_\delta}$, with

$$e_\delta = \sum_{k\delta||n}\mu^*\left(\frac{n}{k\delta}\right). \tag{21}$$

The exponents e_δ are integers that are to be determined. Given a divisor δ of n, we let d be the smallest multiple of δ that is a block divisor of n. Note that if $k\delta \mid\mid n$, then there is an integer m such that $k\delta = md$. The condition $k\delta \mid\mid n$ is in general not equivalent with $k \mid\mid n/\delta$, however the condition $md \mid\mid n$ is equivalent with $m \mid\mid n/d$. Using these observations and (1) we conclude that

$$e_\delta = \sum_{k\delta \mid\mid n} \mu^*\left(\frac{n}{k\delta}\right) = \sum_{md \mid\mid n} \mu^*\left(\frac{n}{md}\right) = \sum_{m \mid\mid n/d} \mu^*\left(\frac{n}{md}\right) = \epsilon\left(\frac{n}{d}\right). \tag{22}$$

It follows that $e_\delta = 0$, except when n is the smallest multiple of δ that is a block divisor of n (which occurs if and only if $\kappa(\delta) = \kappa(n)$), in which case $e_\delta = 1$. □

Remark 3. An alternative form of (19) is

$$\Phi_n^*(x) = \prod_{d \mid \frac{n}{\kappa(n)}} \Phi_{\kappa(n)}(x^d), \tag{23}$$

which is obtained on noting that

$$\Phi_n^*(x) = \prod_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \Phi_d(x) = \prod_{d\kappa(n) \mid n} \Phi_{d\kappa(n)}(x) = \prod_{d \mid \frac{n}{\kappa(n)}} \Phi_{\kappa(n)}(x^d),$$

where in the last step we used repeatedly that $\Phi_{pn}(x) = \Phi_n(x^p)$ if $p \mid n$.

Remark 4. Theorem 1 says that $\Phi_n^*(x)$ is an inclusion-exclusion polynomial associated to the prime power factorization of n. A formula of Bachman giving the factorization of an inclusion-exclusion polynomial in cyclotomic polynomials (Theorem 26), then leads to an alternative proof of Theorem 2 (Section 6.1).

Remark 5. The convolution defined by

$$(f \diamond g)(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} f(d)g(n/d) \qquad (n \in \mathbb{N})$$

was mentioned by Subbarao [23] and investigated by Thrimurthy [25]. It preserves the multiplicativity of functions, although it is noncommutative and nonassociative. However, as it is easy to check, for any arithmetic functions f, g and h,

$$(f \diamond g) \diamond h = f \diamond (g * h), \tag{24}$$

where $*$ is the Dirichlet convolution. See also the review MR0480305 (58 # 478) of [25].

Our next theorem generalizes Theorem 2. Indeed, Theorem 2 follows from (25) on making the choice $g(n) = \log \Phi_n(x)$ (hence $f(n) = \log(x^n - 1)$) and $g^*(n) = \log \Phi_n^*(x)$. In addition, with this choice (26) yields the identity

$$\Phi_n(x) = \prod_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \Phi_d^*(x)^{\mu(n/d)} \qquad (n \in \mathbb{N}),$$

expressing a cyclotomic in terms of unitary cyclotomics. Note that if $d \mid n$, then $\kappa(d) = \kappa(n)$ holds iff $\kappa(n) \mid d$ iff $\kappa(n/d) \mid d$.

Theorem 6. *Let $g, g^* : \mathbb{N} \to \mathbb{C}$ be arbitrary functions. Put $f(n) = \sum_{d|n} g(d)$. Assume that*

$$f(n) = \sum_{d||n} g^*(d) \qquad (n \in \mathbb{N}).$$

Then

$$g^*(n) = \sum_{\substack{d|n \\ \kappa(d)=\kappa(n)}} g(d) \qquad (n \in \mathbb{N}) \tag{25}$$

and

$$g(n) = \sum_{\substack{d|n \\ \kappa(d)=\kappa(n)}} g^*(d)\mu(n/d) \qquad (n \in \mathbb{N}). \tag{26}$$

Remark 7. In Theorem 6 the function g is multiplicative if and only if g^* is multiplicative.

Proof of Theorem 6. By Möbius inversion we have

$$g(n) = \sum_{d|n} f(d)\mu(n/d) \tag{27}$$

and

$$g^*(n) = \sum_{d||n} f(d)\mu^*(n/d). \tag{28}$$

These identities show that given g, the function g^* is uniquely determined and reversely. We have

$$g^*(n) = \sum_{d||n} \mu^*(n/d) \sum_{\delta|d} g(\delta) = \sum_{\delta|n} g(\delta) e_\delta,$$

where e_δ is given by (21). The proof of (25) is now easily completed on invoking (22), cf. the proof of Theorem 2.

Now we prove identity (26). Put $f := g$, $g := \mathbf{1}$ (constant 1 function), $h := \mu$ in identity (24). This gives

$$(g \diamond \mathbf{1}) \diamond \mu = g \diamond (\mathbf{1} * \mu).$$

Here, $g \diamond \mathbf{1} = g^*$ by (25). Also, $\mathbf{1} * \mu = \epsilon$, which is a basic property of the classical Möbius function. Since the function ϵ is the identity for the $\diamond$ operation, we conclude that

$$g^* \diamond \mu = g,$$

completing the proof. □

4 Further properties of unitary cyclotomic polynomials

4.1 Calculation of $\Phi_n^*(\pm 1)$

In this section we determine $\Phi_n^*(\pm 1)$. For completeness and comparison we mention the analogous classical results for $\Phi_n(1)$.

Let Λ^* denote the unitary analog of the von Mangoldt function Λ. It is given by

$$\Lambda^*(n) = \begin{cases} a \log p & \text{if } n = p^a \text{ is a prime power } (a \geqslant 1); \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

and satisfies $\sum_{d||n} \Lambda^*(d) = \log n$ (analogous to the classical identity $\sum_{d|n} \Lambda(d) = \log n$).

Lemma 8. *We have*

$$\Phi_n(1) = \begin{cases} 0 & \text{if } n = 1; \\ p & \text{if } n = p^e; \\ 1 & \text{otherwise,} \end{cases} \text{ and } \Phi_n^*(1) = \begin{cases} 0 & \text{if } n = 1; \\ p^e & \text{if } n = p^e; \\ 1 & \text{otherwise,} \end{cases}$$

with p a prime number and $e \geqslant 1$.

In terms of the (unitary) von Mangoldt function this can be reformulated as follows.

Lemma 9. *We have $\Phi_1(1) = 0$ and $\Phi_1^*(1) = 0$. For $n > 1$ we have*

$$\Phi_n(1) = e^{\Lambda(n)} \text{ and } \Phi_n^*(1) = e^{\Lambda^*(n)}.$$

Proof of Lemma 8. ¿From (5) and (8) we obtain (respectively)

$$\frac{x^n - 1}{x - 1} = \prod_{d|n,\ d>1} \Phi_d(x) \text{ and } \frac{x^n - 1}{x - 1} = \prod_{d||n,\ d>1} \Phi_d^*(x).$$

Thus (respectively)

$$n = \prod_{d|n,\ d>1} \Phi_d(1) \text{ and } n = \prod_{d||n,\ d>1} \Phi_d^*(1). \tag{30}$$

By Möbius inversion the latter identities for all $n > 1$ determine $\Phi_m(1)$ and $\Phi_m^*(1)$ uniquely for all $m > 1$. It is thus enough to verify that the formulae claimed for $\Phi_m(1)$ and $\Phi_m^*(1)$ verify (30), which is evident. □

Remark 10. It is possible to prove Lemma 9 with the (unitary) von Mangoldt function naturally appearing in the proof. To do so one proceeds as in the proof of Lemma 8 and deduces (30) and concludes that these equations uniquely determine $\Phi_m(1)$ and $\Phi_m^*(1)$. It remains then (after taking logarithms) to prove the well-known (trivial) identity $\log n = \sum_{d|n,\ d>1} \Lambda(d) = \sum_{d|n} \Lambda(d)$, and likewise in the unitary case, $\log n = \sum_{d||n,\ d>1} \Lambda^*(d) = \sum_{d||n} \Lambda^*(d)$.

It is not much more difficult to evaluate $\Phi_n^*(-1)$.

Lemma 11. *We have*

$$\Phi_n^*(-1) = \begin{cases} -2 & \text{if } n = 1; \\ 0 & \text{if } n = 2^a; \\ p^b & \text{if } n = 2^a p^b; \\ 1 & \text{otherwise,} \end{cases}$$

with p an odd prime and $a, b \geqslant 1$.

Proof. Follows from the identity (23), Lemma 8 and the well-known result

$$\Phi_n(-1) = \begin{cases} -2 & \text{if } n = 1; \\ 0 & \text{if } n = 2; \\ p & \text{if } n = 2p^e; \\ 1 & \text{otherwise,} \end{cases} \tag{31}$$

with $p \geqslant 2$ a prime number and $e \geqslant 1$.

Assume that $\Phi_n^*(-1) \neq 1$. By (23) it follows that $\Phi_n^*(-1) = \Phi_{\kappa(n)}(-1)^e \Phi_{\kappa(n)}(1)^f$, for some integers $e, f \geqslant 0$. The formulas for $\Phi_n(\pm 1)$ then show that $\kappa(n)|2p$, with p an odd prime, and so $n = 2^a p^b$, $a, b \geqslant 0$. In case $a, b \geqslant 1$, we have $\Phi_n^*(-1) = \Phi_{2p}(-1)^b \Phi_{2p}(1)^{ab-b} = \Phi_{2p}(-1)^b = p^b$, by, respectively, (23), Lemma 8 and (31). The remaining cases are left to the reader. $\square$

Using identity (23) one can likewise immediately evaluate $\Phi_n^*(1)$ from $\Phi_n(1)$.

4.1.1 Some products involving the cos and sin functions

It is known that for any $n \geqslant 2$,

$$\prod_{\substack{j=1 \\ (j,n)=1}}^{n} \sin \frac{\pi j}{n} = \frac{\Phi_n(1)}{2^{\varphi(n)}},$$

$$\prod_{\substack{j=1 \\ (j,n)=1}}^{n} \cos \frac{\pi j}{n} = \frac{\Phi_n(-1)}{(-4)^{\varphi(n)/2}}, \tag{32}$$

proved in [7] (for (32) in case n is odd only) and [28] (for any $n \geqslant 2$) by two different methods, see also [13]. Here we provide the unitary analogs of these products, which in combination with the results of the previous section allow one to explicitly evaluate them.

Theorem 12. *For any $n \geqslant 2$,*

$$\prod_{\substack{j=1 \\ (j,n)_*=1}}^{n} \sin \frac{\pi j}{n} = \frac{\Phi_n^*(1)}{2^{\varphi^*(n)}}, \tag{33}$$

$$\prod_{\substack{j=1 \\ (j,n)_*=1}}^{n} \cos \frac{\pi j}{n} = \frac{\Phi_n^*(-1)}{(-4)^{\varphi^*(n)/2}}. \tag{34}$$

Proof. We adapt the approach in [28] to the unitary case. We need the simple formula

$$S^*(n) := \sum_{\substack{j=1 \\ (j,n)_*=1}}^{n} j = \frac{n\varphi^*(n)}{2} \quad (n \geqslant 2), \tag{35}$$

which can be shown similarly to the usual case. We will only prove (34), the proof of (33) being similar. The product in the left hand side of (34) we denote by $P^*(n)$.

If $n = 2^a$, $a \geqslant 1$, then we note that $(2^{a-1}, 2^a)_* = 1$ and so the product in (34) is zero. By Lemma 11 it follows that also $\Phi_n^*(-1)$ is zero and thus in this case (34) holds. Therefore we may assume that n has an odd prime factor, which implies that $\varphi^*(n)$ is even. By (7) and (35) we then see that

$$\begin{aligned}
\Phi_n^*(-1) &= \prod_{(j,n)_*=1} \left(-1-\zeta_n^j\right) = \prod_{(j,n)_*=1} \left(-\zeta_n^{j/2}\right)\left(\zeta_n^{j/2}+\zeta_n^{-j/2}\right) \\
&= 2^{\varphi^*(n)} P^*(n) \prod_{(j,n)_*=1} \left(-\zeta_n^{j/2}\right) = (-2)^{\varphi^*(n)} \zeta_n^{S^*(n)/2} P^*(n) \\
&= (-2)^{\varphi^*(n)} \zeta_n^{n\varphi^*(n)/4} P^*(n) = (-2i)^{\varphi^*(n)} P^*(n), \\
&= (-4)^{\varphi^*(n)/2} P^*(n),
\end{aligned}$$

completing the proof of (34). □

Remark 13. A completely similar argument leads to a proof of (32). The argument in that case is even easier, as $\varphi(n)$ is even for $n \geqslant 3$ and it is not necessary to deal with the powers of two separately.

4.1.2 Calculation of Φ_n^* at other roots of unity

It is known how to explicitly evaluate $\Phi_n(\zeta_m)$ for $m \in \{3, 4, 5, 6, 8, 10, 12\}$, see [3]. This in combination with identity (23) then allows one to evaluate $\Phi_n^*(\zeta_m)$ for these values of m.

4.2 Unitary version of Schramm's identity

In this section x will be a real variable. It was proved by Schramm [19] that

$$\Phi_n(x) = \prod_{j=1}^{n} \left(x^{(j,n)} - 1\right)^{\cos(2\pi j/n)} \quad (x > 1,\ n \in \mathbb{N}).$$

We will prove the following unitary analog.

Theorem 14. *We have*

$$\Phi_n^*(x) = \prod_{j=1}^{n} \left(x^{(j,n)_*} - 1\right)^{\cos(2\pi j/n)} \quad (x > 1,\ n \in \mathbb{N}).$$

This is, in fact, a corollary of a more general identity concerning the discrete Fourier transform (DFT)

$$F_f^*(m,n) := \sum_{k=1}^{n} f((k,n)_*)\zeta_n^{km} \tag{36}$$

of functions involving the quantity $(k,n)_*$.

Theorem 15. *Let f be an arbitrary arithmetic function. For every $m,n \geqslant 1$,*

$$F_f^*(m,n) = \sum_{d|(m,n)_*} d\,(\mu^* \times f)(n/d). \tag{37}$$

Furthermore, we have

$$F_f^*(m,n) = \sum_{d||n} f(d) c^*_{n/d}(m). \tag{38}$$

Proof. We have by using that $d \,||\, (k,n)_*$ if and only if $d \mid k$ and $d \,||\, n$,

$$F_f^*(m,n) = \sum_{k=1}^{n} \zeta_n^{km} \sum_{d||(k,n)_*} (\mu^* \times f)(d)$$

$$= \sum_{d||n} (\mu^* \times f)(d) \sum_{j=1}^{n/d} \zeta_{n/d}^{jm} = \sum_{\substack{d||n \\ (n/d)|m}} (\mu^* \times f)(d) \frac{n}{d},$$

which proves (37).

By grouping the terms according to the values of $(k,n)_* = d$, we have

$$F_f^*(m,n) = \sum_{d||n} f(d) \sum_{\substack{r=1 \\ (r,n/d)_*=1}}^{r=n/d} \zeta_{n/d}^{rm} = \sum_{d|n} f(d) c^*_{n/d}(m),$$

which proves (38), ∎

Remark 16. By (2) and (3) the unitary Ramanujan sum satisfies $c^*_{\bullet}(m) = \varrho_{\bullet}(m) \times \mu^*$. Thus, the sum in (37) equals $(\varrho_{\bullet}(m) \times \mu^* \times f)(n) = (c^*_{\bullet}(m) \times f)(n)$, leading to another proof of (38).

Remark 17. Identity (37) shows that if f is a real valued function, then so is $F_f^*(m,n)$ and hence in this case the factor ζ_n^{km} in (36) can be replaced by $\cos(2\pi km/n)$. More exactly, if f is a real valued function, then

$$\sum_{k=1}^{n} f((k,n)_*) \cos(2\pi km/n) = \sum_{d|(m,n)_*} d\,(\mu^* \times f)(n/d), \tag{39}$$

$$\sum_{k=1}^{n} f((k,n)_*) \sin(2\pi km/n) = 0.$$

In the special case $f(n) = n$ ($n \in \mathbb{N}$) and $m = 1$ we obtain the following identities:

$$\sum_{k=1}^{n} (k,n)_* \cos(2\pi k/n) = n \sum_{d||n} \frac{\mu^*(d)}{d} = \varphi^*(n) \text{ and } \sum_{k=1}^{n} (k,n)_* \sin(2\pi k/n) = 0.$$

In the classical case where $(k,n)_*$ is replaced by (k,n) and $\varphi^*(n)$ is replaced by $\varphi(n)$, these were pointed out by Schramm [18, 19].

Proof of Theorem 14. By taking $f(n) = \log(x^n - 1)$ we have by (9) that

$$(\mu^* \times f)(n) = \sum_{d||n} \mu^*(d) \log(x^{n/d} - 1) = \log \Phi_n^*(x).$$

The assumption that $x > 1$ ensures that f is real. It then follows from (39) that

$$\prod_{j=1}^{n} \left(x^{(j,n)_*} - 1\right)^{\cos(2\pi jm/n)} = \prod_{d||(m,n)_*} \Phi_{n/d}^*(x)^d \quad (m, n \in \mathbb{N}).$$

The proof is completed on putting $m = 1$. □

5 The coefficients of unitary cyclotomic polynomials

We write

$$\Phi_n^*(x) = \sum_{j=0}^{\infty} a_n^*(j) x^j. \tag{40}$$

This notation looks perhaps strange to the reader, but implicitly defines the coefficients for every j, which serves our purposes. In [11] the following result is proven.

Theorem 18. *Let $m \geqslant 1$ be fixed. We have $\{a_{mn}^*(j) : n \geqslant 1,\ j \geqslant 0\} = \mathbb{Z}$.*

Given any polynomial f, its *height* $h(f)$ is defined as its maximum coefficient in absolute value.

Conjecture 19. *For any given natural number m there is a cyclotomic polynomial having height m.*

This conjecture was put forward by Kosyak et al. [12]. Here we propose the following conjecture.

Conjecture 20. *For any given natural number m there is a unitary cyclotomic polynomial having height m.*

These two conjectures are closely connected.

Proposition 21. *If Conjecture 19 is true, then so is Conjecture 20.*

Proof. Suppose that $h(\Phi_n) = m$. Then, by elementary properties of cyclotomic polynomials, $h(\Phi_{\kappa(n)}) = m$. Now note that $h(\Phi_{\kappa(n)}^*) = h(\Phi_{\kappa(n)})$ (since $\Phi_{\kappa(n)}^* = \Phi_{\kappa(n)}$). □

The best result available to date regarding these two conjectures is the following.

Theorem 22. *Almost all positive integers occur as the height of an (unitary) cyclotomic polynomial. Specifically, for any fixed $\epsilon > 0$, the number of positive integers $\leqslant x$ that do not occur as a height is $\ll_\epsilon x^{3/5+\epsilon}$. Under the Lindelöf Hypothesis this number is $\ll_\epsilon x^{1/2+\epsilon}$.*

Proof. The result is actually a corollary of [12, Theorem 4]. In that theorem only certain special cyclotomic polynomials of the form Φ_{pqr}, with $p < q < r$ primes, feature. As $\Phi_{pqr}^* = \Phi_{pqr}$, we are done. □

This theorem is deep, as it relies on deep results from analytic number theory on gaps between consecutive primes.

Let k be a squarefree integer. Consider the set

$$\mathcal{B}(k) := \{h(\Phi_n^*) : \kappa(n) = k\}. \tag{41}$$

Note that if we replace $h(\Phi_n^*)$ by $h(\Phi_n)$ this set will be $\{h(\Phi_k)\}$.

Lemma 23. *Let k be a squarefree integer. Suppose that k has at most two distinct prime factors. Then $\mathcal{B}(k) = \{1\}$.*

Proof. If k is a prime power, the conclusion follows from (13). If k has precisely two distinct prime factors, say p and q, we can write $\Phi_n^*(x) = Q_{\{p^e,q^f\}}(x)$ by Theorem 1. The polynomial $Q_{\{p^e,q^f\}}(x)$ can be interpreted as the semigroup polynomial associated to the numerical semigroup $\langle p^e, q^f\rangle$ and as such will have height 1 (see Jones et al. [11] or Moree [16]). A shorter, but less conceptual, proof is obtained on merely invoking Lemma 4 of [8]. $\square$

Remark 24. By the above proof and (16) we also have $h(\Phi_{2p^eq^f}^*) = 1$. However, it is not always true that $h(\Phi_{4p^eq^f}^*) = 1$ (for example $h(\Phi_{60}^*) = 2$).

Computer work by Bin Zhang suggests that $\mathcal{B}(k)$ will be large if k has at least three prime factors. This suggests that perhaps there is hope of proving a stronger result on heights assumed by unitary cyclotomic polynomials than is provided by Theorem 22.

Question 25. *Suppose that k has at least three odd prime factors. Is $\mathcal{B}(k)$ unbounded?*

If k has four or more prime factors, we would not be surprised if $\mathcal{B}(k)$ is unbounded. If it has precisely three factors, the situation is not so clear. For example,

$$\max\{h(\Phi_n^*) : n = 2^a3^b5^c,\ a > 0,\ b > 0,\ c > 0,\ 1 < n < 10^7\} = 15.$$

6 More on inclusion-exclusion polynomials

The following result gives the factorization of an inclusion-exclusion polynomial $Q_\rho(x)$ into cyclotomics.

Theorem 26 (Bachman, [2])**.** *Let $\rho = \{r_1, r_2, \ldots, r_s\}$ be a set of increasing natural numbers satisfying $r_i > 1$ and $(r_i, r_j) = 1$ for $i \neq j$. Put*

$$D_\rho = \{d : d | \prod_{i=1}^{s} r_i \text{ and } (d, r_i) > 1 \text{ for all } i\}.$$

Then $Q_\rho(x) = \prod_{d\in D_\rho} \Phi_d(x)$.

Our proof of Theorem 1 makes use of the following basic property of inclusion-exclusion polynomials.

Theorem 27. *Suppose that $Q_{\rho_1}(x) = Q_{\rho_2}(x)$, then $\rho_1 = \rho_2$.*

Its proof rests on the following easy lemma.

Lemma 28. *A Kronecker polynomial can be written as*

$$f(x) = x^s \prod_{d\in\mathcal{D}} (x^d - 1)^{e_d}, \tag{42}$$

where $\mathcal{D}$ is a unique finite set of integers and $s \geqslant 0$ and the $e_d \neq 0$ are unique integers.

First proof. ¿From (11) and (6) we deduce that $f(x)$ can be written as claimed. It remains to show the uniqueness. The integer $s \geqslant 0$ is merely the order of $f(x)$ in $x = 0$. We start by taking $\mathcal{D}$ to be the empty set. We now consider $f(x)/x^s$ around $x = 0$. Either $f(x) = 1$ and we are done, or for some $\epsilon_1 \in \{-1, 1\}$ we have $\epsilon_1 f(x) = -1 + ax^{d_1} + O(x^{d_1+1})$ with $a \neq 0$ an integer. In case a is odd we have $(x^{d_1} - 1)^a = -1 + ax^{d_1} + O(x^{d_1+1})$, in case a is even we have $(x^{d_1} - 1)^a = 1 - ax^{d_1} + O(x^{d_1+1})$. We add d_1 to $\mathcal{D}$ and put $e_{d_1} = a$. Put $g(x) := f(x)(x^{d_1} - 1)^{-a}$. Note that either $g(x) = 1$ or for an appropriate $\epsilon_2 \in \{-1, 1\}$ the identity $\epsilon_2 g(x) = -1 + bx^{d_2} + O(x^{d_2+1})$, with $d_2 > d_1$ and $b \neq 0$ holds. We now add d_2 to $\mathcal{D}$ and put $e_{d_2} = b$. We continue in this way until we arrive at the polynomial 1 (and this will happen as we know a priori that $\mathcal{D}$ is finite). □

Second proof. By (11) we can write $f(x) = x^s(x - 1)^{e_1} g(x)$, with $g(x) = \prod_{d\geqslant 2} \Phi_d(x)^{e_d}$. As $g(x)$ is the product of cyclotomic Φ_d with $d \geqslant 2$, it is selfreciprocal and satisfies $g(0) = 1$. Note that s and e_1 are the order of $f(x)$ in $x = 0$, respectively $x = 1$, and therefore unique. Thus, w.l.o.g., we may assume that $f(x)$ is selfreciprocal and satisfies $f(0) = 1$. Around $x = 0$ we either have $f(x) = 1$ or $f(x) = 1 - ax^{d_1} + O(x^{d_1+1})$ with $a \neq 0$ an integer and $d_1 \geqslant 1$. Note that $f(x) = (1 - x^{d_1})^a + O(x^{d_1+1})$. We start with setting $\mathcal{D}$ to be the empty set. We add d_1 to $\mathcal{D}$ and put $e_{d_1} = a$. Put $g(x) := f(x)(1 - x^{d_1})^{-a}$. Note that either $g(x) = 1$ or $g(x) = 1 - bx^{d_2} + O(x^{d_2+1})$ with $d_2 > d_1$ and $b \neq 0$. We now add d_2 to $\mathcal{D}$ and put $e_{d_2} = b$. We continue in this way until we arrive at the polynomial 1 (and this will happen as we know a priori that $\mathcal{D}$ is finite). We obtain

$$f(x) = \prod_{d\in\mathcal{D}} (1 - x^d)^{e_d} = \prod_{d\in\mathcal{D}} (x^d - 1)^{e_d},$$

with $\mathcal{D}$ and the e_d unique. □

Remark 29. By a similar reasoning one can show that if $f(x) \in \mathbb{Z}[\![x]\!]$ satisfies $f(x) \equiv 1 \pmod{x}$, there exist unique integers $e_1, e_2, \ldots$ such that in $\mathbb{Z}[\![x]\!]$

$$f(x) = \prod_{n=1}^{\infty} (1 - x^n)^{e_n}.$$

This is the so-called *Witt expansion* which arises in many areas of mathematics, see for example Moree [15] for more information.

Proof of Theorem 27. Given an inclusion-exclusion polynomial $f(x)$ in the standard form $f(x) = \sum_i a_i x^i$, we give a procedure leading to a unique $\rho = \{r_1, \ldots, r_m\}$ such that $f(x) = Q_\rho(x)$. As $f(x)$ is a Kronecker polynomial satisfying $f(0) = 1$, by Lemma 28 we can write

$$f(x) = \prod_{d\in\mathcal{D}} (x^d - 1)^{e_d},$$

where $\mathcal{D}$ and the $e_d \neq 0$ are unique and computable. Comparison with (10) then shows that $\mathcal{D} = \{n_0, n_i, n_{ij}, n_{ijk}, \ldots\}$. The question is know which ρ correspond to the set $\mathcal{D}$. We know for example, that whatever ρ is, the product of its r_i is certainly unique and equals $n_0 = \max \mathcal{D}$. The numbers r_i themselves, also turn out to be unique. We order the elements in $\mathcal{D}$ in such a way that $d_1 < d_2 < \ldots$. We put $r_1 = d_1$. If d_2 is coprime with r_1 we put $r_2 = d_2$, if not we consider d_3. If an d_i is coprime to every $r_1, \ldots, r_g$ we have at a certain point, we put $r_{g+1} = d_i$. If r_m is the last r number so found, we have $f(x) = Q_\rho(x)$ with $\rho = \{r_1, \ldots, r_m\}$ uniquely determined. □

6.1 Unitary cyclotomics as inclusion-exclusion polynomials

Armed with Theorem 27 we are now ready to prove Theorem 1.

Proof of Theorem 1. Let $n \geqslant 2$ be an integer and $\prod_{i=1}^{s} p_i^{e_i}$ its canonical factorization with $p_1^{e_1} < p_2^{e_2} < \ldots < p_s^{e_s}$. On comparing formula (9) with (10) it follows that $\Phi_n^*(x) = Q_{\{p_1^{e_1}, \ldots, p_s^{e_s}\}}(x)$. Reversely, given any ascending sequence of prime powers $p_1^{e_1} < \ldots < p_s^{e_s}$ with distinct base primes $p_1, \ldots, p_s$, the polynomial $Q_{\{p_1^{e_1}, \ldots, p_s^{e_s}\}}(x)$ is seen to correspond to $\Phi_n^*(x)$ with $n = \prod_{i=1}^{s} p_i^{e_i}$. The one-to-one part of the claim is a consequence of Theorem 27. □

Theorem 1 together with Theorem 26 can be used to reprove Theorem 2:

Alternative proof of Theorem 2. For $n = 1$ the result is obviously true. Now let $n \geqslant 2$ be an integer and $\prod_{i=1}^{s} p_i^{e_i}$ its canonical factorization with $p_1^{e_1} < p_2^{e_2} < \ldots < p_s^{e_s}$. Put $\rho = \{p_1^{e_1}, \ldots, p_s^{e_s}\}$. Then by Theorem 1, respectively Theorem 26,

$$\Phi_n^*(x) = Q_\rho(x) = \prod_{d \in D_\rho} \Phi_d(x),$$

with $D_\rho = \{d : d \mid n,\ p_1 \cdots p_s \mid d\} = \{d : d \mid n,\ k(d) = k(n)\}$. □

7 Applications of Theorem 6

By selecting $g(n) = c_n(k)$, $g^*(n) = c_n^*(k)$ (Ramanujan, resp. unitary Ramanujan sums), where $f(n) = \varrho_n(k)$, we deduce from Theorem 6 the following identities.

Corollary 30. *For any $n, k \in \mathbb{N}$ we have*

$$c_n^*(k) = \sum_{\substack{d \mid n \\ \kappa(d) = \kappa(n)}} c_d(k) \tag{43}$$

and

$$c_n(k) = \sum_{\substack{d \mid n \\ \kappa(d) = \kappa(n)}} c_d^*(k) \mu(n/d).$$

Let $\mathrm{id}_s(n) = n^s$ ($s \in \mathbb{R}$). In the case $g(n) = J_s(n) := (\mu * \mathrm{id}_s)(n)$ (Jordan function of order s), $g^*(n) = J_s^*(n) := (\mu^* \times \mathrm{id}_s)(n)$, the unitary Jordan function of order s, where $f(n) = n^s$, we deduce the following corollary of Theorem 6.

Corollary 31. *For every $n \in \mathbb{N}$, $s \in \mathbb{R}$ we have*

$$J_s^*(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} J_s(d) \tag{44}$$

and

$$J_s(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} J_s^*(d)\mu(n/d).$$

Identity (43) was deduced by McCarthy [14, Ch. 4], while (44) was obtained by Cohen [5, Lemma 3.1] using different reasonings. If $s = 1$, then $J_1(n) = \varphi(n)$, $J_1^*(n) = \varphi^*(n)$ and we deduce the following corollary.

Corollary 32. *We have*

$$\varphi^*(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \varphi(d) \qquad (n \in \mathbb{N}).$$

This result also follows by setting $k = n$ in Corollary 30, or by comparing the degrees of the polynomials in (19).

We remark that Cohen [5, Lemma 4.1] also showed that the identity

$$\kappa^s(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} d^s \mu^2(d) \qquad (n \in \mathbb{N})$$

holds for any s.

Putting $k = 1$ in Corollary 30 and noting that $\kappa(n)$ is the only squarefree divisor d of n satisfying $\kappa(d) = \kappa(n)$, we obtain the following corollary.

Corollary 33. *We have*

$$\mu^*(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \mu(d) = \mu(\kappa(n)) \qquad (n \in \mathbb{N}).$$

On taking $g(n) = \Lambda(n)$, $g^*(n) = \Lambda^*(n)$ and by the well-known identity $\sum_{d \mid n} \Lambda(d) = \log n$, we see that $f(n) = \log n$ and obtain the final corollary.

Corollary 34. *We have*

$$\Lambda^*(n) = \sum_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \Lambda(d) \qquad (n \in \mathbb{N}).$$

From it we deduce the truth of (29).

8 Connection with unitary Ramanujan sums

Certain formulas concerning the unitary cyclotomic polynomials and unitary Ramanujan sums can be easily deduced from their classical analogues, by using the above identities. For example, we have

Corollary 35. *For any $n > 1$ and $x \in \mathbb{C}$, $|x| < 1$,*

$$\Phi_n^*(x) = \exp\left(-\sum_{k=1}^{\infty} \frac{c_n^*(k)}{k} x^k\right). \tag{45}$$

Proof. It is known that for any $n > 1$ and $|x| < 1$,

$$\Phi_n(x) = \exp\left(-\sum_{k=1}^{\infty} \frac{c_n(k)}{k} x^k\right), \tag{46}$$

see Nicol [17, Cor. 3.2], Tóth [27, Th. 1] or Herrera-Poyatos and Moree [9]. By Theorem 2 and the identity (43) we then obtain

$$\Phi_n^*(x) = \prod_{\substack{d\mid n\\ \kappa(d)=\kappa(n)}} \Phi_d(x) = \exp\left(-\sum_{k=1}^{\infty} \frac{x^k}{k} \sum_{\substack{d\mid n\\ \kappa(d)=\kappa(n)}} c_d(k)\right) = \exp\left(-\sum_{k=1}^{\infty} \frac{c_n^*(k)}{k} x^k\right),$$

completing the proof. □

For any $n > 1$ the series $\sum_{k=1}^{\infty} c_n(k)/k$ converges, see, e.g., Hölder [10]. Therefore, Lemma 9 in combination with (46) gives

$$\log \Phi_n(1) = \Lambda(n) = -\sum_{k=1}^{\infty} \frac{c_n(k)}{k} \quad (n > 1). \tag{47}$$

The second identity was first discovered by Ramanujan and expresses an arithmetic function as an infinite series involving Ramanujan sums (for a different proof see Sivaramakrishnan [21, Theorem 87]). Such expressions are now called *Ramanujan expansions*, see, e.g., Schwarz and Spilker [20, Chapter VIII].

The convergence of $\sum_{k=1}^{\infty} c_n(k)/k$ for $n > 1$ in combination with (43) shows that also $\sum_{k=1}^{\infty} c_n^*(k)/k$ converges for any $n > 1$. Thus, by Lemma 9 again,

$$\log \Phi_n^*(1) = \Lambda^*(n) = -\sum_{k=1}^{\infty} \frac{c_n^*(k)}{k} \quad (n > 1). \tag{48}$$

The second identity in (48) was obtained by Subbarao [22], without referring to unitary cyclotomic polynomials and with an incomplete proof, namely without showing that the corresponding series converges.

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